I use index prices and options to estimate the pricing kernel's elasticity, which approximates the fundamental component of Campbell and Cochrane (1999), the surplus consumption ratio. The approximation relies on an empirically motivated, time-varying relationship between consumption growth and the expected returns on market wealth, therefore avoiding problematic consumption data. I show that my estimate of the surplus consumption ratio: predicts future excess market returns, is highly correlated to business cycle variables and the SVIX of Martin (2017), and is priced in a cross-sectional analysis of equity returns. The results provide new empirical support for consumption-based asset pricing models.

**KEYWORDS:** Empirical pricing kernel elasticity, surplus consumptions ratio, habits, risk aversion, elasticity of intertemporal substitution, return predictability.

**JEL CODES:** E44, G1

## I. INTRODUCTION

One of the main objectives in financial economics is to be able to explain what we observe in financial markets not in isolation, but within the context of the macro-economy. For asset pricing specifically, we want to explain the movements of asset prices by linking them to current and
expected consumption levels. By doing so, we can understand how agents make decisions under uncertainty and therefore make better forecasts of future market returns.

Great progress has been made theoretically to link market returns and the consumption of agents. Constantinides (1990) and Campbell and Cochrane (1999) successfully introduced the idea of consumption habits to asset pricing in order to address the inability of early asset pricing models to explain real world observations. They showed that when an agent’s utility level is non-time-separable, such that it depends on both current consumption and a habit level, their simulated economies were able to resolve the equity premium and risk free rate puzzles (Mehra and Prescott 1985).

While simulations of habit-type models have shown possible reconciliation with historical asset prices, the empirical failure of consumption-based models to predict asset returns over short horizons is well acknowledged (Campbell and Cochrane 2000). One possible reason for this is that consumption data is notoriously plagued with measurement issues, in part caused by the under-reporting and under-representation of survey data, the non-synchronized documentation of expenditure and consumption, and the exclusion of some expenditure categories (Pistaferri 2015). As a consequence of this, the desire for empirical rapprochement between consumption growth and market returns has led to very creative methods to avoid the use of consumption data. For example, Savov (2011) used garbage data in the place of expenditure to acquire a more accurate measure of consumption growth. A notable exception for consumption-based predictors is the consumption-wealth ratio of Lettau and Ludvigson (2001). However, this is only available at a quarterly frequency and therefore not useful for shorter horizon predictions.

In this paper I propose a way around the use of consumption (and garbage) data by using traded market prices - a cleaner and more readily available data source. I do this by assuming a time-varying relationship between consumption growth and the expected returns on wealth. Specifically, I use market index prices and options to calculate the elasticity of the empirical pricing kernel (EPK), which approximates the surplus consumption ratio when incorporating my time-varying elasticity of intertemporal substitution (EIS) assumption into the Campbell and Cochrane (1999) external habits model. The surplus consumption ratio is the only state variable in their model economy, and therefore all variables of interest, such as the equity premium, are functions of that state variable. My EIS specification is motivated by the empirical life-cycle model literature which shows that the EIS is increasing in wealth and across time (Blundell, Browning, and Meghir (1994), Attanasio and Browning (1995) and Crossley and Low (2011)). The stark difference between using market prices instead of consumption data to estimate the surplus consumption ratio can be seen bellow in Figure I.1, with market prices providing a much more variable and economically intuitive estimate. My methodology therefore provides a new, more effec-
tive way to estimate this fundamental latent variable.

![Surplus Consumption Ratio Estimated Using Market Prices Vs Consumption Data](image)

**Figure I.1**

*Two surplus consumption ratio estimates are plotted, the first using market prices, the second using monthly services and non-durable consumption data from the FRED, following the NBER working paper version of Campbell and Cochrane (1999) (p.6, Equation 2.3) to calculate habits.*

The first contribution of this paper is to document significant predictability of excess returns from my estimate of pricing kernel elasticity, which under my assumptions approximates the surplus consumption ratio. The predictability remains statistically significant even after controlling for previously documented predictors such as the variance risk premium (VRP), risk neutral moments, business cycle and accounting measures, as well as variables which proxy for divergences in opinion and overconfidence. This reinforces the importance of consumption-based asset pricing models not only theoretically, but also empirically in explaining asset returns.

The surplus consumption ratio is by definition a cyclical variable. Therefore, the second contribution of this paper is to empirically show that the state variable is indeed time-varying and cyclical, as it is negatively correlated to the term spread, default spread, price-earnings ratio, consumption-wealth ratio and positively correlated to consumer sentiment and risk neutral skewness. The dynamics of my estimate provide a validation of the Campbell and Cochrane (1999) model in an empirical setting, and hence resolve a previous lack of evidence in support of consumption-based asset pricing models.

Other models have also been successful in quantitatively matching market data. Notably, Bansal and Yaron (2004) used the recursive preference set-up of Epstein and Zin (1989) and assumed consumption growth has a small, persistent, and predictable component with time-varying uncertainty. Therefore, both our consumption growth assumptions depart from the i.i.d.
case, but they differ in that my approach provides a micro-founded and economically intuitive mechanism by which the consumption growth dynamics become predictable. Recursive preferences also allow for the separation between risk aversion and the EIS, and therefore for agents to exhibit the preference for the early resolution of uncertainty. However, a limitation of their model is that preferences are not able to change over the business cycle; a key mechanism for explaining my empirical results. The model and assumptions I propose do not have this limitation.

While many papers have explored the predictive power of information and moments estimated from historical and option prices\(^1\) – this study relates most to three papers: Rosenberg and Engle (2002), Bollerslev, Tauchen, and Zhou (2009) and Martin (2017). After specifying two parametric representations of the pricing kernel, Rosenberg and Engle (2002) used the slope of the EPK to estimate time-varying risk aversion and found evidence for counter-cyclical risk aversion over their 1991–1995 sample. This paper differs in that no parametric specification is assumed for the pricing kernel during the elasticity estimation. By using this approach, I show that risk aversion alone cannot explain changes in EPK elasticity over time. This paper also links pricing kernel elasticity to the equity premium and empirically confirms the model implications that the equity risk premium is closely related to pricing kernel elasticity; something they do not do.

Bollerslev, Tauchen, and Zhou (2009) use the second moment of the risk neutral and the physical distribution to test the predictability of the variance risk premium (VRP). The source of information for both our predictors is therefore the same, as is the time-series methodology used to answer the predictability question. However, to support the economic reasoning behind the VRP’s predictability, the authors showed that the VRP can provide an approximation to a component of the equity risk premium when investors have Epstein and Zin (1989) preferences, and when a volatility of volatility process is assumed for consumption growth in combination with some of the Bansal and Yaron (2004) assumptions. I show these assumptions are incompatible with my results as they suggest EPK elasticity should be constant over time.

Similar to this paper, options are employed by Martin (2017) who uses a volatility index (SVIX) to derive a lower bound on the equity premium. The lower bound of the equity risk premium was found to be counter-cyclical and highly volatile, especially during times of market turmoil. Reconciling our results, my estimate of the surplus consumption ratio is at its lowest during bad states of the world and consequently the equity premium is at its highest, a prediction of the model I present in Section II.B. This is intuitive as return volatility is exponentially

higher in bad states of the world, or in low surplus consumption states. I show that EPK elasticity is indeed highly correlated with the SVIX, up to 80%, providing solid economic rationale for the predictive success of the volatility index. Also, using a large dataset of US stock returns, I provide evidence which shows that my estimate is priced in the cross-section, and that stocks which co-varied most with EPK elasticity over the sample earned substantially higher returns, even after controlling for size and value. This reinforces the importance of fundamental macro-economic variables in explaining the cross-sectional disparity of equity returns.

This paper uses the semi non-parametric methodology of Aït-Sahalia and Lo (1998) to estimate the risk-neutral distribution and the GJR-Garch model of Glosten, Jagannathan, and Runkle (1993) to estimate the physical distribution. This methodology results in non-monotonic projections of the pricing kernel onto future returns, the so-called "pricing kernel puzzle". U-shaped and non-monotone pricing kernels have been widely documented in the literature using a variety of methods and assets e.g. Jackwerth and Rubinstein (1996), Jackwerth (2000), Aït-Sahalia and Lo (2000), Rosenberg and Engle (2002), and Bollerslev and Todorov (2011).² Christoffersen, Heston, and Jacobs (2013) show that in the presence of a variance premium, the projection of the pricing kernel onto returns is U-shaped due to a quadratic term in the pricing kernel. Bakshi, Madan, and Panayotov (2010) suggest that heterogeneity in beliefs and investors short the market but long call options can explain the puzzle. No perfect methodology as of yet exists for extracting the pricing kernel. However, information is likely to be more accurate within the tails than out as more options are traded closer to the at-the-money level and physical distribution forecasts tend to differ mostly out in the tails and not close to the mean. I will therefore only use pricing kernel information from within plus and minus 5% of log returns. Song and Xiu (2016) show that the EPK’s projection onto returns is monotonic within the tails of the distribution when they condition on volatility factors. They stress that the EPK’s shape is highly state-dependent, a finding that is in-line with my results and which I am able to explain after linking the EPK’s elasticity to the macro-economy. Hence, my results are not in-line with Linn, Shive, and Shumway (2018), who argue that monthly EPK’s do not contain useful information due to a mismatch in conditional information sets when estimating the physical and risk-neutral distributions separately.

