Mechanism Design with Limited Commitment

Laura Doval 1  Vasiliki Skreta 2

1 California Institute of Technology
2 University of Texas at Austin, University College London, and CEPR
Mechanism Design with Limited Commitment

- Full commitment is the standard assumption in dynamic mechanism design
  - Useful: upper bound on the designer’s payoff.
  - Convenient: *revelation principle* turns the mechanism selection game into a constrained optimization program.

- This *tractability* is lost when the designer has **limited commitment**.

- Limited commitment looms large in many applications of interest:
  - Bargaining, principal-agent (ratchet effect), fiscal policy, social insurance, international relations.

- Trade-off:
  - Optimal mechanism w/finite horizon (Hart & Tirole (1988), Laffont & Tirole (1990), Skreta (2006, 2015), Deb and Said (2015)).
  - Infinite horizon with restrictions (Maestri (2015), Gerardi & Maestri (2017), Strulovici (2017), Acharya and Ortner (2017)).
Revelation principle for mechanism design with limited commitment.

- We study a game between an uninformed designer and an informed agent with persistent private information.
- The designer can commit to today’s contract, but not to the continuation ones.

Result

1. Characterize the minimal class of mechanisms that is sufficient to replicate all equilibrium payoffs of the mechanism selection game.

2. Transform the designer’s problem into a constrained optimization one
   - Usual truthtelling and participation constraints,
   - + designer’s sequential rationality constraint.
Mechanisms (Myerson ’82, Forges ’85)

- $M$ is a set of input messages,
- $S$ is a set of output messages,
- $\beta$ is a communication device,
- $\alpha$ is a (randomized) allocation rule.
Mechanisms (Myerson ’82, Forges ’85)

- $M$ is a set of input messages,
- $S$ is a set of output messages,
- $\beta$ is a communication device,
- $\alpha$ is a (randomized) allocation rule.
Mechanisms (Myerson ’82, Forges ’85)

- $M$ is a set of input messages,
- $S$ is a set of output messages,
- $\beta$ is a communication device,
- $\alpha$ is a (randomized) allocation rule.
Mechanisms

Without loss of generality,

- \( M = V \),
- \( |S| = |M| \),
- \( \beta \) is “invertible”,
- Truth-telling.

Full Commitment
**Limited Commitment 1: Bester & Strausz (ECMA, 2001)**

Assume:

- $|M| = |S|$
- $\beta$ is “invertible”,
- $\alpha$ deterministic.
**Limited Commitment 1: Bester & Strausz (ECMA, 2001)**

Assume:

- $|M| = |S|$
- $\beta$ is “invertible”,
- $\alpha$ deterministic.

Then, for outcomes in the Pareto frontier, it is without loss of generality

- $M = V$, 

However, *Truth telling*. 

**Mechanisms**
Limited Commitment 1: Bester & Strausz (JET, 2007)

Assume:

- $|M| = |S|$
- $\beta$ is invertible
- $\alpha$ deterministic.
Limited Commitment 1: Bester & Strausz (JET, 2007)

Assume:

- \( |M| = |S| \)
- \( \beta \) is invertible,
- \( \alpha \) deterministic.

Then, without loss of generality

- \( M = V \),
- Truth-telling.
Limited Commitment 1: Bester & Strausz (JET, 2007)

Assume:

- \(|M| \neq |S|\) ask: when is it without loss of generality to have \(|M| = |S|\)?
- \(\beta\) is "invertible",
- \(\alpha\) deterministic.

Then, without loss of generality

- \(M = V\),
- Truth-telling.
Revelation principle:

- $S \approx \Delta(V)$.
- In a general class of games, this language allows us to replicate any equilibrium payoff of the interaction between the designer and the agent.
  - No need to assume transfers/time separability/history independence.
- Mechanism serves dual role: allocation today & information tomorrow.
- Mechanisms with $M = V$ and $S = \Delta(V)$ are denoted canonical.

Parsimonious representation:

- In finite horizon, we can write the designer's problem as a sequence of constrained maximization problems.
- Truthtelling + participation + designer’s sequential rationality.
- Constrained Information Design.
Revelation Principle
Mechanism Selection Game: Model

- Two players, the principal and the agent, interact over $T$ periods.
  - $T$ can be infinity.

- The principal holds the bargaining power.

- The agent has private information: type $v \in V$, $|V| < \infty$.

- Each period an allocation $a \in A$ is determined, where $A$ is a compact space.

- Given a sequence of allocations $a^t = (a_0, \ldots, a_{t-1})$, the principal can only choose $a_t \in A(a^t)$.

