Measuring Intangible Capital with Uncertainty

Sungbae An  

Singap or e Management University  

Nan Li*  

Shanghai Jiao Tong University

Abstract

Intangible capital has arguably become an important component of corporate value. However, it is still an open question whether uncertainty associated with investment in intangible capital is higher or lower than physical capital. We estimate the value of intangible capital in a dynamic stochastic general equilibrium model that features capital adjustment costs, investment-specific technological progress and recursive utility. We use the perturbation method up to second order to solve the model and perform Bayesian estimation using particle filter. The unobserved times series of intangible capital is estimated through particle smoother. Data from US economy in the postwar period imply that corporations indeed have formed large amounts of intangible capital as Hall (2001) found. The implied expected return on investment in intangible capital is lower than that of physical capital, which implies that intangible-capital intensive firms have a lower expected return.

JEL Classification: E44 E27 G12

Keywords: Intangible Capital; Tobin’s q; Recursive Preference; Particle Filter; Smoother

*Correspondence: Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai 200052, China. E-mail: nanli@sjtu.edu.cn (Nan Li). School of Economics, Singapore Management University, Singapore 178903. E-mail: sungbae@smu.edu.sg (Sungbae An).
1 Introduction

Intangible capital measures the stock of firm-specific human capital and organizational capital, ownership of a technology or productivity enhancement induced by research and development. People conventionally think that the uncertainty associated with intangible capital is higher than that of the physical capital\(^1\), so it is expensed and written off from the balance sheet of the firms. However, recent research in cross-section stock returns suggest that it may not be the case. Fama and French (1995) and others find that the average return of firms with low book-to-market equity ratio (growth firms) is lower than that of the firms with high book-to-market ratio (value firms). Hansen, Heaton, and Li (2005) view this empirical evidence suggesting possibly important differences in the risk exposure of technologies that feature different mixes of tangible and intangible capital. The value of firm equals the value of capital owned by the firm, so the difference between the market value of the firm and book value of the physical capital owned by the firm measures the value of intangible capital in the absence of adjustment cost of capital investment and mispricing. Furthermore, the difference between stock returns of firms with different book-to-market ratios showed in Fama and French (1995) implies that the uncertainty or risk associated with intangible capital is lower than that of physical capital, if intangible capital is a primary source of divergence in measure of book equity and market equity. However, it is still an open question whether the uncertainty of intangible capital investment is higher or lower than that of physical. To address this question, a proper measure of intangible capital is necessary. Although intangible capital has arguably become an increasingly important component of corporate value, the measurement of intangible capital remains a challenge.

Without an explicit economic model, intangible capital is a residual that is mixed with measurement error or omitted information, and is neither measured nor observed by econometricians. In contrast to the measurement error or omitted information, intangible capital is conceived as an input in production and contributes to the future cash flow of the firm, it is arguably more important in the sector where the investment-specific technology shock prevails. Hence the value of intangible capital is encoded in observed market value and the value of physical capital of the firm. Furthermore, the heterogeneity in the returns to physical and intangible return could potentially be imputed from the observed cross-section variation in returns of firms with different book-to-market equity ratio. On the other hand, ignoring the heterogeneity in the returns to capital biases the estimation of riskiness of the firms.

\(^1\)In our paper, we use tangible capital, physical capital and measured capital interchangeably.
In this paper, we propose to estimate intangible capital in the dynamic stochastic general equilibrium model that features recursive utility and investment-specific technology shock. The model puts an explicit structure on the dynamics of investments, consumption, labor income and market value of the capital stock. Following An and Schorfheide (2007) and Fernández-Villaverde and Rubio-Ramírez (2007) we use second-order approximation to solve the model, particle filters to evaluate the likelihood, and Bayesian procedure to estimate the resulting nonlinear dynamics of capital stocks using the data on observed macroeconomic variables.

McGrattan and Prescott (2000), Atkeson and Kehoe (2002), and McGrattan and Prescott (2010) measure the value of intangible capital or technological capital\(^2\) in the neoclassical growth model without uncertainty by assuming the equality of return on different types of capital. Kapicka (2012) study the importance of technological capital in a two-country model of McGrattan and Prescott (2009) and find that technological capital is about one-third of tangible capital in the US economy. One of the key assumptions in these models is that the net rates of return from investment to both technological capital and tangible capital must be equalized within each firm, which is an equilibrium condition in an economy without uncertainty. Differences in the uncertainty or risk of the intangible capital and tangible capital are the focus of our paper, and we measure intangible capital in a dynamic general equilibrium model with uncertainty.

