Forbearance and Broken Credit Cycles

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Preliminary draft

Abstract

This paper studies how post-financial-crisis economy differs from its pre-crisis period, by providing a microeconomic model of asset price, aggregate output and banks' strategic behaviour. Banks in this model rationally avoid liquidating non-performing loans to make up their balance sheets (i.e. forbearance or zombie lending), when realising loan losses would reduce their capitals lower than the regulatory threshold and force the banks to go bankrupt. But the forbearance comes at a macro-economic cost. Knowing that banks have to liquidate forborne non-performing loans some day, they expect the decrease of collateral asset value in the future, which constrains new lending to productive firms. Forbearance thus decelerates de-leverage of less-productive sectors while it accelerates de-leveraging productive sectors, leading to the reduction of aggregate output. However, we also identify the possibility of output-boosting forbearance when the economy has a strong 'financial accelerator effect'. By studying both banks' incentive of forbearance and its impacts on macro-economy and banking systems, this paper provides a rich model explaining some of post-crisis phenomena, such as low productivity, slow de-leverage in specific sectors, sluggish new lending, excessively resilient property prices, and broken correlation between asset price and output growth. The paper concludes with a welfare analysis of forbearance and a discussion on optimal policy measures.

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1 Introduction

This paper studies how post-financial-crisis economy differs from its pre-crisis period, by providing a microeconomic model of asset price, aggregate output and banks' strategic behaviour. Banks in this model avoid liquidating non-performing loans to make up their balance sheets (i.e. forbearance or zombie lending), when realising loan losses would reduce their capitals lower than the regulatory threshold and force banks to go bankrupt. This is a rational behaviour for the banks, but the forbearance comes at a macro-economic cost. Since banks know that they have to liquidate forborne non-performing loans some day, they expect the decrease of collateral asset value in the future. They therefore raise the haircut of collateral asset when they make new loans, which reduces new lending to productive firms. The forbearance thus decelerates de-leverage of non-performing sectors while it accelerates the de-leveraging of productive sectors, leading to the reduction of aggregate output, and the reduction of banks' profitability as well. By studying both banks' incentive of forbearance and its impacts on macro-economy and banking systems, this paper can provide an explanation some of post-crisis phenomena, such as low productivity, slow de-leverage in specific sectors, sluggish new lending, excessively resilient property prices, and broken correlation between asset price and output growth. The paper concludes with a welfare analysis of forbearance and a discussion on optimal policy measures.

Although financial markets are recovering from the crisis in 2008, it does not necessarily mean that the real economies go back to the status in pre-crisis period. We still have several persistent aftereffects of the crisis. Productivity was lowered after the crisis (Hayashi and Prescott, 2000, show the TFP decline in Japan after the market crash, and Hughes and Saleheen, 2012, show a similar phenomenon globally after the recent crisis). De-leverage of banking sector was slower (Kobayashi et. al., 2003, show lending to non-performing sector was rather increasing after the market crash. Loan amount of real estate sector and others have not decreased compared with manufacturing sectors in the UK as well, as Bank of England, 2011, suggests). Land price (or property price) showed limited adjustment after the rapid boom in pre-crisis period (UK house price increased by 140% from 2000 till 2007 while decreased only by 10% from 2007 till 2012. Japanese land price marked its lowest growth -8.5% in 2005 although the market crashed in 1991). In addition, the positive correlation between output and land price, predicted by Kiyotaki's and Moore's (1997) seminal paper, was broken down after the crisis in Japan (the correlation coefficient from 1960 till 1991 was 0.51, while the coefficient turns to be negative -0.15 from 1991 till 2005 when Japanese banks finished resolving their non-performing loan problem): see Figure 1. (The land price is real.) It also should be noted that these phenomena in Japan and Europe are not observed in the US after the recent crisis.

The aim of the paper is explaining those phenomena as a result of banking crisis, especially forbearance. Forbearance, sometimes called zombie lending (Caballero et. al., 2008), or

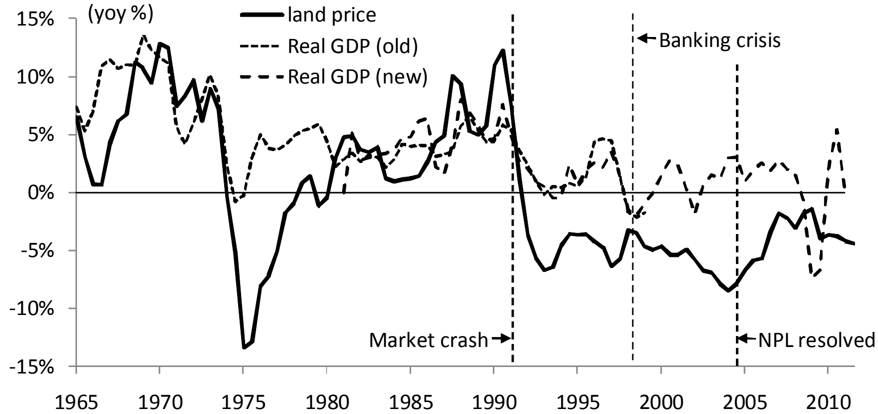


Figure 1: GDP growth and land price growth (Japan)

evergreening loans (Peek and Rosengren, 2005), is the phenomenon that banks do not liquidate non-performing loans (or collateral assets seized from the borrowers) and leave those on their balance sheets. We will mainly discuss the reason why banks forbear and the macroeconomic impact of the forbearance in a single microeconomic model, which has an endogenous asset price, the output of the economy and banks who behave strategically.

The mechanism of forbearance in this paper is as follows. Experiencing a negative macro shock, banks expect the plunge of asset price which reduces the collateral value of their loan portfolios. The plunge of asset price matters for some banks that fail to be repaid and seize borrowers' collateral assets: the plunge lowers the collateral value, i.e. the liquidation value of defaulted loans, and increases the credit losses which reduces the capital of the banks. With fears that the capitals go below its regulatory threshold, which incurs a large non-pecuniary cost to bank managers, banks forbear liquidating non-performing loans (or seized collateral assets). Note that, in this model, banks internalise the cost of liquidation. Forbearance has two private benefits for the banks. First, banks can maintain unliquidated non-performing loans at the book value, i.e. they can hide the deterioration of the loan assets. Second, by minimising the size of liquidation, they can contain the plunge of asset price (and the value of collateral) to some extent which contain the credit loss of liquidated loans. Both contribute to 'make up' the banks' capitals higher than these should be.

However, it is unrealistic to assume that banks can keep the non-performing loans eternally. In this model, therefore, we prohibit banks from 'evergreening' the loans, by forcing banks to liquidate the non-performing loans before the model economy ends. In other words, forbearance is postponing liquidation. This assumption allows us to take it into account that how the expectation of unwinding forbearance affects the outcome of forbearance, which has been ignored in the previous literature. The players in the economy form an expectation that the collateral value will drop in the future when banks liquidate the loans. As banks (lenders) see the expected value of collateral at the maturity date of a loan they are going to extend, the pessimistic expectation tightens the credit constraint of productive firms that need funds to purchase the asset to produce outputs. This is a macroeconomic cost of forbearance in the model. Tightened credit by the pessimistic expectation of collateral value, in conjunction with the higher spot asset price than it should be, reduces productive investments. In addition, unliquidated non-performing borrowers who are not productive at all occupy production capital for nothing.

The negative macroeconomic effect of forbearance is partially internalised by the banks, because tightened credit to productive firms lowers the banks' profits in the long run. This incentivies the banks to choose minimum possible forbearance. Given this structures, we discuss the performance of possible policy options to improve welfare. But we also find that this is not always the case. If the economy has a high 'financial accelerator', i.e. firms' investment is highly sensitive against the capital gains from their asset holdings, then the plunge of asset price is too costly to leave because it leads to a significant decline of the output of the economy. In this case, forbearance can rather boost the output as it supports the asset price and the effect dominates the negative effects explained above.

We also discuss, at the end of this paper, what happens if forbearance is anticipated before crisis occurs. Since forbearance mitigates banks' expected cost of borrowers' defaults, anticipated forbearance eases banks' credit conditions and raise pre-crisis asset price (and output). Therefore, forbearance could amplify boom, by containing the impact of bust.

The most relevant liteature is, clearly, the works on forbearance. Caballero et.al. (2008) study the 'congestion effect' of forbearance: forborne borrowers produce goods (and services) more without maximising their profits, which lowers good prices, reduces the profit margin of the industry and hiders new entry. My model also has the congestion effect in asset price (goods price is exogenous in this model). Asset price is higher than it should be because non-performing borrowers hoard the asset. Focusing on asset price has several benefits. First, as demonstrated above, asset price can bridge the macroeconomic impact of forbearance and banks' incentive of forbearance. Peek and Rosengren (2005) find that banks with small capital ratio are prone to forbear. My model is consistent with this paper and shows how this banks' behaviour interacts with the output of the economy and the asset price. Second, this allows us to consider the link between forbearance and debt overhang. As Hall (2010) and Eggertsson and Krugman (2012) argue, debt overhang is a promising explanation of prolonged stagnation after banking crisis. Forbearance can explain why debt overhang can persist and that expected debt deflation could cause some interesting and realistic phenomena as shown above.

This model can be considered as a reduced-form microeconomic model of Credit Cycles by Kiyotaki and Moore (1997, KM hereafter). As we study the interaction between asset price, output and banks' behaviour, a natural starting point of the discussion would be Financial Accelerator models. The model setup is close to Krishnamurthy (2003) introducing credit-constrained insurers to two-periods reduced-form credit cycles model. Jeanne and Korinek (2010) and Lorenzoni (2008) also build similar models to discuss credit booms and busts. There are a couple of critical differences distinguishing the paper from these reduced-form financial accelerator models. First one is studying price dynamics. For example in Krishnamurthy's model asset price is determined with Financial Accelerator effect only once at t=1 (the other two models above have a similar feature). No amplification mechanism works at t=0 and 2. This is not a convenient feature to consider price dynamism: as described above, it is a key feature of the model that anticipated asset price decline reduce the leverage of the economy. Therefore asset price should be determined under the identical setup at least twice. As we see below, this is not a trivial extension. Second, this model explicitly introduces the defaults of borrowing firms and banks. Banks' loss given default of the firms is endogenously determined.

Theoretically, this paper can be also seen as a model in which banks internalise the negative externalities of liquidating assets, in contrast to Jeanne and Korinek (2010), Lorenzoni (2008) and Brunnermeier and Sannikov (2014) in which externality plays an important role. These models successfully explain amplified boom-bust cycles using an assumption that players are not aware of the negative externality of liquidating assets. But it is not difficult to consider situations that banks do internalise a fraction of the externality of liquidations. In many countries banks are oligopolistic and big banks would consider the cost of their own firesale through the plunge of asset price; By liquidating non-performing loans they have to mark-to-market these loans, which would be costly even if the liquidation does not change the liquidation value. This paper studies how the internalisation of the cost of liquidation changes the outcomes: banks are aware of the cost of liquidation and decide not to liquidate their loan assets (and collateral assets). This reduces the depth of a bust in the asset market, but worsens the slump of the output. Note that this model does not assume oligopolistic banks, although this does not change the result of the model significantly.[1]

Another relevant literature on forbearance is Kocherlakota and Shim (2007). Kocherlakota and Shim (2007) study the optimality of regulatory forbearance, i.e. forbearance by the social planner. In their model, the probability distribution of collateral value plays a crucial role. Lenders are tempted to liquidate their borrowers when collateral value is low since low collateral value cannot incentivise the borrowers without additional reward. The lenders would not liquidate them (forbearance) if the government rescures the lenders by compensating the loss (the reward), but the government is not willing to commit this if the probability of low collateral value is high. The model discussed here is different from their work in various ways. Collateral value

[1] Ota (2009) models the case of a monopolistic bank, and the earlier versions of this paper assume oligopolistic banks. The results are similar to the ones we obtain in this paper.

is endogenously determined, so as the cost of liquidation. Forbearance is the choice of banks, not the government. Forbearance is not committed ex-ante, although correctly anticipated by strategic players. These features provide rich implications on forbearance as we see below.

The paper is structured as follows. Section 2 introduces the benchmark model, which banks do not do forbearance. Section 3 introduces forbearance into the model and section 4 discusses the social welfare and possible policy options. Section 5 concludes, with the discussion considering the reasons why US did not experience forbearance and the post-crisis phenomena cited above. Appendix provides the details of mathematical proofs.

2 Benchmark Model

In this section, we will see the baseline model without forbearance. We first clarify the assumptions and timeline (Section 2.1). We then define three players' utility functions (Section 2.2, 2.3 and 2.4) and derive their demand function of production capital, denoted as land (Section 2.5). We find a unique, or multiple, competitive equilibrium, and see how the equilibrium price (and equilibrium firms' investment) behaves throughout the period (Section 2.6 and 2.7). In this baseline model, equilibrium price and output always have positive correlation, as suggested by Kiyotaki's and Moore's (1997) Credit Cycles.

2.1 Assumptions

There are three sets of players in the economy: firms, dealers and banks. They are all risk neutral, and are born at t=0 and die at t=3. The first three periods represent pre-crisis (t=0), during crisis (t=1), and post-crisis (t=2). At t=3 all players consume everything they have, and die. All players are continuum players with measure one. The economy has one asset, denoted as land, which can serve as production capital and as collateral asset covering loan contracts. The supply of land is fixed at K. Land is initially supplied by an outsider of the economy at the beginning of t=0, and the same outsider purchases at a price (see below) at t=3 when all players die.

At each period, firms and dealers purchase land to initiate their production opportunities. Firms are assumed to be credit constrained: i.e. they do not have a sufficient endowment ω_0 to purchase the optimal amount of land, and borrow from banks up to a fraction of collateral asset values the firms possess to fill the gap. Since banks have no production opportunity, banks' assets are thoroughly loans to the firms. The banks and the dealers are assumed to have an access to an unconstrained funding source at a constant interest rate r coming from the outside of the economy.

Next period, firms obtain harvests from the production opportunities they invested in the previous period. A fraction of firms fail to harvest any (and go bankrupt - this is an idiosyncratic shock). The rest of the firms obtain harvest: the production per unit of land a_t is stochastic which can take two values, $a_H > a_L$ (macro shock). The dealers' productivity is non-stochastic and diminishing return to scale, and the productivity is lower than the expected productivity of the firms. Banks cannot obtain any repayment from the failed firms and seize collateralised land instead. Surviving firms, the dealers and the banks with seized collateral land sell all their land holding, to repay all their debts. At the same time, the surviving firms and the dealers purchase land for the new production opportunities. The Walrasian Auctioneer sets the land price q_t to clear the market.