- Payoffs: $W(a, v)$ for the principal and $U(a, v)$ for the agent for $a \in A^T$, $v \in V$. 

The action set for the principal at time $t$ is given by:

$$\mathcal{M}_t = \{ \mathbf{M}_t = (\langle M^M_t, \beta^M_t, S^M_t \rangle, \alpha^M_t) \}$$

where:

- $M^M_t$ is a finite set of input messages, $|V| \leq |M^M_t|$,  
- $S^M_t$ is a set of output messages, $S^M_t$ contains an image of $\Delta(V)$,  
- $\beta^M_t : M^M_t \mapsto \Delta^*(S^M_t)$ is the communication device,  
- $\alpha^M_t : S^M_t \mapsto \Delta^*(A)$ is the allocation rule.
Mechanism Selection Game: Mechanisms

The action set for the principal at time $t$ is given by:

$$
M_t = \{M_t = (\langle M^M_t, \beta^M_t, S^M_t \rangle, \alpha^M_t)\}
$$

where:

- $M^M_t$ is a finite set of input messages, $|V| \leq |M^M_t|$,
- $S^M_t$ is a set of output messages, $S^M_t$ contains an image of $\Delta(V)$,
- $\beta^M_t : M^M_t \mapsto \Delta^*(S^M_t)$ is the communication device,
- $\alpha^M_t : S^M_t \mapsto \Delta^*(A)$ is the allocation rule.

- A mechanism is canonical if $(V, \Delta(V))$ are its sets of input and output messages.
- Let $\mathcal{M}^C$ denote the set of canonical mechanisms.
- Assume that $\mathcal{M}^C \subseteq M_t$. 
In each period $t$,

- Both players observe a draw from a correlating device $\omega \sim U[0, 1]$.

- The principal offers the agent a mechanism $M_t$.

- The agent observes the mechanism and accepts/rejects:
  - If she rejects, an allocation $a^* \in A$ gets implemented. Move to next period. (Assume $a^* \in \mathcal{A}(a^t)$ for all $t$, $a^t \in A^t$).
  - If she accepts, sends report $m \in M^{M_t}$, unobserved to the principal.
  - $s \in S^{M_t}$ is drawn according to $\beta^{M_t}(.|m)$, observed by the principal.
  - $a \in A$ is drawn according to $\alpha^{M_t}(.|s)$, observed by the principal.
• The above defines an extensive form game.
The above defines an extensive form game.

Strategies:
- For the principal, choose a mechanism for every history $\Gamma$.
- For the agent, when her type is $v \in V$, participation, $\pi_v$, and reporting, $r_v$, for each private history.

Beliefs: at each history, the principal holds beliefs about the agent's:
- payoff relevant type, $v \in V$,
- input messages into the mechanism (payoff irrelevant private history).

Equilibrium
A Perfect Bayesian Equilibrium is a tuple $\langle \Gamma^\ast, (\pi^\ast_v, r^\ast_v)_{v \in V}, \mu^\ast \rangle$ such that:

1. Strategies are sequentially rational,
2. Beliefs are obtained via Bayes' rule whenever possible.
Mechanism Selection Game: Equilibrium

- The above defines an extensive form game.

- Strategies:
  - For the principal, choose a mechanism for every history $\Gamma$.
  - For the agent, when her type is $v \in V$, participation, $\pi_v$, and reporting, $r_v$, for each private history.

- Beliefs: at each history, the principal holds beliefs about the agent’s:
  - payoff relevant type, $v \in V$,
  - input messages into the mechanism (payoff irrelevant private history).
Mechanism Selection Game: Equilibrium

- The above defines an extensive form game.

- Strategies:
  - For the principal, choose a mechanism for every history $\Gamma$.
  - For the agent, when her type is $v \in V$, participation, $\pi_v$, and reporting, $r_v$, for each private history.

- Beliefs: at each history, the principal holds beliefs about the agent’s:
  - payoff relevant type, $v \in V$,
  - input messages into the mechanism (payoff irrelevant private history).

**Equilibrium**

A *Perfect Bayesian Equilibrium* is a tuple $\langle \Gamma^*, (\pi_v^*, r_v^*)_{v \in V}, \mu^* \rangle$ such that:

1. Strategies are sequentially rational,
2. Beliefs are obtained via Bayes’ rule whenever possible.
Alternatively, consider the following **canonical game** where, for all $t$, $\mathcal{M}_t \equiv \mathcal{M}^C$, i.e.,

- $M^{M_t} = V$,

- $S^{M_t} = \Delta(V)$.