Hall (2001) and Laitner and Stolyarov (2003) infer the value of intangible capital in an economy with uncertainty. The key assumption in their models is that physical capital and intangible capital are perfect substitutes of each other, so the value of intangible capital is difference between the market value of firms and the value of physical capital owned by these firms. Intangible capital and physical capital are different types of capital and may play different roles in production. In our model, we explore the implication of the notion that intangible capital is more important in the investment-good production, which is subject to the investment-specific technology shock. Figure 1 shows that the price of the nonresidential fixed investment decreases while the real investment increases relative to the consumption of nondurable goods and services in US economy from 1952Q1 to 2012Q4. The decreases in the relative price of investment is more evident after 1980s, accompanied by the change in the movement of labor supply measured by the average weekly hours worked also changes after 1980s as shown in Figure 2. We investigate the implication of the investment-specific technology shock on the variation of the value of

\(^2\) McGrattan and Prescott (2009) introduce the concept of technological capital in the neoclassical growth model, and in our paper we don’t differentiate technological capital and intangible capital and use them interchangeably.
capital stock and return to investments in this economy. Furthermore, we study the movement in the asset value relative to the measures of capital in this economy and whether the time series variation in observed book-to-value (B/M) ratio of the stock market as shown in Figure 3 can be accounted by the variation in the value of intangible capital. In addition, we identify the difference in the risk exposure of intangible capital and physical capital and test whether the heterogeneity in returns on physical capital and intangible capital help to explain the cross-section returns observed in the market, especially the stock returns of firms with different book-to-market ratios.

Li (2009) introduces intangible capital in the two-sector neoclassical growth model with uncertainty that features investment-specific technology shock and adjustment costs, and uses linear approximation to solve the model and Kalman filter to estimate the value of intangible capital from macroeconomic variables observed in the economy. We make two major extensions to the two-sector model of Li (2009). First, we adopt recursive preferences of Epstein and Zin (1989). This allows us to break the link the risk aversion and intertemporal elasticity of substitution, and generate sizable risk premium with reasonable risk free rate. Li (2009) show that in a model with state-separable CRRA preference and capital adjustment cost, the model implied risk premium is low and the riskiness of intangible capital and physical capital are not significantly different from each other. Secondly, we employ Bayesian procedure so that the estimated parameters and generated capital series are given credible bands.

A recent work by Ai, Croce, and Li (2013) builds a stylized model to investigate formally whether the difference in the contribution of intangible capital to the firms can explain the cross-section difference in the average return of portfolios sorted by book-to-market ratio (value premium). In their model, firms with low book-to-market ratios (growth firms) are those with relatively large amount of intangible capital which are not directly exposed to aggregate risk, therefore earn lower expected returns. Our paper differs from the Ai, Croce, and Li (2013) in two important aspects. First, we examine the difference in risk exposure of tangible and intangible capital which are both exposed to the aggregate shock as well as an independent investment-specific technology shock. This approach is similar to that of Papanikolaou (2011) and Borovicka and Hansen (2011). Secondly, we solve the dynamic general equilibrium of the model using second-order approximation and then estimate the resulting nonlinear dynamics of the model using particle filter. Borovicka and Hansen (2011) solve the model of Ai, Croce, and Li (2013) using second-order approximation and study the term structure of risk exposure associated with consumption and capital dynamics, while we estimate the time-series of intangible capital using particle smoother and examine whether the difference in the risk exposure to aggregate shock and investment-specific technology shock can explain the
cross-section expected return difference observed in the market. Kogan and Papanikolaou (2010) find empirically that difference in the risk exposure to the investment-specific technology shock helps to explain the value premium, while our paper investigate formally in a general equilibrium model whether the intangible capital and tangible capital are exposed differently to the investment-specific technology shock and aggregate shock, and whether this difference can explain the cross-section return difference between the firms with difference capital composition and book-to-market ratios.