The paper is organised as follows. Section II will introduce my time-varying EIS specification and then incorporate it into the Campbell and Cochrane (1999) model. Section III will then explain how to estimate the surplus consumption ratio using market index prices and options. Section IV explores the relationship between my estimate and important macroeconomic variables. Section V will present the results of the predictability analysis along with robustness checks and various applications of the elasticity estimate. Section VI will conclude.

²For a comprehensive and up-to-date survey of the "pricing kernel puzzle" see Cuesdeanu and Jackwerth (2016).
II. A Time-Varying Elasticity of Intertemporal Substitution

This section introduces a new consumption growth specification which has a time-varying relationship with the expected return on market wealth. The magnitude of this relationship is captured by the EIS. Acknowledging this dynamic relationship means I can avoid consumption data issues and therefore more accurately estimate the surplus consumption ratio.

Empirical tests of the relationship between consumption growth and expected returns have rejected the constant EIS assumption, with evidence showing the EIS is both increasing with consumption, and varying across time (Blundell, Browning, and Meghir (1994), Attanasio and Browning (1995) and Crossley and Low (2011)). In light of this evidence, I propose the following conditional specification:

$$E_t \Delta c_{t+1} = g + \psi_t E_t r_{t+1},$$

or, in realized form:

$$\Delta c_{t+1} = g + \psi_t r_{t+1} + e_{t+1},$$

where $E_t$ is the expectation conditional on information available at time $t$, $\Delta c_{t+1} = \log \left( \frac{C_{t+1}}{C_t} \right)$ is the one-period change in log consumption, $g$ is the mean log consumption growth rate, $r_{t+1} = \log (R_{t+1})$ is the log gross return on market wealth, and $e_{t+1} \sim N.i.i.d.(0, \sigma^2)$ a random shock.

Following Blundell, Browning, and Meghir (1994), I define the EIS as:

$$\psi_t = \frac{\partial E_t \Delta c_{t+1}}{\partial E_t r_{t+1}} = \frac{\frac{\partial U(C_t)}{\partial C_t}}{C_t \frac{\partial^2 U(C_t)}{\partial C_t^2}}$$

where $U(.)$ is a period-specific utility function.

Unlike Hall (1988) and Campbell (1993), I allow the EIS to vary over time and wealth levels. Therefore, my specification captures two important aspects which are empirically motivated. Firstly, unlike Campbell and Cochrane (1999) (henceforth CC), I do not model consumption growth as a random walk process. Instead, I assume that current consumption, relative to next period, is influenced by the expected future return on wealth. The lower that $\psi_t$ is, the stronger the preference is to smooth consumption over time. $\psi_t$ less (more) than one corresponds to the income (substitution) effect dominating. Bansal and Yaron (2004) provide empirical evidence for a predictable component in consumption growth, but unlike above, did not allude to what it is. Secondly, the decision of how much to consume now, relative to later, based on expectations of next periods wealth, will not be constant, but will change over time. These aspects allow me to
express the pricing kernel in reduced form, which I show below can be estimated using prices.

A. TIME-VARYING EIS WITH EXTERNAL HABITS

In this section I incorporate my consumption growth specification into CC’s external habit model and show how the negative elasticity of the pricing kernel approximates the surplus consumption ratio. Following CC, let utility at time $t$ be a function of the difference between current consumption and habit level $X_t$:

$$U(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}, \quad (II.4)$$

where $\gamma$ is a parameter. Let the surplus consumption ratio equal $S_t = \frac{C_t - X_t}{C_t}$, where the growth rate of $S_t$, $\Delta S_{t+1} = log\left(\frac{S_{t+1}}{S_t}\right)$, follows an ARMA(1,1) type process:

$$\Delta S_{t+1} = \phi \Delta S_t + \lambda(s_t)(\Delta c_{t+1} - g) - \lambda(s_{t-1})(\Delta c_t - g), \quad (II.5)$$

with $\phi$ a parameter coefficient and $\lambda(s_t)$ the sensitivity function which governs how the surplus consumption ratio reacts to changes in consumption growth. Unlike CC, I assume $\lambda(s_t)$ is a positive, decreasing function of $S_t$, which satisfies the condition:

$$\frac{\partial S_t(1 + \lambda(s_t))}{\partial S_t} > 0. \quad (II.6)$$

This assumption will ensure my estimate is perfectly positively correlated to the surplus consumption ratio. While this no longer results in a constant risk-free rate, an objective of CC, I do not need to impose a constant risk-free rate for my results to hold.

Local curvature was defined by CC to be time-varying and counter-cyclical, and it is therefore fitting to use their consumption-based model as a building block in my analysis. In the context of their model and using (II.3), the time-varying elasticity of intertemporal substitution is:

$$\psi_t = \frac{S_t}{\gamma}, \quad (II.7)$$

which is both time-varying and increasing in consumption, and therefore in-line with the empirical literature. We can re-write (II.2) and (II.5) as:

$$\frac{C_{t+1}}{C_t} = e^{R + \epsilon_{t+1}} R_{t+1}^{S_t/\gamma} \quad (II.8)$$

$$\frac{S_{t+1}}{S_t} = \xi_t \left(\frac{C_{t+1}}{C_t}\right)^{\lambda(s_t)} \quad (II.9)$$

respectively, where $\xi_t = e^{\phi \Delta s_t - \lambda(s_{t-1})(\Delta c_t - g) - \lambda(s_t)g}$. We can therefore also re-write the marginal rate
of substitution, $M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}$, in terms of gross consumption growth by substituting in (II.9):

$$M_{t+1} = \beta \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma(1+\lambda(s_t))},$$

(II.10)

where $\beta$ is a subjective time discount factor. Finally, substituting in (II.8), we observe the reduced form pricing kernel, which unlike (II.10), we can estimate using traded security prices:

$$M_{t+1} = \beta \xi_t^{-\gamma} \left( e^{g_t} + e^{t+1} R_t S_t / \xi_t \right)^{-\gamma(1+\lambda(s_t))}.$$  

(II.11)

(II.11) is in contrast to much of the literature on pricing kernel estimation which simply assumes $C_{t+1} / C_t = R_{t+1}$. For example, Rosenberg and Engle (2002) make this assumption as it allows them to interpret the slope of the pricing kernel's projection onto returns as risk-aversion. I later show that my results are not compatible with this interpretation.

Let $M_t^* = \mathbb{E}_t(M_{t+1} | R_{t+1})$ be the conditional expectation at time $t$ of the pricing kernel for each possible realization of next periods return on wealth. $M_t^*$ can be expressed in closed form as:

$$M_t^* = R_{t+1}^{-\gamma S_t/(1+\lambda(s_t))} \beta \xi_t^{-\gamma(1+g(1+\lambda(s_t)))} e^{\sigma^2 \gamma^2(1+\lambda(s_t))^2}.$$  

(II.12)

The negative elasticity of the pricing kernel's conditional expectation is then:

$$\kappa_t = -R_{t+1} \frac{\partial M_t^*}{\partial R_{t+1}} = S_t (1 + \lambda(s_t)).$$

(II.13)

An estimate of (II.13) will approximate the only state variable of the CC model, from which all variables of interest are a function of. As $\lambda(s_t)$ is defined as a decreasing function of $S_t$ which also satisfies (II.6), the dynamics of $\kappa_t$ will be perfectly positively correlated with $S_t$ by assumption.

**B. THE EQUITY PREMIUM**

The surplus consumption ratio is the only state variable in the CC world, and therefore the dynamics of all important economic measures are functions of the surplus consumption ratio. For example, define $R_f$ as the gross risk-free rate of return. $log(\mathbb{E}_t[M_{t+1}R_{t+1}]) = 0$, and if $log(M_{t+1}R_{t+1})$ is assumed to be normally distributed$^3$, the standard log equity risk premium equation can be derived as,

$$log\left( \frac{\mathbb{E}_t R_{t+1}}{R_f} \right) \equiv -cov\left( log(M_{t+1}), log(R_{t+1}) \right).$$

(II.14)

$^3$Log-normality is not necessary however, the resulting form demonstrates the relationship between surplus consumption and the equity premium more clearly.
Let $\pi_t$ denote the log equity risk premium. From (II.11) we can observe the equity risk premium as:

$$\pi_t = \gamma (1 + \lambda(s_t)) \text{var}(r_{t+1}).$$  \hspace{1cm} (II.15)

The above equation predicts that when keeping $\text{var}(r_{t+1})$ fixed, the equity risk premium is high in bad states of the world (low $S_t$ and high $\lambda(s_t)$) and low in good states because $S_t$ and $\lambda(s_t)$ are negatively correlated. This is in line with empirical evidence e.g. Martin (2017). A lot of research has already looked at estimating $\text{var}(r_{t+1})$ and its relationship with asset returns. I will focus on estimating $S_t$ and testing the prediction that log excess returns can be predicted by the surplus consumption ratio.

### III. Estimation

In this section I introduce an estimate of pricing kernel elasticity which as shown, approximates the surplus consumption ratio. Let $Z_{t+1}$ define a general vector of state variables and therefore $M'_{t+1}(Z_{t+1}) = \frac{\beta U''(Z_{t+1})}{U'(Z_{t+1})}$. In a similar fashion to CC, it holds that curvature with respect to $Z_{t+1}$ is equal to:

$$\kappa_t = -\frac{Z_{t+1}U''(Z_{t+1})}{U'(Z_{t+1})} = -\frac{Z_{t+1}M'_{t+1}(Z_{t+1})U'(Z_t)}{U'(Z_{t+1})\beta} = -\frac{Z_{t+1}M'_{t+1}(Z_{t+1})}{M_{t+1}(Z_{t+1})}. \hspace{1cm} (III.1)$$

The curvature of the utility function is equivalent to the negative elasticity of the pricing kernel. This definition of pricing kernel elasticity allows a more general interpretation of what my statistic measures. For example, assuming $\Delta c_{t+1} = r_{t+1}$ and a power utility function, $\kappa_t = \gamma$ (Appendix.A provides a summary for the interpretation of negative EPK elasticity using the CC, power utility, and recursive preferences set-up under different consumption growth assumptions.). It will however become clear that models which do not allow for time-varying preferences cannot be reconciled with the empirical results of this paper.