This process is repeated three times (t=0,1 and 2) and at t=3, the final period, all the players sell their land holding and die after they consume all the wealth. At t=3 the equilibrium land price should be zero as there is no investment opportunity nor borrowing needs,[2] but we assume that the outside comes back to the economy to purchase all the land K at the equilibrium price at t=2. This is an adhoc assumption, but we will see below that this does not necessarily lose generality much in fact.

We will find equilibria specifying land price q_t, firms' land holding k_t^f, dealers' land holding k_t^b, banks' lending amount D_t and required repayment R_t for $t = \{0, 1, 2\}$.

2.2 Firms' problem

The economy has continuum firms with measure one. They are ex-ante identical, risk neutral players possessing a profitable investment opportunity. The opportunity requires a production capital, denoted as land hereafter (land does not depreciate). By purchasing land at a period t, the firms can produce a_{t+1} per unit of land (constant return to scale) with probability $1-\gamma$ (we denote the successfully-harvesting firms as survived firms). With probability γ firms fails to harvest any (denoted as defaulted firms). The survived firms' productivity a_{t+1} is a stochastic variable taking $a_{t+1} \in \{a_H, a_L\}$ with probability π and $1-\pi$ respectively, updated at each period (i.i.d. over time). The choice of a_{t+1} is a macro shock, i.e. it is the same across firms. a_{t+1} is realised at t+1, but firms (and banks) have expectation of a_{t+1} at t: the expectation is denoted as $E_t[a_{t+1}]$. Note that the firms are exposed to two shocks, idiosyncratic one (with probability

[2]Krishnamurthy (2003) uses the setup that the asset price goes to zero at the final period. Since expected price growth plays an important role, $q_3 = 0$ is inconvenient in this model.

γ) and macro one (with probability π). We also consider a simplified case that players of the economy do not anticipate the macro shock and believe at$=\pi a_H + (1-\pi)\, a_L$ for any t.

All the firms receive an initial endowment ω_0 at the beginning of t=0. Firms purchase land k_0^f, and lever their position by borrowing from banks pledging purchased land as collateral. It is assumed that each firm can borrow from a bank: the monopolistic nature of bank loan plays an important role below. It is assumed that ω_0 is too small so that firms cannot start investment projects without borrowing money from banks. I assume $E_t[a_{t+1}]$ is sufficiently high so that the leverage continues until the following budget and collateral constraints always bind (note that the firms' production function is linear).

We assume that firms can take two "moral hazard" actions to justify collateralisation of loans and the haircut of collateral. First one is walking away with borrowed wealth. Therefore banks have to provide collateralised loans only. Second, firms can take a cheating investment project, which provides a private benefit B to the firms but the default probability is fixed at $\gamma = 1$. Banks thus have to charge haircut over collateral asset, in order to let the firms invest their own wealth to the investment project that is larger than B. In the following argument we treat h as an exogenous parameter since h is anyway the function of exogenous parameter B. Practically, h would represent a loan-to-value ratio (LTV) cap applied by public authorities. Note that firms cannot walk way after they harvest. This is because the harvest is directly stored at the lending banks' current account.[3]

Budget constraint of the firms is

$$q_t k_t^f \leq D_t + \omega_t \tag{1}$$

where D_t is the amount of borrowing from a bank. The constraint implies that the purchase of land is capped by the loan size plus endowment. The budget constraint is always binding in this model, as the firms' project is constant return to scale and the profitability of the project is high. And the loan size is capped by a fraction of expected future value of collateral assets, as the following collateral constraint shows.

$$D_t \leq (1-h) E_t\left[q_{t+1}\right] k_t^f \tag{2}$$

Note that collateral does not cover firms' interest payment since we have assumed that firms can pledge the repayment above. Since D_t is the choice variable of banks, we cannot determine

[3] We need this assumption since we assume very high a_t and that banks take all excess surplus. If collateral has to cover future (possible) repayment, firms' credit constraint is tightened as firms become profitable. This is less intuitive and less convenient for our modelling here. And in reality, LTVs are calculated based on principals, not on the total repayment value.

if this collateral constraint is binding at this stage. We sometimes denote the budget and collateral constraints as credit constraints.

Haircut in real life is applied to the current asset value, not the expected asset value. To incorporate this, we define an "actual haircut" $\tilde{h}$ so that:

$$\begin{aligned} (1-h)E_t\left[q_{t+1}\right] &= (1-\tilde{h})q_t \\ \tilde{h} &= 1-(1-h)\frac{E_t\left[q_{t+1}\right]}{q_t} \end{aligned} \tag{3}$$

The equation (2) can be written as:

$$D_t \leq (1-\tilde{h})q_t k_t^f$$

and if the budget and collateral constraints are binding,

$$D_t = q_t k_t^f - \omega_t = (1-\tilde{h})q_t k_t^f = \frac{1-\tilde{h}}{\tilde{h}}\omega_t \tag{4}$$

Under the assumption of the binding collateral constraint (2), the firms' demand function of land is defined as follows.

$$k_t^f = \frac{D_t+\omega_t}{q_t} = \frac{\omega_t}{\tilde{h}q_t} \tag{5}$$

For a given and fixed wealth ω_t (when $t=0$) the demand function is downward sloping and convex to the origin. But at t>0, firms' wealths are updated with their profits obtained in the previous periods. The following transition function specifies the evolution of firms' wealth.

$$\begin{aligned} E_t[\omega_{t+1}] &= \pi\left(1-\gamma\right)\left(\omega_t + D_t - q_t k_t^f + a_{t+1}^H k_t^f + q_{t+1}^H k_t^f - R_t\right) \\ &\quad + \left(1-\pi\right)\left(1-\gamma\right)\left(\omega_t + D_t - q_t k_t^f + a_{t+1}^L k_t^f + q_{t+1}^L k_t^f - R_t\right) \end{aligned} \tag{6}$$

where R_t is the repayment amount ($R_t = 0$ if a firm defaults with probability γ). With probability γ firms cannot harvest any, but they still have their lands. Surviving firms will sell land to repay the loan, and defaulted firms' land would be seized by the banks. Note also that the first three terms in the brackets are zero since the credit constraints are binding.

Since D_t and R_t are the choices of banks which we have not yet discussed, we cannot specify the variables at this moment. We will come back to the equation (6) after we discuss banks' objective function.

2.3 dealers' problem

We define another set of players dealers: they have unconstrained access to a funding source outside of the economy at the interest rate r, and can purchace land k_t^b for their investment opportunity producing $k_t^b\left(A-k_t^b\right)$ at the next period. A is constant and $A<E[a]$, i.e. dealers are less productive than firms for any k_t^b, and $A=2K$ to ensure that $k_t^b\left(A-k_t^b\right)$ is monotonically increasing everywhere.

We further assume that dealers cannot observe firms' behaviour and their objective functions, so they create their ex-ante belief on the next period land price q_{t+1} randomly around the current price q_t. I.e. they are assumed to be noise traders as assumed by e.g. Kyle (1985) and on average they expect $E_t\left[q_{t+1}\right]=q_t$.

The dealers' payoff function Π, to be maximised, is:

$$\Pi_t=f\left(A,k_t^b\right)-rq_tk_t^b$$

The FOD of $E\left[\Pi_{t+1}\right]$ wrt k_t^b is

$$\frac{\partial E\left[\Pi_{t+1}\right]}{\partial k_t^b}=A-2k_t^b-rq_t=0$$

And the banks' demand function is

$$k_t^b=\frac{A}{2}-\frac{1}{2}rq_t$$

For the notational convenience, we normally define the dealers' demand function on the domain of k_t^f: therefore the dealers' demand function is horizontally flipped and described as an upward sloping curve.

2.4 Bank's problem

Banksare also contiuum players with measure one, and are the only ones who can lend to firms. Banks can obtain funding for the lending from the outside funding source at the interest rate r. Each bank lends to only a bank and each firm can oborrow only from a bank. Banks are not allowed to diversity their loan portfolio since this is the simplest way to introduce the possibility of bank failure. The matchings between banks and firms are reshuffled at each period.

Banks write loan contracts specifying the loan size D_t, and repayment amount $R_t=\left(1+l_t\right)D_t$ (where l_t is the lending rate) i.e. they have perfect negotiation power over borrowing firms, by making a take-it-or-leave-it offer to firms. They can thus take all excess surplus of firms' investment project.

If banks fail to be repaid, they seize collateral assets and liquidate those at the asset market. If the liquidation value (determined endogenously in the asset market) is lower than the lending amount, then those banks incur credit losses. If the loss is large enough for their capital level W_t to go below a threshold $\bar{W}$, then the public authority forces the bank to close their business, which is very costly for the bank managers (incurring a non-pecuniary private cost $X < 0$). The failed banks are eliminated from the economy, and taken over by the depositor (outside of the model) for the period. The depositor would be inexperienced compared with incumbent banks, which permits the firms to enjoy a larger private benefit $\bar{B} \geq B$ when they shirk. But we assume $\bar{B} = B$ for the time being, for the sake of simplicity. I.e. there is no real economic cost of bank failure in this model (depositors lending to the banks would incur a loss, but we do not consider the problem). We will loosen this assumption in Section 4.

The initial capital W_0 is endowed at the beginning of $t = 0$. Note that a bank's capital W_t does not restrict banks' lending activity: i.e. there is no capital adequacy ratio constraint in the economy. The threshold $\bar{W}$ is introduced only to define bank failure. Loan amount can be restricted by firms' collateral constraint only, if it is binding.

Banks choose $\{D_t, R_t\}$ maximising their $E_t[W_3]$ at each period t. The banks' capital transition function, given that the banks survive at the period t (i.e. $W_t \geq \bar{W}$) is defined as follows:

$$\begin{aligned} E_t[W_{t+1}] &= W_t + E_t\left[(1-\gamma)R_t + \gamma q_{t+1}k_t^f - (1+r_t)D_t + (1-I_{t+1})X\right] \\ &= W_t + E_t\left[(1-\gamma)R_t + \gamma q_{t+1}\frac{D_t+\omega_t}{q_t} - (1+r_t)D_t\right] + \Pr\left(W_{t+1} < \bar{W}\right)\cdot X \quad (7) \end{aligned}$$

The first term in the blacket on the RHS of the equation (7) is the repayment revenue, the second term is the liquidation value of failed borrowers' collateral, and the third term is the repayment amount to the dealers. The last term of the RHS is the expected cost of the bank's failure by $W_{t+1} < \bar{W}$. I_{t+1} is the index variable: $I_{t+1} = 1$ when $W_{t+1} \geq \bar{W}$ and $I_{t+1} = 0$ otherwise.

2.4.1 Optimal lending behaviour

First we need to clarify how banks choose loan size D_t. Banks find it optimal to increase D_t if the first order derivative of the equation (7) is strictly positive.

$$\frac{\partial E_t\left[W_{t+1}\right]}{\partial D_t} = (1-\gamma)\left(1+l_t\right) + \gamma \frac{E\left[q_{t+1}\right]}{q_t} + \gamma \frac{\partial E\left[q_{t+1}\right]/q_t}{D_t} - \left(1+r_t\right) + \frac{\partial \Pr\left(W_{t+1} < \bar{W}\right)}{\partial D_t} X$$

It is immediate from the equation above that there exists a unique $\hat{l}$ such that for any $l_t > \hat{l}$ $\frac{\partial E_t[W_{t+1}]}{\partial D_t} > 0$. And as we will see below, we can assume that l_t is always sufficiently high from the assumption of sufficiently high $E\left[a\right]$ (firms' profitability): therefore banks maximise their D_t until the firms' collateral constraint (2) binds. Note that the banks do not make any excessive loans ex-ante. Even though the penality when a bank fails to receive repayment would be large, the expected profitability is still positive ex-ante.

Second, banks choose their l_t. Since we have assumed that banks make a take-it-or-leave-it offer to firms, banks seem to be optimal to offer the highest possible l_t to maximise their profits. But this is not neccesarily the case when banks maximise their long-run payoff, $E_t\left[W_3\right]$. This is because leaving some excess surplus on the hand of firms improves firms' borrowing capacity in the following periods, which increases profitable lending opportunities for the banks. In this case, the optimal l_t does not take the corner solution that takes all excess surplus of the firms.

However, since we have assumed that the matching between banks and firms is reshuffled at each period, we do not need to consider the banks' benefit of reducing l_t. Even if a bank gives some surplus to the borrowers, the probability that they will be matched again is zero as the number of banks and firms are infinite. Suppose first that all the other banks choose maximum l_t: a bank's best responce is joining the strategy since the bank will not be rewarded by being matched by the same firm. On the other hand, if all the other banks choose to leave some surplus on the hand of borrowing firms, a bank finds it better to free-ride on the other banks' effort to improve firms' net wealth ω_{t+1}. Therefore, taking the all excess surplus of the firms is the unique symmetric Nash equilibrium if banks are continuum. Note also that this ensures sufficiently large l_t if $E\left[a\right]$ is high, which we have assumed above. This means that banks are optimal to be myopic and maximise $E_t\left[W_{t+1}\right]$ at each period t.

The following lemma summarises the discussion above.

