EXITING FROM QE

by

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Abstract

We develop a regime-switching SVAR (structural vector autoregression) in which the monetary policy regime, chosen by the central bank responding to economic conditions, is endogenous and observable. QE (quantitative easing) is one such regime. The model incorporates the exit condition for terminating QE. We apply it to Japan, a country that has experienced three QE spells. Our impulse response analysis shows that an increase in reserves raises output and inflation and that exiting from QE can be expansionary.

Keywords: quantitative easing, structural VAR, observable regimes, Taylor rule, impulse responses, Bank of Japan.

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1 Introduction and Summary

Quantitative easing, or QE, is an unconventional monetary policy that combines zero policy rates and positive excess reserves held by depository institutions at the central bank. This paper uses an SVAR (structural vector autoregression) to study the macroeconomic effects of QE. Reliably estimating such a time-series model is difficult because only several years have passed since the adoptation of QE by central banks around the world. We are thus led to examine Japan, a country that has accumulated, by our count, 130 months of QE as of December 2012. Those 130 QE months come in three installments, which allows us to evaluate the effect of exiting from QE.

We will start out by documenting for Japan that reserves are greater than required reserves (and often several times greater) when the policy rate is below 0.05% (5 basis points) per year. We say that the zero-rate regime is in place if and only if the policy rate is below this critical rate. Therefore, the regime is observable and, since reserves are substantially higher than the required level, the zero-rate regime and QE are synonymous. There are three spells of the zero-rate/QE regime: March 1999 - July 2000 (call it QE1), March 2001 - June 2006 (QE2), and December 2008 to date (QE3). They account for the 130 months. For most of those months the BOJ (Bank of Japan) made a stated commitment of not exiting from the zero-rate regime unless inflation is above a certain threshold.

Our SVAR, in its simplest form, has two monetary policy regimes: the zero-rate regime in which the policy rate is very close to zero, and the normal regime of positive policy rates. It is a natural extension of the standard recursive SVAR[1] to accommodate both the zero lower bound on the policy rate and the exit condition. There are four variables: inflation, output (measured by the output gap), the policy rate, and excess reserves, in that order. The first two equations of the system are reduced-form equations describing inflation and output dynamics. The third is the Taylor rule providing a shadow policy rate. Due to the zero lower bound, the actual policy rate

[1] See Christiano, Eichenbaum, and Evans (1999) for the recursive SVAR. Their SVAR orders variables by placing non-financial variables (such as inflation and output) first, followed by monetary policy instruments (such as the policy rate and measures of money), and financial variables (such as stock prices and long-term interest rates).

cannot be set equal to the shadow rate if the latter is negative. The fourth equation specifies the central bank's supply of excess reserves under QE. The exit condition requires that the central bank ends the zero-rate regime only if the shadow rate is positive *and* the inflation rate is above a certain threshold. The regime is endogenous because its occurrence depends on inflation and output through the zero lower bound and the exit condition.

We describe the effects of various monetary policy changes by IRs (impulse responses). The IRs we employ are a generalization, to non-linear systems such as ours, of the standard IRs for linear systems. To calculate the effect of a change in the policy rate, the reserve supply, or the regime that occurs in the base period t, we compare the projected path of inflation and output given the baseline history up to t with the path given the alternative history that differs from the baseline history only with respect to the policy variable in question in t. We find:

- QE is expansionary. That is, when the current regime is the zero-rate/QE regime, the IR of output and inflation to an increase in excess reserves is positive. This is consistent with the finding in the literature on the macro effects of QE to be reviewed in the next section. The significance of our finding is that we allow the regime to vary endogenously in the future.

- The profile of the IR of inflation and output to policy rate cuts is sensitive to the specification of the Taylor rule. If we do not allow for the intercept in the Taylor rule to vary over time, the price puzzle emerges, namely, the response of inflation to a rate cut is *negative* for many periods over the horizon. Furthermore, the output response is negligible. These puzzling results disappear if the intercept is allowed to depend on a measure of the equilibrium real interest rate.

- Surprisingly, exiting from QE can be expansionary. We set t = July 2006, the month the zero-rate/QE regime was terminated, and consider an alternative and counter-factual history of not exiting from QE in t. Since the two histories differ in t not just in the regime but also in the policy rate and excess reserves, we combine IRs to these three differences. We find that output and inflation are *lower* under the counter-factual alternative of extending QE to July 2006. Our analysis of regime changes is free from the Lucas critique because the reduced form is allowed to shift when the regime changes.

All these findings delivered by the simple SVAR model hold up when we extend it to encompass two features about excess reserves found specifically in Japanese data. First, not all QEs are alike. In the “weak” QE, as observed in QE1 (March 1999 - July 2000), excess reserves behave differently than in the “strong” QE, as in QE2 and QE3, when they are large and responsive to inflation and output. Second, there are a few incidents of positive excess reserves under positive interest rates. The IR profiles are similar when these two features are incorporated.

One possible objection to our finding that exiting from the zero-rate regime can be expansionary is a possibility of spurious causality. Perhaps the macro dynamics involves a hidden autonomous regime. Changes in the monetary policy regime appear to have effects only because they act as a signal of the hidden regime. To address this concern, we set up a simple model of output and the policy rate in which the output process, following the hidden-state Markov-switching model, is exogenous. We find that the IR of output to an exit from the zero-rate regime is almost non-existent. Therefore, our finding of the expansionary effect of exiting from QE does not seem spurious.

Of the eight sections forming the rest of the paper, Sections 4-6 contain our analysis of the baseline model. Section 2 reviews the related literature. Section 3 is the case for the monetary policy regime observability. Section 4 describes our four-variable SVAR. Section 5 reports our parameter estimates. Section 6 defines IRs for our regime-switching SVAR, displays estimated IR profiles, and then combines those IRs to calculate the effect of exiting from QE. Section 7 examines robustness to several variations of the model. It also extends the model to incorporate two QE types and positive excess reserves under positive policy rates. Section 8 examines the issue of spurious causality. Section 9 is a brief conclusion.

2 Relation to the Literature

The literature on the macroeconomic effects of QE is growing rapidly.[2] Remarkably, all the studies we came across with report that QE raises inflation and output. In one strand of the

[2] There is already a large literature on the effect of the zero lower bound on the yield curve. For a recent example, see Hamilton and Wu (2012) and Christensen and Rudebusch (2013).

literature, the measure of QE is price-based. Kapetanios *et. al.* (2012) and Baumeister and Benati (2013) include the yield spread in their VARs (vector autoregressions). The QE measure in Wu and Xia (2014) is the shadow policy rate properly defined.

More relevant to our paper are those studies that use quantities as the QE measure. The earliest and also the cleanest is Honda *et. al.* (2007) for Japan. Their QE measure is reserves, which was the target used by the BOJ (Bank of Japan) during the zero-rate period of 2001 through 2006. Their recursive VAR of prices, output, and reserves, estimated on monthly data for the zero-rate period, shows that the IR of prices and output to an increase in reserves is positive. A more elaborate SVAR with the same QE measure, estimated by Schenkelberg and Watzka (2013) on Japanese monthly data for the period of 1995-2010 (when the policy rate was below 1%), yields the same conclusion. The QE measure in Gambacorta *et. al.* (2014) is the level of central bank assets. Their VAR is recursive except that they allow the central bank assets and the financial variable (VIX in their case) to interact contemporaneously in the same month. The sample period is January 2008-June 2011. They overcome the shortness of the sample by utilizing data from eight advanced economies including Japan.

Another way to deal with the small sample problem is to include the normal period of positive policy rates but allow the model parameters to vary over time in some specific ways. Kimura and Nakajima (2013) use quarterly Japanese data from 1981 and assume two QE spells (2001:Q1 - 2006:Q1 and 2010:Q1 on). Their TVAR (time-varying parameter VAR) takes the zero lower bound into account by forcing the variance of the coefficient in the policy rate equation to shrink during QEs. Fujiwara (2006) and Inoue and Okimoto (2008) apply the hidden-state Markov-switching SVAR to Japanese monthly data. They find that the probability of the second state was very high in most of the months since the late 1990s. For those months, the IR of output to an increase in the base money is positive and persistent.

Because the regime is chosen by the central bank to honor the zero lower bound, or more generally, to respond to inflation and output, it seems clear that the regime must be treated as endogenous. And, as will be argued in the next section, a strong case can be made for the observability of the monetary policy regime. None of the papers with quantitative QE measures cited so far treat the regime as observable and endogenous. Furthermore, their IR analysis does not allow the regime to change in the future.

The regime in Iwata and Wu (2006) and Iwata (2010), in contrast, is observable and endogenous. It is necessarily endogenous because the policy rate in their VAR, being subject to the zero lower bound, is a censored variable. Our paper differs from theirs in several important respects. First, our SVAR incorporates the exit condition as well as the zero lower bound. Second and crucially, we consider IRs to regime changes. This allows us to examine the macroeconomic effect of exiting from QE. As already mentioned in the introduction, our paper has a surprising result on this issue. Third, their IR exhibits the price puzzle (see Figure 3 of Iwata (2010)). We show in our paper that, at least for the output gap measure we consider, the price puzzle is to a large extent resolved if we allow the equilibrium real interest rate to vary over time.[3]

3 Identifying the Zero-Rate Regime

Identification by the "L"

We identify the monetary policy regime on the basis of the relation between the policy rate and excess reserves. Figure 1a plots the policy rate against m, the *excess reserve rate* defined as the log of the ratio of the actual to required levels of reserves.[4] Because the BOJ (Bank of Japan) recently started paying interest on reserves, the vertical axis in the figure is not the policy rate r itself but the *net* policy rate $r - \bar{r}$ where $\bar{r}$ is the rate paid on reserves (0.1% since November 2008). It is the cost of holding reserves for commercial banks.

The figure shows a distinct L shape. Excess reserves are positive for all months for which

[3] Braun and Shioji (2006) show that the price puzzle is pervasive for both the U.S. and Japan in the recursive SVAR model. For Japan, they use monthly data from 1981 to 1996 and find that a large and persistent price puzzle arises for a variety of choices for the financial variables including commodity prices, the Yen-Dollar exchange rate, oil prices, the wholesale price index, and the 10-year yield on government bonds. They also find that the puzzle arises when each of those financial variables are placed third after inflation and output.

[4] The policy rate in Japan is the overnight uncollateralized interbank rate called the "Call rate". The level of reserves and the policy rate are the averages of daily values over the reserve maintenance period to be consistent with the required reserve system in place. See the data appendix for more details.

the net policy rate $r - \bar{r}$ is below some very low critical rate, and zero for most, but not all, months for which the net rate is above the critical rate.[5] Those months with $m > 0$ and with very low net policy rates will form the zero-rate period. To examine those months with $m > 0$ but with positive net policy rates, we magnify the plot near the origin in Figure 1b. The dotted horizontal red line is the critical rate of $r - \bar{r} = 0.05\%$ (5 basis points). The dots off the vertical axis (for which $m > 0$) and over the red dotted line can be divided into two groups. The first is composed of the filled squares above the dotted red line. They come from the period July 2006 - November 2008, between spells of very low net policy rates. The observation in this group with the largest value of m is $(m_t, r_t - \bar{r}_t) = (0.21, 0.49\%)$ for $t =$ September 2008, the time of the Lehman crisis. The second group above the red dotted line is indicated by filled circles. Their value of m is much lower than for the first group. They come from the late 1990s and the early 2000s when the Japanese financial system was under stress. The largest m is $(m_t, r_t - \bar{r}_t) = (0.089, 0.22\%)$ for $t =$ October 1998 when the Long-Term Credit Bank went bankrupt.

Because the supply curve of reserves should be horizontal when the policy rate is positive, the second group represents the *demand* for excess reserves when the shock to reserve demand is large for precautionary reasons. Regarding the first group (the filled squares), it appears that, until the Lehman crisis, precautionary demand was not the reason for commercial banks to hold excess reserves. Industry sources indicate that, after several years of near-zero interbank rate with large excess reserves, the response by smaller-scale banks when the policy rate turned positive from essentially zero was to delay re-entry to the interbank market.[6] As more banks returned to the

[5] The two months of significantly positive excess reserves when the policy rate is about 8% are February and March of 1991, when the Gulf war was about to end.

[6] A breakdown of excess reserves by type of financial institutions since 2005, available from the BOJ's homepage, shows that large banks quickly reduced their excess reserves after the termination in July 2006 of the zero-rate policy while other banks (regional banks, foreign banks, and trust banks) were slow to adjust. The average of excess reserves for July 2006 - August 2008 is only 0.1% of the average for January 2005 - June 2006 for large banks and 5.4% for other banks. In order to exploit the arbitrage opportunity presented by the positive interbank rates, banks need to train their employees afresh. The reason commonly cited for the slow adjustment (see, e.g., Kato (2010)) is that medium- to small-scale banks, after several years of near-zero overnight rates, didn't find it profitable to immediately return to the interbank market by incurring this re-entry cost.

interbank market, however, aggregate excess reserves steadily declined. This declining trend continued until Lehman, when smaller banks as well as large ones sharply increased reserves. In the empirical analysis below, we set m to zero for those months leading up to Lehman (or, equivalently, we constrain the lagged m coefficient in the reduced form to zero). On the other hand, we view the positive excess reserves from September 2008 until the arrival of the next zero-rate period as representing demand and leave the excess reserve value as is.

We say that the *zero-rate regime* is in place if and only if the net policy rate $r - \bar{r}$ is below the critical rate of 0.05% (5 annual basis points). Since there are no incidents of near-zero excess reserves when the net rate is below the critical rate (the minimum is 0.041, see Table 3 below), the zero-rate regime is synonymous with QE (quantitative easing). For this reason we will use the term "the zero-rate regime" and "QE" interchangeably. Under our definition, there are three periods of the zero-rate/QE regime in Japan:

QE1: March 1999 - July 2000,

QE2: March 2001 - June 2006,

QE3: December 2008 to date.

Figure 2a has the time-series bar chart of the excess reserve rate m. The three QE spells are indicated by the shades. As just explained, the thin bars between QE2 and the Lehman crisis of September 2008 will be removed in the empirical analysis below.[7] QE1 looks different from QE2 and QE3. The value of m during QE2, much higher than during QE1, was supply-determined because the level of reserves (i.e., the current account balance) during the spell was the BOJ's target. It seems clear that the same was true for QE3 because, although no longer an explicit target, the current account balance was the frequent subject during the BOJ's policy board meetings. QE2 and QE3 will be referred to as the period of "strong" QE. QE1 is the period of "weak" QE because the value of m, although positive, is much lower than under "strong" QE. For the most part, we will treat QE1 as a historical aberration. That is, the SVAR of the next section

[7] The value of m for December 1999 was very high, about 0.9, due to the Y2K problem. This Y2K spike has been replaced by the sample mean of m over QE1 in the bar chart.

and the IR (impulse response) analysis of Section 6 will assume that the only type of QE under the zero-rate regime is the “strong” type. A full analysis of both types of QE is postponed until Section 7.[8]

Consistency with BOJ Announcements

Our dating of the zero-rate regime, which is based solely on the net policy rate, agrees with announced monetary policy changes. To substantiate this claim, we collected relevant announcements of the decisions made by the BOJ’s Monetary Policy Meetings (Japanese equivalent of the U.S. FOMC, held every month and sometimes more often) in Table 1. For example, the end of our QE1 is followed by the 11 August 2000 BOJ announcement declaring the end of a zero-rate policy, and the 14 July 2006 BOJ announcement follows our QE2’s end. The 19 March 2001 announcement marks the start of our QE2. The only discrepancy between our QE darting and the BOJ announcements is the start of QE1. The 12 February 1999 BOJ announcement, which is to guide the policy rate as low as possible, is more than one month before the start of our QE1 (whose first month is the March 1999 reserve maintenance period). It took a while for the BOJ to lower the policy rate averaged over a reserve maintenance period below 0.05%.

The Exit Condition

Several authors have noted that the BOJ’s zero-interest rate policy is a combination of a zero policy rate and a stated commitment to a condition about inflation for exiting from the zero-rate regime.[9] Indeed, the BOJ statements collected in Table 1 indicate that during our three zero-rate/QE spells, the BOJ repeatedly expressed its commitment to an exit condition stated in

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[8] We do this for four reasons. First, the exposition of the SVAR and the definition of the IRs are much more transparent if there are only two regimes, one of which is the normal regime. Second, the model with just one QE type may be adequate for economies other than Japan, notably the U.S. Third, the market’s expectations embedded in the reduced form may well be that the “weak” QE would never be repeated. Fourth, as will be shown in Section 7, the results will not change greatly if the model is extended to two QE types.

[9] See, e.g., Okina and Shiratsuka (2004) and Ueda (2012).

terms of the year-on-year (i.e., 12-month) CPI (Consumer Price Index) inflation rate. For example, during QE1's very first reserve maintenance period (March 16, 1999 - April 15, 1999), the BOJ governor pledged to continue the zero rate "until the deflationary concern is dispelled" (see the 13 April 1999 announcement in the table). To be sure, the BOJ during the first twelve months of QE3 did not publicly mention the exit condition, until December 18, 2009. However, as Ueda (2012), a former BOJ board member, writes about this period: "At that time some observers thought that the BOJ was trying to target the lower end of the understanding of price stability, which was 0-2%." (Ueda (2012, p. 6)) We will assume that the exit condition was in place during this episode as well.

The last several months of QE2 (ending in June 2006) require some discussion. Table 2 has data for those and surrounding months. The 9 March 2006 announcement declared that the exit condition was now satisfied. However, the actual exit from the zero-rate regime did not take place until July 2006. To interpret this episode, we note that the year-on-year CPI inflation rate (excluding fresh food) for March 2006 was significantly above 0%, about 0.5%, if the CPI base year is 2000, but merely 0.1% (as shown in the table) if the base year is 2005. The 2005 CPI series was made public in August 2006. We assume that the BOJ postponed the exit until July because it became aware that inflation with the 2005 CPI series would be substantially below inflation with the 2000 CPI series.

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4 The Regime-Switching SVAR

This section presents our four-variable SVAR (structural vector autoregression). Strictly for expositional clarity, the model here makes two simplifying assumptions about the excess reserve rate m (the log of the actual-to-required reserve ratio). First, it is zero under the normal regime of positive policy rates. Second, the zero-rate regime is equated with "strong" QE. That is, there is only one type of QE and m under QE is supply-determined by the central bank. Those assumptions will be lifted in Section 7.

The Standard Three-Variable SVAR

As a point of departure, consider the standard three-variable SVAR in the review paper by Stock and Watson (2001). The three variables are the monthly inflation rate from month $t-1$ to t (p_t), the output gap (x_t), and the policy rate (r_t).[10] The inflation and output gap equations are reduced-form equations where the regressors are (the constant and) lagged values of all three variables. The third equation is the Taylor rule that relates the policy rate to the contemporaneous values of the year-on-year inflation rate and the output gap. The error term in this policy rate equation is assumed to be uncorrelated with the errors in the reduced-form equations. This error covariance structure, standard in the structural VAR literature (see Christiano, Eichenbaum, and Evans (1999)), is a plausible restriction to make, given that our measure of the policy rate for the month is the average over the reserve maintenance period from the 16th of the month to the 15th of the next month.

As is standard in the literature (see, e.g., Clarida *etl. al.* (1998)), we consider the Taylor rule with interest rate smoothing. That is,

$$\text{(Taylor rule)} \qquad r_t = \rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}, \;\; r_t^* \equiv \alpha_r^* + \underset{(1\times 2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2). \qquad (4.1)$$

Here, π_t, defined as $\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11})$, is the year-on-year inflation rate over the past 12 months. If the adjustment speed parameter ρ_r equals unity, this equation reduces to $r_t = r_t^* + v_{rt}$. We will call r_t^* the *desired Taylor rate*.

Introducing Regimes

The three-variable SVAR just described does not take into account the zero lower bound on the policy rate. Given the interest rate $\bar{r}_t$ (≥ 0) paid on reserves, the lower bound is not zero but $\bar{r}_t$. The Taylor rule with the lower bound, which we call the *censored Taylor rule*, is

$$\text{(censored Taylor rule)} \quad r_t = \max\Big[\underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \bar{r}_t \Big], \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2). \qquad (4.2)$$

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[10] In Stock and Watson (2001), the three variables are inflation, the unemployment rate, and the policy rate. We have replaced the unemployment rate by the output gap, because Okun's law does not seem to apply to Japan. The sampling frequency in Stock and Watson (2001) is a quarter.

Now $\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}$ is a *shadow rate*, not necessarily equal to the actual policy rate.

It will turn out useful to rewrite this in the following equivalent way. Define the monetary policy regime indicator s_t by

$$s_t = \begin{cases} \text{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{4.3}$$

Then the censored Taylor rule can be written equivalently as

$$\text{(censored Taylor rule)} \qquad r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \quad v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } s_t = \text{P}, \\ \bar{r}_t & \text{if } s_t = \text{Z}. \end{cases} \tag{4.4}$$

Note that $r_t - \bar{r}_t = 0$ if and only if $s_t = \text{Z}$. Thus, consistent with how we identified the regime in the previous section, we have $s_t = \text{P}$ (call it the *normal regime*) if the net policy rate $r_t - \bar{r}_t$ is positive and $s_t = \text{Z}$ (the *zero-rate regime*) if the rate is zero. An outside observer can tell, without observing the shadow Taylor rate, whether the regime is P or Z.