In terms of the methodology our paper is closely related to Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012) who investigate a DSGE model with recursive preferences. They solve the model with the third order approximation and estimate it by maximum likelihood constructed with particle filter. Besides the fact that Bayesian estimation procedure is used, our paper delivers the estimates of unobservable time series, e.g., Tobin's \( q \), physical and intangible capital, via the particle smoother, which is the main methodological contribution of our paper.

This paper is organized as follows. Section 2 presents the two-sector DSGE model with intangible capital. Section 3 applies Bayesian estimation procedure with particle filter and smoother to estimate the value of intangible capital stock in the US economy. The value of intangible capital is also estimated for the models of Hall (2001) for comparison, using the approach developed in this paper. Section 4 discusses the model's implications for asset prices and aggregate macroeconomic variables. Section 5 concludes.

2 The Two Sector Model

We introduces heterogeneous capital in a two-sector economy of Greenwood, Hercowitz, and Krusell (1997), Christiano and Fisher (1998) and Whelan (2003), in which investment and capital grow faster than consumption goods. The most important feature of the model in this paper is that there are two types of capital, namely physical and intangible capital, and intangible capital is not perfect substitute but complimentary to physical capital in producing consumption goods and investment goods. The two-sector model presented here shares some common features with that of McGrattan and Prescott (2000) in that the share of intangible capital is different across sectors, but in McGrattan and Prescott (2000)'s economy, two sectors grow at the same rate and the relative price of investment is always 1. It is well documented that there is a downward trend in relative price of investment goods while real investment has grown faster than real consumption in postwar US economy, see for example Greenwood, Hercowitz, and Krusell
(1997) and Whelan (2003). As shown in Figure 1, the relative price of investment to price of consumption has declined since late 1950s and especially after 1980, and the ratio of real investment to real consumption has trended upwards since late 1950s and has risen dramatically since 1991. The average growth rate of real private fixed nonresidential investment is 4.04% per year over the period 1947Q1-2012Q4, 0.8% per year faster than the real consumption. The two-sector model with investment-specific technological shock allows us to exploit the information on relative prices of investment goods to measure intangible capital.

Grunfeld (1960), Lucas and Prescott (1971) and Hayashi (1982) show that capital adjustment costs provide a framework to study the relation between the value of firm and its capital stock. The specification of adjustment costs associated with installation of capital stock allows the shadow price of installed capital to be different from the price of a unit of new investment goods, that is, Tobin’s $q$ to vary over time. Our model features capital adjustment costs, which drive a wedge between the marginal cost of installed capital measured by stock prices and the marginal cost of new capital measured by investment good prices, and allow us to exploit the variation in the stock prices relative to the book value of capital as shown in Figure 3 to measure intangible capital.

2.1 Preferences and Technology

The economy is populated with the infinitely lived households with recursive utility following Epstein and Zin (1989)

$$V_t = \max \left\{ (1 - \beta)U_t^{1-\rho} + \beta \left( E_t \left[ V_{t+1}^{1-\gamma} \right] \right)^{1-\rho} \right\}^{1/1-\rho} \tag{1}$$

where $\beta$ is the discount factor which measures the impatience to consume, $\gamma$ measures the risk aversion to wealth gambles in the next period, and $1/\rho$ measures the intertemporal elasticity of substitution when there is perfect certainty. $V_{t+1}$ is the time $t+1$ utility of investor or continuation value of the consumption stream from time $t+1$ forward. Alternatively, we denote the certainty equivalent of future utility $W_t = E_t \left[ V_{t+1}^{1-\gamma} \right]^{1/(1-\gamma)}$, whose introduction simplifies the notation and approximation drastically. $U_t$ is the atemporal utility non-separable in consumption and leisure

$$U_t = C_t(1 - n_t)^\theta$$
where $C_t$ denotes consumption at date $t$, $n_t$ denotes the sum of the factions of productive time allocated to the two production sectors and $1 - n_t$ is the fraction of productive time allocated to leisure. The parameter $\theta$ is the atemporal elasticity of the substitution between consumption and leisure. The larger is $\theta$, the more the household is willing to substitute consumption for leisure.