Lemma 1 *Since the number of banks is positive infinite, banks choose the largest possible D_t where the firms' collateral constraint binds and the highest possible lending rate l_t (and R_t) where the firms' participation constraint binds.*

2.4.2 Firms' wealth transition with the banks' lending behaviour

The banks choose D_t and $R_t = (1+l_t)D_t$ to maximise $E_t[W_{t+1}]$ subject to the firms' and banks' participation constraints:

$$\begin{aligned} E_t[\omega_{t+1}] &\geq \omega_t \\ E_t[W_{t+1}] &\geq W_t \end{aligned}$$

Note that the firms' participation constraint would be $E_t[\omega_{t+1}] \geq E[a]\,\omega_t/q_t + (E[q_{t+1}] - q_t)\,\omega_t/q_t$ if firms are allowed to invest without borrowing from banks, as firms can purchase $k_t^f = \omega_t/q_t$ without bank loans and banks have to assure the expected income gain and capital gain to satisfy the participation constraint. We come around this issue by assuming that firms' minimum investment size is always larger than ω_t, for the sake of simplicity.[4]

The banks with superior negotiation power choose the maximum possible R_t satisfying the firms' participation constraint as we have seen in Lemma 1. The LHS of the participation constraint (equation 6) is, therefore:

$$\begin{aligned} E_t[\omega_{t+1}] &= \pi(1-\gamma)\left(a_{t+1}^H k_t^f + q_{t+1}^H k_t^f - R_t\right) \\ &\quad + (1-\pi)(1-\gamma)\left(a_{t+1}^L k_t^f + q_{t+1}^L k_t^f - R_t\right) \end{aligned}$$

Following to the theory of optimal debt contracts, we assume that R_t is constant and the firms take all the stochastic residuals as retained earnings. Denote that the firms' expected retained earning conditional on the firms' survival as T_t. Then we can rewrite the equation as follows:

$$E_t[\omega_{t+1}] = (1-\gamma)\left[\begin{array}{c} T_t + \pi\left\{\left(a_{t+1}^H - E_t[a_{t+1}]\right)k_t^f + \left(q_{t+1}^H - E_t[q_{t+1}]\right)k_t^f\right\} \\ +(1-\pi)\left\{\left(a_{t+1}^L - E_t[a_{t+1}]\right)k_t^f + \left(q_{t+1}^L - E_t[q_{t+1}]\right)k_t^f\right\} \end{array}\right]$$

where $\left(a_{t+1}^H - E_t[a_{t+1}]\right)k_t^f$ is the income surplus (gap from the expected income) and $\left(q_{t+1}^H - E_t[q_{t+1}]\right)k_t^f$ is the capital surplus. The equation assumes that the firms will use 100% of capital surplus from their land holding to increase their net wealth ω_{t+1} and then increase their investment k_{t+1}^f. This is, however, not very realistic. Many non-financial firms record their assets in book value, not market value, to segregate their profits from market price volatility which is irrelevant to their own business. To incorporate this, we introduce one more parameter $\eta \in [0,1]$: the firms' net wealth is increased (or decreased) by the fraction η of the capital surplus.

[4] See, for instance, Hirakata et. al. (2013) for the works carefully considering the issue.

For the notational convenience, we use the following transition function, given a_{t+1}, throughout the paper.

$$\begin{aligned} \omega_{t+1}|_{a_{t+1}} &= (1-\gamma)\left\{T_t + \left(a^H_{t+1} - E_t\left[a_{t+1}\right]\right)k^f_t + \eta\left(q_{t+1} - E_t\left[q_{t+1}\right]\right)k^f_t\right\} \\ &= (1-\gamma)\left\{\psi(T_t) + \eta q_{t+1} k^f_t\right\} \end{aligned} \tag{8}$$

$$\text{where } \psi(T_t) = T_t + \left(a^H_{t+1} - E_t\left[a_{t+1}\right]\right)k^f_t - \eta E_t\left[q_{t+1}\right]k^f_t$$

$\psi(T_t)$ is the constant component at t+1. Note that $\psi(T_t)$ can be negative for a large η.

2.5 Firms' demand function of land

Now we can define firms' demand function of land, given the banks' optimal choice of D_t and R_t. Firms try to maximise their final wealth ω_3, under the budget and credit constraints defined above. Since the lemma 1 ensures the corner solution where the firms' budget and credit constraints (the equations 1 and 2) are binding, the constraints define the firms' demand function. By rewriting these equations, we have:

$$q_t k^f_t \leq (1-h)E_t\left[q_{t+1}\right]k^f_t + \omega_t$$

And from the equations (3) and (8), we have the following firms' inverted constrained demand function as follows:

$$k^f_t = \frac{\omega_t}{\tilde{h}q_t} = \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k^f_{t-1}\right\}}{q_t - (1-h)E_t\left[q_{t+1}\right]} \tag{9}$$

If the firms' demand is not constrained, the demand function becomes horizontal as the production opportunity is constant return to scale (we have assumed sufficiently large $E[a]$ to ensure that the uncontrained equilibrium does not emerge. The collateral constraint does not bind anywhere $q_t \leq (1-h)E_t\left[q_{t+1}\right]$ (note that the actual haircut $\tilde{h}$ becomes negative in this region). Therefore the demand curve kinks twice, as Figure 2(a) shows. The curve cannot be lower than $q_t = (1-h)E_t\left[q_{t+1}\right]$ as firms' demand diverges to positive infinite.

It is important that the firms' net wealth ω_t is the positive function of the spot land price q_t, and therefore could boost the firms' demand k^f_t. This is exactly what the Financial Accelerator models assume. Furthermore, η controls the strength of the Financial Accelerator in this model. The following lemma shows that the choice of η is crucial in the shape of the firms' demand function.

Lemma 2 *At t=1, for a given and fixed $E_t[q_{t+1}]$, the demand function is downward sloping and convex to the origin if $E_{t-1}[q_t] < (1-h)E_t[q_{t+1}]$. If $E_{t-1}[q_t] > (1-h)E_t[q_{t+1}]$, there exists $\hat{\eta}$ such that for $^{\forall}\eta < \hat{\eta}$ the function is downward sloping and convex to the origin, and vice versa. At t=2 where $q_{t+1} = q_t$, for any $\eta < \check{\eta}$ the demand curve is downward sloping and convex to the origin, and otherwise upward sloping and convex.*

See the appendix for the proof. When η is small, the Financial Accelerator effect becomes weak and normal price effect (for a given budget, purchasing amount is negatively correlated to price) dominates the Financial Accelerator. The curve is convex to origin since firms are leveraged (the curve becomes linear if $h = 1$). But if η is high enough, the Financial Accelerator effect could dominate the price effect (higher asset price increases firms' wealth and eases their credit constraint, which boosts demand of land and raises asset price further) and higer land price rather increasesthe firms' purchasing power of the asset.

For the time being, we focus on the case where η is small so that the demand function is downward sloping. Small η, i.e. smaller financial accelerator coefficient, would be more plausible assumption, as firms normally do not mark-to-market their physical assets frequently to avoid their profits being exposed to market risks. The small η also helps us simplify the model, as this ensures the uniqueness of the equilibrium.[5] We will come back to the case when η is large in Section 3.4. Figure 2(a) visualises the discussion when η is small. Note that at t=0 when exogenously fixed ω_0 is endowed, the constrained demand function is decreasing and convex to the origin for sure.

2.6 Asset market equilibrium

Asset market equilibrium is determined as the solution to the following equations.

$$\begin{aligned} k_t^f &= \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{q_t - (1-h)E\left[q_{t+1}\right]} \\ k_t^b &= \frac{A}{2} - \frac{1}{2} r q_t \\ K &= k_t^f + k_t^b \end{aligned}$$

Since we have proved that the firms' constrained demand function is monotonically decreasing (note that we have assumed $\eta < \hat{\eta}$ to ensure the downward sloping demand function) and the

[5] Note that this assumption is technically very similar to what Jeane and Korinek (2010) assume on the variable ϕ. Althought the modelling details are different, the variable ϕ determines the size of amplification effect through credit supply, and Jeane and Korinek assume sufficiently small ϕ to ensure the uniqueness of equilibrium.

dealers' demand function (defined on k_t^f) is monotonically increasing, the existence of unique equilibrium is ensured. If the intersect of the functions are below the dotted line in Figure 2(a), i.e. $q_t < (1-h)E[q_{t+1}]$, then the equilibrium price is $q_t^* = (1-h)E[q_{t+1}]$. We will see below that this does not occur on any equilibrium pathes.

At t=2, q_3 is given exogenously as $E_2[q_3] = q_2$.

$$\begin{aligned} k_t^f &= \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{h q_t} \\ k_t^b &= \frac{A}{2} - \frac{1}{2} r q_t \end{aligned} \tag{10}$$

The same argument applies to ensure the existence of unique equilibrium. Trivially unique equilibrium exists at t=0 irrespective of η, as ω_0 is constant. The following proposition is the summary of the results.

Proposition 3 *For each period, the baseline model has unique equilibrium when $\eta < \hat{\eta}$.*

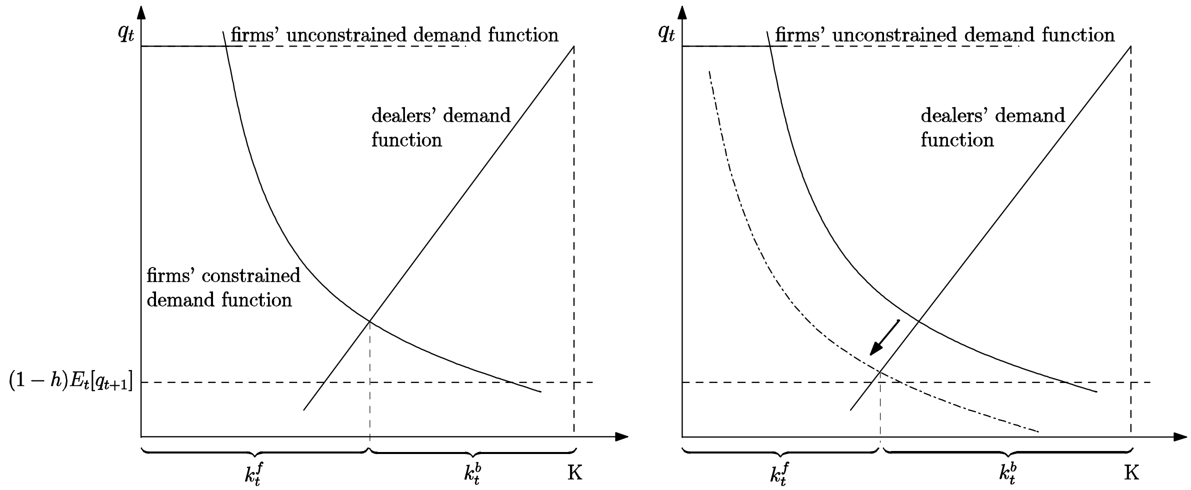
We have the following lemma from the discussion above. The mathematical proof is provided at the appendix.

Lemma 4 *k_t^f and q_t are strictly increasing against $\psi(T_{t-1})$.*

Figure 2(b) summarises the proposition 3 and the lemma 4 (for given $E_t[q_{t+1}]$). The arrow shows what happens to the equilibrium if the economy experiences a negative macro shock. The negative income shock $(a_L - E_{t-1}[a_t])\, k_{t-1}^f$ shifts the demand curve leftward and the following decline of asset price generates the capital loss $\eta(q_t - E_{t-1}[q_t])\, k_{t-1}^f$. The capital loss reduces the net wealth further and lowers the demand curve further (this process will be repeated). This is the Financial Accelerator effect of the model.

2.7 Dynamics of the model

For the given parameter sets, we have seen that there exists unique equilibrium asset price. The model needs to specify the equilibrium profile $\left\{k_t^f, k_t^b, D_t, R_t, q_t\right\}$ for $t \in \{0,1,2\}$. At t=3, there is no production opportunity and all players consume everything and die: therefore we do not need to specify anything other than q_3. In a closed economy, the equilibrium price q_3^* has



(a) Equilib. of two demand functions (b) Equilib. with a negative macro shock

Figure 2: Equilibria with weaker financial accelerators

to be equal to zero since land loses its value as production capital and collateral asset (there is no production opportunity at t=3). This is, however, not a convenient feature to consider asset price dynamism, which we focus on in the following section of forbearance. Therefore we assume an outsider of the economy visiting the economy at t=3 and purchases all land at the equilibrium price of the previous period; $q_3^* = q_2^*$. This is clearly an ad-hoc assumption but it does not lose much generality, since $E_t\left[q_{t+1}^*\right] \simeq q_t^*$ for $t < 2$ as we will see below.

We solve the model by backward induction. Given that $q_3 = q_2^*$, we first solve the equilibrium at t=2. From the proposition 3, the uniqueness of q_2^* is ensured. Equilibrium q_2^* determines k_2^f and k_2^b simultaneously, which determines the equilibrium D_2. From the equation (8) and the firms' credit constraints, $E_2\left[\omega_3\right] = \omega_2 = (1-\gamma)\,T_2$: this specifies equilibrium R_2.

Given that the equilibrium action profile at t=2, we can proceed to the equilibrium at t=1. We do not need to consider complicated problems to maximise $E_1\left[\omega_3\right]$, $E_1\left[W_3\right]$ and $E_1\left[\Pi_3\right]$ by the actions at t=1. This is because banks' optimal decisions take corner solutions as we have seen that firms' decisions are always constrained.

2.7.1 When macro shocks are unanticipated

Suppose first the economy without macro shock, $a_t = a = E\left[a\right]$ for $\forall t$, to simplify the problem. This ensures that the firms' net wealth ω_t is also constant over time, since their participation constraint is always binding. At each period of time, we need to specify a tuple $\{T_{t-1}, E_{t-1}\left[q_t\right]\}$

to have equilibrium q_t and k_t^f satisfying $\omega_t = \omega_{t-1}$ (these two are obvious once we specify the tuple – see the equation 10). An obvious tuple $\{T_{t-1}, E_{t-1}[q_t]\}$ achieving the constant net wealth ω_t is $E_{t-1}[q_t] = q_{t-1}$ and $T_{t-1} = \omega_{t-1}/(1-\gamma)$. The lemma 4 ensures that this is the unique tuple achieving this: for any larger T_{t-1}, the firms' demand function shifts rightward which raises q_t, and vice versa (note that the dealers' demand function is fixed over time and the change of the firms' demand function dictates the change of the equilibrium).

Proposition 5 *If macro shocks on at are unanticipated by the players of the economy,* $E_{t-1}[q_t] = q_{t-1}$ *and* $E_{t-1}\left[k_t^f\right] = k_{t-1}^f$. *If the economy has no macro shock, the equations are correct without the expectation operators.*

This assumption simplifies the model significantly without losing generality much. Specifically, we have $\zeta = \varphi = 0$, which will be defined below. We cannot discuss, however, how the anticipated forbearance changes the behaviour of players at t=0 (pre-crisis period), which is discussed in Section 3.3.

2.7.2 When macro shocks are anticipated

However, when we have anticipated macro shock on a_t, $E_t\left[\pi\omega_t\left(a^H\right) + (1-\pi)\,\omega_t\left(a^L\right)\right]$ is not necessarily equal to ω_{t-1} when $T_{t-1} = \omega_{t-1}/(1-\gamma)$. This is because q_t is not a linear function of ω_t (and of a_t). This is obvious from the proof of the lemma 4. We thus need to adjust T_{t-1} to ensure the participation constraint binding, which resembles to certainty equivalence. In this section we determine the adjustment factor ζ_t. Readers may skip the following lemma and proposition since these are not an essential part of the model: assuming $\zeta_t = 0$ does not change the story.

Lemma 6 *In any possible equilibrium,* k_2^{f*} *and* q_2^* *are strictly increasing and concave with respect to* T_1.