The Exit Condition

We have thus obtained a simple regime-switching three-variable SVAR by replacing the Taylor rule by its censored version. We expand this model to capture the two aspects of the zero-rate regime discussed in the previous section. One is the exit condition, the additional condition needed to end the zero-rate regime when the shadow rate $\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}$ has turned positive. As was documented in the previous section, the condition set by the BOJ is that the year-on-year inflation rate be above some threshold. We allow the threshold to be time-varying.

More formally, we retain the censored Taylor rule (4.4) but modify (4.3) as follows.

$$\begin{cases} \text{If } s_{t-1} = \text{P}, \;\; s_t = \begin{cases} \text{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \\ \text{If } s_{t-1} = \text{Z}, \;\; s_t = \begin{cases} \text{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t \text{ and } \pi_t > \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ threshold}}, \;\; v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma^2_{\overline{\pi}}), \\ \text{Z} & \text{otherwise.} \end{cases} \end{cases} \tag{4.5}$$

We assume that the stochastic component of the threshold ($v_{\overline{\pi}t}$) is i.i.d. over time.[11] It is still the case that $r_t - \bar{r}_t = 0$ if and only if $s_t = \text{Z}$, regardless of whether $s_{t-1} = \text{P}$ or Z. As before, an outside observer can tell the current monetary policy regime just by looking at the net policy rate: $s_t = \text{P}$ if $r_t - \bar{r}_t > 0$ and $s_t = \text{Z}$ if $r_t - \bar{r}_t = 0$.

Adding m to the System

The second extension of the model is to add the excess reserve rate m_t (defined, recall, as the log of actual-to-required reserve ratio) to the system. This variable, while constrained to be zero in the normal regime P, becomes a monetary policy instrument in the zero-rate regime Z. It is a censored variable because excess reserves cannot be negative. If m_{st} is the (underlying) supply of excess reserves, actual m_t is determined as

$$m_t = \begin{cases} 0, & \text{if } s_t = \text{P}, \\ \max\left[m_{st}, 0\right], & \text{if } s_t = \text{Z}. \end{cases} \tag{4.6}$$

Our specification of m_{st} is analogous to the policy-rate Taylor rule and in the spirit of the McCallum rule (McCallum (1988)); it depends on the current value of inflation and output with

[11] If we introduced serial correlation by allowing $v_{\overline{\pi}t}$ to follow the AR(1) (the first-order autoregressive process) for example, we would have to deal with an unobservable state variable (which is $v_{\overline{\pi},t-1}$ for the AR(1) case) appearing only in an inequality. The usual filtering technique would not be applicable.

partial adjustment:

$$\text{(excess reserve supply)} \qquad m_{st} \equiv \alpha_s + \underset{(1\times 2)}{\boldsymbol{\beta}_s{}'} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_s m_{t-1} + v_{st}, \;\; v_{st} \sim \mathcal{N}\left(0, \sigma_s^2\right). \qquad (4.7)$$

The speed of adjustment is $1 - \gamma_s$. We expect the inflation (π_t) and output (x_t) coefficients to be negative, i.e., $\boldsymbol{\beta}_s < \mathbf{0}$, since the central bank would increase excess reserves when deflation worsens or output declines.

Taking Lucas Critique into Account

The central bank sets the policy rate under the normal regime and the excess reserve level under the zero-rate regime. Since the policy rule is different — very different — between the two regimes, the Lucas critique implies that the reduced-form equations describing inflation and output dynamics can shift with the regime. If the private sector in period t sets (p_t, x_t) in full anticipation of the period's regime to be chosen by the central bank, the period t reduced form should depend on the date t regime. Since we view this to be a very remote possibility, we assume that the reduced-form coefficients and error variance and covariances in period t depend, if at all, on the *lagged* regime s_{t-1}.

To Recapitulate

This completes our exposition of the regime-switching SVAR on four variables, p_t (monthly inflation), x_t (the output gap), r_t (policy rate), and m_t (the excess reserve rate). The underlying sequence of events leading up to the determination of the two policy instruments (r_t, m_t) can be described as follows. At the beginning of period t and given the previous period's regime s_{t-1}, nature draws two reduced-form shocks, one for inflation and the other for output, from a bivariate distribution. The reduced-form coefficients and the error variance-covariance matrix may depend on s_{t-1}. This determines (p_t, x_t) and hence the 12-month inflation rate $\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11})$. The central bank then draws three policy shocks $(v_{rt}, v_{\overline{\pi}t}, v_{st})$ from $\mathcal{N}(\underset{(3\times 1)}{\mathbf{0}}, \text{diag}(\sigma_r^2, \sigma_{\overline{\pi}}^2, \sigma_s^2))$. It can now calculate: $\rho_r r_t^* + (1 - \rho_r) r_{t-1} + v_{rt}$ (the shadow Taylor rate given in (4.1)), $\overline{\pi} + v_{\overline{\pi}t}$ (the inflation threshold shown in (4.5)), and m_{st} (excess reserve supply, given in (4.7)). Suppose the previous regime was the normal regime (so $s_{t-1} = \text{P}$). Then the bank picks $s_t = \text{P}$ if $\rho_r r_t^* + (1 - \rho_r) r_{t-1} + v_{rt} > \bar{r}_t$, and $s_t = \text{Z}$ otherwise. Suppose, on the other hand, that $s_{t-1} = \text{Z}$.

Then the bank terminates the zero-rate/QE regime and picks $s_t = \text{P}$ only if

$\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt} > \bar{r}_t$ *and* $\pi_t > \overline{\pi} + v_{\overline{\pi}t}$. If $s_t = \text{P}$, the bank sets r_t to the shadow rate and the market sets m_t to 0; if $s_t = \text{Z}$, the bank sets r_t at $\bar{r}_t$ and m_t at $\max[m_{st}, 0]$.

The model's variables are $(s_t, \mathbf{y}_t)$ with $\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t)$. Assume, as we do in the empirical analysis, that the reduced-form equations involve only one lag. To be clear about the nature of the stochastic process the model generates, assume, only here and temporarily, that the monthly inflation rate p_t rather than the 12-month inflation rate π_t enters the Taylor rule and the excess reserve supply equation and that $\bar{r}_t$ (the rate paid on reserves) is constant (at zero). Then the model with the exit condition is a time-invariant mapping from $(s_{t-1}, \mathbf{y}_{t-1})$ and the i.i.d. date t shocks (consisting of the reduced-form shocks and the policy shocks $(v_{rt}, v_{\overline{\pi}t}, v_{st})$) to $(s_t, \mathbf{y}_t)$. Therefore, the stochastic process generated by the model, $\{s_t, \mathbf{y}_t\}_{t=0}^{\infty}$, is a first-order Markov process. Now, with the 12-month inflation π rather than the 1-month inflation p in the Taylor rule and in the excess reserve supply equation, the number of lags for $\mathbf{y}$ is not 1 but 11 and the mapping is from $(s_{t-1}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ..., \mathbf{y}_{t-11})$ and the date t shocks. With the time-varying exogenous variable $\bar{r}_t$, the mapping is not time-invariant.

For later reference, we shift the time t forward by one period and write the mapping as

$$(s_{t+1}, \mathbf{y}_{t+1}) = f_t(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}; \underset{(2\times 1)}{\boldsymbol{\varepsilon}_{t+1}}, v_{r,t+1}, v_{\overline{\pi},t+1}, v_{s,t+1}; \boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C). \tag{4.8}$$

Here, $\boldsymbol{\varepsilon}_{t+1}$ is the bivariate reduced-form shock in date $t+1$ and v's are the monetary policy shocks. $(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ form the model's parameter vector. The first subset of parameters, $\boldsymbol{\theta}_A$, is the reduced-form parameters describing inflation and output dynamics. Because we allow the reduced form to depend on the (lagged) regime, the parameter vector $\boldsymbol{\theta}_A$ consists of two sets of parameters, one for P and the other for Z. The second subset, $\boldsymbol{\theta}_B$, is the parameters of the Taylor rule (4.5), while the third subset, $\boldsymbol{\theta}_C$, describe the excess reserve supply functions (4.7). More precisely,

$$\boldsymbol{\theta}_B = \left(\alpha_r^*, \underset{(2\times 1)}{\boldsymbol{\beta}_r^*}, \rho_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}} \right), \quad \boldsymbol{\theta}_C = \left(\alpha_s, \underset{(2\times 1)}{\boldsymbol{\beta}_s}, \gamma_s, \sigma_s \right).$$

The mapping is not time-invariant only because of the presence of the exogenous variable.

5 Estimating the Model

This section has three parts: a summary of Appendix 2 about the derivation of the model's likelihood function, a summary of the data description of Appendix 1, and a presentation of the estimation results.

The Likelihood Function (Summary of Appendix 2)

Were it not for regime switching, it would be quite straightforward to estimate the model because of its block-recursive structure. As is well known, the regressors in each equation are predetermined, so the ML (maximum likelihood) estimator is OLS (ordinary least squares). With regime switching, the regressors are still predetermined, but regime endogeneity needs to be taken into account as described below.

Thanks to the block-recursive structure, the model's likelihood function has the convenient property of additive separability in a partition of the parameter vector, so the ML estimator of each subset of parameters can be obtained by maximizing the corresponding part of the log likelihood function. More specifically, the log likelihood is

$$\text{log likelihood} = L_A(\boldsymbol{\theta}_A) + L_B(\boldsymbol{\theta}_B) + L_C(\boldsymbol{\theta}_C). \tag{5.1}$$

The parameter vectors $\boldsymbol{\theta}_A$, $\boldsymbol{\theta}_B$, and $\boldsymbol{\theta}_C$ have been defined at the end of the previous section.

The first term, $L_A(\boldsymbol{\theta}_A)$, being the log likelihood for the reduced-form for inflation and output, is entirely standard, with the ML estimator of $\boldsymbol{\theta}_A$ given by OLS. That is, the reduced-form parameters for regime P can be obtained by OLS on the subsample for which the lagged regime s_{t-1} is P, and the same for Z. There is no need to correct for regime endogeneity because the reduced form errors for period t is independent of the *lagged* regime. Regarding the reserve supply parameters $\boldsymbol{\theta}_C$, which are estimated on subsample with $s_t = \text{Z}$, the censoring implicit in the "max" operator in (4.6) calls for Tobit with m_t as the limited dependent variable. However, since there are no observations for which m_t is zero on subsample Z (which makes the zero-rate regime synonymous with QE as noted in Section 3), Tobit reduces to OLS. There is no need to correct for regime endogeneity because the current regime s_t is independent of the error term of the excess reserve supply equation.

Regime endogeneity *is* an issue for the second part $L_B(\boldsymbol{\theta}_B)$, because the shocks in the Taylor rule and the exit condition, $(v_{rt}, v_{\overline{\pi}t})$, affect regime evolution. If the exit condition were absent so that the censored Taylor rule (4.2) were applicable, then the ML estimator of $\boldsymbol{\theta}_B$ that controls for regime endogeneity would be Tobit on the whole sample composed of P and Z; subsample P, on which $r_t > \overline{r}_t$, provides "non-limit observations" while subsample Z, on which $r_t = \overline{r}_t$, is "limit observations". With the exit condition, the ML estimation is only slightly more complicated because whether a given observation t is a limit observation or not is affected by the exit condition as well as the lower bound.

The Data (Summary of Appendix 1)

The model's variables are p (monthly inflation), x (output gap), r (the policy rate), and m (the excess reserve rate).

The excess reserve rate m is the log of actual to required reserves. We have already mentioned that actual reserves and the policy rate r for the month are the averages over the reserve maintenance period. The graph of m has been shown in Figure 2a. Recall that we defined the zero-rate/QE regime Z as months for which the net policy rate $r_t - \overline{r}_t$ is less than 5 basis points. We ignore the variations of r during the regime by setting $r_t - \overline{r}_t$ to zero for all observations in subsample Z.

The output measure underlying the output gap x is a monthly GDP series obtained by combining quarterly GDP and a monthly comprehensive index of industry activities available only since January 1988. This determined the first month of the sample period. For potential GDP, we use the official estimate by the Cabinet Office of the Japanese government (the Japanese equivalent of the U.S. Bureau of Economic Analysis). It is based on the Cobb-Douglas production function with the HP (Hodrick-Prescott) filtered Solow residual. The output gap is then defined as 100 times the log difference between actual and potential GDP. Monthly GDP and potential GDP are in Figure 2b. It shows the well-known decline in the trend growth rate that occurred in the early 1990s, often described as the (ongoing) "lost decade(s)". It also shows that the output gap has rarely been above zero during the lost decades. The fluctuations in potential output toward the end of the sample period reflect the earthquake and tsunami of March 2011.

The inflation rate p is constructed from the CPI (consumer price index). The relevant CPI component is the so-called “core” CPI (the CPI excluding fresh food), which, as documented in Table 1, is the price index most often mentioned in BOJ announcements. (Confusingly, the core CPI in the U.S. sense, which excludes food and energy, is called the “core-core” CPI.) We made adjustments to remove the effect of the increase in the consumption tax rate in 1989 and 1997 before performing a seasonal adjustment. We also adjusted for large movements in the energy component of the CPI between November 2007 and May 2009.[12] The year-on-year (i.e., 12-month) inflation rate π_t equals $\pi_t = \frac{1}{12}(p_t + \cdots + p_{t-11})$. Figure 2c has π_t since 1988 along with the policy rate r_t and the trend growth rate, defined as the 12-month growth rate of the potential output series shown in Figure 2b.

Simple statistics of the relevant variables are in Table 3. Since we set the net policy rate $r_t - \bar{r}_t$ to zero under Z and since $\bar{r}_t = 0$ during QE1 and QE2 and $\bar{r}_t = 0.1\%$ during QE3, the policy rate r_t itself is 0% during QE1 and QE2 and 0.1% during QE3.

Parameter Estimates

Having described the estimation method and the data, we are ready to report parameter estimates. We start with $\boldsymbol{\theta}_B$.

Taylor rule with exit condition ($\boldsymbol{\theta}_B$).

Most existing estimates of the Taylor rule for Japan end the sample at 1995 because the policy rate shows very little movements near the lower bound since then.[13] In our ML estimation, which can incorporate the lower bound on the policy rate, the sample period can include all the many recent months of very low policy rates. On the other hand, the starting month is January 1988 at the earliest because that is when our monthly output series starts.

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[12] It appears that those large movements were discounted by the BOJ. The monetary policy announcement of August 19, 2008 (http://www.boj.or.jp/en/announcements/release_2008/k080819.pdf), which stated that the policy rate would remain at around 50 basis points, has the following passage: “The CPI inflation rate (excluding fresh food) is currently around 2 percent, highest since the first half of 1990s, due to increased prices of petroleum products and food.”

[13] See Miyazawa (2010) for a survey.

Before commenting on the ML estimate shown in Table 4 below, we state two considerations underpinning our specification of the Taylor rule.

- (variable real interest rates) We have been treating the intercept in the desired Taylor rate r_t^* (the α_r^* in (4.1)) as a constant because of the assumption of the constant real interest rate.[14] This assumption, however, does not seem appropriate for Japan, given the decline in the trend growth rate during the "bubble" period of the late 1980s to the early 1990s, shown in Figure 2c.[15] That the intercept α_r^* may have declined during the period can be seen from the figure. Before the bubble, say in 1988, the 12-month inflation rate was very low, about 0.4% but the policy rate was well above zero, about 4%. In the post-bubble period, both the policy rate and the inflation rate are very low.[16] See Figure 3 below for the behaviour of the desired rate when trend growth is factored in.

- (deviations from the Taylor rule) It is widely agreed that the BOJ under governor Yasushi Mieno set the policy rate at a very high level to quell the asset price bubble.[17] We view this as a prolonged deviation from the Taylor rule and include a dummy variable, to be called the "Mieno dummy". It takes a value of 1 from December 1989, when Mieno became governor, to June 1991, the month before the policy rate was cut. Another deviation seems to have occurred during the banking crisis of the second half of the 1990s. Between September 1995 and July 1998, the policy rate remained low despite improvements in inflation and output. Assuming

[14] In Taylor's (1993) original formulation, the constant term α_r^* equals 1%. It is the difference between the equilibrium real interest rate, which is assumed constant at 2%, and half times the target inflation rate of 2%.

[15] For example, Hayashi and Prescott (2002) document that both the TFP (total factor productivity) and the rate of return on capital declined in the early 1990s. The Taylor rule for Japan in Braun and Waki (2006) allows the equilibrium real rate to vary with the TFP growth.

[16] The decline in the output gap only partially explains the post-bubble low policy rates. The output gap was 0.8% in 1988 and −2.0% in 1995. Even if the output gap coefficient in the desired Taylor rule is as high as 0.5, the decline in the desired rate explained by the output gap is about 1.4% (= $0.5 \times (2.0\% + 0.8\%)$).

[17] See, for example, a booklet for popular consumption by Okina (2013) who was a director of the BOJ's research arm.

that the BOJ refrained from raising the policy rate to help alleviate the Japanese banking crisis, we include a dummy for this period in the equation as well.[18]

Economists at the BOJ were aware of the intercept drift due to changing real interest rates. For example, Okina and Shiratsuka (2002) include the trend growth rate (as measured as the 12-month growth rate of potential GDP) in their Taylor rule.[19] We do the same here. Our specification of the Taylor rule, therefore, is that the intercept α_r^* in the desired Taylor rule is an affine function of the trend growth rate as well as the Mieno (anti-bubble) and banking crisis dummies.

Table 4 reports our ML estimates. The Mieno dummy coefficient of 2.5% in the desired Taylor rate indicates that the policy rate during the height of the bubble was well above what is prescribed by the Taylor rule. As expected, the banking crisis dummy has a negative sign — the desired policy rate would have been higher on average by 37 basis points were it not for the banking crisis. The trend growth rate has a coefficient that is close to unity (0.98) and highly significant ($t = 11.5$). The inflation and output coefficients ($\boldsymbol{\beta}_r^*$ in (4.1)) are estimated to be $(0.75, 0.07)$. Given the low persistence of inflation (to be found in the reduced-form inflation equation below), it is not surprising that the inflation coefficient is below unity. The estimated speed of adjustment per month is 14%. The mean of the time-varying threshold inflation rate affecting the exit condition is mere 0.53% per year.

We should note that it is crucial to include the Mieno dummy if the sample includes the

[18] The Bank of Japan started releasing minutes of the monetary policy meetings only since March 1998 (the 3 March 1998 release is about the meeting on January 16, 1998), so it is not possible for outside observers to substantiate the claim. However, those released minutes of the early part of 1998 do include frequent mentions of the financial system. For example, the minutes of the 16 January 1998 meeting has the following passage: "... a majority of the members commented that the sufficient provision of liquidity would contribute to stabilizing the financial system and to improving household and depositor sentiment."

[19] As is well known in the RBC (real business cycle) calibration literature, the trend growth rate is closely linked to the equilibrium real interest rate. For the case of the power utility $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$, the long-run (log) real interest rate is an affine function of the trend growth rate: $-\log(\beta) + \gamma g$, where g here is the long-run growth rate of output. In the case of the log utility ($\gamma = 1$), the long-run real interest rate and the trend growth rate move one-for-one.

bubble period because without it the inflation coefficient is very imprecisely estimated (0.32, $t = 0.5$), with the run-up of the policy rate during the bubble period almost entirely attributed to the trend growth rate.

The desired Taylor rate r_t^* implied by the ML estimate is shown in the dotted line in Figure 3. It highlights the role of the exit condition. The desired Taylor rate r_t^*, and hence the shadow Taylor rate ($\rho_r r_t^* + (1 - \rho_r) r_{t-1}$), turned positive in the middle of QE2. Yet the QE was not terminated until the inflation rate is slightly above zero (as shown in Table 2).

Excess reserve supply equation (θ_C).

We have noted that the ML estimator can be obtained by regressing m_t on the constant, π_t, x_t, and m_{t-1} on subsample Z. We have also noted earlier that QE1 looks very different from QE2 and QE3, with m_t much lower and less persistent during QE1. For this reason we drop QE1 when estimating the excess reserve supply equation. The estimates are in Table 5. Both the inflation and output coefficients pick up the expected sign. The issue of how to treat m during QE1 will be addressed in Section 7.

Inflation and output reduced-form equations (θ_A).

As mentioned above, the ML estimate of the reduced form can be obtained by OLS on two separate subsamples, "lagged" subsample P (i.e., those t's with $s_{t-1} = \text{P}$) and "lagged" subsample Z (with $s_{t-1} = \text{Z}$). The BIC (Baysian information criterion) instructs us to set the lag length to one on both subsamples.[20] We include the current values of the Mieno (anti-bubble) and the

[20] If n is the lag length and K is the total number of coefficients (including the intercepts) of the bivariate system, we have $K = 2 \times (2 + 4n)$ for lagged subsample Z (there are two regressors bedides (p, x, r, m): the constant and the trend growth rate). For lagged subsample P, we have $K = 2 \times (4 + 3n)$ because lagged m is absent but the Mieno and banking crisis dummy are present. Let T be the sample size and $\widehat{\varepsilon}_t$ be the 2×1 matrix of estimated reduced-form residuals. Given the moderate sample size, we set the maximum lag length at $n_{max} = 6$ and the sample starts from t =July 1988. The information criterion to be minimized over $n = 1, 2, ..., n_{max}$ is

$$\log\left(\left|\frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t\widehat{\varepsilon}_t'\right|\right) + K \cdot C(T)/T, \tag{5.2}$$

where $C(T) = \log(T)$. Under the AIC (Akaike information criterion) which sets $C(T) = 2$, the lag length chosen is 2 for lagged subsample P and 1 for Z.

banking crisis dummies and the trend growth rate in the set of regressors because the Lucas critique implies that those variables affecting the intercept term of the Taylor rule could have shifted the reduced form equations.