### A. Data

Options’ data on the S&P 500 (SPX) from January 1996 to April 2016 is collected from OptionMetrics. Daily closing price data on the underlying S&P 500 covering the same period is also collected from OptionMetrics, along with a dividend correction and a risk free rate proxy. To estimate the VRP, VIX close levels are taken from the CBOE section on WRDS and realized volatilities estimated from high-frequency data are taken from the Oxford-Man Institute of Quantitative Finance’s Realized Library. Data from the Michigan Consumer Sentiment Index (MCSI) is gathered
<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>5%</th>
<th>10%</th>
<th>50%</th>
<th>90%</th>
<th>95%</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option Price, $H ($)</td>
<td>16</td>
<td>28</td>
<td>0.025</td>
<td>0.03</td>
<td>0.1</td>
<td>3.7</td>
<td>48</td>
<td>74</td>
<td>298</td>
</tr>
<tr>
<td>Implied Volatility, $\sigma$</td>
<td>0.29</td>
<td>0.16</td>
<td>0.06</td>
<td>0.11</td>
<td>0.13</td>
<td>0.24</td>
<td>0.50</td>
<td>0.63</td>
<td>1.00</td>
</tr>
<tr>
<td>Strike, $X$</td>
<td>1416</td>
<td>482</td>
<td>50</td>
<td>640</td>
<td>790</td>
<td>1400</td>
<td>2050</td>
<td>2185</td>
<td>3500</td>
</tr>
<tr>
<td>Implied Futures Price, $F$</td>
<td>1587</td>
<td>413</td>
<td>599</td>
<td>9049</td>
<td>1052</td>
<td>1564</td>
<td>2086</td>
<td>2105</td>
<td>2134</td>
</tr>
<tr>
<td>Log Moneyness, $\log(F/X)$</td>
<td>-0.15</td>
<td>0.33</td>
<td>-3.23</td>
<td>-0.74</td>
<td>-0.51</td>
<td>-0.08</td>
<td>0.13</td>
<td>0.22</td>
<td>1.25</td>
</tr>
</tbody>
</table>

Summary statistics for the cleaned Wednesday options sample from January 1996 to April 2016.

from the University of Michigan and the log(P/E) and log(P/D) ratios are calculated from data on Robert Shiller's website\(^4\). The consumption-wealth ratio is taken from the website of Sydney Ludvigson\(^5\), with the most recent quarterly observation available matched to each month. The default spread (DFSP) is calculated using the spread between BAA and AAA Corporate bonds and the term spread (TMSP) is calculated as the difference in yields between the ten-year treasury bond and the three-month treasury bill both from the Federal Reserve Bank of St.Louis. Risk neutral skewness and kurtosis are estimated following Bakshi, Kapadia, and Madan (2003).

Options with zero open interest, implied volatility recorded as "NaN" or those which violate no-arbitrage conditions are dropped. Options with less than a week and over a year to maturity are discarded, and only Wednesday options are used to eschew any beginning or end of week effects on option prices, which also leads to smoother implied volatility surfaces when compared to using all days of the week. The price of the option is taken to be the midpoint between the highest bid and lowest ask price and the dividend adjustment is subtracted from the underlying price to simplify later calculations. The implied futures price is calculated using the put-call parity.

Summary statistics for the cleaned options' panel is displayed in Table III.1. The mean and median log moneyness is -0.15 and -0.08 respectively suggesting many more puts than calls are traded. Figure III.1 displays the histogram of log moneyness for options traded in the final cleaned sample. The histogram has a heavy negative skew and most observations have a log moneyness around zero.

---

\(^4\)http://www.econ.yale.edu/shiller/data.htm
\(^5\)https://www.sydneyludvigson.com/data-and-appendixes/
B. PRICING KERNEL AND ELASTICITY ESTIMATION

For the state process $Z_{t+1}$, the pricing kernel can alternatively be expressed as the Radon-Nikodym derivative,

$$M_{t+1} = \beta \frac{U'(Z_{t+1})}{U'(Z_t)} = \exp(-r_f) \frac{Q(Z_{t+1})}{P(Z_{t+1})}. \tag{III.2}$$

I estimate the ratio of two distributions which as shown in (III.2) is equivalent to the pricing kernel when discounted. The risk-neutral distribution $Q$ is estimated using options’ data within each month following Aït-Sahalia and Lo (1998). The physical distribution $P$ is estimated using a GJR-GARCH model with a two year rolling window of historical S&P 500 returns, as outlined in Appendix B. Similar estimation approaches were used by Rosenberg and Engle (2002) and Christoffersen, Heston, and Jacobs (2013). The pricing kernel is calculated by plotting the smoothed interpolated values from $Q$ divided by the corresponding values from $P$. The values on the y-axis are adjusted so that the two axes correspond. Estimates of each distribution are likely to be less accurate in the tails of each distribution where data is more scarce. To avoid this potential problem, I will use information within plus and minus 5% log moneyness. For diagrams it is more informative to plot log moneyness which is approximately equal to the equivalent net return within the liquid moneyness range.

Figure III.2 displays four representative log pricing kernel surfaces from across the 20 year sample. Non-monotonically decreasing kernels similar to Jackwerth (2000), Aït-Sahalia and Lo (2000), Rosenberg and Engle (2002), Bakshi, Madan, and Panayotov (2010), Golubev, Härdle, and Timofeev (2014) and Härdle, Okhrin, and Wang (2015) are observed, who all documented convex...
Figure III.2 Log Kernel Plots

Four indicative log kernel surfaces are displayed above from different months across the 20 year sample.

regions in the pricing kernel, each using a variety of methods. As can be seen, the shape and slope of the kernel is changing markedly throughout time. The axes which present the log pricing kernel’s values are also changing a lot from one month to the next demonstrating the variability of the pricing kernel’s slope. The slope can be seen to steepen and flatten depending on the data sample used, motivating the empirical investigation as to whether changes in the pricing kernel elasticity across time contains information pertinent to return predictability.

Let $\kappa_t(R_{t|\tau})$ denote the negative pricing kernel elasticity evaluated at gross return $R_{\tau}$ with horizon $t = \tau$, $M^*(R_{t|\tau}, F_t)$ denote the time $t$ projection of the pricing kernel on future returns $R_{\tau}$, and $F_t$ the filtered probability space at time $t$. To estimate this value at each realization of $R_{\tau}$,
the below estimation is used:

\[
\kappa_t(R_t|\tau) = -\frac{R_t}{M^*(R_t|\tau,F_t)} \frac{\partial M^*(R_t|\tau,F_t)}{\partial R_t} \approx \frac{M^*(0%|\tau,F_t) - M^*(R_t|\tau,F_t)}{M^*(R_t|\tau,F_t)}.
\] (III.3)

Figure III.3 plots the negative EPK elasticity for four sample months. The surface's display differing shapes across the sample, including an increasing function across horizons and very steep slopes in others. The magnitude of the elasticity is also changing substantially across time.

As only out-of-the-money options are used, positive returns correspond to out-of-the-money call options and the price of options is known to increase along with uncertainty or variance. Christoffersen, Heston, and Jacobs (2013) show that including a variance premium in the risk neutral dynamics of the return process generates a U-shaped pricing kernel projection onto returns and therefore a non-monotonic elasticity function also. With heterogeneous beliefs similar to Bakshi, Madan, and Panayotov (2010), the characteristics and the wealth levels of investors trading in puts and calls could be different in a way that explains their respective surplus consumption ratio levels.

C. ESTIMATING THE ELASTICITY OVER TIME

To test the relationship between information from the EPK elasticity surfaces and other important macroeconomic and financial data, one observation per month needs to be extracted to create a time-series. One possibility is to extract for each month, the same \( \kappa_t(R_t|\tau) \) for a specific \( R_t \) and horizon. As Figure III.1 demonstrates, puts closest to the at-the-money level are the most traded, as well as options with up to 90 days time to maturity. \( \kappa(-2\%|\tau = 60) \) will therefore be used for the predictability analysis however, to ensure results are robust, multiple combinations across different returns and horizons are tested. I find results are robust to different combinations of returns and horizons but the correlations are highest when estimating the elasticity using the most liquidly traded options. In addition to this, principal component analysis of \( \kappa_t \) for a given horizon across multiple returns shows that the first principal component explains a high percentage of the variation in \( \kappa_t \). Specifically, for the 30 day time-to-maturity horizon between a log return of -5% and 5%, the first principal component explains 72% of the series variation, and between a log return of -2% and 2%, this increases to 78%. For the 60 day horizon, the variation explained by the first principal component is 90% and 95% respectively, and for the 90-day time-to-maturity 88% and 93%. Figure III.4 plots the time series of EPK elasticity across return levels for the 60 day time-to-maturity horizon. The function is varying substantially, with the maximum and minimum points oscillating across time.
Figure III.3 EPK Elasticity Surfaces

*Four indicative elasticity surface plots are displayed above from different months across the sample.*

**IV. RELATIONSHIP BETWEEN EPK ELASTICITY AND MACROECONOMIC VARIABLES**

In this section I will explore the variability of EPK elasticity and how it relates to macro-economic conditions. Using the first principal component of the 60 day horizon EPK elasticity, Figure's
IV.1, IV.2, IV.3, and IV.4 display the relationship between $\kappa_t$ and the default spread, Michigan Consumer Sentiment Index, log price-earnings and consumption-wealth ratio respectively. In Figure's IV.1, IV.2 and IV.4, we observe a strong negative correlation with the default spread, log price-earnings ratio and the consumption-wealth ratio, along with a strong positive correlation with the sentiment index in Figure IV.3. Comparing the behaviour of the elasticity with important macroeconomic variables across time guides the intuition to what is driving its variability and what it represents.