See appendix for the proof. Since q_2^* is a concave mapping of T_1, q_2^* is also concave against the production shock $\Delta a_1 = a_1 - E[a]$. This means that ω_2 is a concave function of a_1, which ensures that $\pi\omega_t\left(a^H\right) + (1-\pi)\,\omega_t\left(a^L\right) < \omega_1$ when $T_1 = \omega_1/(1-\gamma)$. We thus need to increase T_1 to satisfy the participation constraint and the risk premium is denoted as ζ_1.

Since $E_1[\omega_2]$ is monotonically increasing wrt ζ_1, through itself and through a higher $E[q_2]$, there exists unique $\zeta_1^* > 0$ such that $E_1[\omega_2] = \omega_1$. With ζ_1^*, $E[q_2]$ has to be still smaller than

q_1. This is because, if $E[q_2] \geq q_1$ with the ζ_1^*, there is no capital loss to be compensated, then we have to have $\zeta_1^* \leq 0$ in order to have $E_1[\omega_2] = \omega_1$. This is contradiction. The difference $\varphi_1 = E[q_2] - q_1$ is the function of the curvature of the demand curve, ω_1, and Δa_1: i.e. φ_1 is fixed when the auctioneer determines q_1. The same argument applies for $t = 0$ (see the appendix for the proof). We have proved the following proposition.

Proposition 7 *The asset price q_t follows a random walk process with a small negative drift $\varphi_t < 0$ under the optimal loan contract. $\varphi_2 = 0$ since there is no uncertainty in this period.*

Given that $E_t[q_{t+1}] = q_t + \varphi_t$, we can rewrite the land demand functions as follows, for any $t \in \{0, 1, 2\}$:

$$\begin{aligned} k_t^f &= \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k_{t-1}^f\right\}}{hq_t - (1-h)\varphi_t} \\ k_t^b &= \frac{A}{2} - \frac{1}{2} r q_t \\ K &= k_t^f + k_t^b \end{aligned}$$

At the period zero, no investment has been made and $k_{-1}^f = 0$. Note that, on the equilibrium path, q_t^* and $y_t^* = a_t k_{t-1}^f$ are positively correlated, as Kiyotaki and Moore (1997) model.

Next, we will consider the banks' choice of D_0, in other words, consider if and when the banks' participation constraint is satisfied. Banks rationally expect the possibility that they will go bankrupt and be penalised by X. Banks therefore lend only when the expected cost is dominated by the expected benefit, i.e. the expected profit of loan contracts. This condition is always satisfied since we assume sufficiently large $E[a]$ (since the firms' participation constraint is binding and the banks take all excess surplus, the assumption of large $E[a]$ is equivalent to the assumption of a large l_t or large R_t). Note that banks do not make any excessive lending. The equilibrium D_0 is fair given the expected profit of the lending opportunity.

3 Modelling forbearance

In the benchmark model, we have assumed that banks have to liquidate all seized collateral immediately, together with surviving firms and dealers. This has ensured that the total supply of land is always K, which is allocated by Walrasian auctioneer to firms and new-born dealers at the market price q_t. Banks could fail as a result, if the banks' capitals are scarce and the asset price declines: the collateral value is not sufficient to cover the loan losses and the capitals W_t go below the regulatory threshold $\bar{W}$. Whether banks fail or not is not important to specify equilibrium shown above, since banks are assumed to liquidate eerything even when the liquidation triggers their own bankruptcy.

Now we introduce a new action as follows, to introduce the cost of liquidating non-performing assets (i.e. liquidating their collateral). The banks' managers, who incur a private utility loss X, do not have to liquidate seized collateral for a fraction $\theta \in [0, 1]$. I.e. the banks keep their non-performing borrowers knowing that they do not recover. This is called as forbearance, zombie lending, or evergreening loans. The bank managers would choose $\theta > 0$ hoping for the recovery of the land market in the next period so that they can liquidate the 'toxic' assets at a better price.

$\theta = 0$ is equivalent to the benchmark case where banks do not forbear liquidation at all. $\theta = 1$ means that the bank does not liquidate any of the failed borrower's asset. $\theta \in (0, 1)$ represents a partial liquidation. This would represent a couple of situations. First, banks do not liquidate bad borrowers, and the non-performing loan contracts are valued by the collateral value (e.g. if LTV ratio is 1.2, the bank has to realise 20% loss of the loan). Second, banks liquidate bad borrowers but do not liquidate the collateral asset, as observed in Spain recently. The un-liquidated collateral is valued at mark-to-market value. Note that mathematically those two are equivalent. In the following we take the former assumption.

To simplify the banks' strategic behaviour, we assume that banks can choose forbearance only at t=1, and they have to unwind at t=2. This is to ensure that banks unwind all forborne loans at a market price, otherwise we cannot observeproper price dynamics under forbearance.

Forbearance in this model allows insolvent firms to hold (a fraction of) their production capital θk_0^f. The market clearing condition at time t=1 is now $K = k_1^f + k_1^b + \theta\gamma k_0^f$, i.e. forbearance squeezes total land supply. In other words, this reduces the total supply of land temporarily at t=1 to $K - \theta\gamma k_0^f$, which shifts the dealers' demand curve to the left. We assume that θ is public information. For the time being we do not consider how the expectation of θ changes the equilibrium at t=0 for simplicity. I.e., we assume that the possibility of forbearance was not anticipated at all at t=0, which is not an unrealistic assumption.

Figure 3 describes how the equilibrium shifts by forbearance. The figure only show the initial impact alone assuming $E_1[q_2]$ is unaffected by θ, therefore the new crossing point is not necessarily the new equilibrium. Note that we have assumed small η to ensure the downward sloping firms' demand curve. The details are discussed below.

The insolvent firms' recovery ratio is assumed to be zero for simplicity. Note that this is a neutral assumption, as we do not assume any recovery of these 'zombie' firms, nor the further

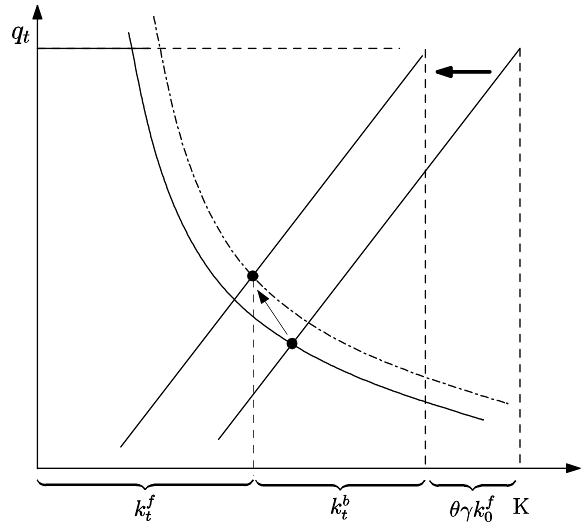


Figure 3: The impact of forbearance

deterioration of their credit quality. The expected change of the zombie firms' credit quality matters for the optimal choice of θ, but the absolute level of their credit quality does not.

3.1 Impact of θ

3.1.1 Price impacts

First we consider the impact of θ on q_1 and q_2.[6]

$$\begin{aligned} k_1^f &= \frac{(1-\gamma)\left\{\psi(T_0)+\eta q_1(\theta)\,k_0^f\right\}}{q_1(\theta)-(1-h)E\left[q_2(\theta)\right]} \\ k_1^b &= \frac{A}{2}-\frac{1}{2}rq_1(\theta) \\ K &= k_1^f(\theta)+k_1^b(\theta)+\theta\gamma k_0^f \end{aligned}$$

$$\begin{aligned} k_2^f &= \frac{(1-\gamma)\left\{\psi(T_1(\theta))+\eta q_2(\theta)\,k_1^f(\theta)\right\}}{hq_2(\theta)} \\ k_2^b &= \frac{A}{2}-\frac{1}{2}rq_2(\theta) \\ K &= k_2^f(\theta)+k_2^b(\theta) \end{aligned}$$

[6] Note that $q_1[\theta] \neq E_1[q_2(\theta)]$ as $\varphi_1 < 0$.

By solving the set of equations for t=1, we have the following implicit function of $q_1(\theta)$, for given $E[q_2(\theta)]$:

$$\begin{aligned} & 2(1-\gamma)\psi(T_0) - 2\theta\gamma(1-h)E[q_2(\theta)]\, k_0^f \\ = \; & r\{q_1(\theta)\}^2 - \left\{r(1-h)E[q_2(\theta)] + 2\theta\gamma k_0^f + 2\eta(1-\gamma)k_0^f\right\} q_1(\theta) \end{aligned} \tag{11}$$

Since the dealers' demand curve is always upward sloping, the leftward shift of the dealers' demand curve (by increasing θ) raises q_1 for sure, irrespective of the slope of the demand curve, for given and fixed $E[q_2(\theta)]$.

Since ω_1 is the positive function of q_1, ω_1 must be higher by forbearance and therefore it raises $E_1[\omega_2]$ to satisfy the firms' participation constraint $E_1[\omega_2] = \omega_1$. Higher $E_1[\omega_2]$ unambiguously raises $E_1[q_2]$ from the lemma 4, thus $E_1[q_2(\theta > 0)] > E_1[q_2(\theta = 0)]$ is ensured, which shifts the firms' demand curve upward at t=1 and t=2.

Next, we proceed to consider whether $E[q_2(\theta)]$ is bigger or smaller than $q_1(\theta)$, which is the crucial part of this section. Note first that, if θ kept constant at t=2, $E[q_2(\theta)] = q_1(\theta) + \varphi_1$ and $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$ as we have seen in the previous section. Note also that φ_1 and ζ_1 are independent from θ as θ does not change the curvature of the firms' demand function. Now consider $\theta = 0$ at t=2. $E[q_2(\theta)]$ has to be smaller than the case above as the dealers' demand function shifts rightward. This reduces $E_1[\omega_2]$ as the firms' capital loss is increased, which violates the firms' participation constraint followed by the increase of T_1. However, we also know that if T_1 is large enough to achieve $E[q_2(\theta)] = q_1(\theta) + \varphi_1$, then there is no capital loss and T_1 should go back to $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$. I.e. $T_1 = \omega_1(\theta)/(1-\gamma) + \zeta_1$ is too small as it violates the firms' participation constraint, and $T_1 > \omega_1(\theta)/(1-\gamma) + \zeta_1$ achieving $E[q_2(\theta)] = q_1(\theta) + \varphi_1$ is too large as it does not maximise banks' payoff (since the firms' participation constraint is not binding). Since $E[q_2(\theta)]$ is a continuous and monotonically increasing function of T_1 from the lemma 4, there exists unique T_1^* such that te firms' participation constraint is binding. And from the monotonicity of $E[q_2(\theta)]$ against T_1, we have the following proposition.

Proposition 8 *$q_1(\theta)$ and $E[q_2(\theta)]$ are strictly increasing against θ. For any $\theta > 0$, $E[q_2(\theta)] < q_1(\theta)$.*

Since the actual haircut $\tilde{h}$ is defined as $\tilde{h}(\theta) = 1 - (1-h)\frac{E[q_2(\theta)]}{q_1(\theta)}$, we have the following corollary.

Corollary 9 *The banks' actual haircut $\tilde{h}$ is increasing against θ.*

This corollary plays an important role when we consider how forbearance affects lending and output.

3.1.2 Output impacts

When we consider the impact of θ on y, we need to think two factors: one is the productive sector's k_t^f and the other is the decline of capital $(1-\theta)K$. In the following discussion we focus on the impact on k_t^f, since $\partial k_t^f/\partial\theta < 0$ is a sufficient condition of $\partial y/\partial\theta < 0$, and the size of k_t^f is crucial when we consider banks' profit function below. The impact on output y is shown by numerical exercise following.

We could see the sign of $\frac{\partial q_1}{\partial \theta}$ without endogenising $E[q_2;\theta]$ above, since upward shift of the demand curves by higher $E[q_2;\theta]$ unambiguously raise q_1. But it is unclear if the upward shift of the curves increases k_t^f, since it relies on the slope of the firms' demand curve: the sign of $\frac{\partial k_1^f}{\partial \theta}$ depends on how large those two demand curves shifts against the increase of θ.

If we endogenise $E[q_2;\theta]$ of the firms' demand function, then the firms' demand function no longer shifts against the change of θ, since θ affects firms' demand only through current and expected prices q_1 and $E_1[q_2]$. Now θ shifts the dealers' demand function only, and clearly the slope of the firms' demand function determines the sign of $\frac{\partial k_1^f}{\partial \theta}$. If the firms' demand curve is downward sloping, the leftward shift of the dealers' demand curve reduces the equilibrium output k_1^f, and vice versa. Therefore, $\frac{\partial k_1^f}{\partial \theta} < 0$ iif $\frac{\partial k_1^f(\theta)}{\partial q_1(\theta)} < 0$ and $\frac{\partial k_1^f}{\partial \theta} > 0$ iif $\frac{\partial k_1^f(\theta)}{\partial q_1(\theta)} > 0$. And we have the following proposition:

Proposition 10 *There exists $\check{\eta}$ such that for any $\eta < \check{\eta} < \hat{\eta}$ k_1^f is decreasing against θ. k_2^f is always increasing against θ.*

See the appendix for the proof. The reason why k_1^f can be increasing against θ nevertheless the firms' demand curve is downward sloping is that a higher θ raises ω_1 and shifts the demand curve toward upper right (see the dotted line in Figure 3), which could outweigh the decline of k_1^f by the leftward shift of the dealers' demand curve. In other words, forbearance has two effects: the wealth effect increasing firms' wealth by raising capital gain (than it should be), and the haircut effect tightening firms' credit constraint by raising the actual haircut $\tilde{h}$. When η is high the wealth effect dominates the haircut effect (graphically, when the firms' demand curve is steep), k_1^f can be increased by forberance. k_2^f is always increasing against θ as higher θ increases $E_1[\omega_2]$ for sure.

This proposition implies the possibility of welfare-boosting forbearance. If η is high, i.e. the financial accelerator effect of the economy is strong, the plunge of land price q_1 is too costly for the economy to bear, which justifies forbearance. Maintaining land price higher could boost investment (than it should have been) and output. We will come back to this issue in the section 3.4.