In this economy, the consumption and investment goods are produced in separate sectors. Sector $c$ produces consumption goods with constant return to scale technology as following,

$$C_t \leq (\eta_{m,t} K_{m,t-1})^{\alpha_m} (\eta_{u,t} K_{u,t-1})^{\alpha_u} (A_t n_t^c)^{1-\alpha_m-\alpha_u}$$  \hspace{1cm} (2)

where $K_{m,t-1}$ and $K_{u,t-1}$ are the total (measured) physical capital and (unmeasured) intangible capital carried from date $t-1$ into date $t$ and used in the production; $\eta_{m,t}$ and $\eta_{u,t}$ are the share of physical capital and intangible capital used in consumption production sector, respectively. $n_t^c$ is the labor input in this sector at date $t$.

New investment goods in physical capital and intangible capital are produced in sector $x$. The technology for producing new investment goods at date $t$ is

$$X_{m,t} + X_{u,t} \leq \left( (1 - \eta_{m,t}) K_{m,t-1} \right)^{\alpha_m} \left( (1 - \eta_{u,t}) K_{u,t-1} \right)^{\alpha_u} (A_t \Xi_t n_t^x)^{1-\alpha_m-\alpha_u}$$  \hspace{1cm} (3)

where $X_{m,t}$ and $X_{u,t}$ are the investment goods in physical and intangible capital produced at date $t$, respectively. $n_t^x$ is the labor input in sector $x$ at date $t$.

Note that the technological process $A_t$ is an economy-wide aggregate productivity shock while the investment-specific shock $\Xi_t$ only affects the investment good producing sector. We assume the logarithm of both shocks follow random walk with positive drift and AR(1) innovation,

$$\log A_t = g_A + \log A_{t-1} + \log z_t$$  \hspace{1cm} (4)

$$\log z_t = \rho_z \log z_{t-1} + \epsilon_{z,t}$$  \hspace{1cm} (5)

and

$$\log \Xi_t = g_\Xi + \log \Xi_{t-1} + \log \xi_t$$  \hspace{1cm} (6)

$$\log \xi_t = \rho_\xi \log \xi_{t-1} + \epsilon_{\xi,t}$$  \hspace{1cm} (7)

where $\{\epsilon_{z,t}\}$ and $\{\epsilon_{\xi,t}\}$ are iid normal random variables with zero means and standard deviations of $\sigma_z$ and $\sigma_\xi$, respectively, and independent of each other. Parameters $g_A$ and $g_\Xi$ are the mean growth rates of technological changes $A_t$ and $\Xi_t$, respectively.
To keep things as simple as possible without loss of the interesting aspects of the model, we assume that the total capital share is same in two sectors, that is 

\[ \alpha^c_m + \alpha^c_u = \alpha^x_m + \alpha^x_u = \alpha \]

where \( \alpha \) is the total capital share and \( 0 < \alpha < 1 \).

There are adjustment costs associated with the installation of new investment goods of both types of capital, so capital stock evolves according to the following:

\[ K_{j,t} = (1 - \delta_j)K_{j,t-1} + \Phi_j \left( \frac{X_{j,t}}{K_{j,t-1}} \right) K_{j,t-1}, \quad \text{for } j = m, u \]  

(8)

where \( \delta_j \) is the depreciation rate and \( K_{j,t} \) is the total stock of type-\( j \) capital. \( \Phi_j(\cdot) \) is a positive concave function in investment-capital ratio with \( \Phi_j(\cdot) > 0, \Phi'_j(\cdot) > 0 \) and \( \Phi''_j(\cdot) < 0 \).

### 2.2 Balanced Growth Path

In this economy, along the balanced growth path, investment and capital stock of both types follow the same stochastic trend, which is different from the stochastic trend followed by consumption. Let \( g^c \) denote the growth rate of investment and capital stock, and \( g^c \) denote that of consumption. Then from log-differencing equations (2) and (3), we get

\[ g^c = g_A + \alpha g_x \]
\[ g^x = g_A + g_z \]

Thus, the stochastic trend of consumption \( \Psi^c_t \) is given as following:

\[ \log \Psi^c_t = \log A_t + \alpha \log \Xi_t \]
\[ = g^c + \log \Psi^c_{t-1} + \log z_t + \alpha \log \xi_t \]

Similarly, the stochastic trend of capital stocks \( \Psi^x_t \) is given as following:

\[ \log \Psi^x_t = \log A_t + \log \Xi_t \]
\[ = g^x + \log \Psi^x_{t-1} + \log z_t + \log \xi_t \]
The growth rate of real investment goods exceeds that of real consumption by \((1 - \alpha)\sigma\Xi\), and the relative price of the investment goods has a downward stochastic trend of 
\[
\Psi_t^i/\Psi_t^x = \Xi_t^{\alpha-1}.
\]

The second welfare theorem holds in this two-sector economy, so the quantities in a competitive equilibrium of the model can be computed by solving the social planner’s problem, and the relative prices can be computed using the Lagrangian multipliers from a solution to the planner’s problem. The social planner’s problem is to satisfy (1) subject to resource constraints (2), (3) and capital evolution process (8) with \(K_{m,0}\) and \(K_{u,0}\) given.