The surplus consumption ratio is a cyclical variable; consumption is high relative to habit in good states of the world, and low relative to habit in bad states. This is exactly the dynamic behaviour observed in Figure's IV.1, IV.2, IV.3, and IV.4 as good states of the world are also characterised by low default spreads, high sentiment, low price-earnings ratios and low consumption-wealth ratios. The average $\kappa_t$ across the sample was 0.0644. This is very close to the steady state surplus consumption ratio used by CC in their calibrations. As $\kappa_t(R_t|\tau)$ approximates $S_t$, we can also approximate the EIS depending on our assumption of $\gamma$. For example, $\gamma = 2$ would mean an EIS equal to 0.0322. $\gamma = 0.0429$ would mean an EIS equal to 1.5. My empirical estimate of $S_t$ using traded market securities is therefore both in-line with the theoretical range of values posited by Campbell and Cochrane (1999), and is also able to accommodate a range of values for the EIS, both below and above 1 depending on the value of $\gamma$. 
Figure IV.1: The first principal component of the 60 day horizon EPK elasticity between -2% and 2% (blue solid line) plotted against the default spread (red dashed line). Default rate divided by 50 to scale both series.

Figure IV.2: The first principal component of the 60 day horizon EPK elasticity between -2% and 2% (blue solid line) plotted against the Michigan Consumer Sentiment Index (red dashed line). MCSI divided by 650 to scale both series.
Figure IV.3: The first principal component of the 60 day horizon EPK elasticity between -2% and 2% (blue solid line) plotted against the log price-earnings ratio (red dashed line). $\kappa_t$ is multiplied by 40 to scale both series.

Figure IV.4: The first principal component of the 60 day horizon EPK elasticity between -2% and 2% (blue solid line) plotted against the consumption-wealth ratio (red dashed line).
When looking at stockholders consumption data Vissing-Jorgensen and Attanasio (2003) conclude the EIS must be above 1, a range also used by Bansal and Yaron (2004) and Drechsler and Yaron (2011) for long-run risks models. Other studies such as Hall (1988) estimated the EIS to be close to zero. Interestingly Epstein and Zin (1991) also found the EIS to be less than one and for agents to prefer the late resolution of uncertainty.

Estimating the surplus consumption ratio using traded market securities circumvents the issues which plagued estimation using official consumption survey (NIPA) data, notably the issues concerning accurate measurement and the timing of data collection. Investors trade and value securities based partly on their current and expected wealth realizations. This directly feeds through into asset prices and therefore, using (II.13), we can approximate investor's $S_t$. It is therefore interesting to observe a strong negative correlation, -60%, with the consumption-wealth ratio of Lettau and Ludvigson (1999), which uses data on consumption, asset holding and labour income, as they had predicted a negative correlation with the surplus consumption ratio due to the consumption-wealth ratio declining during expansions and rising prior to the onset of a recession. My estimate of the surplus consumption ratio therefore supports their prediction. Whereas the consumption-wealth ratio is available at a quarterly frequency, my estimate of the surplus consumption ratio can be calculated for much shorter frequencies, and it is hence more useful for predicting more imminent macroeconomic events.

My estimate is not consistent with a power utility function or the recursive preferences of Epstein and Zin (1989) for two reasons. Firstly, my estimate is varying over time, which is not consistent with a constant $\gamma$ or $\psi$ assumption. Secondly, if risk aversion was changing over time it is intuitive to have counter-cyclical risk aversion whereas my statistic is cyclical. I explore these alternative preference specifications and consumption growth assumptions in Appendix.A.

Table IV.1 summarises the time-series data and shows the high persistence of the levels when compared to their differences as measured by the AR(1) coefficient. Granger and Joyeux (1980) address the issues associated with highly persistent dependent variables and therefore the first difference of the EPK elasticity is used for regression analysis. A similar transformation was proposed by Baker and Wurgler (2006) and Baker and Wurgler (2007) for the change in the first principal component of their sentiment index level.

V. Return Predictability

This section looks at the relationship between the change in the negative elasticity of the EPK, which approximates changes in the surplus consumption ratio, and the market risk premium. Let $r_t$ denote the monthly log excess return of the S&P 500 at time $t$, $\alpha$ a constant, $X_{t-1}$ either a vector or matrix of predictors one month before, $\beta$ their OLS coefficients, and $\epsilon_t$ the error term. Then assume,

$$ r_t = \alpha + \beta X_{t-1} + \epsilon_t. \tag{V.1} $$

Figure V shows the time series relationship between $\Delta \kappa (-2\%)$ and the next months log excess return. Table V.1 displays the regression results for the negative EPK elasticity estimated at
Table IV.1: Correlation Matrix and Predictor’s Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>κ</th>
<th>Δκ</th>
<th>VRP</th>
<th>Log(P/D)</th>
<th>Log(P/E)</th>
<th>TRSP</th>
<th>DFSP</th>
<th>MCSI</th>
<th>Skew</th>
<th>Kurt</th>
<th>Cay</th>
</tr>
</thead>
<tbody>
<tr>
<td>κ</td>
<td>1.00</td>
<td>-0.21</td>
<td>-0.18</td>
<td>0.11</td>
<td>-0.53</td>
<td>-0.38</td>
<td>-0.53</td>
<td>0.30</td>
<td>0.38</td>
<td>0.12</td>
<td>-0.60</td>
</tr>
<tr>
<td>Δκ</td>
<td>1.00</td>
<td>0.06</td>
<td>0.03</td>
<td>-0.02</td>
<td>-0.07</td>
<td>-0.05</td>
<td>0.02</td>
<td>0.05</td>
<td>-0.11</td>
<td>-0.05</td>
<td></td>
</tr>
<tr>
<td>VRP</td>
<td>1.00</td>
<td>-0.16</td>
<td>-0.13</td>
<td>-0.68</td>
<td>0.76</td>
<td>0.47</td>
<td>0.56</td>
<td>0.22</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log(P/D)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log(P/E)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TRSP</td>
<td>1.00</td>
<td>-0.04</td>
<td>-0.36</td>
<td>0.28</td>
<td>-0.10</td>
<td>0.12</td>
<td>-0.13</td>
<td>0.11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DFSP</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MCSI</td>
<td>1.00</td>
<td>0.26</td>
<td>0.57</td>
<td>-0.05</td>
<td>-0.09</td>
<td>-0.04</td>
<td>0.35</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skew</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kurt</td>
<td>1.00</td>
<td>0.27</td>
<td>-0.39</td>
<td>-0.29</td>
<td>-0.18</td>
<td>0.06</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cay</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

AR(1)  0.99  -0.20  0.34  1.00  1.00  0.99  0.99  0.98  0.93  0.94
Mean   0.64  0.02  -0.04  40.33  31.72  18.02  10.47  858.97 -1.43  391.50
Std.dev. 0.02  0.07  0.04  0.22   0.41   1.19  0.45  13.52  0.04  23.79
Skewness 0.08  1.12  6.72  -0.13  1.77  -0.34  2.85  -0.17  -0.52  2.39
Kurtosis 2.36  7.89  68.18  3.65  7.02  1.97  13.14  2.40  2.93  14.48

Sample covers December 1997 to April 2016 except for the VRP which begins January 2000. Mean values have been multiplied by 100. κ and Δκ are the negative elasticity and one period change in negative elasticity respectively. VRP is the variance risk premium, Kurt and Skew the risk-neutral kurtosis and skewness, Log(P/D) and Log(P/E) the log price to dividend and price to earnings ratios. MCSI is the Michigan Consumer Sentiment Index, TRSP and DFSP the termspread and default spread. Cay is the consumption-wealth ratio matched to monthly data.

a return of -2% and a horizon of 60 days across the full sample. Δκ(−2%) is statistically significant at the 2% confidence level in predicting next months log excess returns with a t-statistic of 2.40 and an $\hat{R}^2$ of 3.72%. Over the most recent 10 year sample in which data quality and quantity were drastically improved, Δκ(−2%) has an $\hat{R}^2$ of 7.19%. Newey and West (1994) automatically selected standard errors with a Bartlett kernel are used which are relatively more conservative than other rule based lag selectors. For example, using the AIC information criterion to select standard error lags results in a t-statistic for Δκ(−2%) of 3.05. Likewise, Newey-West standard errors with an arbitrary 10 lags results in a t-statistic of 3.06. In terms of magnitude, a one unit increase in Δκ(−2%) predicts an annualized increase in the (un-logged) equity premium of 2.24%, and a one standard deviation increase in Δκ(−2%) predicts an annualized increase of 11.23%.