Another important point we can find here is that forbearance improves output in the long run (k_2^f is always increasing against θ, for any $\eta > 0$). This is because forbearance keeps incumbent firms' net wealth that would have been impaired by the plunge of asset price.[7] Therefore, we have a trade-off between the negative short-term effect (reducing k_1^f) and the positive long term effect (increasing k_2^f), when $\eta < \check{\eta}$. It is not easy to judge which effect dominates in our simple three periods model: forberance is assumed to be unwound in a period, and the post-forbearance economy lasts only a period as well. But forbearance continued for almost a decade in Japan, therefore the positive effect of post-forbearance economy should be, if any, properly discounted to the present value to compare the cost of forbearance. As the model exogenously fixes the length of forbearance and the discount factor is assumed to be one, this model is not appropriate to consider the trade-off. See Section 5 for the further discussion on this.

For the technical convenience, however, we have the following lemma which we use in the following section.

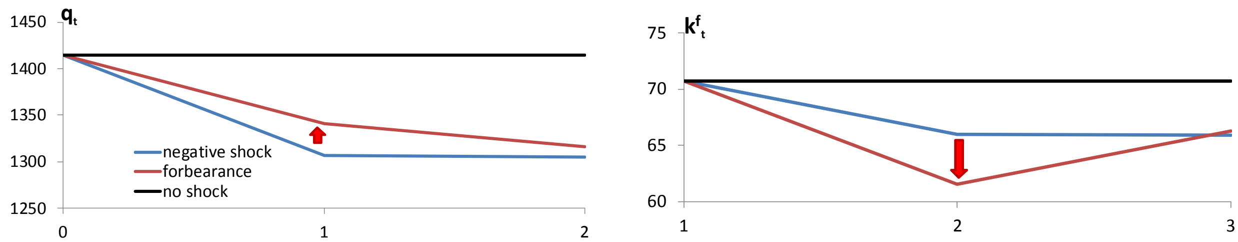
Lemma 11 $k_1^f + E_1\left[k_2^f\right]$ *is strictly decreasing against* θ *if* $\eta < \bar{\eta} < \check{\eta}$ *and if* $E\left[a\right]$ *is sufficiently large, as assumed before.*

See the appendix for the proof. The outline of the proof is simple: if the financial accelerator effect is small, then both $k_1^f(\theta)$ and $k_2^f(\theta)$ decrease for a given θ. This ensures the unique threshold $\bar{\eta}$ so that for any $\eta < \bar{\eta}$ the aggregate output effect of forbearance is negative.

Impulse response functions plotted in the Figures 4 (a) and (b) summarise the arguments above (see the footnote[8] for the parameters used for this numerical simulations). The solid (black) lines represent the equilibrium paths when the economy does not experience any macro shocks (as Proposition 5 shows, the asset price and the output stay constant over time). If the economy experiences a negative macro shock but forbearance is not allowed (i.e. the baseline

[7] To be precise, incumbent firms are benefited from contained capital loss by forbearance at t=1. Expecting the unwinding of forbearance followed by capital loss of land holding, the incumbent firms request a larger retained earning to satisfy the participation constraint, which allows them to maintain the benefit at t=2. If firms cannot anticipate the unwinding of forbearance, the incumbents' long-term benefit fades away at t=2.

[8] The numerical simulations used the following parameters: $a_H = 1500$, $a_L = 1225$, $A = 200$, $\gamma = 0.3$, $\omega_0 = 15000$, $h = 0.15$, $r = 0.1$, $\pi = 0.8$, $\eta = 0.125$, and $\theta = 0.63$. The θ used here is optimal for the banks (see Section 4).



(a) Equilibrium paths of land price (small η) (b) Euilibrium paths of output (small η)

Figure 4: Impulse responses with and without forbearance

model), both the land price and the output decreases at t=1 as the blue lines of the figures show. But when banks forbear liquidation, the equilibrium land price increases but the output decreases as the red lines show. Readers find that the land price goes down further when forbearance is unwinded in the following period, while the output recovers. The figures show how forbearance breaks credit cycles (positive correlation between asset price and output).

3.2 Banks' incentive

We have seen above the impact of forbearance on the economy for a given θ. Next we will endogenise the choice of θ, as an optimal choice of banks. Banks choose θ to maximise their future payoffs, as described below. In short, banks choose θ to maintain their capital level W_t higher than the regulatory threshold. W_1 decreases when banks face loan losses, i.e. the borrowers of the bank go bankrupt under recession. And the regulatory threshold $\bar{W}$ is prone to be tightened when banks experience these negative shocks.[9] Both create an incentive for banks to make up their capital level W_t by forbearance, otherwise these banks are forced to close their business which is assumed to be costly for bank managers.

This section consists of three parts. First, we see that if and how forbearance can 'make up' banks' current inefficient capital W_1. Second, we will see the impact of forbearance to the banks' long term profit $E_1[W_3]$, which is important for the optimal level of forbearance. Third, we will specify banks' Nash equilibrium actions. Section 4 sees if the Nash equilibrium is socially optimal.

[9] For example, Basel I was implemented in Japan in 1993, two years after the market crash. Basel III will be implemented following to the crisis in 2008.

3.2.1 Defining stricken banks' capital

We have equilibrium q_t and k_t^f for given θ, knowing the functions of equilibrium $q_t(\theta)$ and $k_t^f(\theta)$. Banks will choose their optimal θ after a_1 is realised, to maximise their final wealth W_3. First we define the capital of banks that experience default of the borrowers (stricken banks hereafter):

$$W_1(\theta) = W_0 - (1+r)D_0 + (1-\theta)\, q_1(\theta)\, k_0^f + \theta E_1\left[(1-\gamma)\, R_0 + \gamma q_2 k_0^f\right] \tag{12}$$

where the lending amount $D_t = q_t k_t^f - \omega_t$, and $E_1\left[(1-\gamma)\, R_0 + \gamma q_2 k_0^f\right]$ is the expected profit of the loan contract D_0, i.e. the book value of loan contracts. Note that γ does not appear in the equation 12 since this is the payoff function of the stricken banks only. The second term is the repayment amount from the banks to the depositors, and the third term is the value of liquidated loans (seized collateral). The last term is the value of forborne loans. This term assumes that stricken banks that fail to be repaid can 'hide' the default of the borrowers: i.e. the banks can behave as if the fraction θ of the borrowers that fail to repay at t=1 would repay in full in the following period with the probability $1-\gamma$.[10] I.e. the value of forborne loans is the same as new loans. This assumption is based on the 'snow-balling loans' a lá Sekine et.al. (2003), which stricken banks provide additional loans to defaulted firms to let them service the borrowers' debts, so that the banks can pretend as if they are performing well. This assumption is consistent to Peek and Rosengren (2005) as well. Note that, since the borrowers have already gone bankrupt and are assumed not to recover for sure, the last term of the equation (12) is never realised at t=2. The funding cost to support the forborne loans will be charged at the end of the period (the beginning of t=2).

Some, however, would consider that evaluating forborne loans by the 'book value', i.e. banks can iminate bad loans as if they are normal loans, is a strong assumption. It is possible to consider an alternative assumption, e.g. the public authorities are aware of the non-performing loans and force the banks to value the loans by the current value of the collateral assets. This assumption would be consistent to the cases where stricken banks foreclose the borrowers but do not liquidate the seized collateral hoping for the recovery of asset price, for instance. We have confirmed that this does not change the outcomes of the model significantly, although this assumption complicates the model further.[11]

[10] As the discount rate is assumed to be zero, the delay of repayment is costless in this model.

[11] If we assume that forborne loans are valued by the collateral value, credit losses that the stricken banks have to bear are thoroughly determined by the asset price. Since we have assumed that the banks are atomlessly many, each bank's liquidation does not change the asset price and therefore the banks cannot internalise the cost of liquidation. Subsequently we need to assume finitely many banks, which complicates the model definitions. See an old version of the paper for the details as the paper had assumed this.

Lemma 12 *$W_1(\theta)$ is strictly increasing against θ, and therefore there exists a unique $\hat{\theta}$ so that $W_1(\theta) = \bar{W}$.*

This is obvious from the equation 12. Higher θ raises q_1, which contains the banks' credit loss $(1-\theta)\left\{(1+r)D_0 - q_1(\theta)k_0^f\right\}$, as well as it permits the banks to avoid realising the failure of the borrowers. This lemma means that banks can make up their insufficient capital levels by forbearance. The monotonicity of $W_1(\theta)$ ensures the existence of $\hat{\theta}$, although this does not necessarily guarantee that $\hat{\theta}$ takes a feasible value within the range $[0,1]$.

3.2.2 Forberance and long run profitability

Next, we will see how forbearance θ changes stricken banks' long term payoff W_3, which is important to consider the optimal level of θ. The stricken banks choose $\theta \in [0,1]$ to maximise their final expected payoff $E_1[W_3]$. $E_1[W_3]$ consists of three components, conditional on the bank's survival at t=1. The components are; the payoff from the forborne loans (the first line of the equation 13 below); the expected payoff from new lending at t=1 (the second line of the equation); and lastly, the expected payoff from new lending at t=2. The following equation summarises the components.

$$\begin{aligned}
E_1[W_3] = {} & W_0 - (1+r)D_0 + (1-\theta)q_1(\theta)k_0^f + \theta E_1[q_2(\theta)]k_0^f - r\theta D_0 \\
& + (1-\gamma)\left(E[a_2]k_1^f(\theta) - T_1 - rD_1\right) + \gamma\left\{E_1[q_2(\theta)]k_1^f(\theta) - (1+r)D_1\right\} \\
& + \Pr\left[W_2 \geq \bar{W}\right]\left[\begin{array}{c} (1-\gamma)\left(E[a_3]k_2^f(\theta) - T_2(\theta) - rD_2(\theta)\right) \\ +\gamma\left\{E_1[q_3(\theta)]k_2^f(\theta) - (1+r)D_2(\theta)\right\} \end{array}\right] \\
& + \Pr\left[W_2 < \bar{W}\right] \cdot X
\end{aligned} \tag{13}$$

Note that we do not consider the possibility of $W_3 < \bar{W}$, since the assumption of $q_3 = q_2$ assures zero credit loss at t=3.

Based on the payoff function, we have the following lemma:

Lemma 13 *If $E[a_t]$ is sufficiently large and $\eta < \bar{\eta} < \check{\eta}$, $E_1[W_3]$ is strictly decreasing against θ.*

See the appendix for the proof. The condition $\eta < \bar{\eta}$ means that forbearance is loss-making for the banks if and ony if forberance is output-reducing in the long run (see Lemma 11). This is because the sufficiently large $E[a_t]$ dominates the other factors such as the short term benefit of forbearance (liquidating collateral assets at a higher price at t=1).

3.2.3 Optimal choice of forbearance

The stricken banks chooses θ by maximising the following objective function.

$$\max_{\theta^* \in [0,1]} E_1 \left[W_3 (\theta^*) ; W_1 (\theta^*) \geq \bar{W} \right]$$
$$s.t. \quad W_1 (\theta^*) \geq \bar{W}$$

The function $E_1 \left[W_3; W_1 \geq \bar{W} \right]$ is defined by the equation (13). If $W_1 < \bar{W}$ the utility function is fixed at the following:

$$E_1 \left[W_3; W_1 < \bar{W} \right] = W_0 - (1+r)D_0 + q_1 (\theta) k_0^f - r\theta D_0 + X$$

Note that this function is independent from θ, as it is assumed that failed banks are forced to liquidate all loans and close their business. The sufficiently large $E[a]$ and the sufficiently large penalty of default X assures $E_1 \left[W_3; W_1 \geq \bar{W} \right] > E_1 \left[W_3; W_1 < \bar{W} \right]$ and that all stricken banks try their best to satisfy the constraint $W_1 \geq \bar{W}$ whenever possible. If they find it impossible, i.e. $\hat{\theta} > 1$, the banks have to choose $\theta = 1$. On the other hand, if $\hat{\theta} < 0$, the banks can survive without forbearance and $\theta^* = 0$ for sure, since $E_1 \left[W_3; W_1 \geq \bar{W} \right]$ is decreasing against θ from Lemma 13.

We will limit our attention to the cases where $\hat{\theta} \in [0, 1]$ below. Given this, $W_1 (\theta^*) = \bar{W}$ from Lemma 13. Since the long-term profit is decreasing against θ, the banks have no incentive to increase $W_1 (\theta^*)$ strictly higher than $\bar{W}$: i.e. the constraint is always binding when $\hat{\theta} \in [0, 1]$. And from Lemma 12, $\theta^* = \hat{\theta}$. Summarising the discussions above, $\theta^* = \hat{\theta}$ if $\hat{\theta} \in [0, 1]$, and $\theta^* = 0$ otherwise.

However, we have to consider here a potential strategic problem. In the previous argument we have implicitly assumed that all stricken banks choose the same θ so that to achieve $W_1 (\theta) = \bar{W}$, but there is no guarantee that $\theta^* = \hat{\theta}$ is supported as a Nash equilibrium. A stricken bank could be tempted to choose a smaller θ when the other banks choose $\theta^* = \hat{\theta}$. In the following, we will see what kind of θ can be supported as Nash equilibrium. Since all stricken banks are identical to each other at t=1, we focus only on symmetric Nash equilibria in the following.

First, we consider if $\theta^* = \hat{\theta}$ is Nash equilibrium. Given that all the other stricken banks choose $\theta^* = \hat{\theta}$, the bank's optimal choice would be $\theta^{dev} = 0$ since forbearance is loss-making in the long run, if the deviating bank can maintaing $W_1 \left(\theta^{dev}; \theta^* \right)$ higher than $\bar{W}$. Since the bank's non-performing loans are valued at $E_1 \left[(1-\gamma) R_0 + \gamma q_2 \left(\hat{\theta} \right) k_0^f \right]$, and $q_1 \left(\hat{\theta} \right)$ and $q_2 \left(\hat{\theta} \right)$ are independent from the deviating bank's action, $W_1 \left(\theta^{dev}; \theta^* \right) \geq \bar{W}$ if and only if

$E_1\left[(1-\gamma)R_0+\gamma q_2\left(\hat{\theta}\right)k_0^f\right]\leq q_1\left(\hat{\theta}\right)k_0^f$. The assumption of a sufficiently large $E[a]$, howver, ensures $E_1\left[(1-\gamma)R_0+\gamma q_2k_0^f\right] > q_1k_0^f$: therefore, any small liquidation of non-performing loans reduces W_1. Since $\theta^*=\hat{\theta}$, the deviating bank's capital $W_1\left(\theta^{dev};\theta^*\right)$ is not smaller than $\bar{W}$ only when $\theta^{dev}\geq\hat{\theta}$. Subsequently, the optimal choice of the 'deviating bank' is $\theta=\hat{\theta}$ and $\theta^*=\hat{\theta}$ is a Nash equilibrium.