That is, the principal only chooses $\beta$ and $\alpha$. 
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \( \langle \Gamma^*, (\pi_v^*, r_v^*)_{v \in V}, \mu^* \rangle \).

Then there exists a payoff-equivalent PBE of the canonical game, \( \langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle \), such that
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, $\langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle$.

Then there exists a payoff-equivalent PBE of the canonical game, $\langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle$, such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \(\langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle\).

Then there exists a payoff-equivalent PBE of the canonical game, \(\langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle\), such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \( \langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle \).

Then there exists a payoff-equivalent PBE of the canonical game, \( \langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle \), such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs \( t + 1 \):

\[
\mu'_{MC_t}(\mu | v) = \frac{\mu'(v')}{\sum_{v' \in V} \mu'(v')} \quad \beta_{MC_t}(\mu | v)
\]
The Revelation Principle for Limited Commitment

Theorem
Fix any PBE of the mechanism-selection game, $\langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle$.

Then there exists a payoff-equivalent PBE of the canonical game, $\langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle$, such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs $t + 1$:

$$
\mu'(v) = \sum_{v' \in V} \mu'(v')
$$
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \( \langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle \).

Then there exists a payoff-equivalent PBE of the canonical game, 
\( \langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle \), such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs \( t + 1 \):

\[
\mu'(v) \beta^{MC}_{t}(\mu | v)
\]
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, $\langle \Gamma^*, (\pi^*_v, r^*_v)_{v \in V}, \mu^* \rangle$.

Then there exists a payoff-equivalent PBE of the canonical game, $\langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle$, such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs $t + 1$:

$$\frac{\mu'(v)\beta^{MC}_{t}(\mu|v)}{\sum_{v' \in V} \mu'(v')\beta^{MC}_{t}(\mu|v')}$$
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \( \langle \Gamma^*, (\pi_v^*, r_v^*)_{v \in V}, \mu^* \rangle \).

Then there exists a payoff-equivalent PBE of the canonical game, \( \langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle \), such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs \( t + 1 \):

\[
\mu'(M^C_t, 1, \mu)(v) = \frac{\mu'(v) \beta^M_t(\mu|v)}{\sum_{v' \in V} \mu'(v') \beta^M_t(\mu|v')}
\]
The Revelation Principle for Limited Commitment

**Theorem**

Fix any PBE of the mechanism-selection game, \( \langle \Gamma^*, (\pi_v^*, r_v^*)_{v \in V}, \mu^* \rangle \).

Then there exists a payoff-equivalent PBE of the canonical game, 
\( \langle \Gamma', (\pi'_v, r'_v)_{v \in V}, \mu' \rangle \), such that

1. At all histories, the agent participates with probability 1 if her type has positive probability,

2. At all histories, the agent reports her type truthfully,

3. At all histories, recommended beliefs coincide with realized beliefs \( t + 1 \):

\[
\mu'(M_t^C, 1, \mu)(v) = \frac{\mu'(v)\beta^{M_t^C}(\mu|v)}{\sum_{v' \in V} \mu'(v')\beta^{M_t^C}(\mu|v')} = \mu(v)
\]
Canonical input messages: \[ M = V \]
**Lemma 1**

There is a payoff equivalent PBE s.t. the agent conditions her strategy on $v$ and the public history alone.
Proof Sketch

\[ M = V \]

**Lemma 1**

There is a payoff equivalent PBE s.t. the agent conditions her strategy on \( v \) and the public history alone.

- Otherwise, input messages could not be just the \( v \)'s.

- The agent has two pieces of private information:
  - her payoff relevant type, \( v \in V \),
  - her past interactions with the mechanism.
Proof Sketch

\[ M = V \]

**Lemma 1**

There is a payoff equivalent PBE s.t. the agent conditions her strategy on \( v \) and the public history alone.

- Otherwise, input messages could not be just the \( v \)'s.

- The agent has two pieces of private information:
  - her payoff relevant type, \( v \in V \),
  - her past interactions with the mechanism.

- It implies that the principal cannot peak into his past devices.
Proof Sketch

\[ M = V \]

**Lemma 1**

There is a payoff equivalent PBE s.t. the agent conditions her strategy on \( v \) and the public history alone.

- Otherwise, input messages could not be just the \( v \)'s.

- The agent has two pieces of private information:
  - her payoff relevant type, \( v \in V \),
  - her past interactions with the mechanism.

- It implies that the principal cannot peak into his past devices.

- It follows from:
  - If the agent conditions on past input messages, then she is indifferent.
  - It is possible to construct a strategy for the agent that gives the principal the same payoff.
Canonical output messages: $S = \Delta(V)$
Proof Sketch

\[ S = \Delta(V) \]

- Let \( M_t \) be a mechanism on the support of \( \Gamma^* \) and \( s \in S^{M_t} \).