These results provide empirical evidence for the predictive ability of the surplus consumption ratio, the only state variable in the CC model. This is important as the empirical success of consumption based predictors has been very limited over recent decades, and therefore
Figure V: $\Delta \kappa (-2\%)$ is plotted against next months log excess return across the full sample.
Table V.1 Monthly return regressions - all predictors lagged one period

<table>
<thead>
<tr>
<th></th>
<th>(10Years)</th>
<th>Simple</th>
<th>Multiple</th>
<th>(10Years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>-0.58</td>
<td>0.17</td>
<td>0.33</td>
<td>-0.06</td>
</tr>
<tr>
<td>( \kappa ) (-2%)</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta \kappa ) (-2%)</td>
<td>0.19</td>
<td>0.24</td>
<td></td>
<td>0.19</td>
</tr>
<tr>
<td>VRP</td>
<td>-0.30</td>
<td></td>
<td></td>
<td>-0.29</td>
</tr>
<tr>
<td>( \log(P/E) )</td>
<td>-0.005</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \log(P/D) )</td>
<td></td>
<td>-0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skew</td>
<td>-0.21</td>
<td></td>
<td></td>
<td>-0.19</td>
</tr>
<tr>
<td>Kurt</td>
<td></td>
<td></td>
<td></td>
<td>-0.01</td>
</tr>
<tr>
<td>Cay</td>
<td></td>
<td></td>
<td></td>
<td>-0.20</td>
</tr>
<tr>
<td>( \bar{R}^2 ) (%)</td>
<td>-0.02</td>
<td>3.72</td>
<td>7.19</td>
<td>5.84</td>
</tr>
</tbody>
</table>

Sample covers December 1997 to April 2016. HAC standard errors (following Newey and West (1994)) are used and t-statistics are reported in parentheses below the beta coefficients. \( \kappa, \Delta \kappa, \) constant and kurtosis multiplied by 100. All regressors are lagged one period with a frequency of one month. The VRP data begins in January 2000 and all multiple regressor regressions begin in January 2000 unless ten years is stated which begins January 2006 and ends April 2016. The regression specification is \( r_t = \alpha + \beta X_{t-1} + \epsilon_t. \)
reinforces the importance of consumption based predictors in explaining future asset returns. The economic motivation for the surplus consumption ratio is simple: in bad states of the world, investors require higher compensation for holding a risky asset as they are more averse to potential negative returns. As the surplus consumption ratio increases along the business cycle, risk aversion decreases and the EIS increases, and agents are therefore more willing to increase their expected consumption growth with less compensation needed to hold risky assets due to their higher wealth levels. This is in-line with Attanasio and Browning (1995) who find wealthier households find it easier to substitute consumption intertemporally.

It is possible that the significance of the predictors is due to the absence of other important variables which have been highlighted in the predictability literature. This possibility is addressed in Table V.1 by the inclusion of numerous popular predictor variables to see if $\Delta \kappa$ is robust and not merely a proxy for these other predictors. $\Delta \kappa$ remains significant even after including variables such as the variance risk premium, the consumption-wealth ratio, and risk neutral skewness, whilst confirming some previous studies which found these variables have predictive power for future returns but with a more recent sample. The empirical exercise of Conrad, Dittmar, and Ghysels (2013) was for cross sectional returns over time, and therefore Table V.1 provides new evidence to support the predictability of risk neutral skewness but in a time-series rather than cross-sectional setting. The t-statistics of the VRP are more pronounced than those of Bollerslev, Tauchen, and Zhou (2009). This suggests a stronger relationship between the VRP and future log excess returns over my more recent sample than their sample which ended in December 2007 and which did not include the bulk of the financial crisis and the subsequent important economic events. Similarly to Bollerslev, Tauchen, and Zhou (2009), including the consumption-wealth ratio alongside $\Delta \kappa$ actually increases the t-statistic of my measure, while the consumption-wealth ratio is not statistically significant in predicting future excess returns at the monthly horizon. The higher t-statistic for $\Delta \kappa$ may be due to the consumption-wealth ratio controlling for certain risk factors which then allows for better estimation, or the possibility that both consumption-based measures are jointly capturing the relevant information needed to predict future excess returns. It is also notable that the beta coefficient remains relatively stable in magnitude throughout all specifications. The last column of Table V.1 shows that over the most recent 10 years, 17% of the variation in log excess returns can be explained by $\Delta \kappa$, the VRP and the risk neutral skewness.

Relating these results to the empirical predictability literature, they are in-line with Bollerslev, Tauchen, and Zhou (2009), Bollerslev, Marrone, Xu, and Zhou (2014) and Bollerslev, Todorov, and Xu (2015) which show that the distance between the RND and historical density has predictive power for the underlying as well as other recent empirical literature which looked at statistics and measurements from the RND such as Driessen, Maenhout, and Vilkov (2009), Buss and Vilkov (2012), and Conrad, Dittmar, and Ghysels (2013). For completeness, column one of Table V.3 runs the same test using the first principal component of $\Delta \kappa(R_i|\tau = 60)$ for $R_i = [-5\%, 5\%]$. $\Delta \kappa(pca)$ is statistically significant in predicting future log excess returns over the full sample with a t-statistic of -2.06 and an $R^2$ of 2.21%. The negative regression coefficient for $\Delta \kappa(pca)$ is in line with the theoretical model presented in Section II: a positive change in $\kappa_t$ means the surplus consumption ratio has increased, risk aversion has therefore decreased, and the required rate of return for holding the risky asset has reduced.

I have used existing methods to estimate the risk-neutral and physical distribution of mar-
ket returns, from which I’ve extracted an economically meaningful statistic, the surplus consumption ratio. Similar to Martin (2017), I’ve expressed the equity premium in a simple manner and I have shown that by using forward looking options data, along with historical price information, future realized excess returns can be predicted.

A. RELATIONSHIP BETWEEN EPK ELASTICITY AND THE SVIX

Martin (2017) derives a lower bound for the equity premium using option prices. He finds that the equity premium is extremely volatile and that it increases during market downturns. If the negative correlation condition (NCC) holds, he argues that the SVIX is the lower bound of the equity premium. Moreover, evidence is provided to suggest that without having to estimate any model parameters, the SVIX directly proxies for the equity risk premium.

It was shown in Section II that after incorporating a time-varying elasticity of substitution into the Campbell and Cochrane (1999) model, the negative elasticity of the conditional pricing kernel approximates a variable which is perfectly positively correlated to the surplus consumption ratio. As the surplus consumption is the only state variable in the Campbell and Cochrane (1999) model, all variables of interest, such as the equity premium, are functions of it. This produces two interesting questions. Firstly, is $\kappa_t$ correlated to the SVIX? Given $\kappa_t$ estimates the only state variable, the expected answer would be yes. Secondly, if the lower bound is indeed tight and the SVIX is measuring the equity premium, is $\kappa_t$ still significant in predicting future returns when included alongside the SVIX? If the lower bound is tight, $\kappa_t$ would be expected to no longer be significant in predicting future realized log excess returns or for the $R^2$ to not be higher when $\kappa_t$ is included in the regression as all information about the variation in log excess returns is already contained within the SVIX.
Using data from the website of Ian Martin⁶, Figure V.A plots the two-month SVIX against the scaled inverse of $\kappa_t$, because local curvature, or risk aversion, in the Campbell and Cochrane (1999) model is equal to $\gamma/\hat{S}_t$. Since the SVIX is calculated each weekday, only the last recorded entry each month is used for comparison as it will contain the most relevant information. The two-month SVIX corresponds to the 60-day EPK used for the main empirical study of this paper. A strong positive correlation with the two-month SVIX is observed, precisely a 70% correlation. The two variables can be seen to move closely in tandem across the time-series. For the 6-month and 12-month SVIX, the correlation increases to 77% and 78% respectively. Therefore the answer to the first question is yes, they are closely related. This suggests that the SVIX, a model-free estimate of the equity premium, is very closely related to the surplus consumption ratio. In bad states of the world, when surplus consumption is low, risk aversion is higher than in good states and agents require higher expected returns as compensation for holding the risky market portfolio. These results provide a direct link between a consumption-based asset pricing model and a model-free estimate of the market risk premium, which reinforces the relevance of fundamental economic mechanisms in driving market returns.

Table V.2 contains the predictability regression results for the month’s last two-month SVIX, the first period change of this variable, and $\Delta \kappa_t$. The first regression column shows that using the SVIX in this way is not useful in predicting returns. Martin (2017)’s predictability analysis used daily data and overlapping returns therefore the two results are not directly comparable. Applying the same transformation as used for $\kappa_t$, I calculate $\Delta SVIX$ which is the one period change. Column two shows this variable is also not significant in predicting future realized log excess returns. The last column shows the regression when including $\Delta SVIX$ and $\Delta \kappa(-2\%)_{t-1}$ in the same predictive regression:

$$r_t = \alpha + \beta_1 \Delta \kappa(-2\%)_{t-1} + \beta_2 \Delta SVIX_{t-1} + \epsilon_t.$$  \hspace{1cm} (V.2)

The $\bar{R}^2$ decreases from 3.73% to 3.34%. This is not the same methodology used in Martin (2017) as daily overlapping returns are not used however, it may suggest the surplus consumption ratio is the fundamental economic variable driving the success of the SVIX as a predictor of the market risk premium.

A.1. Parametric Relationship Between $\kappa_t$ and the SVIX

A strong relationship between $\kappa_t$ and the SVIX was established by estimating the correlation of the two variables over time. By using parametric representations for the pricing kernel we can also provide a more detailed characterisation of the relationship between $\kappa_t$ and the SVIX.