Second, if all the other stricken banks choose any $\theta^*<\hat{\theta}$, a bank cannot avoid bankruptcy when the bank chooses the same θ. The bank can survive for sure if it chooses a higher θ so that they can hide the credit loss by valuating the loans at the book value. Third, if all the other banks choose any $\theta^*>\hat{\theta}$, a bank always finds it optimal to choose a smaller θ, since $W_1(\theta^*;\theta^*)>\bar{W}$ and there exists $\theta^{dev}<\theta^*$ so that $W_1\left(\theta^{dev};\theta^*\right)=\bar{W}$. Subsequently, we have the unique symmetric Nash Equilibrium $\theta^*=\hat{\theta}$. The following proposition summarises the discussion.

Proposition 14 *If $\eta<\bar{\eta}$, and if $\hat{\theta}>0$, the stricken banks have a unique symmetric Nash equilibrium, $\theta^{NE}=\hat{\theta}$. If $\hat{\theta}\leq 0$, $\theta^{NE}=0$ is the unique equilibrium.*

The proposition above considers only the case where output is decreasing against θ: but we have seen that it is possible that output can be increasing when η is high. In this case, from the lemma 13, forbearance becomes profitable in the long run. The unique Nash equilibrium is trivially $\theta=1$, irrespective of the level of the stricken banks' capital.

Corollary 15 *If $E_1[W_3]$ is monotonically increasing against θ, the stricken banks' unique optimal choice is $\theta=1$ irrespective of $\hat{\theta}$. If $E_1[W_3]$ has a local maximum at $\theta\in\left(\hat{\theta},1\right)$,*

This corollary also shows the feasibility of the 'output-boosting' forbearance. If the economy finds an opportunity of 'output-boosting' forbearance, the banks do it irrespective of their capital ratios. This is not intuitive and contradicts with the observation by Peek and Rosengren (2005).

3.3 Anticipated forbearance before crisis

Throughout the discussions above, we have assumed that any players of the economy do not anticipate the possible forbearance at $t=0$. Without anticipating the forbearance, banks lend D_0 at t=0 when the participation constraint is satisfied, i.e. when the expected profit of loan contract outweighs the expected cost of bankruptcy. This condition is always satisfied since we assume sufficiently large $E[a]$. In this section we consider a situation when players are aware of the possibility of forbearance.

Anticipated forbearance changes the equilibrium at t=0 through a higher $E_0[q_1]$. Higher $E_0[q_1]$ raises expected wealth $E_0[\omega_1]$ of firms and reduces banks' actual haircut $\tilde{h}$. First, banks react by lowering firms' retained earning T_0, since the firms' participation constraint is not binding. From Lemma 4 we have unique T_0 achieving $E_0[q_1] = q_0 + \varphi_0$, but with such T_0 $E_0[\omega_1] < \omega_0$ since T_0 is lower than the previous cases nevertheless the additional capital gain was wiped off (this is analogous to the discussion of Proposition 8 using the intermediate value theorem). Therefore we have unique T_0 such that $E_0[\omega_1] = \omega_0$ and $E_0[q_1] > q_0 + \varphi_0$.[12] Note that q_0 is higher than the case when banks do not anticipate the possibility of forbearance, since higher $E_0[q_1]$ raises D_0 which shifts the firms' demand curve to the upper-right. K_0^f is also increased. Subsequently, The anticipation of forbearance will increase leverage before the crisis occurs, if the collateral constraint binds, in other words, if we maintain our assumption that $E[a]$ is sufficiently large.

The following proposition summarises the above.

Proposition 16 *Anticipated forbearance will ease credit conditions before a crisis. To be precise, anticipated forbearance will raise land price q_0 and k_0^f, as well as borrowers' debt outstanding D_0.*

Note that a higher D_0 should be followed by a higher $\hat{\theta}(= \theta^*)$ (see the equation 12). In some marginal cases, the raised $\hat{\theta}$ could be higher than 1, so that $\theta^* = 0$ (as banks cannot help themselves even when they maximise forbearance). In this case, banks find it optimal to contain the size D_0 so that $\hat{\theta} \leq 1$. We do not discuss the marginal cases in this paper further.

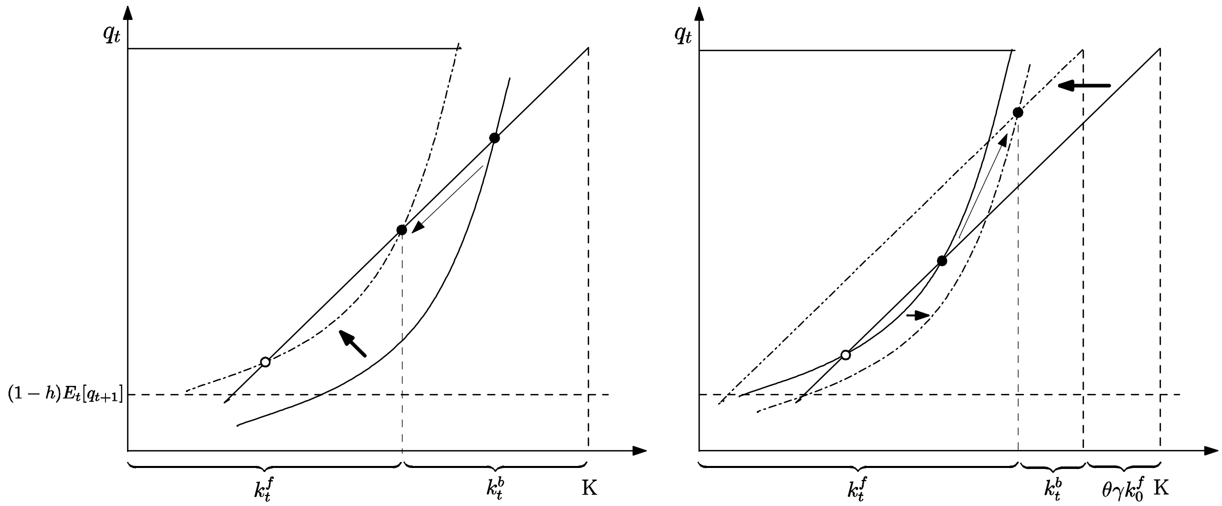
3.4 Extension: in case demand function is upward sloping

So far, we have limited our attention to the case η is small, ensuring the downward sloping firms' demand curve. In this section we see the other cases to complete the analysis.

As the lemma 2 shows, small η ensures downward sloping demand curve. If, instead, $\eta > \hat{\eta}$ and $E_{t-1}[q_t] > (1-h)E_t[q_{t+1}]$, we have the following lemma.

Lemma 17 *If the firms' demand curve is upward sloping, the curve is convex ($\partial^2 q_t / \partial k_t^{f2}$).*

[12] Note that q_0 is independent from the choice of T_0 in this model.



(a) equilib. with a negative macro shock

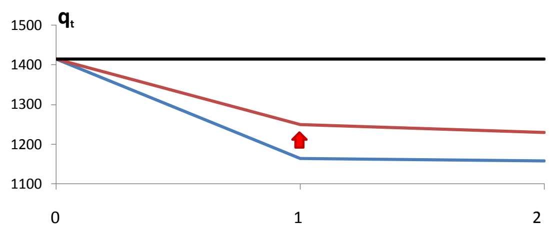
(b) the impact of forbearance

Figure 5: Equilibria with stronger financial accelerators

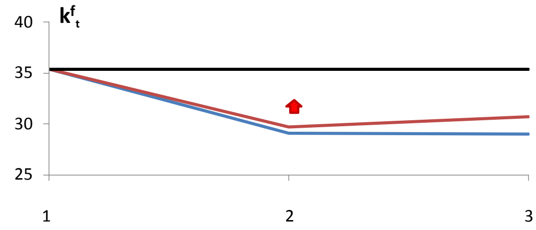
See the appendix for the proof (the proof of Lemma 2 has the result). Note first that $\hat{\eta}$ could be larger than one, i.e. the demand function is always downward sloping. The demand curve becomes upward sloping since the wealth effect (higher land price increases firms' net wealth) dominates the price effect (higher land price reduces the amount of land firms can purchase). With the upward demand function, we could have multiple equilibria: the upper-right crossing is the stable equilibrium, and the lower-left crossing is the unstable equilibrium (Figure 5(a)). The behaviours of equilibrium land price q_t and investment k_t^f against a negative macro shock are similar to the low η cases: the shock shifts the demand curve to the left, and both q_t and k_t^f decreases (see Figure 5(a)). I.e. the positive correlation betwenn the asset price and the output is maintained.

With the upward-sloping firms' demand curve, forbearance is unambiguously welfare-improving as it boosts both land price q_1 and the firms' investment k_1^f (see Figure 5(b)), which ensures the optimal θ^* to be one (see the Corollary 15). In this sense, the cases of upward-sloping demand curve do not add much on the findings above. The mechanism of forbearance is basically identical to the case where the firms' demand curve is downward sloping but η is higher than $\check{\eta}$.

Figure 6, the impulse response functions when the economy experiences a negative macro shocck at the end of t=0, summarises the arguments above. Compared with Figure 4, the output also increases by forbearance.



(a) Equilibrium paths of land price (high η)



(b) Equilibriu paths of output (high η)

Figure 6: Impulse responses with and without forbearance

An interesting case of upward sloping demand curve is that if the demand curve shifts largely to the left then we would lose the intersection with the dealers' demand curve. The equilibrium will jump discontinuously to $q_t = (1-h)E_t[q_{t+1}]$ or $q_t = 0$.[13] These discontinuous jumps can be considered as a catastrophic equilibrium, or crisis, although we do not study the details in this paper. This gives an additional role to forbearance. Forbearance shifting the dealers' demand function recovers the intersection of the two demand curves so that the economy can avoid a market crash (the discontinuous plunge of asset price).

4 Forbearance and social welfare

We next see the social welfare of the identified equilibria to discuss possible policy options. For the sake of simplicity, we do not consider the extensions we have seen in the previous section (higher η and banks' capital constraint).

In fact, the welfare effect of forbearance in this model is a trivial question as we do not assume any real cost of bank failure. Banks with $W_t < \bar{W}$ goes bankrupt when the equilibrium land price q_t is chosen and W_t is realised. Faild banks are eliminated from the economy immediately, but healthy banks replace the failed banks' loan costlessly (since there is no constraint on the lending amount in this model and each bank can lend to the infinite number of borrowers).[14] Since banks' failure is not costly for firms and the economy, and the cost of the closure of banks

[13] The former exists only when we assume $E_t[q_{t+1}]$ as given and fixed, and it is not supported by the dynamic equilibrium, as $E_t[q_{t+1}] = q_t + \varphi_t$ on the equilibrium path. The latter is the case firms cannot borrow any from banks and stop investing at all (the assumption $A = 2K$ ensures that the dealers' demand curve goes through the origin, and firms are assumed not to invest without bank loans).

[14] Of course, in real life we may have real cost of banks' failure: e.g. borrowers may not be able to find a new lender in a timely fashion and they might need to cancel their investments. The operational cost of closing banks would not be negligible as well, and the depositors would incur the loss. Those factors would provide other reasons for social planner to choose regulatory forbearance. This is currently left for future study.

X is a private cost for bank managers, social planner's welfare function is thoroughly determined by the output y_t. Note that maximising y_t and maximising k_t^f are nearly equivalent if we assume $E[a]$ is sufficiently large as we have assumed.

In reality, bank failure incurs some real economic cost and it is not difficult to introduce the real cost to the model. For instance, as Diamond and Rajan (2000) assumes, the depositors of a failed banks seize the loan assets and directly lend to borrowing firms. Since the depositors are inexperienced as lenders, firms are allowed to extract a larger private benefit $\bar{B} > B$ by cheating the depositors (see the section 2.2). Knowing the possibility of cheating, the depositors need to raise the haircut $\bar{h} > h$, which reduces leverage of the economy. The reduced leverage lowers k_t^f as long as collateral constraint is binding as we have assumed, which reduces social welfare.

Since forberance action θ has the corner solution $\theta^* \in \left\{0, \hat{\theta}\right\}$ (if $\eta < \breve{\eta}$), the socially optimal θ can be either 0 or $\hat{\theta}$. If $\bar{B} - B$ is small, the replacement cost of lenders does not lowers productive investment k_t^f significantly, and the social planner will find forbearance more costly than closing failed banks. If $\bar{B} - B$ is sufficiently large, the social planner finds it optimal to avoid bank failure, which could make forbearance welfare-improving. Of course, the social planner might find the other policy options optimal, such as public capital injection, although those options are out of the focus of this paper. Since k_t^f and y_t are independent from $\bar{B}$ if $\theta \geq \hat{\theta}$ and the increase of h (to $\bar{h}$) monotonically lowers k_t^f (and $\bar{h}$ is strictly increasing against $\bar{B}$), we have the following proposition:

Proposition 18 *There exists a unique threshold of $\bar{B}$ (or $\bar{h}$) such that for any $\bar{B}$ (or $\bar{h}$) larger than the threshold, forbearance is welfare-decreasing.*

Note that the threshold of $\bar{B}$ could be positive infinite (or $\bar{h} = 1$), since if the fraction of failed banks γ is negligibly small the economic cost of bank failure would also be negligibly small irrespective of the size of $\bar{h}$. Public authorities would find it optimal to minimise forbearance in this case. Otherwise, the authorities should rescue the failed banks to maintain the firms' investment and output. In the latter cases, authorities would be able to justify forbearance, or they can ensure that banks forbear by regulatory forbearance. To be precise, when the authorities find bank failure costly, regulatory forbearance is not the only option: public capital injection to failing banks would be a straightforward measure to avoid bank failure. We do not consider those options here to limit our attention to forbearance.

Going back to our initial assumption that the replacement cost of failed banks is zero, we will consider what public authorities can do when forbearance is welfare-decreasing. Since the socially optimal θ is zero in this case, public authorities would be required to consider policy options to discourage forbearance. The following section discusses three policy options.

4.1 Policy options

We have seen that social planner finds it optimal to discourage or eliminate forbearance if the economy's financial accelerator is relatively weak, i.e. η is low. Here we discuss what kind of policy options the social planner has.