To solve the social planner’s problem, we follow King, Plosser, and Rebelo (1988) to transform the economy by removing the stochastic trends of the variables that are not stationary, and characterize the equilibrium of the transformed economy by the first-order conditions. We solve this nonlinear system by second-order approximation around the steady state using perturbation method, apply Bayesian method to obtain the posterior parameter distribution, then use particle smoother to estimate the time series of the unobserved capital stocks from the observed variables. Our focus is to study the implied return and risk of intangible and tangible investment and intangible capital. We summarize the prices and rates of return in this economy in the next subsection.

### 2.3 Asset Prices and Rates of Return

Before we present the estimates of the intangible capital in the model, we would like to discuss the implication of intangible capital on the measurement of prices and rates of return of the investment goods, capital stock, and stocks of firms.

- **Period-\(t\) price of type-\(j\) capital which will be installed at beginning of period \(t + 1\),** \(\tilde{P}_{j,t}\), equals to the inverse of the marginal adjustment cost with respect to the investment at date \(t\),

\[
\tilde{P}_{j,t} = \frac{P_t^x}{\Phi_j' \left( \frac{X_{j,t}}{K_{j,t-1}} \right)} \quad \text{for } j = m, u
\]

- **Period-\(t\) price of type-\(j\) capital previously installed at beginning of period \(t\) is given
by

\[
P_{j,t} = \tilde{P}_{j,t} \left[ (1 - \delta_j) + \Phi_j \left( \frac{X_{j,t}}{K_{j,t-1}} \right) - \Phi'_j \left( \frac{X_{j,t}}{K_{j,t-1}} \right) \frac{X_{j,t}}{K_{j,t-1}} \right] \quad \text{for } j = m, u \quad (9)
\]

- Market value of firm equals to the market value of capital owned by this firm. Denote \( MV_t \) as the aggregate market value of the firms in the two sectors, then

\[
MV_t = \tilde{P}_{m,t}K_{m,t} + \tilde{P}_{u,t}K_{u,t} = P_t^x \left[ \frac{K_{m,t}}{\Phi'_m \left( \frac{X_{mt}}{K_{m-1}} \right)} + \frac{K_{u,t}}{\Phi'_u \left( \frac{X_{ut}}{K_{u-1}} \right)} \right]
\]

- The book value of firm, \( BV_t \), is book value of total capital owned by this firm, which is same as the replacement cost of the total capital, that is

\[
BV_t = P_t^x (K_{m,t} + K_{u,t})
\]

- Tobin’s \( q \) of type-\( j \) is the ratio of price of installed capital goods to the price of investment goods of type-\( j \) capital,

\[
q_{j,t} = \frac{\tilde{P}_{j,t}}{P_t^x} = \left( \Phi_j \left( \frac{X_{j,t}}{K_{j,t-1}} \right) \right)^{-1} \quad \text{for } j = m, u \quad (10)
\]

- Aggregate Tobin’s \( q \) is the weighted average of the Tobin’s \( q \) of intangible and tangible capital, weighted by the relative quantity of the capital stocks

\[
q_t = \frac{MV_t}{BV_t} = \frac{K_{m,t}q_{m,t} + K_{u,t}q_{u,t}}{K_{m,t} + K_{u,t}} \quad (11)
\]

- Due to the existence of intangible capital, Tobin’s \( q \) in (11) is not observable. The market-to-book ratio plotted in Figure 3 is the ratio of market value of total capital to the replacement cost of physical capital, which is denoted as \( q^* \) and defined as following

\[
q^*_t = \frac{MV_t}{P_t^x K_{m,t}} = q_{m,t} + q_{u,t} \frac{K_{u,t}}{K_{m,t}} \quad (12)
\]