Table 6 shows the estimates. First consider lagged subsample P. We exclude lagged m in order to be consistent with the model's current assumption that $m = 0$ under regime P; in Section 7, when we recognize occasionally positive excess reserves, the effect of lagged m will be taken into account.

On lagged subsample P, Andrew's (1993) sup F test finds no structural break for the inflation equation but a structural break for the output equation occurring in March 1995.[21] We show in Table 6 the reduced-form estimates for the two P subsamples split at March 1995. The output gap (x) equation indeed looks very different before and after the break, particularly for the lagged x and the lagged r coefficients. The output persistence measured by the lagged x coefficient is much lower before the break. The output effect of the policy rate (the lagged r coefficient) is similar in magnitude but the sign is reversed.

The monthly inflation (p) equation on lagged subsample P, with no significant structural breaks, exhibits two notable features, observable before and after the break. First, inflation persistence is non-existent, as indicated by the lagged p coefficient of about -0.1. Second, the lagged r coefficient is positive and large. The estimated value of the coefficient of 0.44 after the break means that a 1 percentage point cut in the policy rate *lowers* inflation by 0.44 percentage points in the next period. The estimate, however, is not statistically significant with a t-value of

[21] In testing for structural breaks, we allow all coefficients to shift except for the Mieno and banking crisis dummies. We exclude Mieno and crisis dummies because their values are zero for many possible break dates.

0.8.[22] [23]

Turn now to lagged zero-rate subsample Z. By the Lucas critique, the difference in the policy rule for excess reserves between QE1 and QE2&QE3 mean that the reduced-form coefficients during QE1 could be different. For this reason the sample excludes QE1 and combines QE2 and QE3. The regressors include r_{t-1} because, although it is constant in each QE spell, it differs across spells (see Table 3). Therefore, if r_{t-1} were replaced by the QE3 dummy, the lagged r coefficients (of 0.49 and -0.66, both statistically insignificant) in the inflation and output equations would be scaled down by a factor of 10, with the coefficients of the other variables unchanged.

The positive lagged m codata appendixefficients on lagged subsample Z imply that both inflation and output rise as excess reserves are increased. The coefficient of 0.40 in the output equation, for example, means that the response of the output gap in the next period to a unit increase in m (an increase by 100 percentage points) is 0.40 percentage points.

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[22] The positive r_{t-1} coefficient may be due to the fact that r_{t-1} is the average over the period of the 16th of month $t-1$ and the 15th of month t. If the central bank can respond to price increases of the month by raising the policy rate in the first 15 days of the month, there will be a positive correlation between p_t and r_{t-1}. To check this, we replaced r_{t-1} by r_{t-2} and found a very similar coefficient estimate.

[23] The positive lagged r coefficient in the monthly inflation equation emerges on U.S. data as well. The inflation and output reduced form estimated on U.S. monthly data is as follows (t-values in brackets):

$$p_t = \underset{[1.93]}{0.94} + \underset{[12.1]}{0.45}\, p_{t-1} - \underset{[-1.5]}{0.12}\, x_{t-1} + \underset{[7.9]}{0.34} r_{t-1},$$

$$x_t = \underset{[0.8]}{0.023} + \underset{[1.6]}{0.0039} p_{t-1} + \underset{[187]}{0.99}\, x_{t-1} + \underset{[2.4]}{0.0067} r_{t-1},$$

t = March 1960,..., August 2008. Here, p_t is the monthly CPI inflation rate from month $t-1$ to t in annual percentage rates, x_t is the unemployment rate (not the output gap) of month t in percents, and r_t is the average from the 16th day of month t to the 15th day of month $t+1$. The data appendix includes a documentation of the U.S. monthly data. The estimated lagged r_{t-1} coefficient in the inflation equation declines as the sample becomes more recent: it is 0.31 [$t = 6.4$] if the sample starts from 1970, 0.27 [$t = 5.4$] if from 1980, and -0.08 [$t = -0.6$] if March 1995. Because the x here for the U.S. data is the unemployment rate, not the output gap, the positive lagged r coefficient of 0.0067 in the x equation is not surprising.

6 Impulse Response (IR) and Other Counter-Factual Analyses

With the estimates of our model parameters in hand, we turn to the IR (impulse responses) and other counter-factual analyses. For linear models, the IR analysis is well known since Sims (1980). Our model, however, is nonlinear because the dynamics depends on the regime and also because of the nonnegativity constraint on excess reserves. In this section, we state the definition of IRs for our model and calculate responses of inflation and output to changes in monetary policy variables including the regime.

IRs for Nonlinear Processes in General

Consider for a moment a general strictly stationary process $\underset{(n\times 1)}{\mathbf{y}_t} \equiv (y_{1t}, y_{2t}, ..., y_{nt})$. Gallant, Rossi, and Tauchen (1993, particularly pp. 876-877) proposed to define an IR as the difference in conditional expectations under two alternative possible histories with one history being a perturbation of the other. The IR of the i-th variable to the j-th variable k-period ahead is defined as

$$\begin{aligned} &\mathrm{E}\Big(y_{i,t+k} \,|\, \underbrace{(y_{1t}, ..., y_{j-1,t}, y_{jt} + \delta, y^{(a)}_{j+1,t}, ..., y^{(a)}_{n-1,t}, y^{(a)}_{nt})}_{\mathbf{y}_t \text{ in the alternative history}}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big) \\ &\quad - \mathrm{E}\Big(y_{i,t+k} \,|\, \underbrace{(y_{1t}, ..., y_{j-1,t}, y_{jt}, y^{(b)}_{j+1,t}, ..., y^{(b)}_{n-1,t}, y^{(b)}_{nt})}_{\mathbf{y}_t \text{ in the baseline history}}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ... \Big), \;\; k = 1, 2, ..., \end{aligned} \tag{6.1}$$

where δ is the size of perturbation, $y^{(a)}_{\ell,t}$ $(\ell = j+1, ..., n)$ is the conditional expectation of $y_{\ell,t}$ conditional on the alternative history up to and including $y_{jt} + \delta$, and $y^{(b)}_{\ell,t}$ similarly is the expectation conditional on the baseline history up to and including y_{jt}. These expected values are "filled in" for the remaining elements $(\ell = j+1, ..., n)$ of $\mathbf{y}_t$ to trace out the effects of the shock to the j-th variable through the contemporaneous correlation among the variables.[24] This definition, when applied to linear processes, reduces to the orthogonalized IR of variable i to variable j, which for (block) recursive linear VARs is the standard IR.[25]

Adaptation to Our Model

Our model's variables are $(s_t, \mathbf{y}_t)$ where $\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t)$ and s_t is the monetary policy regime. The adaptation of the IR defined above to our model is easiest to see for the last variable of the system, m_t, because no "filling-in" is needed.

m-IR (IRs to Changes in m)

Since the central bank has control over m only under the zero-rate regime, we assume $s_t = \mathrm{Z}$ and define the IR to a change in m (labeled "m-IR") as:

$$(m\text{-IR}) \qquad \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z},\ \underbrace{(p_t, x_t, \bar{r}_t, m_t + \delta_m),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the alternative history}}\ \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big)$$
$$- \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z},\ \underbrace{(p_t, x_t, \bar{r}_t, m_t),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the baseline history}}\ \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big), \quad y = p, x, r, m. \tag{6.2}$$

The conditional expectations here are defined by the mapping (4.8). It thus suffices to include the

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[24] It may appear that a more natural definition is without the filling-in. That is, we could alternatively define an IR as

$$\mathrm{E}\Big(y_{i,t+k}\,|\,(y_{1t}, \dots, y_{j-1,t}, y_{jt} + \delta), \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots\Big) - \mathrm{E}\Big(y_{i,t+k}\,|\,(y_{1t}, \dots, y_{jt}), \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots\Big).$$

The two definitions are equivalent if the process $\{\mathbf{y}_t\}$ is linear, but not necessarily so with nonlinear processes. We chose the definition (6.1) for two reasons (if you are interested). First, the difference is very minor for our model. Second, there is a subtlety in the above alternative definition when applied to Markov processes. To illustrate, consider a bivariate process with the conditional distribution of $\mathbf{y}_{t+1}$ that depends on at most two lags $(\mathbf{y}_t, \mathbf{y}_{t-1})$. In the IR of variable i to variable 1, look at the conditional expectation under the baseline history for example. In definition (6.1), it is: $\mathrm{E}\Big(y_{i,t+k}|(y_{1t}, y_{2t}^{(b)}), \mathbf{y}_{t-1}\Big)$. In the alternative definition, the conditioning information must be $(y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2})$, not $(y_{1t}, \mathbf{y}_{t-1})$. Otherwise the alternative definition is not equivalent to definition (6.1) for linear processes. This is because in (6.1) the expected value $y_{2t}^{(b)}$ depends on $(\mathbf{y}_{t-1}, \mathbf{y}_{t-2})$.

[25] For a proof, see, e.g., Hamilton (1994, Section 11.4 (particularly equation [11.4.19]) and Section 11.6).

current value of $(s, \mathbf{y})$ and ten lags of $\mathbf{y}$ in the history. In both the baseline and alternative histories, we set $r_t = \bar{r}_t$ because that is what is implied by the regime $s_t = \mathrm{Z}$. To calculate the conditional expectation given the history, we use estimated parameters $(\widehat{\boldsymbol{\theta}}_A, \widehat{\boldsymbol{\theta}}_B, \widehat{\boldsymbol{\theta}}_C)$ for the parameter vectors in (4.8). The estimated reduced-form parameters $\widehat{\boldsymbol{\theta}}_A$ are in Table 6, the estimated Taylor rule parameters $\widehat{\boldsymbol{\theta}}_B$ are in Table 4, and the excess reserve supply parameters $\widehat{\boldsymbol{\theta}}_C$ are in Table 5. Two further aspects of the calculation of conditional expectations need to be mentioned at this junction:

- (Monte Carlo integration) We compute numerically the conditional expectations by simulating a large number of sample paths generated by the mapping (4.8) and then taking the average of those simulated sample paths. In the estimated IRs and counter-factual simulations to be reported below, 10,000 simulated paths are generated.

- (projected paths of exogenous variables) There are four exogenous variables in the system: $\bar{r}$ (the rate paid on reserves), the banking crisis dummy (for September 1995-July 1998), the Mieno (anti-bubble) dummy (for December 1989-June 1991), and the trend growth rate (the 12-month growth rate of potential GDP). Each simulated sample path of $(s, \mathbf{y})$ from the base period t depends on the projected path from t on of those exogenous variables.[26] The IR, which compares two simulated sample paths, is invariant to the projected exogenous variables path with linear systems, but not so with nonlinear systems such as ours. The projected path of the exogenous variables we use for the IR calculations reported below is their actual path (with the values beyond the sample period set equal to the value at the end of the period).

r-IR (IRs to Changes in r)

A change in the policy rate is possible only under regime P. The IR to a policy rate change,

[26] Therefore, the expectations operator should have a subscript t (E_t rather than E). We won't carry this sub t for notational simplicity.

labeled "r-IR", then, is

$$(r\text{-IR}) \qquad \begin{aligned} &\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, \underbrace{(p_t, x_t, r_t + \delta_r, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the alternative history}} \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big) \\ &- \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, \underbrace{(p_t, x_t, r_t, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the baseline history}} \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big), \quad y = p, x, r, m. \end{aligned} \tag{6.3}$$

Under the assumption (to be relaxed in Section 7) of zero excess reserve demand, the excess reserve rate m is zero under P. So the "filled-in" value of m_t is 0 in both the baseline and alternative histories.

PZ-IR (IRs to a Change in Regime from P to Z)

To define IRs to changes in the regime s_t, we require that the regime be the only difference between the two possible histories. So set r_t to $\bar{r}_t$ in both histories because that is the rate set by the central bank under $s_t = \mathrm{Z}$ and set m_t to 0, the value of m under P. Thus,[27]

$$(\text{PZ-IR}) \qquad \begin{aligned} &\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, \underbrace{(p_t, x_t, \bar{r}_t, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the alternative history}} \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big) \\ &- \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, \underbrace{(p_t, x_t, \bar{r}_t, 0),}_{\mathbf{y}_t \equiv (p_t, x_t, r_t, m_t) \text{ in the baseline history}} \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\Big), \quad y = p, x, r, m. \end{aligned} \tag{6.4}$$

Estimated IRs

In the next several figures, we display estimated IRs with error bands. The error bands are obtained as follows. Draw a parameter vector from the estimated asymptotic distribution and do

[27] It is true that, in our model, the policy rate r_t is greater than the rate paid on reserves $\bar{r}_t$ under P, so the baseline history in the second conditional expectation in the definition (6.4) is not possible. We can, however, make this conditional expectation well-defined as the limit as the policy rate falls arbitrarily close to $\bar{r}_t$:

$$\text{the second conditional expectation in (6.4)} \equiv \lim_{r \downarrow \bar{r}_t} \mathrm{E}\left(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, r, 0), \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-10}\right).$$

the Monte Carlo integration (with 1,000 simulations given the parameter vector).[28] Continue this until we accumulate 400 "valid" IRs.[29] Pick the 84 and 16 percentiles for each horizon (so the coverage rate is 68%, corresponding to one-standard error bands).

m-**IRs**

The m-IR does not depend very much on the choice of the base period t. Figure 4a shows the m-IR profile for $k = 0, 1, 2, ..., 60$ for the base period of February 2004 (the peak QE month) when $m_t = 1.849$, about 6.4 (= $\exp(1.849)$) times required reserves.[30] The lower-left panel shows the response of m, so its intercept at horizon $k = 0$ (the base period) equals the perturbation δ_m. Its size is chosen so that its ratio to the estimated standard deviation of the reserve supply shock v_{st} (which is 0.132 from Table 5) roughly equals the ratio of $-\delta_r$ (the perturbation in r-IR) to the estimated standard deviation of the policy rate shock v_{rt} (0.134 from Table 4). We will set $\delta_r = -1$ percentage point in the r-IRs below and $\delta_m = 1.0$.

The estimated IR profile of the output gap (x) is shown in the upper-right panel of the figure.

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[28] The likelihood function is additively separable as shown in (5.1) where $\boldsymbol{\theta}_A$ is the bivariate reduced-form parameters (including the error variance-covariance matrix), $\boldsymbol{\theta}_B$ is the Taylor rule parameters, and $\boldsymbol{\theta}_C$ describes the excess reserve supply function. Consequently, if $\widehat{\boldsymbol{\theta}}_A$ is the ML estimator of $\boldsymbol{\theta}_A$, for example, and if $\text{Avar}(\widehat{\boldsymbol{\theta}}_A)$ is its asymptotic variance, a consistent estimator, $\widehat{\text{Avar}(\widehat{\boldsymbol{\theta}}_A)}$, of the asymptotic variance is the inverse of $1/T$ times the Hessian of $L_A(\boldsymbol{\theta}_A)$ where T is the sample size. For $\boldsymbol{\theta}_B$, we draw the parameter vector by generating a random vector from $\mathcal{N}\left(\widehat{\boldsymbol{\theta}}_B, \frac{1}{T}\widehat{\text{Avar}(\widehat{\boldsymbol{\theta}}_B)}\right)$. We do the same for and $\boldsymbol{\theta}_C$. For $\boldsymbol{\theta}_A$, we draw the parameter vector according to the RATS manual. That is, let $\widehat{\boldsymbol{\Sigma}}$ here be the ML estimator of the 2×2 variance-covariance matrix $\boldsymbol{\Sigma}$ of the bivariate error vector in the reduced form. It is simply the sample moment of the bivariate residual vector from the reduced form. We draw $\boldsymbol{\Sigma}$ from the inverse Wishart distribution with $\widehat{\boldsymbol{\Sigma}}$ and $T - K$ as the parameters, where T is the sample size and K is the number of regressors. Let $\widetilde{\boldsymbol{\Sigma}}$ be the draw. We then draw reduced-form coefficient vector from $\mathcal{N}(\mathbf{b}, \widetilde{\boldsymbol{\Sigma}} \otimes \mathbf{S}_{XX}^{-1})$, where $\mathbf{b}$ here is the estimated reduced-form coefficients and $\mathbf{S}_{XX}$ is the sample moment of the reduced-form regressors.

[29] Let $IR(i, k)$ be the k-period ahead IR of variable i and let n be the IR horizon. For each i, define $v_{1i} \equiv \sum_{k=1}^{\ell}(IR(i, k))^2$ and $v_{2i} \equiv \sum_{k=\ell+1}^{n}(IR(i, k))^2$ where ℓ is the largest integer not exceeding $0.8n$. We declare the IR "valid" if $\min_i v_{2i}/v_{1i} \leq 0.1$. We set n (the IR horizon) to 120.

[30] Because the base period t is after the structural break date of March 1995, the estimated reduced-form parameter vector $\widehat{\boldsymbol{\theta}}_A$ used for simulating sample paths for the Monte Carlo integration comes from the reduced-form estimate for the post-break period (the middle panel in Table 6).

Its impact effect (the IR at $k = 1$) is about 0.40% (which is the lagged m coefficient in the output equation of 0.40 shown in Table 6 times $\delta_m = 1$). Because of the persistence in the output dynamics exhibited in the estimated reduced form, the IR builds on the impact effect and goes up to nearly 2% in 12 months or so. For monthly inflation (p), the impact effect (at $k = 1$) is greater, but the effect tapers off due to the lack of persistence. Because both output and inflation rise, regime P is more likely to occur under the alternative scenario. This is why the response of the policy rate (r) gradually rises from zero with the response of m turning negative. This also explains why the average duration from the base period of the initial regime (which is Z in both the base and alternative scenarios) is shorter under the alternative scenario with 10 months than under the base scenario with 28 months.

r-IRs

For the r-IR, we wish to examine, as we did with the m-IR, expansionary monetary policies. So we take the policy rate perturbation δ_r to be negative 1 percentage point ($\delta_r = -1$). In order to calculate the response of the cut in the policy rate, however, the base period has to be May 1995 or before, when the policy rate is above 1 percent. We take the base period t to be the earliest month after the structural break, March 1995, when the policy rate, at $r_t = 2.0\%$, was comfortably above zero.

Figure 4b has the IR profiles. The 1 percentage point rate cut can be read off from the intercept of the lower-left panel, which shows the response of r. In the response of p, shown in the upper-left panel, the impact effect (the IR at $k = 1$) is *negative*, at −0.44%, but the wrong sign is quickly reversed in several months. The error band shows that, for all k, the response is not significantly different from zero. The output gap response shows that the rate cut has a strong expansionary effect, reaching a peak of about 2.4% in 12 months. Because of the high initial policy rate of 2.0%, the system rarely switches to QE in the simulations (the average duration of the initial regime of P is about 3 years under either scenario, baseline or alternative), which explains the almost no response of m as shown in the south-east panel of the figure. Therefore, the IR would look very similar even if we allowed for two QE types, “weak” and “strong” QEs. That this is indeed the case will be shown in Section 7.

Counter-factual Analysis

More interesting counter-factual analyses are possible if we combine the three IRs. To illustrate, we examine the episode of the winding-down of QE2. The data on $(s_t, m_t, r_t, p_t, \pi_t, x_t)$ during the episode are in Table 2.

The last month of QE2 is June 2006 and the normal regime P resumed in July 2006. If QE2 were allowed to continue until July 2006, what difference would it have made? We can answer the question by setting t = July 2006 (when the regime was P) and taking Z as the counter-factual alternative regime. The difference we calculate, then, is

$$\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t^e), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, r_t, 0), \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big). \quad (6.5)$$

Thus, the perturbation occurs to not just one but three variables: r_t, m_t, and s_t. Here, m_t^e is the "filled-in" value for m, namely the level of m_t that can be expected given the history leading up to (p_t, x_t) and given the current regime is s_t = Z (so m_t is supply-determined). It can be written as[31]

$$m_t^e \equiv \mathrm{E}\Big[\max[m_{st}, 0]\,|\,p_t, x_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big] = \mathrm{E}_{v_{st}}\Big[\max[m_{st}, 0]\,|\,\pi_t, x_t, m_{t-1}\Big] \quad \text{with } m_{st} \text{ given by (4.7).} \quad (6.6)$$

The estimate of this m_t^e for t = July 2006 is 0.45, which is about 1.6 (= exp(0.45)) times required reserves, about a quarter of the ratio (of 6.4) observed at the peak QE month of February 2002.

The estimated profile of the difference (6.5) for $y = p, x, r, m$ is in Figure 4c. The perturbations to m of $\delta_m = 0.45$ and to r of $\delta_r = -0.26$ ($r_t = 0.26\%$ and $\bar{r}_t = 0\%$ in July 2006) can be read off from the intercepts in the two lower panels. Surprisingly, despite the increase in m from $m_t = 0$ to $m^e = 0.45$, both inflation and output *decline*.