4.1.1 Capital injection

Since banks forbear liquidation to avoid the situation $W_1 < \bar{W}$, public capital injection increasing W_1 directly (or loosening capital requirement $\bar{W}$) would resolve the problem. The marginal effect of capital injection is, however, non-monotonic. In the equation (12) the capital injection can be described as increasing W_0. From Lemma 12 $\hat{\theta}$, the threshold to have $W_1 = \bar{W}$ is monotonically decreasing against the capital injection ΔW. Since the optimal $\theta^* = \hat{\theta}$ only when $\hat{\theta} \in [0,1]$ and $\theta^* = 0$ otherwise, $\hat{\theta}(\Delta W)$ is a discontinuous, non-monotonic function.

Insufficient capitalisation, or minimum capitalisation to keep stricken banks survive, therefore incentivises banks' forbearance. This is consistent to Philippon's and Schnabl's (2013) model and Giannetti's and Simonov's (2013) empirical findings.

4.1.2 Tight provisioning

Penalising forbearance, by tight provisioning or higher funding cost (to maintain non-performing loans), would have non-monotonic effects too. If public authorities require banks to increase loan loss provisions proportional to their zombie loans, we can rewrite the equation (12) as follows:

$$W_1(\theta) = W_0 - (1+r)D_0 + (1-\theta)\, q_1(\theta)\, k_0^f + (1-\beta)\,\theta E_1\left[(1-\gamma)\, R_0 + \gamma q_2 k_0^f\right]$$

where β is the provisioning rate for a unit of zombie loan. Higher β obviously reduces the marginal benefit of forbearance $\partial W_1/\partial\theta$, and increases the threshold $\hat{\theta}$. Therefore the optimal θ^* is increasing against β as long as $\hat{\theta} \leq 1$, and then plunges to zero.

We are able to discuss the impact of higher interest rate in a similar way. By raising r for banks' liability D_0, with a small modification to the model,[15] we have an analogous result to the tight provisioning.

[15] Currently the funding cost of forbearance is charged at the following period t=2 (i.e. implicitly assuming long-term funding), and does not appear in the equation above. In addition, interest rate also make dealers' demand curve flatter thus the impact would not be intuitive. But if we assume that higher interest rate affects banks alone and the funding cost is immediately charged at t=1, we have an analogous argument as the tight provisioning.

Although both tight provisioning and capital injection have non-monotonic effects to forbearance, the nature of the effects is different. As the social planner increase capital injection to a bank, the optimal θ^* jumps to 1 from zero, and then gradually decreases until zero. Gradual tightening of credit loss provisions increases θ^* continuously, and then θ^* plunges from one to zero. Clearly, the social planner need to be cautious when it aims at discouraging forbearance by tightening provisioning or by higher funding cost, as it could trigger a sudden unwinding of forbearance in a massive scale.

5 Conclusion

We have seen that why banks do forbearance and how it affects the economy using the reduced-form model of KM's (1997) Credit Cycles. Stricken banks cannot bear the cost of writing off bad borrowers and forbear liquidating their non-performing loans to make up their balance sheets for the time being. This raises the price of collateral asset than it should be, and contains the negative equity of bad loans, allowing banks to make up their balance sheets.

Whether forbearance could decrease the output of the economy or not relies on the structure of the economy. We have seen that if the financial accelerator effect of an economy is relatively weak, i.e. firms do not actively realise capital gains and losses of their asset holdings and maintain their investment levels stable, forbearance would lower the economy's output. This is because forbearance maintains land price higher than it should be, which reduces productive firms' "purchasing power" of land (price effect), and because the expectation of land price decline when banks unwind forbearance in the future tightens the credit constraint of firms through a higher haircut of collateral (haircut effect). Note that we have two channels of negative externalities here. First one is to productive firms: the haircut effect tightens the credit constraint of both productive and zombie firms. Second, banks lose their profitable lending opportunities to the productive firms — not only forbearing banks but healthy banks have the same problem.

The haircut effect plays an important role to endogenise leverage of the economy. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), firms' net wealth is crucial in determining the amount of loans, as the leverage is given and fixed. This ensures the positive correlation between asset price and output. But in this model, anticipated decline of collateral value lowers leverage, as banks require a higher haircut of collateral to compensate the deterioration of collateral value. In other words, supported land price by forbearance creates pessimism over future land price, which reduces lending and output, resulting in the broken correlation.

But if the financial accelerator effect is strong, forbearance can rather boost the output. Forbearance under recession contains the capital loss of the firms and support the firms' net wealth and their borrowing capacity. In the economy with a strong financial accelerator effect, the net wealth would be significantly higher than it should have been, and the 'wealth effect' would dominate the price and haircut effects, which increases the output of the economy.

It is not easy to determine if an economy has a strong financial accelerator effect: it would be different across sectors (the accelerator of manufacturing industries would be low, while that of real estate sector would be high), different over time (e.g. the accelerator could be lower in recession) and different across countries. But if forbearance boosts output, banks should find it optimal to maximise forbearance as much as possible, which is not realistic. This could be a supporting evidence implying that forbearance is output reducing in most countries. We then extend the model to incorporate banks' capital adequacy ratio requirement constraining bank lending. The extension does not change the results significantly: forbearance would support asset price but reduce output, while forbearance can improve the output of the economy when majority of banks are capital-constrained (this is because forbearance will support asset price and ease banks' lending constraint).

We have also seen banks' incentive mechanism of forbearance. Banks choose forbearance since it is too costly for the banks to realise loan losses, since the loss, lending amount minus the liquidation value of collateral, is large enough to lower their capitals below a regulatory threshold and the banks go bankrupt. Forbearance is one of multiple equilibria, which are "no bank forbears" and "all stricken banks forbear". This could explain a reason why forbearance is not observed in the US.[16] Another possible reason of the domination of foreclosure in the US would be the less-monopolistic nature of lending contracts. We have assumed that banks can lend many firms but each firm can only borrow from a bank. This enables a bank to decide forbearance by itself. But if a firm is borrowing from many lenders, possibly through securitisation, each forbearance decision needs coordination of all lenders, which is extremely difficult. Some lenders would be well-capitalised and do not share the incentive of forbearance. In an extreme case, identifying all lenders would be difficult once a loan is securitised. Less-securitised lending contracts in Europe and Japan would make it easier for banks to forbear.

We consider another extension to study the welfare effect of forbearance. In the baseline model, bank failure is assumed to be costless for the economy for simplicity. Under this assumption, forbearance lowering output is always welfare-reducing as the social planner does not care the private cost of bank failure incurred by the bank manager. But once we introduce a friction to replace failed lending banks, we can find a possibility of welfare-enhansing forberance even when forbearance reduces output, since bank failure itself lowers output and forbearance

[16] In the US, loan outstanding to commercial real estate sectors declined significantly in 2009 and 2010. Even the outstanding of residential mortgage decreased during the period. In the UK or Japan in 1990s loans to these sectors showed strong resilience, which are treated as a clear evidence of forbearance.

can rescue failing banks. If forbearance is output-boosting it is always welfare-enhansing, although this is not very realistic as we discussed above.

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6 Appendix

6.1 Proof of Lemma 2

The FOD is,

$$\begin{aligned}
\frac{\partial k_t^f}{\partial q_t} &= \frac{\eta(1-\gamma)k_{t-1}^f}{q_t-(1-h)E_t\left[q_{t+1}\right]} - \frac{(1-\gamma)\left\{\psi(T_{t-1})+\eta q_t k_{t-1}^f\right\}}{\left\{q_t-(1-h)E_t\left[q_{t+1}\right]\right\}^2} \\
&= \frac{(1-\gamma)}{q_t-(1-h)E_t\left[q_{t+1}\right]}\left\{\eta k_{t-1}^f - \frac{\psi(T_{t-1})+\eta q_t k_{t-1}^f}{q_t-(1-h)E_t\left[q_{t+1}\right]}\right\} \\
&= \frac{(1-\gamma)}{q_t-(1-h)E_t\left[q_{t+1}\right]}\left\{\frac{\eta\left\{-(1-h)E_t\left[q_{t+1}\right]\right\}k_{t-1}^f-\psi(T_{t-1})}{q_t-(1-h)E_t\left[q_{t+1}\right]}\right\}
\end{aligned}$$

The (constrained) demand function is downward sloping if $-\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^f-\psi(T_{t-1})<0$ and if $q_t-(1-h)E\left[q_{t+1}\right]\neq 0$.

$$\begin{aligned}
& -\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^f-\psi(T_{t-1}) \\
=\ & -\eta(1-h)E\left[q_{t+1}\right]k_{t-1}^f - T_{t-1} - \left(a_t - E_{t-1}\left[a_t\right]\right)k_{t-1}^f + \eta E_{t-1}\left[q_t\right]k_{t-1}^f \\
=\ & \eta\left\{E_{t-1}\left[q_t\right]-(1-h)E_t\left[q_{t+1}\right]\right\}k_{t-1}^f - \left(a_t-E_{t-1}\left[a_t\right]\right)k_{t-1}^f - T_{t-1} \qquad (14)
\end{aligned}$$

Since we have assumed that $T_{t-1}+(a_t - E_{t-1}[a_t])\, k^f_{t-1} - \eta\,(q_t - E_{t-1}[q_t])\, k^f_{t-1} > 0$ (to ensure that surviving firms with negative macro shock possess positive net wealth ω_t), if the first term is sufficiently small the demand function is downward sloping. If $E_{t-1}[q_t] < (1-h)E_t[q_{t+1}]$ the function is downward sloping irrespective of η. If $E_{t-1}[q_t] > (1-h)E_t[q_{t+1}]$, there exists unique $\hat{\eta}$ such that for any $\eta < \hat{\eta}$ the FOD is negative and vice versa. Note that $\hat{\eta}$ does not necessarily takes a feasible value, i.e. $\hat{\eta} \leq 1$.

The SoD is,

$$\frac{\partial k^{f2}_t}{\partial^2 q_t} = (-2)\frac{(1-\gamma)}{\{q_t - (1-h)E[q_{t+1}]\}^3}\left[\eta\,\{-(1-h)E[q_{t+1}]\}\, k^f_{t-1} - \psi(T_{t-1})\right]$$

The SOD is positive if the FOD is negative, and vice versa. I.e., the demand curve is concave if it is upward sloping, and is convex (to the origin) if it is downward sloping.

If $E_t[q_{t+1}] = q_t$, at t=2,

$$k^f_t = \frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k^f_{t-1}\right\}}{hq_t}$$

The FOD is,

$$\begin{aligned}
\frac{\partial k^f_t}{\partial q_t} &= \frac{(1-\gamma)\eta k^f_{t-1}}{hq_t} - h\frac{(1-\gamma)\left\{\psi(T_{t-1}) + \eta q_t k^f_{t-1}\right\}}{(hq_t)^2} \\
&= \frac{1-\gamma}{hq_t}\left\{\eta k^f_{t-1} - h\frac{\psi(T_{t-1}) + \eta q_t k^f_{t-1}}{hq_t}\right\} \\
&= \frac{1-\gamma}{(hq_t)^2}\cdot(-h)\cdot\psi(T_{t-1})
\end{aligned}$$

The FOD is negative if $h\psi(T_{t-1}) > 0$, i.e.

$$\begin{aligned}
-hT_{t-1} - h\,(a_t - E_{t-1}[a_t])\, k^f_{t-1} + h\eta E_{t-1}[q_t]\, k^f_{t-1} &< 0 \\
\{\eta E_{t-1}[q_t] - (a_t - E_{t-1}[a_t])\}\, k^f_{t-1} - T_{t-1} &< 0
\end{aligned}$$

For any parameter set, there exists a threshold $\hat{\eta}$ such that for any $\eta < \hat{\eta}$, the FOD is negative. Note again that there is no guarantee that $\hat{\eta} < 1$. In other words, if $\psi(T_{t-1}) > 0$, then the demand function is downward sloping.

The SOD is,

$$\frac{\partial k_t^{f2}}{\partial^2 q_t} = \left(-2h^2\right) \frac{1-\gamma}{(hq_t)^3} \cdot (-h) \cdot \psi(T_{t-1})$$

The SOD is positive if $\psi(T_{t-1}) > 0$, i.e. if FOD is negative. This means that if the demand curve is upward sloping, it is convex, and if the curve is downward sloping it is convex to the origin. ■

6.2 Proof of Lemma 4

This is graphically obvious. We first prove $\partial k_2^f / \partial\psi\,(T_1)$ and then prove $\partial k_1^f / \partial\psi\,(T_0)$ by backward induction. From the Quadratic formula, the equilibrium k_2^{f*} should be:

$$k_2^{f*} = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c}}}{2\tilde{a}}$$

The sign of the second term of the numerator should be plus in any cases, since if $\tilde{c} > 0$, $\frac{-\tilde{b}-\sqrt{\tilde{b}^2+4\tilde{a}\tilde{c}}}{2\tilde{a}}$ becomes negative, and if $\tilde{c} < 0$, $\tilde{b} < 0$ and $\tilde{b}^2 + 4\tilde{a}\tilde{c} > 0$, $\frac{-\tilde{b}-\sqrt{\tilde{b}^2+4\tilde{a}\tilde{c}}}{2\tilde{a}}$ is positive but unstable equilibrium we do not consider further. Clearly $\tilde{c}$ is strictly increasing against ψ, the lemma is proved for k_2^f and q_2.

At t=1,

$$\tilde{c} = (1-\gamma)\left\{\psi(T_{t-1}) + \eta\frac{E\left[q_{t+1}\right]}{1+r}k_{t-1}^f\right\}$$

Since we have just proved that $\partial E_1\left[k_2^f\right] / \partial\psi\,(T_1)$, $\tilde{c}$ is again strictly increasing against ψ. ■

6.3 Proof of Lemma 6

First we see the concavity of q_t against T_{t-1}. Since q_t is a linear function of k_t^f at the equilibrium (since the supply curve is linear) this is equivalent to see the FOD and SOD of k_t^f w.r.t. T_{t-1}. We see the equilibrium k_2^{f*} first as follows:

$$h\frac{2}{r}\left(k_2^{f*}\right)^2 - \eta(1-\gamma)\frac{2}{r}k_1^f k_2^{f*} = (1-\gamma)\psi(T_1)$$

Since we know that the sign of FOD is positive from Lemma 4, we will concentrate on the sign of SOD. The FOD of the implicit function is:

$$h\frac{4}{r}k_2^{f*}\frac{\partial k_2^{f*}}{\partial T_1} - \eta(1-\gamma)\frac{2}{r}k_1^f\frac{\partial k_2^{f*}}{\partial T_1} = (1-\gamma)\frac{\partial\psi(T_1)}{\partial T_1}$$

Note first that $\partial k^f_{t-1}/\partial T_{t-1} = 0$, since when a_t is realised k^f_{t-1} has been chosen and fixed. In addition, $\partial\psi(T_{t-1})/\partial T_{t-1} = 1$ by definition. Rearranging this, we have:

$$\left(h\frac{4}{r}k_2^{f*} - \eta(1-\gamma)\frac{2}{r}k_1^f\right)\frac{\partial k_2^{f*}}{\partial T_1} = 1-\gamma$$

The inside of the bracket of LHS has to be positive from Lemma 4. The SOD is:

$$h\frac{4}{r}\frac{\partial k_2^{f*}}{\partial T_{t-1}} + \left(h\frac{4}{r}k_2^{f*} - \eta(1-\gamma)\frac{2}{r}k_1^f\right)\frac{\partial^2 k_2^{f*}}{\partial T_{t-1}^2} = 0$$

If the FOD is positive the SOD has to be strictly negative at t=2.