- Upon observing \( s \), two things happen:
  - The allocation is drawn from \( \alpha^{M_t}(\cdot|s) \).
  - Principal updates his beliefs about \( V \) and past inputs using \( \beta^{M_t} \) and \( r^*_v \):
    \[ \mu^*_s(v, \cdot). \]

- Lemma 1 implies that \( \mu^*_s(v, \cdot) \) is constant.
  \[ \Rightarrow \text{relevant part of beliefs are about the agent's type!} \]

- Natural conjecture: relabel \( s \simeq \mu^*_s \).
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:

- Offer the agent a richer set of allocations
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:

- Offer the agent a richer set of allocations

\[ p \quad s \quad \mu, a \]

\[ 1 - p \]

\[ s' \quad \mu, a' \]
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:

- Offer the agent a richer set of allocations \textit{(randomized allocations)}

\[
\begin{align*}
\text{s} & \rightarrow \mu, a \\
p & \quad \text{(coordinate continuation play)} \\
1 - p & \\
\text{s'} & \rightarrow \mu, a'
\end{align*}
\]
$S = \Delta(V)$

However, the principal can be using $s$ to:

- Offer the agent a richer set of allocations (randomized allocations)
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:

- Offer the agent a richer set of allocations (randomized allocations)

  - Coordinate continuation play

\[ p \xrightarrow{s} \mu, a \]
\[ (1-p) \xrightarrow{s'} \mu, a' \]

\[ p \xrightarrow{s''} \mu, a \]
\[ (1-p) \xrightarrow{s'} \mu, a' \]
Proof Sketch

$S = \Delta(V)$

However, the principal can be using $s$ to:

- Offer the agent a richer set of allocations (randomized allocations)

- Coordinate continuation play
$S = \Delta(V)$

However, the principal can be using $s$ to:

- Offer the agent a richer set of allocations (randomized allocations)

  \[
  p \quad s \rightarrow \mu, a \\
  1 - p \quad s' \rightarrow \mu, a'
  \]

- Coordinate continuation play (correlating device)

  \[
  p \quad s \rightarrow \mu, a, eqbm1 \\
  1 - p \quad s' \rightarrow \mu, a, eqbm2
  \]

  \[
  s'' \rightarrow \mu, a
  1 - p \quad s'' \rightarrow \mu, a'
  \]
Proof Sketch

\[ S = \Delta(V) \]

However, the principal can be using \( s \) to:

- Offer the agent a richer set of allocations (randomized allocations)

\[ p \quad s \rightarrow \mu, a \]
\[ 1 - p \quad s' \rightarrow \mu, a' \]

- Coordinate continuation play (correlating device)

\[ p \quad s \rightarrow \mu, a, eqbm1 \]
\[ 1 - p \quad s' \rightarrow \mu, a, eqbm2 \]
\[ \omega_1 \rightarrow eqbm1 \]
\[ \omega_2 \rightarrow eqbm2 \]
Proof Sketch

\[ S = \Delta(V) \]

**Lemma 2**

There is a one-to-one mapping between output messages and equilibrium beliefs.
Truth telling and participation with probability 1
Proof Sketch

**Truthtelling and participation with probability 1**

Fix a history and a $M_t \in \text{supp } \Gamma^*$. Let

$$
\sigma(M_t) : S^{M_t} \mapsto \Delta(V) \\
\sigma(M_t)(s) = \sum_{h^t, m \in M^{M_t}} \mu^*(h^t, M_t, 1, s)(\cdot, m),
$$

we can define for each $\mu \in \Delta(V)$,

$$
\alpha^{M_t^C}(a|\mu) = \alpha^{M_t}(a|\sigma^{-1}(M_t)(\mu)) \\
\beta^{M_t^C}(\mu|v) = \sum_{m \in M^{M_t}} \beta^{M_t}(\sigma^{-1}(M_t)(\mu)|m)r^*_v(M_t, 1)(m),
$$

Participation with probability 1:

- As usual, we can have the agent participate, but
  - not only need to guarantee she receives the same allocation, but also,
  - make sure that this can be done without altering the continuation mechanism for the agent.
The theorem says that all equilibrium payoffs of the mechanism selection game are also equilibrium payoffs of the canonical game.

However, canonical game has a smaller set of deviations.

It turns out that this does not matter. Indeed,
• The theorem says that all equilibrium payoffs of the mechanism selection game are also equilibrium payoffs of the canonical game.