To see why continuing QE2 would have been contractionary (namely, terminating QE2 was

[31] This conditional expectation can be computed analytically by one of the standard Tobit formulas. Consider the Tobit model $y = \max[\mathbf{x}'\boldsymbol{\beta} + u, c]$ where $u \sim \mathcal{N}(0, \sigma^2)$. We have: $\mathrm{E}(y|\mathbf{x}) = [1 - \Phi(v)] \times [\mathbf{x}'\boldsymbol{\beta} + \sigma\lambda(v)] + \Phi(v)c$, where $v \equiv (c - \mathbf{x}'\boldsymbol{\beta})/\sigma$ and $\lambda(v) \equiv \phi(v)/[1 - \Phi(v)]$. Here, ϕ and Φ are the pdf and cdf of the standard normal distribution.

expansionary), decompose the (overall) difference (6.5) as the sum of m-IR, PZ-IR, and r-IR:

$$\begin{aligned}(6.5) = &\underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t^e), \ldots\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big)}_{m\text{-IR}} \\ &+ \underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big)}_{\text{PZ-IR}} \\ &+ \underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, r_t, 0), \ldots\Big)}_{r\text{-IR}}.\end{aligned} \tag{6.7}$$

The culprit is the pure regime change effect represented by the PZ-IR. Its profile, shown in Appendix Figure 2, is very similar to the overall profile in Figure 4c. As we know from Figure 4a, the m-IR component is expansionary, which means that the r-IR component for the same base period is *contractionary* in spite of the decline in the policy rate from $r_t = 0.26\%$ to $\bar{r}_t = 0\%$. This is because lowering the rate from an already very low level makes it more likely that the regime switches from P to Z in the future with all the contractionary effect of the pure regime change effect. That the r-IR component is almost a mirror image of the m-IR component is shown in Appendix Figure 3.

A question arises: if exiting from QE by switching to P in July 2006 was expansionary, would it have been better to end it earlier? We can answer this question by considering the opposite of (6.5) for the base period t before July 2006 when the excess reserve rate m_t was greater. That is, take Z, not P, as the baseline regime and take P, not Z, as the counter-factual alternative regime. So the difference we calculate is

$$\begin{aligned}&\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), \mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-10}\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t), \mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-10}\Big) \\ &\quad = -\Big[\underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{P}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big)}_{\text{PZ-IR}}\Big] \\ &\quad\quad - \Big[\underbrace{\mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, m_t), \ldots\Big) - \mathrm{E}\Big(y_{t+k}\,|\,s_t = \mathrm{Z}, (p_t, x_t, \bar{r}_t, 0), \ldots\Big)}_{m\text{-IR}}\Big]\end{aligned} \tag{6.8}$$

for any of the Z months preceding July 2006. There is no r-IR component because we set the policy rate at $\bar{r}$ in both the baseline and alternative scenarios. The first component of the difference, which is the negative of the PZ-IR, is positive for both p and x because the PZ-IR is, as just seen, contractionary. Whether the overall difference (6.8) is positive or not (namely, whether ending QE would have been expansionary or not) depends on the strength of the m-IR component which, in turn, depends on the size of m_t. If m_t is not large enough, the PZ-IR component dominates and the profiles of inflation and output responses would be the opposite of those in Figure 4c. This is indeed the case for t = June 2006 (with $m_t = 0.46$ as shown in Table 2), May 2006 (with $m_t = 0.55$) and April 2006 ($m_t = 1.0$ or the actual-to-required reserve ratio of 2.7), but not for March 2006 with $m_t = 1.51$ or with the actual-to-required reserve ratio of 4.5. Exiting from QE2 in March 2006 and hence reducing m from 1.51 to zero would have been contractionary.

Why is PZ-IR Contractionary?

By way of answering the question, we focus on the the impact effect on (p, x), namely their IR at $k = 1$ (one period ahead), because it can be calculated analytically. Write the reduced form for period $t + 1$ as[32]

$$\begin{bmatrix} p_{t+1} \\ x_{t+1} \end{bmatrix} = \Big\{ \underset{(2\times1)}{\mathbf{c}(s_t)} + \underset{(2\times1)}{\boldsymbol{\phi}_g(s_t)} g_{t+1} \Big\} + \underset{(2\times1)}{\boldsymbol{\phi}_p(s_t)} p_t + \underset{(2\times1)}{\boldsymbol{\phi}_x(s_t)} x_t + \underset{(2\times1)}{\boldsymbol{\phi}_r(s_t)} r_t + \underset{(2\times1)}{\boldsymbol{\phi}_m(s_t)} m_t + \underset{(2\times1)}{\boldsymbol{\varepsilon}_{t+1}}, \tag{6.9}$$

where g_{t+1} is the concurrent trend growth rate (the 12-month growth rate of potential output to month $t + 1$). We can interpret the term in braces, $\mathbf{c}(s_t) + \boldsymbol{\phi}_g(s_t)g_{t+1}$, as the time-varying intercept. Our estimates of the coefficients can be read off from Table 6. For example,

$$\mathbf{c}(\mathrm{P}) = \begin{bmatrix} 0.12 \\ -0.88 \end{bmatrix}, \ \ \mathbf{c}(\mathrm{Z}) = \begin{bmatrix} -0.57 \\ -0.99 \end{bmatrix}, \ \ \boldsymbol{\phi}_g(\mathrm{P}) = \begin{bmatrix} -0.51 \\ 1.31 \end{bmatrix}, \ \ \boldsymbol{\phi}_g(\mathrm{Z}) = \begin{bmatrix} -0.24 \\ 0.03 \end{bmatrix}.$$

The impact effect of the regime change from P to Z comes from the change in the reduced-form coefficients. Since $r_t = \bar{r}_t$ and $m_t = 0$ in the PZ-IR, we have:

$$\begin{aligned} \begin{bmatrix} \text{PZ-IR of } p \text{ at } k = 1 \\ \text{PZ-IR of } x \text{ at } k = 1 \end{bmatrix} &= \Big\{ [\mathbf{c}(\mathrm{Z}) - \mathbf{c}(\mathrm{P})] + \big[\boldsymbol{\phi}_g(\mathrm{Z}) - \boldsymbol{\phi}_g(\mathrm{P})\big] g_{t+1} \Big\} \\ &\quad + \big[\boldsymbol{\phi}_p(\mathrm{Z}) - \boldsymbol{\phi}_p(\mathrm{P})\big] p_t + \big[\boldsymbol{\phi}_x(\mathrm{Z}) - \boldsymbol{\phi}_x(\mathrm{P})\big] x_t + \big[\boldsymbol{\phi}_r(\mathrm{Z}) - \boldsymbol{\phi}_r(\mathrm{P})\big] \bar{r}_t. \end{aligned} \tag{6.10}$$

[32] There is no need to include the Mieno (anti-bubble) and banking crisis dummies in the reduced form because their values are zero for the base period in question.

For the base period of t =July 2006, we have $p_t = -0.3$, $x_t = -0.8$ from Table 2. Also, $\bar{r}_t = 0$ and $g_{t+1} = 0.86$. Thus, for t =July 2006,

$$\begin{bmatrix} \text{PZ-IR of } p \text{ at } k=1 \\ \text{PZ-IR of } x \text{ at } k=1 \end{bmatrix} = \left\{ \underbrace{\begin{bmatrix} -0.57 - 0.12 \\ -0.99 - (-0.88) \end{bmatrix}}_{\mathbf{c}(\mathrm{Z})-\mathbf{c}(\mathrm{P})} + \underbrace{\begin{bmatrix} -0.24 - (-0.51) \\ 0.03 - 1.31 \end{bmatrix}}_{\phi_x(\mathrm{Z})-\phi_x(\mathrm{P})} \times \underbrace{0.86}_{g_{t+1}} \right\}$$
$$+ \underbrace{\begin{bmatrix} -0.06 - (-0.09) \\ 0.08 - (-0.01) \end{bmatrix}}_{\phi_p(\mathrm{Z})-\phi_p(\mathrm{P})} \times \underbrace{-0.3}_{p_t} + \underbrace{\begin{bmatrix} 0.12 - 0.13 \\ 0.79 - 0.98 \end{bmatrix}}_{\phi_x(\mathrm{Z})-\phi_x(\mathrm{P})} \times \underbrace{(-0.8)}_{x_t} = \begin{bmatrix} -0.46 \\ -1.1 \end{bmatrix}. \tag{6.11}$$

This shows that the primary source of the impact effect of $(-0.46, -1.1)$ is the difference between the regimes in the time-varying intercept. More specifically for p, the difference is due to the constant term $\mathbf{c}(s_t)$; for x, it is due to the difference in the trend growth coefficient in the reduced form.

7 Robustness and Extensions

In this section, we examine how the inflation and output responses shown in Figure 4a-4c are affected to various changes to the model. It will be shown that: (i) allowing for two QE types makes very little difference, (ii) turning the demand for excess reserves on dampens the r-IR shown in Figure 4b, and (iii) changing the measure of potential GDP to HP (Hodrick-Prescott) filtered GDP brings about the price puzzle in the r-IR.

HP-Filtered GDP as Potential GDP

So far, the measure of potential GDP that underlies the output gap and the trend growth rate has been the official estimate from the Cabinet Office. We now change the measure to the HP-filtered GDP which, as shown in Figure 2b, tracks actual GDP more closely than the Cabinet Office measure. For example, the output gap has been mostly positive since July 2011.

Figure 5a-5c show the monthly inflation (p) and output (x) responses with the alternative measure of potential GDP. To save space, the IR profiles of the policy rate (r) and the excess reserve rate (m) are not shown because they look very similar to those in Figure 4. Of the three major conclusions stated in the introduction, two of them hold up: QE is expansionary and the exit from QE2 was expansionary. The conclusion about the effect of policy rate cuts does not fare so well, however. Recall from Figure 4b about the r-IR that the response of p to a 1 percentage point rate cut is, although negative initially, positive for most of the rest of the horizon and that the output response is positive and strong. The r-IR for p in Figure 5b exhibits the price puzzle, with the inflation response never recovering from the initial negative effect. The output response is about a half in size. The error bands are generally wider.

Excluding the Trend Growth Rate from the System

If we exclude the trend growth rate from both the reduced form shown in Table 6 and the Taylor rule in Table 4, the model becomes the one studied in Hayashi and Koeda (2014). The IR profiles shown in Figure 4 remain more or less the same except for Figure 4b about the r-IR. The price puzzle emerges and output shows virtually no response.

The reason for this is well understood since Sims (1992). If there is a variable (the trend growth rate in the present case) that the central bank responds to but is not included in the Taylor rule, then what the econometrician regards as the monetary policy shock will include not only the true policy shock but also the effect of this missing variable. If this variable is also missing in the inflation and output reduced form, then the IR to the incorrectly identified policy shock will be contaminated by the effect of the missing variable. In the case of a rate cut in the r-IR, the contaminated policy shock contains not only a genuine unexpected decrease in the policy rate but also a decline in the trend growth rate. For output, the expansionary effect of a rate cut is offset by the contractionary effect of a decline in trend growth. This explains the virtual non-response of output to a rate cut found in Hayashi and Koeda (2014).

Turning Excess Reserve Demand On

In all the simulations underlying the Monte Carlo integration, we turned the demand for excess reserves off by setting m to zero under regime P. We now relax this assumption. It entails three changes. First, replace the zero excess reserve under P in (4.6) by $\max[m_{dt}, 0]$ where m_{dt} is the demand for excess reserves to be specified below. Second, include lagged m in the reduced-form equations for lagged subgsample P. The upper panel of Table 7 has the reduced-form estimates for the post break period from March 1995. The lagged m coefficient comes out with a negative sign in both the p and x equations. Third, the definition of r-IR in (6.3) and the PZ-IR in (6.4) needs to be modified as follows: In (6.3), replace the zero for m_t in the alternative history by $m_t^{(a)}$ (the expected value of $\max[m_{dt}, 0]$ given the history up to $r_t + \delta_r$). Likewise, replace the zero for m_t in the baseline history by $m_t^{(b)}$ (the expected value of $\max[m_{dt}, 0]$ given the history up to r_t). Similarly in (6.4), replace the zero for m_t in both the baseline and alternative histories by $m_t^{(b)}$ (the expected value given the history up to $\bar{r}_t$).

The specification of m_{dt} we consider relates the excess reserve demand to the current values of π (the 12-month inflation rate), x (the output gap), r (the policy rate) and the lagged value of m. The equation is to be estimated on the subsample in which m is demand-determined. There is no need to correct for regime endogeneity because the excess reserve demand shock is independent of the regime. The estimation method is Tobit because of the censoring in $\max[m_{dt}, 0]$.

We argued in Section 3 that m was supply-determined during QE2 and QE3. Regarding QE1, based on our reading of the summary of discussions at the BOJ policy board meetings, we assume that m is demand-determined during QE1.[33] Thus the subsample for the excess reserve demand equation consists of those months under regime P between January 1988 and December 2012 (170 months) and QE1 (17 months).[34] We define the limit observations as the months for which $m < 0.5\%$. There are 141 such months.[35] The estimated equation is (t-values in

[33] In almost all the board meetings during QE1, one board member proposed to increase the current account balance far beyond what is required to guide the interbank rate to zero. The proposal was invariably voted down.

[34] Excluding the 17 QE1 months from the sample produces very similar estimates.

[35] Recall that we have set $m_t = 0$ for months between QE2 and QE3 (except the Lehman crisis months of September to November 2008), on the ground that banks postponed re-entry to the interbank market and held on to excess reserves. So those months, indicated by the thin bars in Figure 2a, are limit observations.

brackets)[36]

$$
\begin{aligned}
m_{dt} &= \underset{[-0.2]}{-0.005} + \underset{[0.5]}{0.011}\pi_t - \underset{[-2.6]}{0.015}\,x_t - \underset{[-2.4]}{0.12}\,r_t + \underset{[4.4]}{0.60}\,m_{d,t-1} + \underset{[2.7]}{1.01}\,\mathrm{GULF}_t, \\
&\text{estimated standard deviation of the error} = 0.053 \text{ (s.e.} = 0.0057\text{)}, \\
&\text{sample size} = 187, \text{number of limit observations} = 141.
\end{aligned} \tag{7.1}
$$

The last regressor, GULF_t, is a Gulf war dummy for February to April 1991.[37] The output coefficient is negative, perhaps because commercial banks desire excess reserves in recessions. The estimated error size (measured by its standard deviation) of 0.053 should be compared to the average fitted value of m_{dt} of about -0.25 (banks on average would have liked to hold only 75% of required reserves). So m_t under P, which is $\max[m_{dt}, 0]$, is positive only occasionally.

When the excess reserve demand is turned on, only the r-IR, displayed in Figure 6, is affected noticeably. As in Figure 4b's r-IR, the initial regime, which is P in both the baseline and alternative scenarios, lasts for about 3 years. During those years, m is positive occasionally, which is contractionary because the lagged m coefficient, as shown in Table 7, is negative in both the inflation and output equations under P . The contractionary effect is greater under the alternative scenario because the lower policy rate increases m when it is positive. Thus the response of p and x is dampened.

Allowing for Two QE Types

Finally, we extend the model to allow for two QE types, while the excess reserve demand is kept on. The zero-rate/QE regime Z is now composed of two sub-regimes. Under the "strong" QE, as

[36] The regime is P in July 2006, but the previous month is the last month of QE2 when m is far above 0. We assume that the excess reserve demand in that previous month is zero. So $m_{d,t-1} = 0$ for t = July 2006.

[37] The value of m was about 2% in February, 5% in March, and 1% in April 1991. We include the Gulf dummy because we suspect there was some technical reason for excess reserves. At that time, there was a huge increase in the deposit by the Japanese treasury at the Bank of Japan. Most of it was for the payment of 13 billion dollars by the Japanese government to the U.S to help defray the cost of the Gulf war (which ended in February 1991). The output gap then was well above 2%, the policy rate was above 8%, and the financial system was apparently sound. There was no reason for commercial banks to hold excess reserves and the desired m would have been well below zero.

in QE2&QE3 and labeled "S", the policy rate is zero and m is determined by the excess reserve supply equation (4.7). Under the "weak" QE, as in QE1 and labeled "W", the policy rate is zero but m is set by demand. Thus the censored Taylor rule (4.4) remains valid with $\mathrm{Z} = \mathrm{S}, \mathrm{W}$, but the equation determining m_t, (4.6), is now

$$m_t = \begin{cases} \max\left[m_{dt}, 0\right], & \text{if } s_t = \mathrm{P}, \mathrm{W} \\ \max\left[m_{st}, 0\right], & \text{if } s_t = \mathrm{S}. \end{cases} \tag{4.6'}$$

Regarding the regime evolution (4.5), we assume that the central bank chooses between "weak" and "strong" QEs randomly, with probability q for "weak" QE ("W") and $1 - q$ for "strong" QE ("S"). That is, (4.5) is modified as

$$\begin{cases} \text{If } \; s_{t-1} = \mathrm{P}, \;\; s_t = \begin{cases} \mathrm{P} & \text{if } \; \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t, \\ \mathrm{Z} = \begin{cases} \mathrm{S} & \text{with probability } 1-q \\ \mathrm{W} & \text{with probability } q \end{cases} & \text{otherwise.} \end{cases} \\ \text{If } \; s_{t-1} = \mathrm{Z}, \;\; s_t = \begin{cases} \mathrm{P} \quad \text{if } \; \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t \; \text{ and } \; \pi_t > \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ threshold}}, \;\; v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma_{\overline{\pi}}^2), \\ \mathrm{Z} \quad \text{otherwise, } \mathrm{Z} = \mathrm{S}, \mathrm{W}. \end{cases} \end{cases} \tag{4.5'}$$

Thus we do not allow for the regime to change from S to W or from W to S; if the previous regime is S, for example, then the current regime is either P or S. In the IR calculations below, we set $q = 1/3$. If we set $q = 0$, this model reduces to the one studied in the preceding subsection, with only one QE type and with the excess reserve demand turned on.

The last piece of the model is the reduced form under W, which needs to be estimated on lagged subsample W (those t's for which $s_{t-1} = \mathrm{W}$ or $t-1$ is in QE1). QE1 has only 17 observations. The shortness of the sample forces us to impose two restrictions on the reduced form. First, because r is constant (at 0) during QE1, the lagged r coefficient cannot be identified. We constrain the coefficient to be zero. Second, there is not much variation in the trend growth rate g during QE1, which creates near multi-collinearity between the constant and g. We subsume the effect of trend growth rate in the constant by dropping g from the reduced form.[38] The lower panel of Table 7 has the estimates. Unlike the reduced form estimated on lagged subsample Z (consisting of QE2&QE3) in Table 6, the lagged m coefficient in the inflation equation is negative.

Allowing for two QE types makes so little difference that the IR profiles are not shown here. Figure 4a and 4c remain virtually unaffected. The r-IR looks similar to Figure 6's r-IR, which is for the case of one QE type and the active excess reserve demand.

8 Spurious Causality?

Our finding, exemplified in the PZ-IR, that changing the regime from P to Z by itself causes a contraction, is surprising. One possible explanation is that the inflation and output dynamics is not adequately captured by the autoregressive model of Table 6. A change in the monetary policy regime may appear to cause output if an underlying persistence not captured by lagged output influences the regime.

To address this concern, we examine in this section a simple model of three variables, x (the output gap), r (the policy rate), and s (the monetary policy regime). In the model, x is exogenous to r. The x process is the hidden-state Markov-switching model of Hamilton (1989):

$$x_t = \mu(a_t) + \phi(a_t)x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2(a_t)). \tag{8.1}$$

Here, $\{a_t\}$ is a two-state Markov chain with $a_t = 1, 2$. Unlike the monetary policy regime s_t, it is a hidden regime that is unobservable to both the central bank and the econometrician. The model is

[38] One way of avoiding those restrictions is to assume the reduced form is the same under W and P. But this amounts to assuming that the exit condition had no effect during QE1.

completed by the censored Taylor rule:

$$r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \quad r_t^* \equiv \alpha_r^* + \beta_r^* x_t, & \text{if } s_t = \mathrm{P}, \\ 0 & \text{if } s_t = \mathrm{Z}. \end{cases} \tag{8.2}$$

$$s_t = \begin{cases} \mathrm{P} \quad \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r) r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > 0, \quad v_t \sim \mathcal{N}(0, \sigma_r^2), \\ \mathrm{Z} \quad \text{otherwise.} \end{cases} \tag{8.3}$$

The model's parameters are: those describing the x process ($\mu(1)$, $\mu(2)$, $\phi(1)$, $\phi(2)$, $\sigma^2(1)$, $\sigma^2(2)$ and the 2×2 transition probability matrix of the Markov chain) and those describing the Taylor rule ($\alpha_r^*, \beta_r^*, \rho_r, \sigma_r$).

For the Taylor rule parameters, we set $\alpha_r^* = 1\%$, $\beta_r^* = 0.5$, $\rho_r = 15\%$, and $\sigma_r = 0.13$. For the x process parameters, we set

$$\mu(1) = -0.063,\ \mu(2) = 0.233,\ \phi(1) = 0.938,\ \phi(2) = 0.930,\ \sigma^2(1) = 0.385,\ \sigma^2(2) = 7.45,$$

and the diagonal elements of the transition matrix is $(0.976, 0.640)$. This is the maximum likelihood estimate on the actual output gap data for $t =$ February 1988, ..., December 2012.[39] We use this calibrated model as the DGP (data generating process).