Given this, we can consider the same concavity issue at t=1. Equilibrium k_1^{f*} is determined by the following equations:

$$\begin{aligned}
k_1^f &= \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1 k_0^f\right\}}{q_1 - (1-h)E\left[q_2\right]} \\
k_1^b &= \frac{A}{2} - \frac{1}{2}rq_1 \\
K &= k_1^f + k_1^b
\end{aligned}$$

Substituting $E\left[q_2\right] = q_1 + \varphi_1$,

$$\begin{aligned}
k_1^f &= \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1 k_0^f\right\}}{hq_1 - (1-h)\varphi_1} \\
k_1^b &= \frac{A}{2} - \frac{1}{2}rq_1 \\
q_1 &= \frac{A - 2\left(K - k_1^f\right)}{r}
\end{aligned}$$

Substituting this into the demand function, we have the equilibrium k_t^{f*}.

$$k_1^{f*} = \frac{(1-\gamma)\left\{\psi(T_0) + \eta\frac{A-2\left(K-k_1^f\right)}{r}k_0^f\right\}}{h\frac{A-2\left(K-k_1^f\right)}{r} - (1-h)\,\varphi_1}$$

$$\left\{h\frac{A-2\left(K-k_1^f\right)}{r}-(1-h)\,\varphi_1\right\}k_t^{f*}=(1-\gamma)\left\{\psi(T_0)+\eta\frac{A-2\left(K-k_1^f\right)}{r}k_0^f\right\}$$

$$\begin{aligned}&\left\{\frac{A-2K}{1+r}-(1-h)\,\varphi_1-\eta\frac{2}{r}(1-\gamma)k_0^f\right\}k_t^{f*}+h\frac{2}{r}\left(k_1^{f*}\right)^2\\ =\;&(1-\gamma)\left\{\psi(T_{t-1})+\eta\frac{A-2K}{r}k_{t-1}^f\right\}\end{aligned}$$

Since $A=2K$,

$$\left\{-(1-h)\,\varphi_1-\eta\frac{2}{r}(1-\gamma)k_0^f\right\}k_t^{f*}+h\frac{2}{r}\left(k_1^{f*}\right)^2=(1-\gamma)\psi(T_0) \tag{15}$$

Taking derivative,

$$\left\{-(1-h)\,\varphi_1-\eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}+h\frac{4}{r}E_0\left[k_1^{f*}\right]\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}=(1-\gamma)\frac{\partial E_1\left[\psi(T_0)\right]}{\partial T_0}$$

$$\begin{aligned}\left\{h\frac{4}{r}E_0\left[k_1^{f*}\right]-(1-h)\,\varphi_1-\eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}&=1-\gamma\\ \left\{h\frac{4}{r}E_0\left[k_1^{f*}\right]-(1-h)\,\varphi_1-\eta\frac{2}{r}(1-\gamma)k_0^f\right\}\frac{\partial^2 E_0\left[k_1^{f*}\right]}{\partial T_0^2}+h\frac{4}{r}\left(\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}\right)^2&=0\end{aligned}$$

$\frac{\partial E_0\left[k_1^{f*}\right]}{\partial T_0}>0$ only when the bracket on LHS is strictly positive, and this is the sufficient condition for the second order derivative to be strictly negative. $E_0\left[k_1^{f*}\right]$ is therefore increasing and strictly concave wrt T_0. And from the firms' wealth transition function,

$$\begin{aligned}E_0\left[\omega_1\right]&=(1-\gamma)\left\{\frac{\omega_0}{1-\gamma}+\eta E_0\left[q_1(a_0)-q_0\right]k_0^f\right\}\\&<\omega_0\end{aligned}$$

a negative drift φ_0 is obtained. Since $E_0\left[\omega_2\right]=E_0\left[\omega_1\right]$, $E_0\left[\varphi_1\right]=\varphi_0$, although this $E_0\left[\varphi_1\right]$ never be realised since $\omega_2\gtreqless\omega_1$. ■

6.4 Proof of Proposition 10

For a fixed $a_1 (= a_L)$ and a given $\theta > 0$, the dealers' demand curve shifts leftward and k_1^f decreases as long as $\eta < \hat{\eta}$, i.e. the firms' demand curve is downward sloping. However, we need to consider one more influence of the forbearance: the raised $q_1(\theta)$ increasees ω_1 and shifts the firms' demand curve rightward (i.e. financial accelerator effect). This raises k_1^f for sure, since the dealers' demand curve is upward sloping. If this demand curve's shift dominates the leftward shift of the dearlers' demand function, k_1^f can be increasing against a higher θ.

To find out the condition that k_1^f is decreasing against θ, all we need to see is the behaviour of the firms' demand function. Since the dealers' demand function is fixed, k_1^f increases when $\check{\theta} > 0$ is chosen if the following condition is satisfied: the firms' demand function $D_1\left(S_1^{-1}\left(k_1^f(\theta = 0); \check{\theta}\right)\right) \geq k_1^f(\theta = 0)$. D_1 is firms' demand function as a function of q_1. S_1^{-1} is the dealers' inverse demand function withe a given θ. I.e. $S_1^{-1}\left(k_1^f(\theta = 0); \check{\theta}\right)$ is the asset price that the dealers demand $K - \theta\gamma k_0^f - k_1^f(\theta = 0)$. If the firms' demand is higher than $k_1^f(\theta = 0)$ with such a price, k_1^f is increasing against θ. In this proof, we try to find a condition that $k_1^f\left(\check{\theta}\right) < k_1^f(\theta = 0)$ at the price above.

For a fixed a_1, θ and q_1, the firms' demand function is determined by the parameter η and the expected price $E_1[q_2(\theta)]$.

$$k_1^f = \frac{(1-\gamma)\left\{\psi(T_0) + \eta q_1(\theta) k_0^f\right\}}{q_1(\theta) - (1-h)E[q_2(\theta)]}$$

Since the dealers' demand function goes back to the original position at t=2, $E_1[q_2(\theta)]$ is determined by the change of $E_1[\omega_2] = \omega_1(\theta)$. It is obvious that $\omega_1(\theta)$ is increasing against θ as long as $\eta > 0$, and $\partial\omega_1(\theta)/\partial\theta$ is increasing against η, from the definition of the firms' wealth. Subsequently, $\partial\ E_1[q_2(\theta)]/\partial\theta$ is increasing against η as well.

Since $\partial\omega_1(\theta)/\partial\theta$ and $\partial\ E_1[q_2(\theta)]/\partial\theta$ are increasing against η, the firms' demand $k_1^{f,D}$ increases more by a higher θ when η is high, i.e. $\Delta k_1^{f,D}$ by a higher θ is monotonically increasing against η. This ensures the unique existence of the threshold $\check{\eta}$ such that for any $\eta < \check{\eta}$ the equilibiurm firms' investment k_1^f is decreasing against θ. Note also that $\check{\eta} < \hat{\eta}$, since if the firms' demand function is upward sloping k_1^f is always increasing against θ.

Since $\partial E_1[\omega_2(\theta)]/\partial\theta$ is increasing against η, we also know that $\partial E_1\left[k_2^f(\theta)\right]/\partial\theta$ is increasing against η, for any $\eta \geq 0$. ■

6.5 Proof of Lemma 11

If $\eta = 0$, a higher θ does not increase ω_1 and therefore $k_2^f(\theta > 0)$ stays the same as the case without forbearance. I.e. $k_1^f + k_2^f$ is decreasing against θ for sure. On the other hand, since $k_1^f(\theta)$ and $k_1^f(\theta)$ are increasing against θ if $\eta > \check{\eta}$, $k_1^f + k_2^f$ is decreasing against θ if $\eta \geq \check{\eta}$. And we have seen that $\partial k_1^f(\theta)/\partial\theta$ and $\partial E_1\left[k_2^f(\theta)\right]/\partial\theta$ is increasing against η, if $\eta < \check{\eta}$. From the Intermediate Value Theorem, there exists a unique threshold $\bar{\eta} < \check{\eta}$ such that for any $\eta < \bar{\eta}$, $k_1^f + k_2^f$ is decreasing against θ. ■

6.6 Proof of Lemma 13

$$\begin{aligned}
E_1[W_3] \;=\;& W_0 - (1+r)D_0 + (1-\theta)\, q_1(\theta)\, k_0^f + \theta E_1[q_2(\theta)]\, k_0^f - r\theta D_0 \\
& + (1-\gamma)\left(E[a_2]\, k_1^f(\theta) - T_1 - rD_1\right) + \gamma\left\{E_1[q_2(\theta)]\, k_1^f(\theta) - (1+r)\, D_1\right\} \\
& + \Pr\left[W_2 \geq \bar{W}\right]\left[(1-\gamma)\left(E[a_3]\, k_2^f(\theta) - T_2(\theta) - rD_2(\theta)\right) + \gamma\left\{E_1[q_3(\theta)]\, k_2^f(\theta) - (1+r)\, D_2(\theta)\right\}\right] \\
& + \left(1 - \Pr\left[W_2 \geq \bar{W}\right]\right)\cdot X
\end{aligned} \tag{16}$$

The FOD of $E_1[W_3]$ w.r.t. θ is (note that $E_1[q_2(\theta)] = E_1[q_3(\theta)]$ by assumption):

$$\begin{aligned}
\frac{\partial E_1[W_3]}{\partial\theta} \;=\;& \left(E_1[q_2(\theta)] - q_1(\theta)\right) k_0^f + (1-\theta)\frac{\partial q_1}{\partial\theta} k_0^f + \theta\frac{\partial E_1[q_2]}{\partial\theta} k_0^f - rD_0 \\
& + \left\{(1-\gamma)\, E[a_2] + \gamma E_1[q_2(\theta)]\right\}\frac{\partial k_1^f(\theta)}{\partial\theta} + \gamma\frac{\partial E_1[q_2(\theta)]}{\partial\theta} k_1^f(\theta) - (r+\gamma)\frac{\partial D_1}{\partial\theta} \\
& + \Pr\left[W_2 \geq \bar{W}\right]\left[\left\{(1-\gamma)\, E[a_3] + \gamma E_1[q_2(\theta)]\right\}\frac{\partial k_2^f(\theta)}{\partial\theta} + \gamma\frac{\partial E_1[q_2(\theta)]}{\partial\theta} k_2^f(\theta) - (r+\gamma)\frac{\partial D_2}{\partial\theta}\right] \\
& + \frac{\partial \Pr\left[W_2 \geq \bar{W}\right]}{\partial\theta}\left[\begin{array}{c} (1-\gamma)\left(E[a_3]\, k_2^f(\theta) - T_2(\theta) - rD_2(\theta)\right) \\ +\gamma\left\{E_1[q_2(\theta)]\, k_2^f(\theta) - (1+r)\, D_2(\theta)\right\} - X \end{array}\right]
\end{aligned}$$

First we will see the sign of $\frac{\partial \Pr\left[W_2 \geq \bar{W}\right]}{\partial\theta}$. Since $E_1[W_2]$ is defined by the first two lines of the equation (16), the first order derivative is the first two lines of $\frac{\partial E_1[W_3]}{\partial\theta}$ defined above. The sign is not trivially determined as various different factors are involved in. Some factors reduce the bank's capital W_2: expected capital loss of collateral assets by postponing to liquidate (knowing that the asset price would decline in the next period); additional funding cost to maintain the forborne loans; and the reduction of profitable new lending (the decline of k_1^f). But some factors incrases the capital: benefit coming from contained capital loss by forbearance (at t=1); and the sustained wealth effect $\frac{\partial E_1[q_2]}{\partial\theta} > 0$ at t=2 (see Proposition 8). It would not be impossible to find a set of parameters making $\frac{\partial \Pr\left[W_2 \geq \bar{W}\right]}{\partial \theta}$. But throughout the paper we assume that $E\left[a\right]$ is sufficiently large: this ensures that the loss of new lending opportunity outweighs the positive factors above and that $\Pr\left[W_2 \geq \bar{W}\right]$ is decreasing against θ.

Second, we will see whether $\frac{\partial E_1[W_3]}{\partial \theta}$ is decreasing w.r.t $E\left[a\right]$ or not. The relevant terms are the following: $(1-\gamma) E\left[a_1\right] \frac{\partial k_1^f(\theta)}{\partial \theta} + \Pr\left[W_2 \geq \bar{W}\right] (1-\gamma) E\left[a_1\right] \frac{\partial k_2^f(\theta)}{\partial \theta} + \frac{\partial \Pr\left[W_2 \geq \bar{W}\right]}{\partial \theta} (1-\gamma) E\left[a_3\right] k_2^f(\theta)$. We can rearrange this as follows:

$$(1-\gamma) E\left[a_1\right] \left\{ \frac{\partial k_1^f(\theta)}{\partial \theta} + \Pr\left[W_2 \geq \bar{W}\right] \frac{\partial k_2^f(\theta)}{\partial \theta} \right\} + \frac{\partial \Pr\left[W_2 \geq \bar{W}\right]}{\partial \theta} (1-\gamma) E\left[a_3\right] k_2^f(\theta)$$

From Lemma 11, $\frac{\partial k_1^f(\theta)}{\partial \theta} + \frac{\partial k_2^f(\theta)}{\partial \theta} < 0$ if $\eta < \bar{\eta}$. Therefore the first term of the above is strictly negative if $\eta < \bar{\eta}$, and the second term is negative from the argument in the previous paragraph. This ensures that $\frac{\partial E_1[W_3]}{\partial \theta}$ is decreasing w.r.t $E\left[a\right]$. Since the terms above is a linear function of $E\left[a\right]$, the decline of the terms is not bounded below, which ensures negative $\frac{\partial E_1[W_3]}{\partial \theta}$ for sufficiently large $E\left[a\right]$ and $\eta < \bar{\eta}$. ■