A good candidate for a parametric representation of the data generating process and pricing kernel is the GARCH option pricing model of Christoffersen, Heston, and Jacobs (2013). By introducing a variance premium, they were able to distinguish between the risk-neutral and physical variance, which otherwise would not be possible (see Heston and Nandi (2000)). This will allow a direct link between the negative elasticity of the pricing kernel and the risk-neutral variance. Another advantage of their specification is that it produces a U-shaped projection of the pricing kernel onto future log returns.

---

⁶http://personal.lse.ac.uk/martiniw/
Table V.2 SVIX monthly return regressions - all predictors lagged one period

<table>
<thead>
<tr>
<th></th>
<th>-0.24</th>
<th>0.14</th>
<th>0.17</th>
<th>0.14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>(-0.33)</td>
<td>(0.45)</td>
<td>(0.53)</td>
<td>(0.44)</td>
</tr>
<tr>
<td>SVIX</td>
<td>0.04</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>∆SVIX</td>
<td>-0.27</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>∆κ(-2%)</td>
<td>0.19</td>
<td>0.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>∼R² (%)</td>
<td>-0.43</td>
<td>1.77</td>
<td>3.73</td>
<td>3.34</td>
</tr>
</tbody>
</table>

Sample covers January 1998 to January 2012. HAC standard errors (following Newey and West (1994)) are used and t-statistics are reported in parentheses below the beta coefficients. ∆κ(-2%) and constant multiplied by 100. All regressors are lagged one period with a frequency of one month. The regression specification is \( r_t = \alpha + \beta X_{t-1} + \epsilon_t \).

Following Christoffersen, Heston, and Jacobs (2013) let the log pricing kernel be defined as,

\[
\ln \left( \frac{M_{t+1}}{M_t} \right) = \frac{\xi \alpha}{\sigma^2_{t+1}} (r_{t+1}-r_f)^2 - \mu (r_{t+1}-r_f) + \left( \eta + \xi (\beta - 1) + \xi \alpha (\mu - \frac{1}{2} + \gamma) \right) \sigma^2_{t+1} + \delta + \xi \omega + \phi r_f, \quad (V.3)
\]

and therefore the negative elasticity of the un-logged pricing kernel is,

\[
- \frac{r_{t+1}}{\left( \frac{M_{t+1}}{M_t} \right) \partial r_{t+1}} = \left( \mu - \frac{2 \xi \alpha (r_{t+1}-r_f)}{\sigma^2_{t+1}} \right) r_{t+1} = \left( \mu - \frac{2 \xi \alpha (r_{t+1}-r_f)}{\sigma^2_{t+1}(1-2\alpha \xi)} \right) r_{t+1}, \quad (V.4)
\]

where \( \xi \) is the variance premium, \( \alpha \) a parameter in the physical variance square-root return process, \( \sigma^2_{t+1} \) the physical variance at \( t+1 \), \( \mu \) the equity premium and \( \eta, \beta, \gamma, \delta, \omega, \phi \) parameters in the physical and risk neutral data generating process (see Christoffersen, Heston, and Jacobs (2013) for more detail). Defining the risk-neutral variance as \( \hat{\sigma}^2_{t+1} = \sigma^2_{t+1}(1-2\alpha \xi) \) and substituting this into (V.4), the direct relationship between \( \kappa_t \) and the risk neutral variance is observed. Martin (2017) used the SVIX as an estimate of the conditional risk-neutral variance,

\[
\hat{\sigma}^2_{t+1} = R_{f,t+1}^2 SVIX^2.
\]

For negative excess returns, the negative elasticity of the pricing kernel is decreasing in absolute terms when the risk-neutral variance is increasing. It is becoming less negative. For positive excess returns, the negative elasticity is increasing as risk-neutral variance increases. The model therefore predicts the opposite relationship observed between \( \kappa_t \) and the SVIX in FigureVA, providing a challenge for future option-pricing models which seek to explain not only the
Table V.3 Monthly return regressions - all predictors lagged one period

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.14</td>
<td>0.35</td>
<td>0.41</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>(0.42)</td>
<td>(1.23)</td>
<td>(1.48)</td>
<td>(0.53)</td>
</tr>
<tr>
<td>Δκ(pca)</td>
<td>-7.21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-2.06)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Δκ(-2%)</td>
<td>0.16</td>
<td>0.15</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.43)</td>
<td>(2.06)</td>
<td>(2.32)</td>
<td></td>
</tr>
<tr>
<td>d1</td>
<td>-0.32</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-0.70)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d2</td>
<td>-4.10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-2.74)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d3</td>
<td>1.15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.64)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ΔDIVE</td>
<td>2.70</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.62)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{R}^2$ (%)</td>
<td>2.21</td>
<td>7.03</td>
<td>1.60</td>
<td>3.40</td>
</tr>
</tbody>
</table>

Sample covers December 1997 to April 2016 except column three where 2008 is excluded. d1, d2 and d3 are dummy variables which equal one if an observation is in 2007, 2008 or 2009 respectively. ΔDIVE is a proxy for divergence in opinion. HAC standard errors (following Newey and West (1994)) are used and t-statistics are reported in parentheses below the beta coefficients. All regressors and constant multiplied by 100. All regressors are lagged one period with a frequency of one month. The regression specification is $r_t = \alpha + \beta X_{t-1} + \epsilon_t$.

static representation of the EPK, but also its dynamics across the business cycle.

B. No-Arbitrage Breakdown

The financial crisis triggered an observed breakdown in the covered interest rate parity condition in which it is also argued that no-arbitrage conditions were also violated (Du, Tepper, and Verdelhan (2017)). To test whether the main results are driven only by the financial crisis and a potential breakdown in no-arbitrage conditions, dummy variables d1, d2, d3 are introduced for years 2007, 2008 and 2009 respectively. Column two in Table V.3 shows that the year 2008 is statistically significant but $\Delta \kappa_t$ remains significant. Removing all data from 2008 reduces the $\bar{R}^2$ but $\Delta \kappa_t$ remains significant as shown in column three of Table V.3. This shows that while the financial crisis was an exceptional episode for return predictability, it is not the driving force of my results.

C. Is Exposure to $\kappa_t$ Priced in the Cross-Section?

If $\kappa_t$ approximates the surplus-consumption ratio, then it is reasonable to ask whether exposure to this business cycle risk is priced in the cross-section. The intuition is that stocks which co-vari
The scatter plots show the average annualized returns of each portfolio from January 1998 to April 2016. The portfolio’s correspond to each decile of the sorted $\hat{\beta}_i$ vector. The first decile portfolio corresponds to the lowest ranked 10% of stock betas.
highly with $S_t$ over time are riskier as a positive correlation would indicate that stocks have higher returns when consumption is high and lower returns when consumption is low. By definition $S_t$ is low in bad states of the world, and therefore assets with the highest betas are the riskiest as they mostly pay out in good times and not in the bad times when higher returns are most desired. Such behaviour would demand a risk premium in order to compensate investors for holding these risky assets. To test this hypothesis, I run the following regression for all stocks in the CRSP database from January 1998 - April 2016:

$$R_{i,t} = \hat{\alpha}_i + \hat{\beta}_i \kappa_t,$$

where $R_{i,t}$ is the one month holding period return at time $t$ for stock $i$. I will run the same regression with and without dividends as well as for $\alpha_i \neq 0$ and $\alpha_i = 0$. The CRSP database contains 18,814 unique tickers for the $k_t$ sample period. I then only keep those stocks for which a holding period return exists for the whole sample. This leaves 1,447 stocks.

I then sort the vector of $\hat{\beta}_i$’s into decile portfolios and calculate that portfolio’s equally weighted return. The empirical prediction is that stock’s which co-vary most with $\kappa_t$ will earn higher average returns in order to compensate investors for exposure to business cycle risk. Figure V.3 plots the results. When setting $\alpha = 0$, the top two panels of Figure V.3 show a higher decile portfolio is always associated with higher average returns over the sample period. This is consistent with the initial hypothesis that stocks which co-vary more with an estimate of the surplus consumption ratio are riskier and therefore need to have higher average returns to compensate investors for their exposure to this risk. No relationship is observed when a regression is included in the specification as shown by the bottom two panels of Figure V.3.

C.1. **IS EXPOSURE TO $S_t$ STILL PRICED AFTER CONTROLLING FOR SIZE AND VALUE?**

Fama and French (1992) showed how stocks sorted by their relative size and value can explain a large proportion of the return differential observed in the cross-section of stock returns. It is therefore important to check whether stocks with a higher $\hat{\beta}_i$ from the previous section still earn higher returns once their relative size and value have been accounted for. To test this, I form 12 portfolios, triple sorted by their size (share price multiplied by common shares outstanding) and their value (book value of equity divided by market value), and their $\hat{\beta}_i$. Following Fama and French (1992), I sort on small and big firms, determined by whether their size is below or above the median sample size, and low, middle and high value stocks, which correspond to stocks in the bottom 30%, middle 40%, and highest 30% of value respectively. Lastly, I sort the stocks on whether their $\hat{\beta}_i$ is below or above the median value from the sample, using the no-constant specification and with dividends included in the holding period returns. I merge CRSP and COMPUSTAT databases on WRDS, and once again keep only those stocks which have a data-point for each month over my sample period, leaving 625 stocks. This inevitably introduces a survival bias into my sample, as stocks which survived are possibly "better" than those that did not. However, since I am sorting within this stronger sample, it is still interesting to observe if there is a spread in returns for those with relatively higher $\hat{\beta}_i$, once size and value have been accounted for.
As shown above, the portfolio return is always higher for higher $\hat{\beta}_i$ stocks relative to lower $\hat{\beta}_i$ stocks even after controlling for the size and value of the stocks within each portfolio. This provides strong evidence that the exposure of stocks to $\kappa$, or the surplus consumption ratio is priced in the cross-section.