• However, canonical game has a smaller set of deviations.

• It turns out that this does not matter. Indeed,

**Proposition**

Any equilibrium payoff of the canonical game can be attained in an equilibrium of the mechanism selection game.
Constrained Optimization

- In the canonical game, not all deviations are to mechanisms that induce truth-telling and participation.

- It follows from the proof of the proposition that these are all the deviations that matter.

- Hence, in finite horizon, can write the principal’s problem as selecting between mechanisms such that
  - Agent participates with probability 1.
  - Agent tells the truth.
  - Recommended beliefs are realized beliefs.
  - Continuation mechanisms satisfy sequential rationality.
Indeed, once $S \sim \Delta(V)$, we can think of

- Principal in period $t$: Sender,
- Principal in period $t + 1$: Receiver.

with some special features:

- Sender also takes actions: designs allocation,
- Not all information structures are available: only those that satisfy the PC and IC constraints $\Rightarrow$ Constrained Information Design.

We exploit the connection to ID to provide a program in the finite horizon case that solves for the principal’s optimal mechanism:

- Extend the one-inequality constraint result in Le Trest and Tomala (2017) to allow for any number of equality and inequality constraints.
- Characterize the number of posteriors the principal induces.
- Available in a short paper.
Conclusions
Conclusions

- Revelation principle for mechanism design with limited commitment:
  - **Canonical outputs: beliefs.**
  - Single agent.
  - Finite types. (continuum in Appendix)

- Separate allocation from information revelation.

- Beliefs: non self-referential language.

- Parsimonious representation of the equilibrium payoffs of the mechanism selection game.
Conclusions

- Revelation principle for mechanism design with limited commitment:
  - **Canonical outputs: beliefs.**
  - Single agent.
  - Finite types. (continuum in Appendix)

- Separate allocation from information revelation.

- Beliefs: non self-referential language.

- Parsimonious representation of the equilibrium payoffs of the mechanism selection game.

**Not in the talk:**

- Application to infinite horizon sale of a durable good:
  - Foundation for dynamic bargaining with one-sided offers and one-sided incomplete information.
Thank you!
Mechanisms
Mechanisms

We endow the principal with a collection $(M_i, S_i)_{i \in I}$ such that

- $M_i$ is finite and $|V| \leq |M_i|$ for all $i \in I$,
- $S_i$ contains an image of $\Delta(V)$ for all $i \in I$,
- $(V, \Delta(V))$ is an element of the collection.

Denote by $\mathcal{M}$ the set of all mechanisms with message sets $(M_i, S_i)_{i \in I}$.

Hence, the action set for the principal at time $t$ is given by:

$$\mathcal{M} = \{ M_t = (\langle M^M_t, \beta^M_t, S^M_t \rangle, \alpha^M_t) \}$$

where:

- $(M^M_t, S^M_t) = (M_i, S_i)$ for some $i \in I$,
- $\beta^M_t : M^M_t \mapsto \Delta^*(S^M_t)$ is the communication device,
- $\alpha^M_t : S^M_t \mapsto \Delta^*(A)$ is the allocation rule.
Participation
Why do we guarantee participation only for those types that have positive probability?
Why do we guarantee participation only for those types that have positive probability?

- We can guarantee all types of the agent participate with probability 1,
- This may require using messages $m^*, s^*$ that are only sent by the 0–probability types, $v^*$.
- PBE (and SE) do not impose any restrictions on the principal’s belief at $s^*$

  
  *in the original equilibrium when $v^*$ did not participate, the principal could have believed it was $v' \neq v^*$!*

- Potentially, the belief at $s^*$ coincides with the belief after $s'$, for some $s'$ that shows up on path.
**Participation**

Why do we guarantee participation only for those types that have positive probability?

- We can guarantee all types of the agent participate with probability 1,
- This may require using messages $m^*, s^*$ that are only sent by the 0—probability types, $v^*$.
- PBE (and SE) do not impose any restrictions on the principal’s belief at $s^*$
  
  *in the original equilibrium when $v^*$ did not participate, the principal could have believed it was $v' \neq v^*$!*

- Potentially, the belief at $s^*$ coincides with the belief after $s'$, for some $s'$ that shows up on path.

$\Rightarrow$ This endangers the one-to-one map between used outputs and beliefs.
How do we deal with this?

- We “remove” input messages $m^*$ that lead to output messages that are used only by 0-probability types.

- This removes deviations for the positive probability types, but may violate participation for the 0-probability types.

- Consequently, the only output messages that have positive probability under some device are those that have positive probability under the agent’s reporting strategy and the principal’s beliefs.