The question of interest is, how would the PZ-IR look if calculated by the same procedure as in the IR analysis of Section 6. The procedure given a sample $\{x_t, r_t, s_t\}_{t=1}^T$ generated by the DGP is: (i) fit the regime-dependent autoregressive model:

$$x_t \ \text{ is regressed on } \begin{cases} \text{the constant, } x_{t-1}, r_{t-1} & \text{if } s_{t-1} = \mathrm{P}, \\ \text{the constant, } x_{t-1} & \text{if } s_{t-1} = \mathrm{Z}, \end{cases} \tag{8.4}$$

(ii) estimate by Tobit the Taylor rule parameters, and (iii) calculate PZ-IR by the Monte Carlo integration. The definition of PZ-IR for the model is

$$\text{(PZ-IR)} \qquad \mathrm{E}\Big(x_k \,|\, s = \mathrm{Z}, \underbrace{(\bar{x}, 0)}_{(x_t, r_t) \text{ under the alternative}}\Big) - \mathrm{E}\Big(x_k \,|\, s = \mathrm{P}, \underbrace{(\bar{x}, 0)}_{(x_t, r_t) \text{ under the baseline}}\Big), \ k = 1, 2, ..., \tag{8.5}$$

---

[39] We used a Matlab package written by Marcelo Perlin downloadable from the web.

where $\bar{x}$ is the value of x in the base period.

We use the DGP to create a large number of samples and for each sample we calculate the PZ-IR profile by following steps (i)-(iii).[40] This generates a distribution of the PZ-IR profiles from which we create the error band by picking the 84% and 16% percentiles for each k. Figure 7 shows the error band. It is narrow and contains the horizontal line. The model of this section does not generate the sort of the IR profiles that we found in Section 6.

9 Conclusions

We have constructed a regime-switching SVAR in which the regime is determined by the central bank responding to economic conditions. The model was used to study the dynamic effect of not only changes in the policy rate and the reserve supply but also changes in the regime chosen by the central bank. Our impulse response analysis yields three major conclusions.

- By including a measure of the real interest rate in the Taylor rule, we provide a resolution of the price puzzle for Japan. The response of inflation to a policy rate cut, while initially negative, eventually becomes positive.
- Consistent with the existing literature, we find that an increase in the reserve supply under QE raises output and inflation.
- However, there is an entry cost to QE. That is, the effect of entering QE with no significant increase in the reserve supply is contractionary. If the central bank wishes to raise inflation and output by entering QE, it has to aggressively raise the reserve supply upon entry. The flip side of the entry cost is an exit bonus that exiting from QE is expansionary if the reserve supply at the time of the exit is not too large. Our evidence indicates that the critical level of the

[40] Further details are as follows. The number of datasets generated by the DGP is 500. The initial condition for the DGP is $(a_0, x_0, r_0, s_0) = (1, 0.781, 3.665, \mathrm{P})$. The initial values for (x, r) are their January 1988 values. The sample size is $T = 300$, the number of months between January 1988 and December 2012. We set $\bar{x} = 0$. The number of simulations for the Monte Carlo integration is $1,000$.

actual-to-required reserve ratio below which exiting from QE is expansionary is somewhere between 3 and 4.5.

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Table 1: Policy Announcements by the Bank of Japan, 1999-2012

date	quotes and URLs
1999.2.12	"The Bank of Japan will provide more ample funds and encourage the uncollateralized overnight call rate to move as low as possible." http://www.boj.or.jp/en/announcements/release_1999/k990212c.htm/
1999.4.13	"(The Bank of Japan will) continue to supply ample funds until the deflationary concern is dispelled." (A remark by governor Hayami in a Q & A session with the press. Translation by authors.) http://www.boj.or.jp/announcements/press/kaiken_1999/kk9904a.htm/
1999.9.21	"The Bank of Japan has been pursuing an unprecedented accommodative monetary policy and is explicitly committed to continue this policy until deflationary concerns subside." http://www.boj.or.jp/en/announcements/release_1999/k990921a.htm/
2000.8.11	"... the downward pressure on prices ... has markedly receded. ... deflationary concern has been dispelled, the condition for lifting the zero interest rate policy." http://www.boj.or.jp/en/announcements/release_2000/k000811.htm/
2001.3.19	"The main operating target for money market operations be changed from the current uncollateralized overnight call rate to the outstanding balance of the current accounts at the Bank of Japan. Under the new procedures, the Bank provides ample liquidity, and the uncollateralized overnight call rate will be determined in the market ... The new procedures for money market operations continue to be in place until the consumer price index (excluding perishables, on a nationwide statistics) registers stably a zero percent or an increase year on year." http://www.boj.or.jp/en/announcements/release_2001/k010319a.htm/
2003.10.10	"The Bank of Japan is currently committed to maintaining the quantitative easing policy until the consumer price index (excluding fresh food, on a nationwide basis) registers stably a zero percent or an increase year on year." http://www.boj.or.jp/en/announcements/release_2003/k031010.htm/
2006.3.9	"... the Bank of Japan decided to change the operating target of money market operations from the outstanding balance of current accounts at the Bank to the uncollateralized overnight call rate... The Bank of Japan will encourage the uncollateralized overnight call rate to remain at effectively zero percent. ... The outstanding balance of current accounts at the Bank of Japan will be reduced towards a level in line with required reserves. ... the reduction in current account balance is expected to be carried out over a period of a few months.... Concerning prices, year-on-year changes in the consumer price index turned positive. Meanwhile, the output gap is gradually narrowing. ... In this environment, year-on-year changes in the consumer price index are expected to remain positive. The Bank, therefore, judged that the conditions laid out in the commitment are fulfilled." http://www.boj.or.jp/en/announcements/release_2006/k060309.htm/
2006.7.14	"... the Bank of Japan decided ... to change the guideline for money market operations... The Bank of Japan will encourage the uncollateralized overnight call rate to remain at around 0.25 percent." http://www.boj.or.jp/en/announcements/release_2006/k060714.pdf/
2008.12.19	"... it (author note: meaning the policy rate) will be encouraged to remain at around 0.1 percent (author note: which is the rate paid on reserves)..." http://www.boj.or.jp/en/announcements/release_2008/k081219.pdf
2009.12.18	"The Policy Board does not tolerate a year-on-year rate of change in the CPI equal to or below 0 percent." http://www.boj.or.jp/en/announcements/release_2009/un0912c.pdf
2010.10.5	"The Bank will maintain the virtually zero interest rate policy until it judges, on the basis of the 'understanding of medium- to long-term price stability' that price stability is in sight..." http://www.boj.or.jp/en/announcements/release_2010/k101005.pdf
2012.2.14	"The Bank will continue pursuing the powerful easing until it judges that the 1 percent goal is in sight..." http://www.boj.or.jp/en/announcements/release_2012/k120214a.pdf

Table 2: Winding-down of QE2, March to August 2006

	March	April	May	June	July	August
regime (P for normal, Z for zero-rate/QE)	Z	Z	Z	Z	P	P
ratio of actual to required reserves	4.5	2.7	1.7	1.6	1.0	1.0
m, log of the above ratio	1.51	1.00	0.55	0.46	0	0
r, the policy rate (% per year)	0.0	0.0	0.0	0.0	0.26	0.25
p, monthly inflation rate (% per year)	1.1	−1.4	0.9	0.1	−0.3	0.5
π, year-on-year inflation rate (% per year)	0.1	−0.1	0.0	0.2	0.2	0.3
x, output gap (%)	−0.7	−0.4	−0.6	−0.5	−0.8	−0.5

Note: The ratio of actual to required reserves for July and August 2006, which was 1.2 (July) and 1.1 (August) in data, is set to 1.0. The policy rate under the zero-rate regime is set equal to $\bar{r}$ (the rate paid on reserves) which before November 2008 is 0%.

Table 3: Simple Statistics, January 1988 - December 2012

	p (monthly inflation rate, % per year)	π (12-month inflation rate, %)	x (output gap, %)	r (policy rate, % per year)	m (excess reserve rate)	trend growth rate, %
subsample P (sample size = 170)						
mean	0.802	0.847	−0.219	2.640	0.007	2.129
std. dev.	1.569	1.003	1.929	2.582	0.022	1.474
max	5.565	3.229	4.868	8.261	0.206	4.796
min	−3.917	−0.904	−4.482	0.075	0.0	0.355
QE1 (March 1999-July 2000, sample size= 17)						
mean	−0.230	−0.104	−2.996	0.0	0.098	0.725
std. dev.	0.529	0.086	0.919	0.0	0.069	0.025
max	0.938	0.014	−1.354	0.0	0.275	0.755
min	−1.069	−0.224	−4.328	0.0	0.041	0.679
QE2 (March 2001-June 2006, sample size= 64)						
mean	−0.299	−0.408	−2.184	0.0	1.379	0.990
std. dev.	1.106	0.390	1.159	0.0	0.545	0.070
max	2.273	0.196	−0.395	0.0	1.849	1.126
min	−2.911	−1.066	−4.335	0.0	0.078	0.863
QE3 (December 2008-December 2012, sample size= 49)						
mean	−0.531	−0.498	−3.783	0.1	0.941	0.499
std. dev.	1.418	0.462	2.136	0.0	0.417	0.577
max	3.477	0.270	−1.130	0.1	1.701	1.963
min	−3.705	−1.279	−9.494	0.1	0.349	−0.840

Note: The last column is the 12-month growth rate of potential GDP, defined as 100 times the log difference between the potential GDP of the current month and that of 12 month prior.

Table 4: Taylor Rule, January 1988 - December 2012 (sample size = 300)

Mieno dummy coefficient (% per year)	banking crisis dummy coefficient (% per year)	trend growth rate coefficient (% per year)	inflation coefficient	output coefficient	speed of adjutment (ρ_t, % per month)	std. dev. of error (σ_r, % per year)	mean of threshold ($\overline{\pi}$, % per year)	std. dev. of threshold ($\sigma_{\overline{\pi}}$, % per year)
2.5 [7.0]	−0.37 [−1.8]	0.98 [11.5]	0.75 [5.2]	0.07 [1.4]	14.1 [6.1]	0.134 (0.0073)	0.53 (0.43)	0.33 (0.25)

Note: Estimation by the ML (maximum likelihood) method described briefly in the text and more fully in Appendix 2. t-values in brackets and standard errors in parentheses. The Taylor rule is described as follows:

$$\text{(censored Taylor rule)} \quad r_t = \begin{cases} \underbrace{\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}}, \ v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } s_t = \mathrm{P}, \\ \bar{r}_t & \text{if } s_t = \mathrm{Z}, \end{cases} , \quad \text{(desired Taylor rate)} \ r_t^* \equiv \alpha_r^* + \underset{(1\times 2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix},$$

where the regime s_t is given by

$$\begin{cases} \text{If } s_{t-1} = \mathrm{P}, \ s_t = \begin{cases} \mathrm{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t, \\ \mathrm{Z} & \text{otherwise.} \end{cases} \\ \text{If } s_{t-1} = \mathrm{Z}, \ s_t = \begin{cases} \mathrm{P} & \text{if } \underbrace{\rho_r r_t^* + (1-\rho_r)r_{t-1} + v_{rt}}_{\text{shadow Taylor rate}} > \bar{r}_t \text{ and } \pi_t > \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ threshold}}, \ v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma_{\overline{\pi}}^2), \\ \mathrm{Z} & \text{otherwise.} \end{cases} \end{cases}$$

The intercept α_r^* in the desired Taylor rate depends on: the Mieno dummy (1 for December 1989-June 1991, 0 otherwise), the banking crisis dummy (1 for September 1995-July 1998, 0 otherwise), and the trend growth rate (the 12-month growth rate of potential output). The inflation and output coefficients are the first and second element of $\boldsymbol{\beta}_r^*$. The speed of adjustment is the ρ_r in the shadow rate defined above.

Table 5: Excess Reserve Supply Equation

t is in	coefficient of				R^2	σ_s (%)
	const	π_t	x_t	m_{t-1}		
QE2 & QE3	−0.013	−0.009	−0.018	0.98	0.94	0.132
(113 obs.)	[−0.2]	[−0.2]	[−2.2]	(0.033)		(0.0088)

Note: Estimation by OLS. t-values in brackets and standard errors in parentheses. m_t is the exces reserve rate, π_t is the 12-month inflation rate to month in percents t, x_t is the output gap in percents, σ_s (standard deviation of the error) is estimated as $\widehat{\sigma_s} = \sqrt{SSR/n}$ where n is the sample size. The standard error of $\widehat{\sigma_s}$ is calculated as $\frac{\widehat{\sigma_s}}{\sqrt{2n}}$.

Table 6: Inflation and Output Reduced Form

lagged subsample P, February 1988 - February 1995

s_{t-1} is in	dependent variable	const.	g_t	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
P (85 obs.)	inflation (p_t)	−0.36	−0.027	−0.09	0.02	0.45		0.32
		[−0.4]	[−0.2]	[−0.8]	[0.1]	[2.6]		
	output (x_t)	−3.69	0.48	−0.09	0.31	0.53		0.83
		[−5.9]	[3.9]	[−1.1]	[3.0]	[4.5]		

lagged subsample P, March 1995 - December 2008

s_{t-1} is in	dependent variable	const.	g_t	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
P (85 obs.)	inflation (p_t)	0.12	−0.51	−0.09	0.13	0.44		0.07
		[0.3]	[−0.7]	[−0.7]	[1.5]	[0.8]		
	output (x_t)	−0.88	1.31	−0.01	0.98	−0.52		0.79
		[−2.8]	[2.8]	[−0.2]	[17]	[−1.5]		

lagged subsample Z

s_{t-1} is in	dependent variable	const.	g_t	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
QE2 & QE3 (112 obs.)	inflation (p_t)	−0.57	−0.24	−0.06	0.12	0.49	0.56	0.10
		[−1.0]	[−0.8]	[−0.6]	[1.6]	[0.2]	[2.2]	
	output (x_t)	−0.99	0.03	0.08	0.79	−0.66	0.40	0.80
		[−2.6]	[0.1]	[1.3]	[15]	[−0.3]	[2.3]	

Note: Estimation by OLS. t-values in brackets. p is the monthly inflation rate in percents per year, x is the output gap in percents, r is the policy rate in percents per year, m is the excess reserve rate (defined as the log of the ratio of actual to required reserves), and g is the trend growth rate (the 12-month growth rate in percents of potential output). The Mieno (anti-bubble) dummy (1 if December 1989 $\leq t \leq$ June 1991) and the banking crisis dummy (1 if September 1995 $\leq t \leq$ July 1998) are included in the regressions on lagged subsample P but their coefficients are not reported here; they are not significantly different from zero. There is no need to include those dummies on lagged subsample Z because their value is zero. The value of r_{t-1} is 0 (percent) for (QE1 and) QE2, and 0.1 (percent) for QE3.

Table 7: Inflation and Output Reduced Form, with Occasionally Positive Excess Reserve Demand

lagged subsample P, March 1995 - December 2008								
		coefficient of						
s_{t-1} is in	dependent variable	const.	g_t	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
P (85 obs.)	inflation (p_t)	0.31	−0.73	−0.08	0.10	0.48	−4.1	0.08
		[0.6]	[−1.0]	[−0.7]	[1.0]	[0.9]	[−0.8]	
	output (x_t)	−0.56	0.95	−0.01	0.92	−0.46	−6.9	0.80
		[−1.6]	[1.9]	[−0.2]	[15]	[−1.3]	[−2.0]	

lagged subsample W								
		coefficient of						
s_{t-1} is in	dependent variable	const.	g_t	p_{t-1}	x_{t-1}	r_{t-1}	m_{t-1}	R^2
QE1 (17 obs.)	inflation (p_t)	0.46		−0.19	0.11		−3.9	0.19
		[0.7]		[−0.7]	[0.6]		[−1.7]	
	output (x_t)	−2.2		−0.03	0.45		6.5	0.62
		[−3.0]		[−0.1]	[2.3]		[2.5]	

Note: Estimation by OLS. t-values in brackets. p is the monthly inflation rate in percents per year, x is the output gap in percents, r is the policy rate in percents per year, m is the excess reserve rate (defined as the log of the ratio of actual to required reserves), and g is the trend growth rate (the 12-month growth rate in percents of potential output). The banking crisis dummy (1 if September 1995 $\leq t \leq$ July 1998) is included in the regressions on lagged subsample P but its coefficient is not reported here. The trend growth rate g_t is excluded for lagged subsample W to avoid near-multicollinearity with the constant.

Figure 1a: Plot of Net Policy Rate against Excess Reserve Rate, 1988-2012

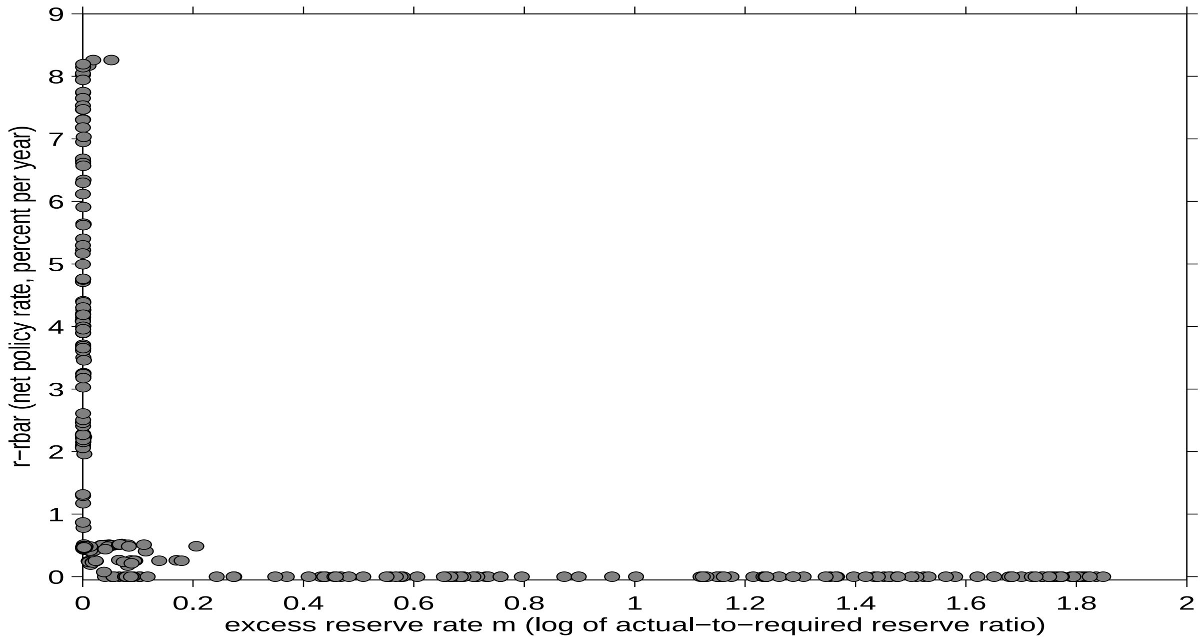


Figure 1b: Plot of Net Policy Rate against Excess Reserve Rate, Near Origin

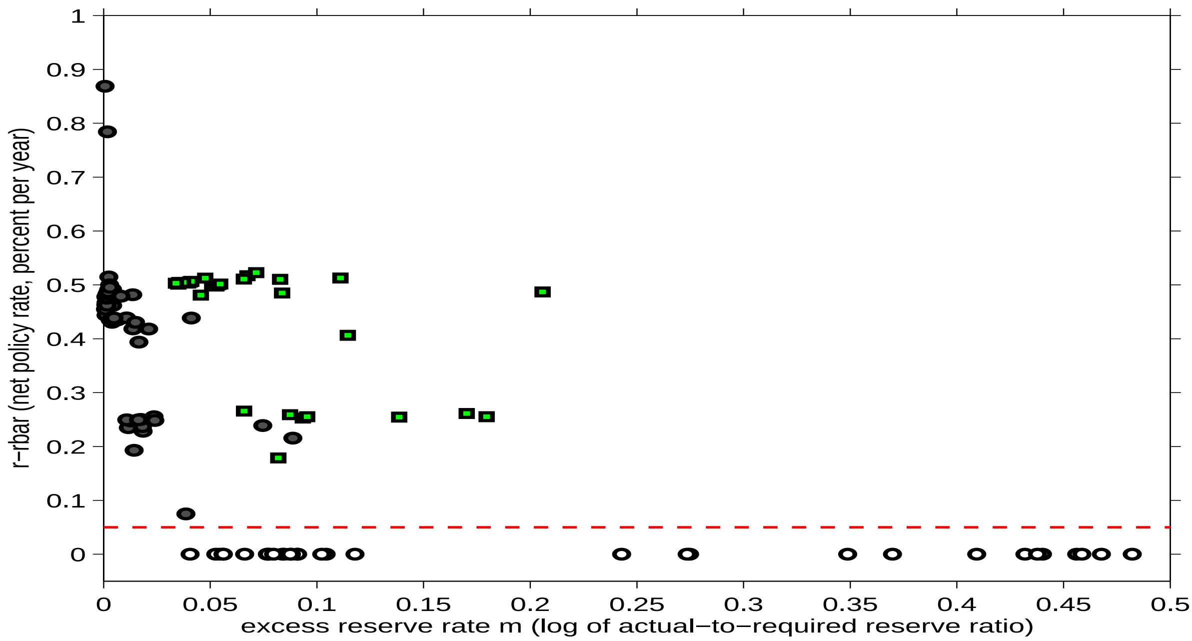
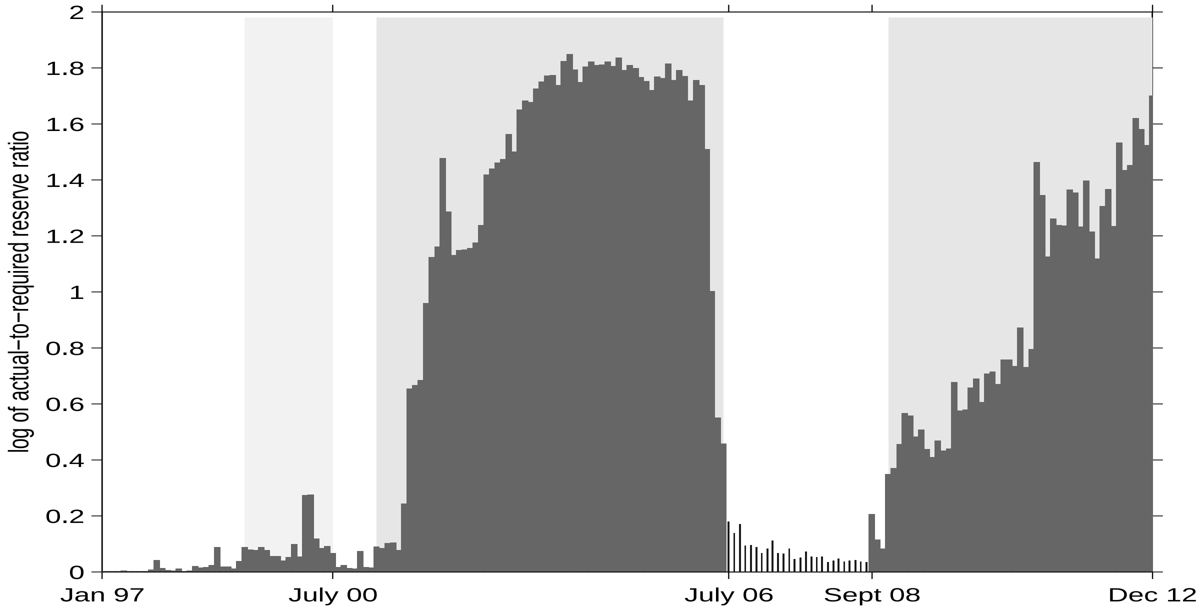


Figure 2a: Excess Reserve Rate, 1997 - 2012



Note: The shades indicate the three spells of the zero-rate period.