D. AN ALTERNATIVE EXPLANATION

There may be alternative theories which can explain the predictability and dynamics of $\kappa_t$. I discuss three of these below.

D.1. PRICEY PUTS AND PORTFOLIO CONSTRAINTS

Many large-scale investors who are long equity will hold put options in order to hedge downside risk and limit potential losses from a large left tail event. How far out-of-money the out-of-the-money protective puts are will depend on the level of the investors aversion to loss which in turn will dictate how much potential gains will be dampened due to expensive close to at-the-money puts (for example married puts will be held by investors with very strict limits to the losses they can incur on their portfolios and therefore be a lot more expensive to hold).

Bakshi, Madan, and Panayotov (2010) used the idea of the pricing kernel slope - in their paper measured by aggregate shorting - to test for a relationship between changing call prices and call option returns. They provided a theoretical model in which an increasing steepness of the pricing kernel indicates an increase in the associated derivative price.

Assuming the above setting, suppose there is an increase in the left slope, or $\kappa(-2\%)$, of the pricing kernel from one period to the next, meaning protective puts are now relatively more expensive. Assume the cause of this change is independent of the underlying and due, for example, to exogenous intermediary constraints. Acknowledging that investors have budget and risk constraints, the latter ensuring portfolios are adequately hedged, investors may need to reduce their holding of the underlying in order to readjust their portfolio to hold the relatively more expensive puts. In aggregate, this offloading of the underlying will see the asset price drift from its fundamental value. Assuming that there are heterogeneous agents simultaneously trading within a market, other agents who have different risk and budget constraints may then buy the underlying as it is now undervalued, potentially leading to an overreaction in the sense of Hong and Stein (1998), leading to higher returns for the underlying asset in that period. This process will repeat itself every period, contingent on when the more risk and budget constrained investors rebalance their portfolios. A similar process but for investors short the market and long call op-
tions could be argued for the upward sloping region of the pricing kernel. This hypothesis would need data on the portfolio holdings of large investors to test its validity.

D.2. Time-Varying Market Participation

\( \kappa_t \) is a cyclical variable as it is negatively correlated to the term spread, default spread, price-earnings ratio, and positively correlated to consumer sentiment and risk neutral skewness. These dynamics meant risk aversion alone could not explain the behaviour of \( \kappa_t \).

However, an alternative explanation which fits with just risk aversion is that during a market downturn, the most risk-averse investors drop out of the market by liquidating their holding of the market portfolio. The only remaining participants in the market are therefore those with lower levels of risk-aversion as they are still willing to maintain their holding of the risky asset without needing current prices to drop to reflect the higher equity premium which is required by the other more risk-averse agents to hold the risky asset. This time-varying market participation would lead to the estimated market risk aversion to be higher during normal or good states of the world, and lower during bad states of the world. Brunnermeier and Nagel (2008) have explored this possibility using household data but didn’t find a strong portfolio composition reaction by investors to changes in their wealth levels. The existing evidence therefore suggests this hypothesis cannot explain the strong cyclical dynamics of \( \kappa_t \).

D.3. Is it divergence in opinion or overconfidence?

Miller (1977) addressed the possibility of heterogeneous beliefs among investors when short-sale constraints exist. Put simply, if investors face rigidities when shorting assets - possibly due to investment platform restrictions, higher costs, or a preventative mandate - investors who hold pessimistic views of an asset will not have their views fully incorporated into the asset’s price as they are prevented from entering the market without frictions. Therefore, the views of those with an optimistic outlook will prevail resulting in overpriced assets when compared to the true equilibrium price which would have resulted if all investors views were taken into account through unconstrained supply and demand.

The literature has proposed using trading volume as a proxy for divergence of opinion as well as a gauge for the level of overconfidence in the market which might also impact future returns (see Hong and Stein (2003) or Odean (1998), Gervais and Odean (2001) and Statman, Thorley, and Vorkink (2006) for examples of its applications). Using both the monthly trading volume of S&P 500 index options and e-mini futures contracts to test whether the changes in the kernel slope are in fact measuring divergences in opinion or overconfidence, \( \Delta \kappa \) is found to be robust to their inclusion in the predictability regressions, with the above variables insignificant in predicting future realized monthly log excess returns.

Moreover, assuming volatility traders don’t dominate the S&P 500 options market, options data allows a unique experiment to evaluate divergence in opinion when looking at the difference in volume traded between calls and puts as both derivatives inherently diverge in outlook for future realizations of the underlying. Therefore, define the variable Divergence (DIVE) as,
\[ DIVE_t = \frac{\sum_{i=1}^{30} (CallOptionsVolume_i - PutOptionsVolume_i)}{\sum_{i=1}^{30} (CallOptionsVolume_i + PutOptionsVolume_i)}, \quad (V.5) \]

and therefore,

\[ \Delta DIVE_t = DIVE_t - DIVE_{t-1}, \quad (V.6) \]

where \( DIVE_t \) is the total monthly difference in the number of calls and puts traded, weighted by the total volume of options traded within that month in order to take into account the growing volume of options traded in more recent years. The same options data sample is used to calculate this variable. Once again, because of the highly persistent nature of \( DIVE_t \), the first difference provides a stationary variable for regression analysis. The last column of Table V.3 shows this variable is not significant and \( \Delta \kappa \) remains significant.

**VI. CONCLUSION**

When it comes to predicting future returns over short horizons, consumption-based asset pricing models have so far failed empirically. By introducing a new time-varying relationship between consumption growth and expected market returns, I avoid the use of inaccurate consumption data. After incorporating this relationship into the Campbell and Cochrane (1999) model, I show that the negative elasticity of the pricing kernel approximates the surplus consumption ratio, the only state variable in their economy. Using index prices and options, I estimate the empirical pricing kernel and its elasticity, and then test the surplus consumption ratio’s dynamics and its ability to predict future market returns.

I find that the surplus consumption ratio is cyclical and highly correlated to business cycle variables. I also show that it can predict future excess market returns, on-top of other established predictors. These findings reinforce the importance of consumption-based asset pricing models in explaining real market data. I also showed that the surplus consumption ratio is strongly related to the SVIX index of Martin (2017), a model-free estimate of the market risk premium. A cross-sectional analysis which grouped stocks based on their co-variance with the surplus consumption ratio showed stocks which co-varied the most earned substantially higher returns than those stocks which co-varied the least. This is in-line with the assumption that risky stocks are those which pay out in good states and not bad states, and therefore require a premium to be held by investors.

The pricing kernel is an extremely powerful tool which can help to increase our understanding of how agents make investment decisions as well as to provide important information on the preferences of agents which is of fundamental importance to all fields of economics. Further research is needed to understand what other information the empirical pricing kernel holds, and
how this information can be applied to other open questions in economics.

APPENDIX A. ALTERNATIVE UTILITY FUNCTION SPECIFICATIONS

The external habits model of Campbell and Cochrane (1999), along with my time-varying EIS assumption, permits a consumption-based theoretical explanation of my empirical pricing kernel statistic. However, for completeness, I will also explore the interpretation of my statistic for the power utility function and the recursive preferences specification under the following consumption growth assumptions: independence, equality, fixed relationship, time-varying relationship.

a) Power Utility Function:

\[ U_t = \frac{C_t^{1-\gamma} - 1}{1-\gamma} \]  \hspace{1cm} (1)

\[ M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \]  \hspace{1cm} (2)

b) External Habits:

\[ U_t = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma} \]  \hspace{1cm} (3)

\[ M_{t+1} = \beta \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \]  \hspace{1cm} (4)

c) Recursive Preferences:

\[ V_t = \left( (1-\beta) C_t^{1-\gamma} + \beta (E_t U_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}} \right)^{\frac{\theta}{1-\gamma}} \]  \hspace{1cm} (5)

\[ M_{t+1} = \beta^\theta R_{t+1}^{\theta-1} \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \]  \hspace{1cm} (6)

<table>
<thead>
<tr>
<th>Consumption Growth Assumption</th>
<th>a) Power Utility</th>
<th>b) External Habits</th>
<th>c) Recursive Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}<em>t [\Delta c</em>{t+1} r_{t+1}] = \mathbb{E}<em>t [\Delta c</em>{t+1}] \mathbb{E}<em>t [r</em>{t+1}] )</td>
<td>N/A</td>
<td>N/A</td>
<td>( 1 - \theta )</td>
</tr>
<tr>
<td>( \Delta c_{t+1} = r_{t+1} )</td>
<td>( \gamma )</td>
<td>( \gamma(1 + \lambda(s_t)) )</td>
<td>( \theta - 1 - \frac{\theta}{\psi} )</td>
</tr>
<tr>
<td>( \Delta c_{t+1} = g + \psi r_{t+1} + e_{t+1} )</td>
<td>( \gamma \psi )</td>
<td>( \gamma \psi(1 + \lambda(s_t)) )</td>
<td>1</td>
</tr>
<tr>
<td>( \Delta c_{t+1} = g + \psi t r_{t+1} + e_{t+1} )</td>
<td>( \gamma \psi t )</td>
<td>( S_t (1 + \lambda(s_t)) )</td>
<td>N/A</td>
</tr>
</tbody>
</table>

The above table shows that without a time-varying separation between consumption growth and the return on market wealth, the cyclical dynamics of the empirical pricing kernels negative elasticity cannot be explained as the interpretation is either a constant parameter, or counter-
cyclical. $\gamma$ and $\psi$ are fixed over time for recursive preferences, whereas the power-utility function alone lacks the business cycle foundations which the external habits model accommodates.