Figure 2b: Actual and Potential Monthly GDP, 1988 - 2012

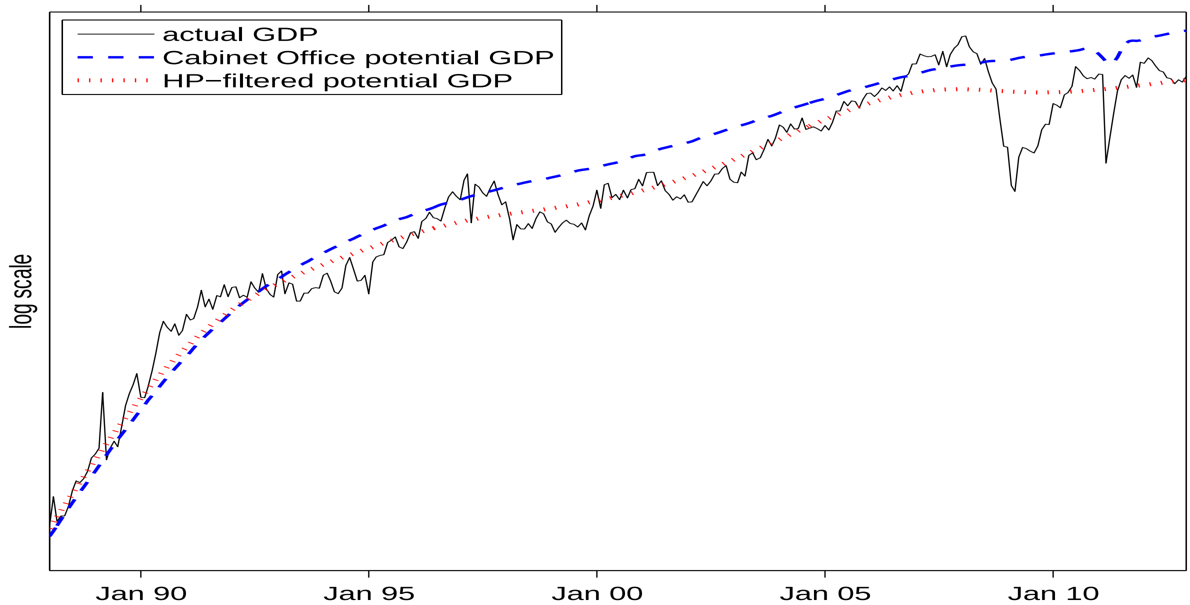
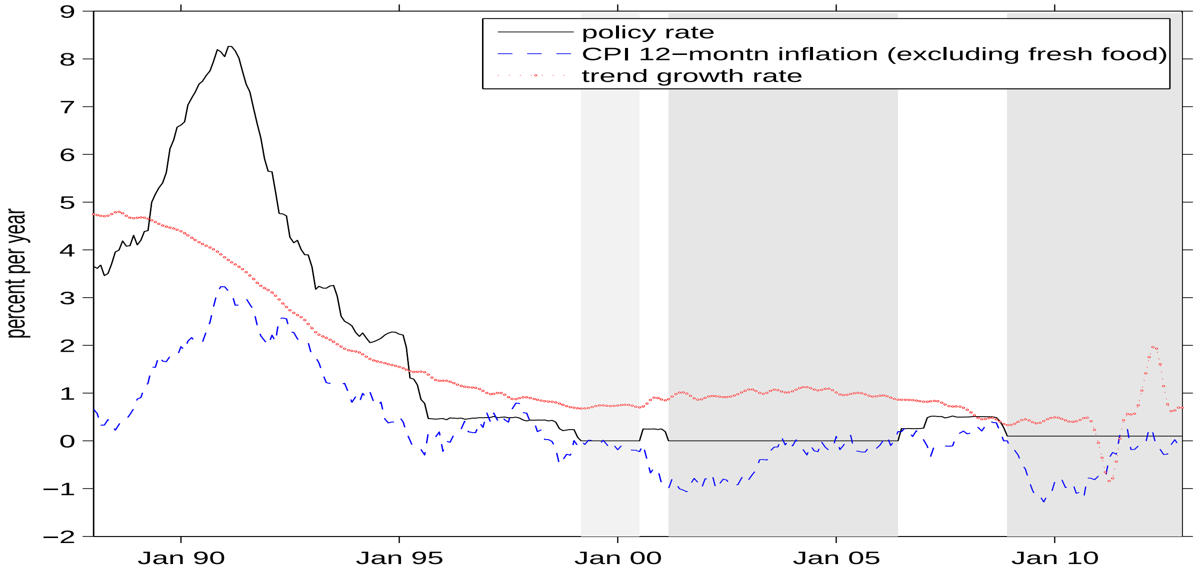
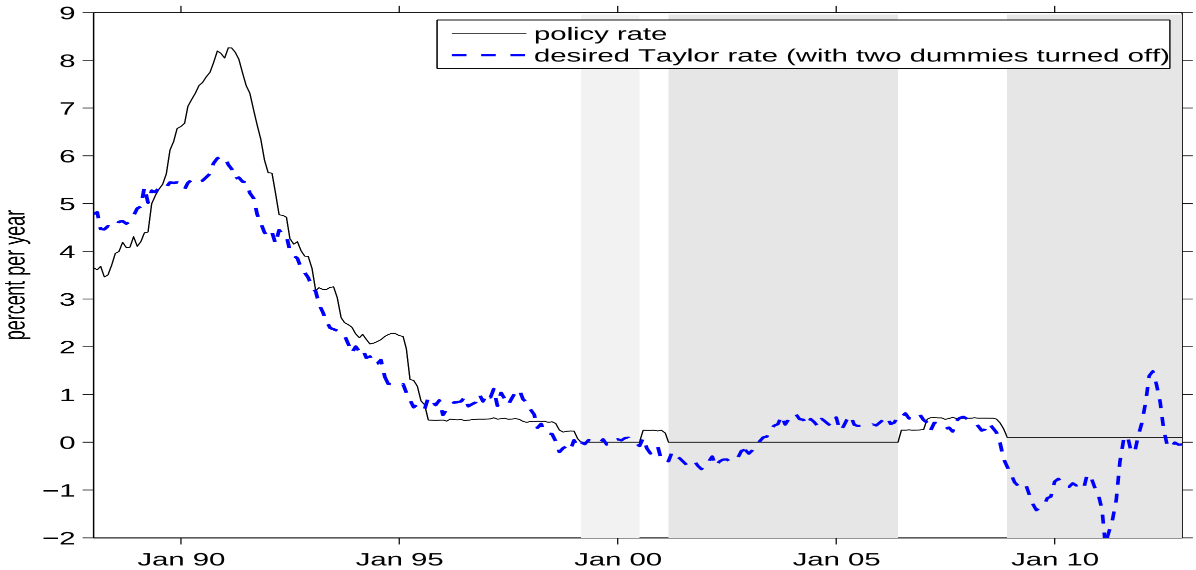


Figure 2c: Policy Rate, Inflation, and Trend Growth Rate, 1988-2012



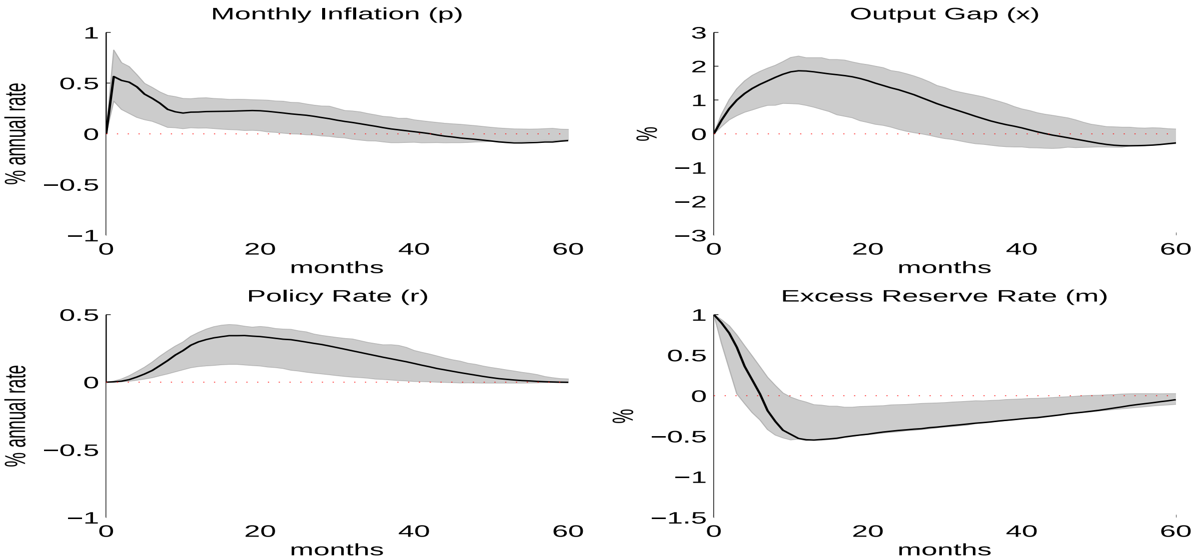
Note: The shades indicate the three spells of the zero-rate period.

Figure 3: Policy Rate and Desired Taylor Rate, 1988 - 2012



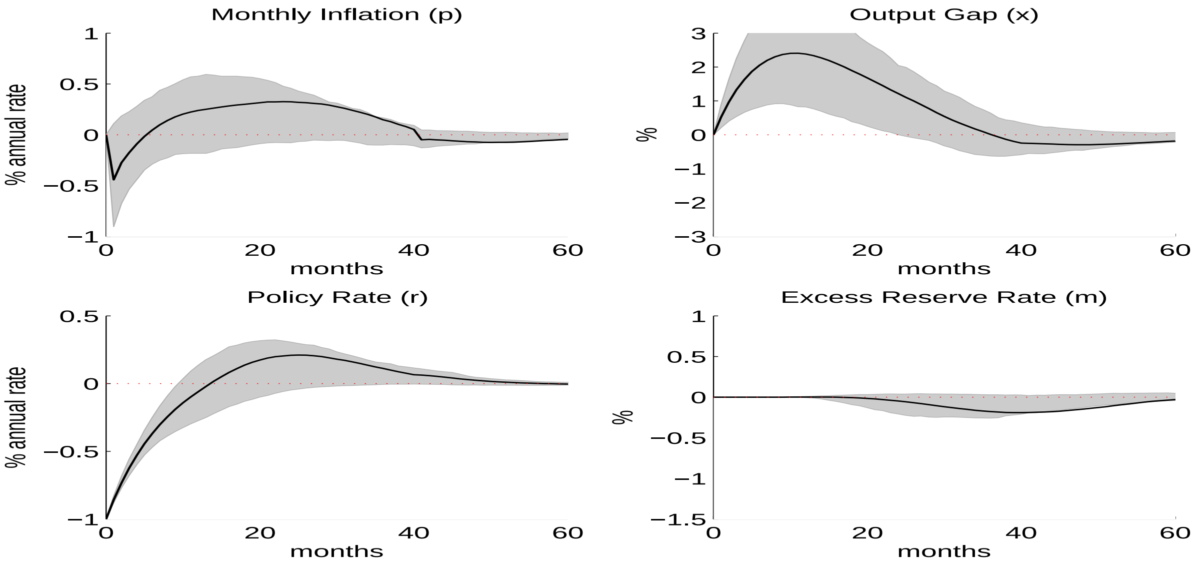
Note: The desired Taylor rate is the r_t^* defined in (4.1). The intercept α_r^* depends on the trend growth rate and the two dummies (the Mieno dummy for December 1989-June 1991) and the banking crisis dummy for September 1995 and July 1998). In the plotted desired Taylor rate, the two dummies are set to zero for all months. The shades indicate the three spells of the zero-rate period.

Figure 4a: m-IR (Impulse Response to m), the base period is February 2004



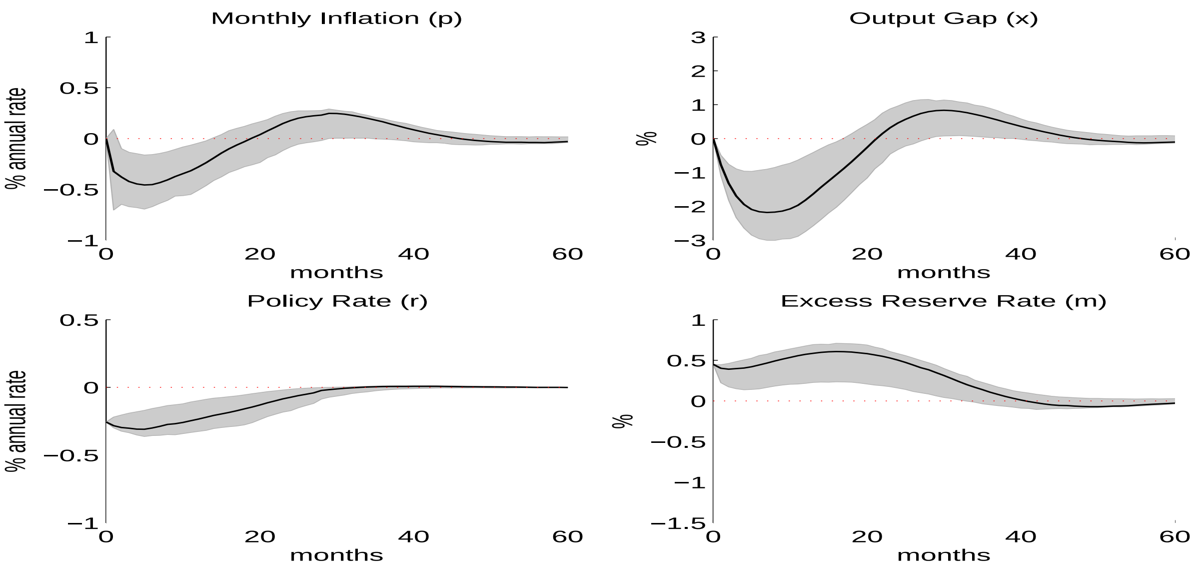
Note: The m-IR is defined in (6.2). The perturbation size is 1 as indicated in the lower-right panel. The 68% probability bands in shades.

Figure 4b: r-IR (Impulse Response to r), the base period is March 1995



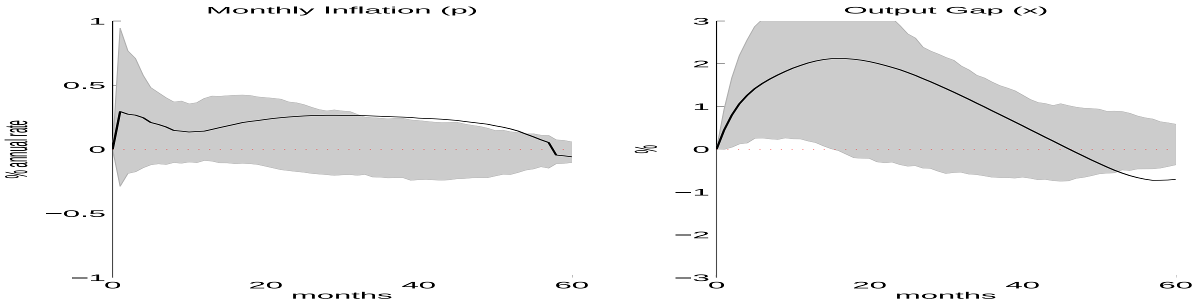
Note: The r-IR is defined in (6.3). The perturbation size is -1 percentage point as indicated in the lower-left panel. The 68% probability bands in shades.

Figure 4c: Effect of Extending QE2 to July 2006



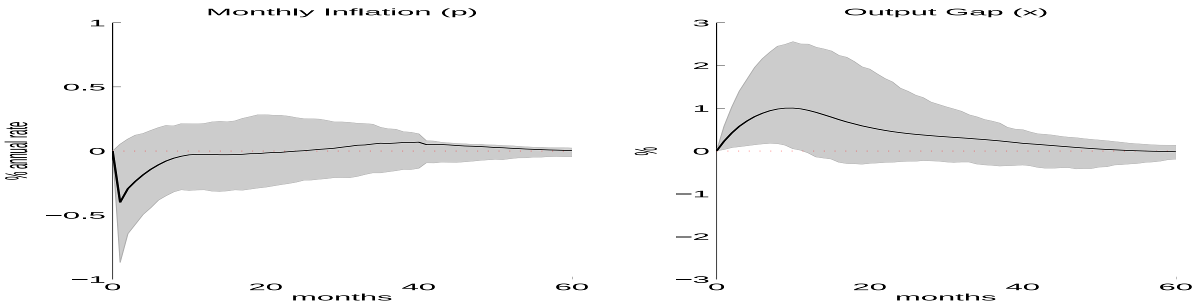
Note: The impulse response is defined in (6.5). The perturbation occurs to *m*, *r*, and the regime. The 68% probability bands in shades.

Figure 5a: m-IR, February 2004, HP-Filtered Potential Output



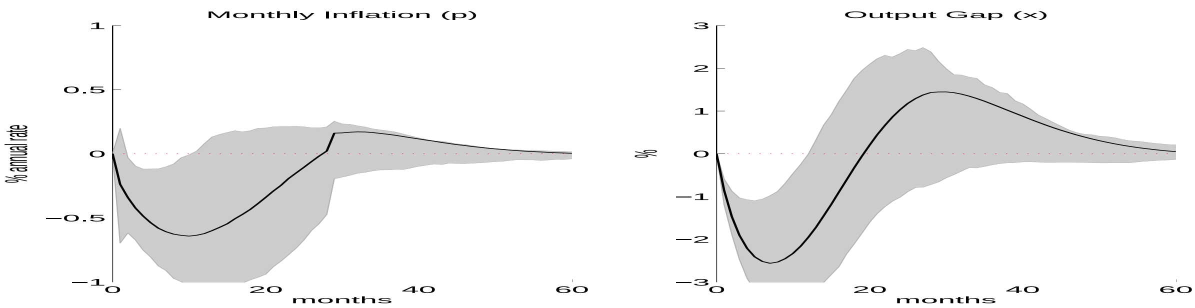
Note: See note to Figure 4a. Only the upper panels are shown.