**Appendix B. Estimating The Pricing Kernel**

As with any estimation process, a degree of error is produced and there is no perfect or even unequivocal method to which the literature agrees. Fortunately, there exists a large and fruitful literature on estimating both the physical distribution $P$, and the risk-neutral distribution (RND) $Q$, in part driven by the empirically observed pricing kernel and risk aversion puzzle’s.

**Appendix B.1. Estimating the Risk-Neutral Density**

Figlewski (2008) pins down two main problems for estimating the RND: firstly, the market does not provide a continuum of exercise prices from which to calculate the distribution because options only trade in a relatively small number of discrete prices in the market. Interpolation and smoothing is required to mitigate the effects of market microstructure noise on estimation. Secondly, due to market options’ strikes being within a certain range, estimation is constrained for the RND within the tails.

Two main strands have developed for the estimation process, broadly defined within the spheres of parametric methods and non-fully-parametric methods. One of the most popular methods in the literature to estimate the RND (or state price density), and the method used in this paper is the non-parametric and semi-parametric methods proposed by Aït-Sahalia and Lo (1998). As the RND is forward looking, options data and more specifically the S&P 500 options data is generally used to extract this information because it is the most liquid index. Instead of assuming a model for the volatility process, the authors non-parametrically estimate the implied volatility function $\hat{\sigma}(F_t, \tau, X, \tau)$ using the three-dimensional Nadaraya-Watson kernel estimate,

$$\hat{\sigma}(F_t, \tau, X, \tau) = \frac{\sum_{i=1}^{n} k_F(F_t - F_{t_i}, \tau) k_X(X - X_i) k_t(\tau - t_i)}{\sum_{i=1}^{n} k_F(F_t - F_{t_i}, \tau) k_X(X - X_i) k_t(\tau - t_i)},$$

where $\sigma_i$ is volatility implied by the option price $H_i$, $F_{t_i}$ is the implied futures price of the underlying, $X$ is the associated strike price of the option, and $\tau$ the option’s time-to-maturity. $k_\xi$ with $\xi \in \{F, X, \tau\}$ is the kernel used for weighting the observed data and $h_\xi$ the chosen bandwidth$^7$.

Therefore, $\hat{\sigma}(F_t, X, \tau)$ is the non-parametrically estimated implied volatility for the option with underlying $F_t$, strike $X$ and time to maturity $\tau$. The option pricing function is then estimated as,

$$\hat{H}(S_t, X, \tau, r_{t, \tau}, \delta_{t, \tau}) = H_{BS}(F_t, X, \tau, r_{t, \tau}; \hat{\sigma}(F_t, X, \tau))^8,$$

where $H_{BS}$ refers to Black-Scholes option pricing formula, $r_{t, \tau}$ the risk free rate from time $t$ till time $\tau$ and $\delta_{t, \tau}$ a continuous dividend stream from time $t$ till time $\tau$. From this the RND can

$^7$The choice of kernel and the optimal bandwidth to minimise the MSE is explained within the paper’s appendix.

$^8$BS refers to Black Scholes.
be estimated using the Breeden and Litzenberger (1978) result that the second derivative of the pricing function with respect to the strike price evaluated at the terminal payoff value, \( X = S_T \) is equal to \( q(S_T) \),

\[
q(S_T) = e^{r_t \tau} \left[ \frac{\partial^2 \hat{H}(S_t, X, r_t, \delta_t, \tau)}{\partial X^2} \right]_{X=S_T}.
\]

This is an extremely powerful result because it is completely model free. For the Black Scholes pricing formula,

\[
H_{BS}(S_t, X, r_t, \delta_t, \tau; \sigma) = S_t e^{-\delta_t \tau} \Phi(d_1) - X e^{-r_t \tau} \Phi(d_2)
\]

where,

\[
d_1 = \frac{\ln(S_t/X) + (r_t - \delta_t + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}}
\]

and \( d_2 \equiv d_1 - \sigma \sqrt{\tau} \), the associated risk neutral density is a lognormal density with mean \((r_t - \delta_t - \sigma^2/2) \tau \) and variance \( \sigma^2 \tau \) for \( \ln(S_T/S_t) \):

\[
q_{BS,t}(S_T) = e^{r_t \tau} \frac{\partial^2 H}{\partial X^2} \bigg|_{X=S_T} = \frac{1}{S_T \sqrt{2\pi \sigma^2 \tau}} \exp \left[ - \frac{\left( \ln(S_T/S_t) - (r_t - \delta_t - \sigma^2/2) \tau \right)^2}{2\sigma^2 \tau} \right]. \tag{7}
\]

The non-parametrically estimated implied volatilities are then used to interpolate an implied volatility surface from which slice's for specific maturities can be extracted in order to build the RND which represents implied probabilities of terminal wealth for a given future date. This slice and the other associated option variables are then simply plugged into the above result to generate an estimate for the RND.

This method benefits from not having to assume a volatility process but has been criticized for its variability and its wild oscillation from one data sample to the next. Most notably Jackwerth (2004) argued the asymptotic error bounds of the method were “too tight” due to gaps from observations in discrete time. Also, as less options are traded far out and into the money, the tails of the distributions are relatively weak and not as reliable. This has motivated the use of hybrid methods which make a tail adjustment with the aim to mitigate the effects of data restrictions. For example, because only a limited range of exercise prices are traded, Beare and Dossani (2016) use the Aït-Sahalia-Lo estimator with Figlewski (2010) tails, which aims to improve estimation in the tails of the distribution by appending tails from a Generalized Pareto Distribution. Figlewski asserts that the fitted tails contain the correct total probability and are a better fit for the extreme portions of the empirical RND. This study does not analyse far out-of-the money options and therefore the addition of Figlewski tails would be moot\(^9\). See Cuesdeanu and Jackwerth (2016) for a detailed survey of the different methods used.

\(^9\)Chen, Joslin and Ni (2016) study the effects of demand for far out-of-the money puts and its relation to risk premia for a wide range of financial assets.
Appendix B.2. Estimating the Historical Density

Golubev, Härdle and Timofeev (2008) explain that the main difficulty in estimating the objective density is that it depends on the historical time series of prices $S_t$ and therefore estimation of the density is complicated by model specification and data scarcity. Only one path of prices is observed from the time-series which without specifying an underlying model which can generate the series, renders the objective density "unknown".

Non-parametric methods have also been proposed for the historical distribution. Aït-Sahalia and Lo (2000) criticised the parametric methods used at the time stating that the models were inconsistent with the behaviour of financial data. They suggest non-parametric estimation of the S-VaR (Statistical VaR) by first calculating the $\tau$ period continuously compounded return, $u_{\tau} \equiv \log(S_T/S_I)$ and then construct a kernel estimator of the density function $g(.)$ of continuously compounded returns,

$\hat{g}(u_{\tau}) \equiv \frac{1}{N H_u} \sum_{i=1}^{N} k_{u_{\tau}} \left( \frac{u_{\tau} - u_{\tau,i}}{H_u} \right)$.

From the density of continuously compounded returns the price density can be recovered, resulting in the following estimator for the S-VaR,

$\hat{f}_{\tau}(S_T) = \frac{\hat{g}(\log(S/S_I))}{S}$,

which can be directly estimated from the kernel estimator. Another method to estimate the historical distribution was proposed by Rosenberg and Engle (2002) to utilise three discrete time models which incorporate stochastic volatility into the returns process. They use Engle’s (1982) ARCH specification, Bollerslev’s (1986) GARCH and Glosten, Jagannathan, and Runkle’s (1993) GJR-GARCH specification which allows asymmetry in the volatility process aiming to capture the observed relationship in financial data that a negative shock at time $t-1$ has a stronger impact in the variance than a positive shock, sometimes referred to as the leverage effect.

The excess returns are modelled by,

$\ln(S_t/S_{t-1}) - r_f = \mu + \epsilon_t$,

where $\epsilon_t = \sigma_t z_t$, $\epsilon_t \sim f(0, \sigma^2_{\epsilon_t|t-1})$, $r_f$ is the risk free of return and the conditional volatility process follows,

$\sigma^2_{\epsilon_t|t-1} = \omega_1 + \omega_2 I + \alpha \epsilon^2_{t-1} + \beta \sigma^2_{\epsilon_{t-1}|t-2} + \delta \text{Max}[0, -\epsilon_{t-1}]^2$.

For ARCH, $\beta = \delta = 0$ and for GARCH, $\delta = 0$. $\omega_2$ permits a shift in long-run volatility using an indicator variable to mark the different time periods. Following Bollerslev and Wooldridge (1992) Quasi-Maximum Likelihood estimation is used to estimate the parameters which was shown to be consistent even when the underlying distribution is wrongly specified as Guassian; it is asymptotically efficient. Once the parameters have been estimated using two years of historical returns data, future volatility is forecasted assuming the appropriate number of trading days in...
between option maturity and then a Gaussian distribution with this foretasted volatility and the GJR-GARCH mean of returns is used to plot. Both innovations from the normal and t-distribution are used with the latter’s difference negligible. Interestingly, Deflefsen et al. (2007) notice that the estimation of the historical distribution does not change the kernel estimate much when comparing the GARCH, Discrete Heston or just taking the historical return series without a model, i.e. directly observed returns over half a year,

\[ \tilde{R}_t = S_{t_l}/S_{t_l-126}. \]

REFERENCES