Figure 5b: r-IR, March 1995, HP-Filtered Potential Output



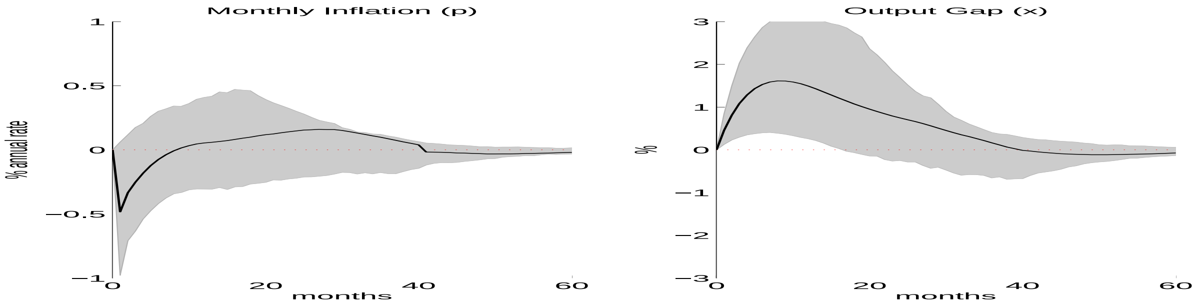
Note: See note to Figure 4b. Only the upper panels are shown.

Figure 5c: Effect of Extending QE2 to July 2006, HP-Filtered Potential Output



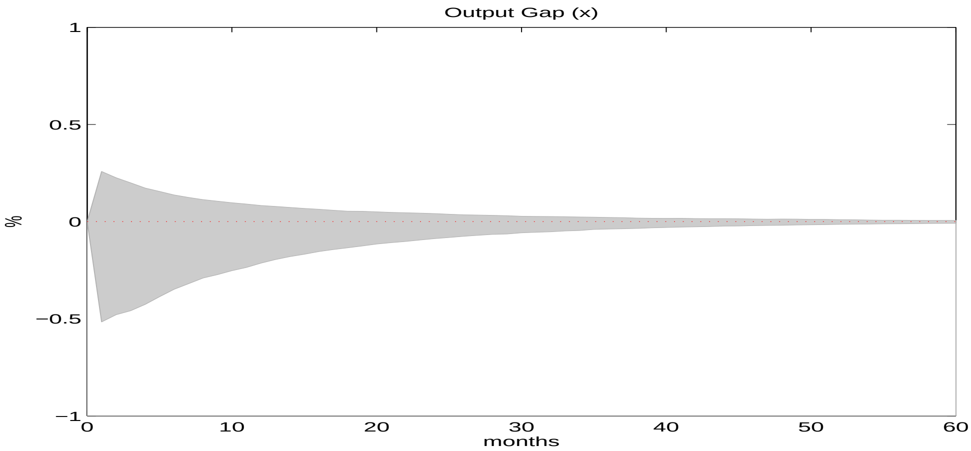
Note: See note to Figure 4c. Only the upper panels shown.

Figure 6: r-IR, March 1995, with Demand for Excess Reserves



Note: The size of perturbation to r is -1 percentage point. The excess reserve rate is not constrained to be 0 under the normal regime P. It is given by the excess reserve demand equation (7.1).

Figure 7: PZ-IR from the Simple Bivariate Model



Note: The profile of output response to a regime change from P to Z for the bivariate model of Section 8. The 68% probability bands in shades.

Appendix 1 Data Description

This appendix describes how the variables used in the paper — p (monthly inflation), π (12-month inflation), x (output gap), r (the policy rate), $\bar{r}$ (the interest rate paid on reserves), and m (the excess reserve rate) — are derived from various data sources.

Monthly and Twelve-Month Inflation Rates (p and π)

The monthly series on the monthly inflation rate (appearing in the inflation and output reduced-form) and the 12-month inflation rate (in the Taylor rule and the excess reserve supply equation) are constructed from the CPI (consumer price index). The Japanese CPI is compiled by the Ministry of Internal Affairs and Communications of the Japanese government. The overall CPI and its various subindexes can be downloaded from the portal site of official statistics of Japan called"e-Stat". The URL for the CPI is
`http://www.e-stat.go.jp/SG1/estat/List.do?bid=000001033702&cycode=0`.
This page lists a number of links to CSV files. One of them,
`http://www.e-stat.go.jp/SG1/estat/Csvdl.do?sinfid=000011288575`
has the "core" CPI (CPI excluding fresh food), the "core-core" CPI (CPI excluding food and energy), and other components from January 1970. They are seasonally *un*adjusted series and combine different base years from January 1970. For how the Ministry combines different base years, see Section III-6 of the document (in Japanese) downloadable from
`http://www.stat.go.jp/data/cpi/2010/kaisetsu/index.htm#p3`
Briefly, to combine base years of 2005 and 2010, say, the Ministry multiplies one of the series by a factor called the "link factor" whose value is such that the two series agree on the average of monthly values for the year 2005.

Twelve-month inflation rates constructed from the (seasonally unadjusted) "core" CPI and the "core-core" CPI are shown in Appendix Figure 1. The two humps for 1989 and 1997 are due to the increases in the consumption tax. The two inflation rates behave similarly, except for the period November 2007 - May 2009.

The above URL has another CSV file, whose link is
`http://www.e-stat.go.jp/SG1/estat/Csvdl.do?sinfid=000011288581`,
has *seasonally adjusted* series for various subindexes (including the "core-core" CPI), but only from January 2005. As explained below, we use the "core-core" CPI between November 2007 and May 2009 that is seasonally adjusted, along with the seasonally unadjusted "core" CPI, in order to construct p (monthly inflation) and π (12-month inflation). The construction involves three steps.

Adjustment for Consumption Tax Hikes. The consumption tax rate rose from 0% to 3% in April 1989 and to 5% in April 1997. We compute the 12-month inflation rate from the seasonally unadjusted index (as the log difference between the current value of the index and the value 12 months ago) and subtract 1.2% for t = April 1989,..., March 1990 (to remove the effect of the April 1989 tax hike) and 1.5% for t = April 1997,..., March 1998 (to remove the effect of the April 1997 tax hike). These two numbers (1.2% and 1.5%) are taken from *Price Report* (various years) by the Economic Planning Agency of the Japanese government (which became a part of the Cabinet Office). We then calculate the index so that its implied 12-month inflation agrees with the tax-adjusted 12-month inflation.

Seasonal Adjustment. We apply the U.S. Census X12-ARIMA method to the seasonally unadjusted (but consumption tax-adjusted) "core" index from January 1987 through December 2012 (26 years). The Census's program can be downloaded from:
`https://www.census.gov/srd/www/winx12/winx12_down.html`
The specification for the seasonal adjustment is the same as the one used by the Ministry (of Internal Affairs and Communications of the Japanese government) for seasonally adjust various CPI subindexes mentioned above. Their spec file for the Censu's X12-ARIMA program is available from
`http://www.stat.go.jp/data/cpi/2010/kaisetsu/pdf/3-7.pdf`.
For example, the ARIMA order is $(0, 1, 1)$. There is no adjustment for the holiday effect.

Adjustment for the 2007-2008 Energy Price Swing. Let CPI_{1t} be the seasonally adjusted "core" CPI obtained from this operation for t = January 1970,..., December 2012. Let CPI_{2t} be the seasonally adjusted "core-core" CPI for t = January 2005,..., December 2012 that is directly available from the above CSV file. Our CPI measure (call it CPI) is CPI_1, except that we switch from CPI_1 to CPI_2 between November 2007 and May 2009 to remove the large movement in the energy component of the "core" CPI. More precisely,

$$CPI_t = \begin{cases} CPI_{1t} & \text{for } t = \text{January 1970}, ..., \text{October 2007}, \\ CPI_{t-1} \times \dfrac{CPI_{2t}}{CPI_{2,t-1}} & \text{for } t = \text{November 2007}, ..., \text{May 2009}, \\ CPI_{t-1} \times \dfrac{CPI_{1t}}{CPI_{1,t-1}} & \text{for } t = \text{June 2009}, ..., \text{December 2012}. \end{cases} \tag{A1.1}$$

That is, the "core" CPI (the CPI excluding fresh food) monthly inflation rate is set equal to that given by the "core-core" CPI (the CPI excluding food and energy) for those months. This is the only period during which the two CPI measures give substantially different inflation rates, see Appendix Figure 1.

Finally, the monthly inflation rate for month t, p_t, is calculated as

$$p_t \equiv 1200 \times [\log(CPI_t) - \log(CPI_{t-1})]. \tag{A1.2}$$

The 12-month inflation rate for month t, π_t, is

$$\pi_t \equiv 100 \times [\log(CPI_t) - \log(CPI_{t-12})]. \tag{A1.3}$$

Excess Reserve Rate (m)

Monthly series on actual and required reserves are available from September 1959. The source is the BOJ's portal site `http://www.stat-search.boj.or.jp/index_en.html/`. The value for month t is defined as the average of daily balances over the reserve maintenance period of the 16th day of month t to the 15th day of month $t+1$. We define the excess reserve rate for month t (m_t) as

$$m_t \equiv [\log(\text{actual reserve balance for month } t) - \log(\text{required reserve balance for month } t)]. \tag{A1.4}$$

We make two changes on the series. First, as was argued in Section 3, observed reserves after QE2 (which ends June 2006) and before the Lehman crisis of September 2008 do not seem to represent demand. For this reason we set $m_t = 0$ for t =July 2006,..., August 2008. Second, there is a Y2K spike in m for t =December 1999 (which is for the reserve maintenance period of December 16, 1999 thruough January 15, 2000). We remove this spike by the average of m over the QE1 months (March 1999 - July 2000) excluding December 1999.

Interest Rate paid on Reserves ($\bar{r}$)

$\bar{r}_t$ is 0% until October 2008 and 0.1% since November 2008.

The Policy Rate (r)

We obtained daily data on the uncollateralized overnight "Call" rate (the Japanese equivalent of the U.S. Federal Funds rate) since the inception of the market (which is July 1985) from *Nikkei* (a data vendor maintained by a subsidiary of *Nihon Keizai Shinbun* (the Japan Economic Daily)). The policy rate for month t, r_t, for t = August 1985,...,December 2012 is the average of the daily values over the reserve maintenance period of the 16th of month t to the 15th of month $t+1$.

In Section 3 of the text, we defined the zero-rate period as months for which the net policy rate $r_t - \bar{r}_t$ is less than 5 basis points. We ignore variations within the 5 basis points by setting $r_t - \bar{r}_t = 0$ for the zero-rate periods.

Monthly Output Gap (x)

The Three Series. Three quarterly series go into our monthly output gap construction: (i) quarterly seasonally adjusted real GDP (from the National Income Accounts (NIA), compiled by the Cabinet Office of the Japanese government), (ii) the monthly "all-industry activity index" (compiled by the Ministry of Economy, Trade, and Industry of the Japanese government (METI) available from January 1988), and (iii) the quarterly GDP gap estimate by the Cabinet Office of the Japanese government. We first provide a description of those series along with their sources.

(i) **Quarterly NIA GDP**

Japanese NIA in general. The Japanese national accounts adopted the chain-linking method in 2004. Quarterly chain-linked real GDP series (seasonally-adjusted) are available from the Cabnet Office. The relevant homepage is
`http://www.esri.cao.go.jp/en/sna/sokuhou/sokuhou_top.html`.

Quarterly GDP from 1994:Q1 ($GDP1$). The current quarterly estimates are continuously revised by the Cabinet Office. We used the "Quarterly Estimates of GDP Jan.-Mar. 2014 (The Second Preliminary)(Benchmark year=2005)", released on June 9, 2014 and available from the above homepage. The CSV file holding this series is:
`http://www.esri.cao.go.jp/jp/sna/data/data_list/sokuhou/files/2014/qe141_2/__icsFiles/afieldfile/2014/06/04/gaku-jk1412.csv`.
The latest quarter is 2014:Q1 (the first quater of 2014). For later reference, call this series "$GDP1$". The series goes back only to 1994:Q1.

Quarterly GDP from 1980:Q1 ($GDP2$). Recently, the Cabinet Office released the chain-linked GDP series (for the same benchmark year of 2005) since 1980. The homepage from which this series can be downloaded is
`http://www.esri.cao.go.jp/jp/sna/sonota/kan-i/kan-i_top.html`,
which unfortunately is in Japanese. The URL for the Excel file holding this series is
`http://www.esri.cao.go.jp/jp/sna/data/data_list/kan-i/files/pdf/gaku-jk_kan-i.xls`.
The URL for the documentation (in Japanese) is
`http://www.esri.cao.go.jp/jp/sna/data/data_list/kan-i/files/pdf/gaiyou.pdf`.
This series, call it "$GDP2$", is from 1980:Q1 to the 1995:Q1.

Linking $GDP1$ and $GDP2$. Because the seasonal adjustment underlying the continuously revised current GDP series, whose first quarer is 1994:Q1, is retroactive and alters the whole series at each release, there is a slight difference between $GDP1_t$ (at $447,159.1$ trillin yen) and $GDP2_t$ (at $447,168.3$ trillin yen) for t = first quarter of

1994. We link the two series at 1994:Q1 as follows.

$$GDP_t = \begin{cases} GDP2_t \times \lambda \ \text{ for } t = \text{1980:Q1 - 1993:Q4}, \\ GDP1_t \ \text{ for } t = \text{1994:Q1 - 2014:Q1}, \end{cases} \tag{A1.5}$$

where λ is the ratio of $GDP1_t$ for t = 1994:Q1 to $GDP2_t$ for t = 1994:Q1.

(ii) **METI's Monthly All-Industry Activity Index.** This index is a Laspeyres index combining four subindexes: a construction industry index, the IP (the Index of Industrial Producion), a services industry index, and a government services index. It therefore excludes agriculture. The latest base year is 2005, with a weight of 18.3% for the IP. METI has released two series, one whose base year is 2005 and the other (called the "link index") that combines various past series with different base years, and the latter series is adjusted so that the two series can be concatenated to form a consistent series. The two seasonally adjusted series, along with a very brief documentation, can be downloaded from `http://www.meti.go.jp/statistics/tyo/zenkatu/index.html`.

(iii) **GDP Gap Estimate by the Cabinet Office.** In constructing potential quarterly GDP underlying their GDP gap estimate, the Cabinet Office uses a production function approach . A documentation (in Japanese) can be found in:
`http://www5.cao.go.jp/j-j/wp/wp-je07/07f61020.html`.
To summarize the document, the production function is Cobb-Douglas with 0.33 as capital's share. Capital input is defined as an estimate of the capital stock (available from the National Income Accounts) times capacity utilization. Labor input is the number of persons employed times hours worked per person. The TFP (total factor productivity) level implied by this production function and actual quarterly, real, seasonally adjusted GDP is smoothed by the HP (Hodrick-Prescott) filter. Potential GDP is defined as the value implied by the production function with the smoothed TFP level. The capital and labor in this potential GDP calculation is also HP smoothed. The (quarterly) GDP gap is defined as: 100×(actual GDP - potential GDP)/potential GDP.

The Cabinet Office does not release their potential GDP series, but they provide their current GDP gap series upon request. The GDP gap series we obtained is for 1980:Q1 - 2014:Q1. We verified, through email correspondences with them, that this series is to be paired with the quarterly GDP series released on June 9, 2014 (the GDP series described above). The GDP gap series is reproduced here (137 numbers):
0.3 -1.3 0.0 1.2 0.9 1.0 -0.2 -0.5 0.4 -0.2 -0.7 -0.4 -1.4 -1.4 -1.0 -1.2 -1.1 -0.5 -0.5 -1.4 -0.1 0.2 1.1 1.4 0.6 -0.8 -1.2 -1.3 -2.8 -2.0 -1.2 0.1 1.2 0.1 0.9 0.9 2.5 0.0 0.6 2.6 0.8 2.8 3.7 2.5 2.5 2.8 2.0 1.9 1.3 0.6 0.5 -0.7 -0.2 -1.4 -2.3 -2.2 -1.8 -3.2 -1.7 -3.1 -2.9 -1.8 -1.5 -1.7 -1.3

-0.5 -0.8 0.5 1.0 -0.2 0.0 -0.3 -2.4 -3.1 -3.0 -2.6 -3.6 -3.4 -3.8 -3.4 -2.0 -2.0 -2.5 -2.1 -1.6 -2.1 -3.4 -3.7 -4.1 -3.4 -3.0 -2.9 -3.7 -2.7 -2.6 -1.8 -1.2 -1.4 -1.5 -2.0 -2.0 -1.0 -0.9 -0.9 -0.7 -0.5 -0.8 0.3 1.1 1.0 0.5 1.2 1.8 0.5 -0.7 -4.0 -7.9 -6.5 -6.5 -5.0 -3.7 -2.8 -1.5 -2.1 -3.2 -3.6 -2.4 -2.4 -1.6 -2.4 -3.3 -3.4 -2.3 -1.7 -1.6 -1.7 -0.2 .

Construction of Potential Quarterly GDP. We can back out the Cabinet Offic's estimate of potential quarterly GDP by combining this series with the actual GDP series. For quarer t, let GDP_t be (real, seasonally adjusted) GDP described in (i) above and let v_t be the GDP gap shown in (iii) above. The implied potential GDP for quarter t, GDP^*_t, satisfies the relation

$$v_t = 100 \times \frac{GDP_t - GDP^*_t}{GDP^*_t}. \tag{A1.6}$$

Construction of Monthly Series. Given the two quarterly series, GDP_t (actual GDP) and GDP^*_t (potential GDP), we create the monthly output gap series x_t for January 1988-December 2012 as follows.

(i) **Monthly Interpolation of GDP_t.** Using the METI all-industry activity index described in (ii) above, the allocation of quarterly GDP between the three months constituting the quarter is done by the method of Chow and Lin ("Best Linear Unbiased Interpolation, Distribution, and Extrapolation of Time Series by Related Series", *Review of Economics and Statistics*, Vol. 53, pp. 372-375, 1971). Quarterly GDP at annual rate for 1988:Q1-2012:Q4 is treated as the low frequency data, and the METI all-industry activity index for January 1988-December 2012 as the high frequency (monthly) indicator. The quarterly averages of interpolated series are constrained to be equal to the corresponding quarterly series. The estimation method is weighted least squares. Actual computation is done using Mr. Enrique M. Quilis's Matlab code available from: `http://www.mathworks.com/matlabcentral/fileexchange/authors/28788`.

(ii) **Monthly Interpolation of GDP^*_t.** We used the spline method. A spline is fitted to GDP^*_t for t = 1980:Q1 to 2012:Q4. The value of the interpolated monthly series for the middle month of the quarter is constrained to be equal to the quarterly series. We used the Matlab function "spline" for this operation.

(iii) **Calculation of x_t for January 1988-December 2012.** Finally, using this smoothed monthly potential GDP and the monthly actual GDP, we define the monthly output gap for month t, x_t, as

$$x_t \equiv 100 \times [\log(\text{actual GDP for month } t) - \log(\text{potential GDP for month} t)]. \tag{A1.7}$$

HP-filtered GDP as Measure of Potential GDP In the other GDP gap series used in the paper, potential GDP is the HP-filtered actual GDP. To construct this GDP gap series, we first apply the HP (Hodrick-Prescott) filter to the log of actual *quarterly* GDP for 1980:Q1-2012:Q4. The smoothness parameter is the customary 1600. The exponent of this HP-filtered series is the potential quarterly GDP series. We then apply the same spline method to this series for 1980:Q1-2012:Q4, to obtain the monthly potential GDP series. Output gap for 1988:Q1-2012:Q4 is then calculated by the formula (A1.7).

U.S. Monthly Data on Inflation, Unemployment Rate, and the Policy Rate

The price index used to compute inflation is the consumer price index for all urban consumers (all items, 1982-84=100) available from the BLS (Bureau of Labor Statistics). The BLS series id is CUSR0000SA0. This series is seasonally adjusted and available at monthly frequency. The unemployment rate is the civilian unemployment rate obtained from the BLS. The series id is LNS14000000. This series is seasonally adjusted and available at monthly frequency. It is expressed in percent. The policy rate is the effective federal funds rate from the Board of Governors of the Federal Reserve System. We take the average of daily values over the 16th day of the month to the 15th day of the following month. All 3 series are available fromt the FRED database website:
`http://www.research.stlouisfed.org/fred2/`.

Appendix 2 The Model and Derivation of the Likelihood Function

This appendix has two parts. The first is a self-contained exposition of the model with two regimes (P and Z) and with the excess reserve demand. The second part derives the likelihood function for the model.

The Model

The state vector of the model consists of a vector of continuous state variables $\mathbf{y}_t$ and a discrete state variable s_t $(= \mathrm{P}, \mathrm{Z})$. The continuous state $\mathbf{y}_t$ has the following elements:

$$\underset{(4\times1)}{\mathbf{y}_t} \equiv \begin{bmatrix} \underset{(2\times1)}{\mathbf{y}_{1t}} \\ r_t \\ m_t \end{bmatrix}, \quad \underset{(2\times1)}{\mathbf{y}_{1t}} \equiv \begin{bmatrix} p_t \\ x_t \end{bmatrix}, \tag{A2.1}$$

where p = monthly inflation rate, x = output gap, r = policy rate, and m = excess reserve rate. The model also involves a vector of exogenous variables, $\mathbf{x}_t$. It includes $\overline{r}_t$, the rate paid on reserves. It can include other variables (such as the banking crisis dummy), but the identity of those other exogenous variables is immaterial in the derivation of the likelihood function below.

The model is a mapping from

$$(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11}, \mathbf{x}_t, \underset{(2\times1)}{\boldsymbol{\varepsilon}_t}, v_{rt}, v_{\overline{\pi}t}, v_{st}, v_{dt})$$

to $(s_t, \mathbf{y}_t)$. Here, $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\overline{\pi}t}, v_{st}, v_{dt})$ are mutually and serially independent shocks. We need to include 11 lags of $\mathbf{y}$ because of the appearance of the 12-month inflation rate in the model, see (A2.3) below. The mapping is defined as follows.

(i) ($\mathbf{y}_{1t}$ determined) $\underset{(2\times1)}{\boldsymbol{\varepsilon}_t}$ is drawn from $\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}(s_{t-1}))$ and $\mathbf{y}_{1t}$ (the first two elements of $\mathbf{y}_t$) is given by

$$\underset{(2\times1)}{\mathbf{y}_{1t}} = \underset{(2\times1)}{\mathbf{c}(s_{t-1})} + \mathbf{A}(s_{t-1})\mathbf{x}_t + \underset{(2\times4)}{\boldsymbol{\Phi}(s_{t-1})}\underset{(4\times1)}{\mathbf{y}_{t-1}} + \underset{(2\times1)}{\boldsymbol{\varepsilon}_t}. \tag{A2.2}$$

Here, only one lag is allowed, strictly for expositional purposes; more lags can be included without any technical difficulties. The matrix $\mathbf{A}(s_{t-1})$ has two rows. The number of its columns equals the dimension of the vector of exogenous variables $\mathbf{x}_t$.

(ii) (s_t determined) Given $\mathbf{y}_{1t}$ and $(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$, the central bank calculates (through $(p_t, ..., p_{t-11}, x_t, r_{t-1})$)

$$\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11}), \quad r_t^e \equiv \alpha_r + \boldsymbol{\delta}_r'\mathbf{x}_t + \boldsymbol{\beta}_r' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_r r_{t-1}. \tag{A2.3}$$

The central bank draws $(v_{rt}, v_{\overline{\pi}t})$ from $\mathcal{N}(\mathbf{0}, \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_{\overline{\pi}}^2 \end{bmatrix})$, and determines s_t as

$$\text{If } s_{t-1} = \text{P}, \quad s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} > \overline{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.4a}$$

$$\text{If } s_{t-1} = \text{Z}, \quad s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} > \overline{r}_t \text{ and } \pi_t > \overline{\pi} + v_{\overline{\pi}t}, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.4b}$$

(iii) (r_t determined) Given s_t, r_t is determined as

$$\text{If } s_t = \text{P}, \text{ then } r_t = r_t^e + v_{rt}. \tag{A2.5a}$$

$$\text{If } s_t = \text{Z}, \text{ then } r_t = \overline{r}_t. \tag{A2.5b}$$

Note that r_t in (A2.5a) is guaranteed to be $> \overline{r}_t$ under P because by (A2.4a) and (A2.4b) $r_t^e + v_{rt} > \overline{r}_t$ if $s_t = \text{P}$.

(iv) (m_t determined) Finally, the central bank draws v_{st} from $\mathcal{N}(0, \sigma_s^2)$ and the market draws v_{dt} from $\mathcal{N}(0, \sigma_d^2)$. The excess reserve rate m_t is determined as

$$\text{If } s_t = \text{P}, \text{ then } m_t = \max\left[m_{dt}^e + v_{dt}, 0\right], \quad v_{dt} \sim \mathcal{N}(0, \sigma_d^2), \tag{A2.6a}$$

$$\text{If } s_t = \text{Z}, \text{ then } m_t = \max\left[m_{st}^e + v_{st}, 0\right], \quad v_{st} \sim \mathcal{N}(0, \sigma_s^2), \tag{A2.6b}$$

where,

$$m_{dt}^e \equiv \alpha_d + \boldsymbol{\delta}_d' \mathbf{x}_t + \boldsymbol{\beta}_d' \begin{bmatrix} \pi_t \\ x_t \\ r_t \end{bmatrix} + \gamma_s m_{d,t-1}, \quad m_{st}^e \equiv \alpha_s + \boldsymbol{\beta}_s' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_s m_{s,t-1}. \tag{A2.7}$$

When $s_t = \text{P}$ and $s_{t-1} = \text{Z}$, we set $m_{d,t-1} = 0$; otherwise both $m_{s,t-1}$ and $m_{d,t-1}$ are equal to m_{t-1}. Thus, formally, $m_{s,t-1}$ and $m_{d,t-1}$ are functions of (s_t, s_{t-1}, m_{t-1}).

Let $\boldsymbol{\theta}$ be the model's parameter vector. It consists of four groups:

$$\begin{cases} \boldsymbol{\theta}_A &= (\mathbf{c}(s), \mathbf{A}(s), \boldsymbol{\Phi}(s), \boldsymbol{\Omega}(s), s = \text{P}, \text{Z}), \\ \boldsymbol{\theta}_B &= \left(\alpha_r, \boldsymbol{\delta}_r, \boldsymbol{\beta}_r, \gamma_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}}\right), \\ \boldsymbol{\theta}_C &= \left(\alpha_s, \boldsymbol{\beta}_s, \gamma_s, \sigma_s\right), \\ \boldsymbol{\theta}_D &= \left(\alpha_d, \boldsymbol{\delta}_d, \boldsymbol{\beta}_s, \gamma_d, \sigma_d\right). \end{cases} \tag{A2.8}$$

Derivation of the Likelihood Function

The likelihood of the data is (with its dependence on the parameter vector left implicit)

$$\mathcal{L} \equiv p\left(s_1, ..., \ s_T, \mathbf{y}_1, ..., \mathbf{y}_T \,|\, \mathbf{x}, \mathcal{Z}_0\right), \tag{A2.9}$$

Here, $\mathbf{x} \equiv (\mathbf{x}_T, \mathbf{x}_{T-1}, ...)$, $\mathcal{Z}_t \equiv (s_t, s_{t-1}, ..., \mathbf{y}_t, \mathbf{y}_{t-1}, ...)$, and $p(.|.)$ is the joint density-distribution function of $(s_1, ..., s_T, \mathbf{y}_1, ..., \mathbf{y}_T)$ conditional on $(\mathbf{x}, \mathcal{Z}_0)$. The usual sequential factorization yields

$$\mathcal{L} = \prod_{t=1}^{T} p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}, \mathcal{Z}_{t-1}\right). \tag{A2.10}$$

Consider the likelihood for date t, $p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}, \mathcal{Z}_{t-1}\right)$ in (A2.10). Since $\{\mathbf{x}_t\}$ is exogenous, it can be written as

$$p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}, \mathcal{Z}_{t-1}\right) = p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right). \tag{A2.11}$$

Recalling that $\mathbf{y}_t = (\mathbf{y}_{1t}, r_t, m_t)$, we rewrite this date t likelihood as

$$\begin{aligned} p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right) = {} & p\left(m_t \,|\, r_t, s_t, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right) \\ & \times p\left(r_t \,|\, s_t, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right) \\ & \times \mathrm{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right) \\ & \times p\left(\mathbf{y}_{1t} \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right). \end{aligned} \tag{A2.12}$$

In what follows, we rewrite each of the four terms on the right hand side of this equation in terms of the model parameters.

The Fourth Term, $p\left(\mathbf{y}_{1t} \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right)$

This term is entirely standard:

$$p\left(\mathbf{y}_{1t} \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right) = b\Big(\mathbf{y}_{1t} - \big(\mathbf{c}(s_{t-1}) + \mathbf{A}(s_{t-1})\mathbf{x}_t + \mathbf{\Phi}(s_{t-1})\mathbf{y}_{t-1}\big); \mathbf{\Omega}(s_{t-1})\Big), \tag{A2.13}$$

where $b(.;\mathbf{\Omega})$ is the density of the bivariate normal with mean $\underset{(2\times1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times2)}{\mathbf{\Omega}}$.

The Third Term, $\mathrm{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right)$

This is the transition probability matrix for $\{s_t\}$. The probabilities depend on $(r_t^e, \pi_t, \overline{r}_t)$ (which in term can be calculated from $(\mathbf{y}_{1t}, \mathbf{x}_t, \mathcal{Z}_{t-1})$, see (A2.3)). They are easy to derive from (A2.4a) and (A2.4b):

s_{t-1} \ s_t	P	Z
P	P_{rt}	$1 - P_{rt}$
Z	$P_{rt}P_{\pi t}$	$1 - P_{rt}P_{\pi t}$

Here,

$$P_{rt} \equiv \text{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e, \bar{r}_t\right) = \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \tag{A2.14}$$

$$P_{\pi t} \equiv \text{Prob}\left(\pi_t > \overline{\pi} + v_{\overline{\pi} t} \,|\, \pi_t\right) = \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right), \tag{A2.15}$$

where $\Phi(.)$ is the cdf of $\mathcal{N}(0,1)$.

The First Term, $p\left(m_t \,|\, r_t, s_t, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right)$

m_t is given by (A2.6a) and (A2.6b). The right-hand-side variables in those equations, including $m_{d,t-1}$ and $m_{s,t-1}$, are functions of $(r_t, s_t, \mathbf{y}_{1t}, \mathbf{x}_t, \mathcal{Z}_{t-1})$. So this term is the Tobit distribution-density function given by

$$h_{jt} \equiv \left[\frac{1}{\sigma_j}\phi\left(\frac{m_t - m_{jt}^e}{\sigma_j}\right)\right]^{1(m_t>0)} \times \left[1 - \Phi\left(\frac{m_{jt}^e}{\sigma_j}\right)\right]^{1(m_t=0)}, \quad j = d \text{ if } s_t = \text{P and } j = s \text{ if } s_t = \text{Z}, \tag{A2.16}$$

where $1(.)$ is the indicator function, $\phi(.)$ and $\Phi(.)$ are the density and the cdf of $\mathcal{N}(0,1)$.

The Second Term, $p\left(r_t \,|\, s_t, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right)$

If $s_t = $ Z, then $r_t = \bar{r}_t$ with probability 1, so this term can be set to 1. If $s_t = $ P, there are two cases to consider.

- For $s_{t-1} = \mathrm{P}$,

$$
\begin{aligned}
&p(r_t \,|\, s_t = \mathrm{P}, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}) \\
&\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, r_t^e, \bar{r}_t\right) \\
&\qquad\qquad \text{(by (A2.4a) and (A2.5a) and since } (r_t^e, \bar{r}_t) \text{ is a function of } (\mathbf{y}_{1t}, \mathbf{x}_t, \mathcal{Z}_{t-1})) \\
&\quad = \frac{p\left(r_t^e + v_{rt} \,|\, r_t^e\right)}{\mathrm{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e, \bar{r}_t\right)} \\
&\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{v_{rt}}{\sigma_r}\right)}{\mathrm{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e, \bar{r}_t\right)} \qquad \text{(b/c } r_t^e + v_{rt} \sim \mathcal{N}\left(r_t^e, \sigma_r^2\right)) \\
&\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \qquad \text{(b/c } P_{rt} = \mathrm{Prob}\left(r_t^e + v_{rt} > \bar{r}_t \,|\, r_t^e\right)) \qquad \text{(A2.17)}
\end{aligned}
$$

- For $s_{t-1} = \mathrm{Z}$,

$$
\begin{aligned}
&p(r_t \,|\, s_t = \mathrm{P}, \mathbf{y}_{1t}, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}) \\
&\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, \pi_t > \bar{\pi} + v_{\bar{\pi}t}, r_t^e, \bar{r}_t, \pi_t\right) \\
&\qquad\qquad \text{(by (A2.4b) and (A2.5a) and since } (r_t^e, \bar{r}_t, \pi_t) \text{ is a function of } (\mathbf{y}_{1t}, \mathbf{x}_t, \mathcal{Z}_{t-1})) \\
&\quad = p\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} > \bar{r}_t, r_t^e, \bar{r}_t\right) \qquad \text{(b/c } v_{rt} \text{ and } v_{\bar{\pi}t} \text{ are independent)} \\
&\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \qquad \text{(as above).} \qquad \text{(A2.18)}
\end{aligned}
$$

Putting All Pieces Together

Putting all those pieces together, the likelihood for date t, (A2.12), can be written as (with $\mathcal{X}_t$ here denoting $(\mathbf{x}_t, \mathbf{x}_{t-1}, ...,)$ for brevity)

$s_t \vert s_{t-1}$	$p\left(m_t \,\vert\, r_t, s_t, \mathbf{y}_{1t}, \mathcal{X}_t, \mathcal{Z}_{t-1}\right)$	$p\left(r_t \,\vert\, s_t, \mathbf{y}_{1t}, \mathcal{X}_t, \mathcal{Z}_{t-1}\right)$	$\mathrm{Prob}\left(s_t \,\vert\, \mathbf{y}_{1t}, \mathcal{X}_t, \mathcal{Z}_{t-1}\right)$	$f\left(\mathbf{y}_{1t} \,\vert\, \mathcal{X}_t, \mathcal{Z}_{t-1}\right)$
P|P	h_{dt}	$\frac{g_t}{P_{rt}}$	P_{rt}	$f_{\mathrm{P}t}$
P|Z	h_{dt}	$\frac{g_t}{P_{rt}}$	$P_{rt}P_{\pi t}$	$f_{\mathrm{Z}t}$
Z|P	h_{st}	1	$1 - P_{rt}$	$f_{\mathrm{P}t}$
Z|Z	h_{st}	1	$1 - P_{rt}P_{\pi t}$	$f_{\mathrm{Z}t}$

Here,

$$f_{\mathrm{P}t} \equiv b\Big(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{P}) - \mathbf{a}(\mathrm{P})d_t - \boldsymbol{\Phi}(\mathrm{P})\mathbf{y}_{t-1}; \boldsymbol{\Omega}(\mathrm{P})\Big),$$
$$f_{\mathrm{Z}t} \equiv b\Big(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{Z}) - \mathbf{a}(\mathrm{Z})d_t - \boldsymbol{\Phi}(\mathrm{Z})\mathbf{y}_{t-1}; \boldsymbol{\Omega}(\mathrm{Z})\Big),$$
$$g_t \equiv \frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right), \;\; P_{rt} \equiv \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \;\; P_{\pi t} \equiv \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right),$$

h_{jt} is defined in (A2.16) and $b(.;\boldsymbol{\Omega})$ is the density function of the bivariate normal distribution with mean $\underset{(2\times1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times2)}{\boldsymbol{\Omega}}$.

Dividing it into Pieces

Taking the log of both sides of (A2.10) while taking into account (A2.11) and (A2.12) and substituting the entries in the table, we obtain the log likelihood of the sample:

$$L \equiv \log(\mathcal{L}) = \sum_{t=1}^{T} \log\left[p\left(s_t, \mathbf{y}_t \,|\, \mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathcal{Z}_{t-1}\right)\right] = L_A + L_1 + L_2 + L_D,$$

where

$$L_A = \sum_{s_{t-1=\mathrm{P}}} \log\left[f_{\mathrm{P}t}\right] + \sum_{s_{t-1=\mathrm{Z}}} \log\left[f_{\mathrm{Z}t}\right], \tag{A2.19}$$

$$L_1 = \sum_{s_t=\mathrm{P}} \log\left[P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log\left[P_{\pi t}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log\left[1 - P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log\left[1 - P_{rt}P_{\pi t}\right], \tag{A2.20}$$

$$L_2 = \sum_{s_t=\mathrm{P}} \left[\log(g_t) - \log(P_{rt})\right] + \sum_{s_t=\mathrm{Z}} \log\left[h_{st}\right], \tag{A2.21}$$

$$L_D = \sum_{s_t=\mathrm{P}} \log\left[h_{dt}\right]. \tag{A2.22}$$

The terms in $L_1 + L_2$ can be regrouped into L_B and L_C, as in

$$L = L_A + \underbrace{L_B + L_C}_{=L_1+L_2} + L_D, \tag{A2.23}$$

where

$$L_B = \sum_{s_t=\mathrm{P}} \log\left[g_t\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log\left[P_{\pi t}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log\left[1 - P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log\left[1 - P_{rt}P_{\pi t}\right], \tag{A2.24}$$

$$L_C = \sum_{s_t=\mathrm{Z}} \log\left[h_{st}\right]. \tag{A2.25}$$

L_A, L_B, L_C and L_D can be maximized separately, because L_j depends only on $\boldsymbol{\theta}_j$ $(j = A, B, C, D)$ $((\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C, \boldsymbol{\theta}_D)$ was defined in (A2.8) above).

As a special case, consider simplifying step (ii) of the mapping above by replacing (A2.4a) and (A2.4b) by

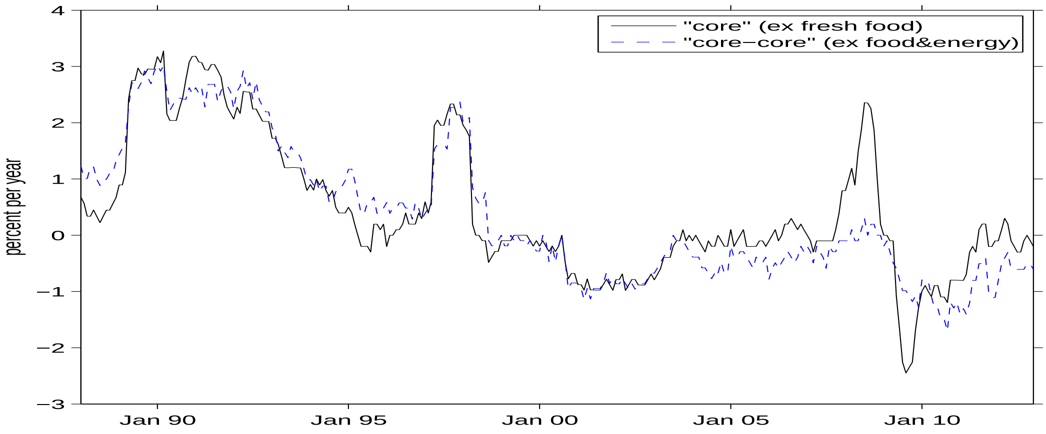
$$s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} > \bar{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.26}$$

Namely, drop the exit condition. This is equivalent to constraining $P_{\pi t}$ to be 1, so L_B becomes

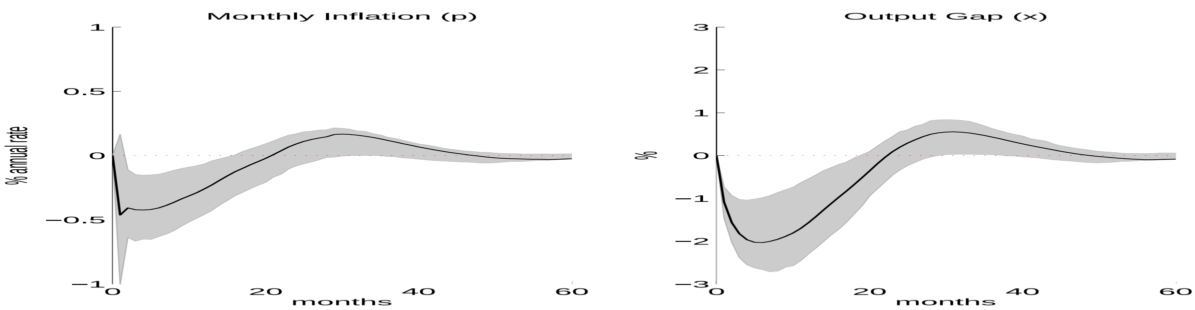
$$L_B = \sum_{s_t=\text{P}} \log\left[g_t\right] + \sum_{s_t=\text{Z}} \log\left[1 - P_{rt}\right], \tag{A2.27}$$

which is the Tobit log likelihood function.

Appendix Figure 1: Twelve-Month CPI Inflation Rate, 1988 - 2012

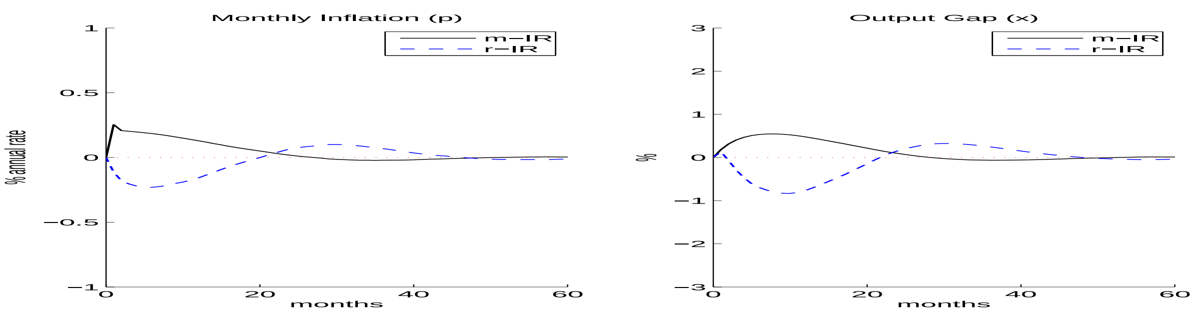


Appendix Figure 2: PZ-IR, base period is July 2006



Note: The PZ-IR is defined in (6.4). The only perturbation is a change in the regime from P to Z.

Appendix Figure 3: IRs, base period is July 2006



Note: The m-IR and r-IR here are defined in (6.7). In the m-IR, the perturbation size to m is 0.45. In the r-IR, the perturbation size is -0.26 percentage point.