Similarly, we can define the observed Tobin’s \( q \) for consumption good and investment good producing firms.

\[
q_{c,t}^{c,*} = \frac{\eta_{m,t}K_{m,t}q_{m,t} + \eta_{u,t}K_{u,t}q_{u,t}}{\eta_{m,t}K_{m,t}} = q_{m,t} + q_{u,t} \frac{\eta_{u,t}K_{u,t}}{\eta_{m,t}K_{m,t}}
\]

\[
q_{x,t}^{c,*} = \frac{(1 - \eta_{m,t})K_{m,t}q_{m,t} + (1 - \eta_{u,t})K_{u,t}q_{u,t}}{(1 - \eta_{m,t})K_{m,t}} = q_{m,t} + q_{u,t} \frac{(1 - \eta_{u,t})K_{u,t}}{(1 - \eta_{m,t})K_{m,t}}
\]
The steady-state value of $q$ is 1, while the steady-state value of $q^*$ is $1 + \frac{K_u}{K_m} > 1$. If there is no adjustment costs, then $q^*$ will always be greater than one. From equation (12) we know that the variations in $q^*$ is partly driven by the variation in the share of intangible capital in total capital, the more is the intangible capital relative to physical capital, the larger is $q^*$. Firms with high $q^*$ or low book-to-market ratio are firms with more intangible capital if there is no cross-sectional differences in the capital adjustment costs.

Another factor that drives the variation in $q^*$ is the adjustment costs. High marginal adjustment cost could drive $q$ as well as $q^*$ below and above 1. The elasticity of the aggregate investment to the aggregate Tobin’s $q$ is a weighted average of the elasticities of investment with respect to Tobin’s $q$ of each type of capital, that is,

$$\phi \equiv \frac{\partial \log X_t}{\partial \log q_t} = \left[ \sum_{j=m,u} w_j \phi_j \right]^{-1}$$

where $w_j = \frac{K_j/X_j}{K/X}$ is the weight. If the elasticity of the investment to Tobin’s $q$ of both types of capital are the same, then the elasticity of the aggregate investment to the aggregate Tobin’s $q$ is simply $\phi = \phi_m = \phi_u$.

Note that market-to-book value of capital ($q^*$) is not same as market equity-to-book-equity ratio which is commonly used in the literature to define growth firms and value firms, unless firms only issue equity. The relation between $q^*$ and $ME/BE$ is

$$q^*_t = \frac{MV_t}{BV_{m,t}} \frac{ME_t}{BE_t} = \frac{MV_t/ME_t}{BV_{m,t}/BE_t}.$$ 

If $\frac{MV_t/ME_t}{BV_{m,t}/BE_t}$ is stable over time, then firms with high $q^*$ can be identified by high market equity-to-book equity. Figure 4\(^3\) shows that this is approximately true in aggregate for US nonfarm and nonfinancial corporate firms during postwar periods except for 1980s. The top panel of Figure 4 shows that for these firms, market equity-to-book equity tracks market-to-book value of capital pretty well, and the correlation coefficient of these two ratios is 0.995 during the period of 1947Q1-2012Q4. The bottom panel plot the ratios of equity to total value of firms measured in market value and book value, and the ratio of these two ratios. It shows that the $\frac{MV_t/ME_t}{BV_t/BE_t}$ is not too far away from 1 and not very

\(^3\)Market value of equity and book value of bond for nonfarm and nonfinancial corporate firms are from Federal Reserve Board Flow Funds Accounts. Book value of equity is calculated as the replacement cost of plant and equipment minus book value of bonds. For details of the data construction and calculation of market value of bonds, see data appendix of Hall (2001).
volatile for most of the periods. The average of this ratio is 1.093 and standard deviation is 0.075.

### 2.3.1 Rates of Return

We can derive the Euler equation from the equilibrium conditions

$$\mathbf{E}_t \left[ m_{t,t+1} r_{j,t+1} \right] = 1$$

where $m_{t,t+1}$ is the stochastic discount factor defined as

$$m_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\rho} \left( \frac{1-n_{t+1}}{1-n_t} \right)^{\theta(1-\rho)} \left( \frac{V_{t+1}}{W_t} \right)^{\rho-\chi}$$

and $r_{j,t+1}$ is the investment return of type-$j$ capital in terms of consumption goods,

$$r_{j,t+1} = \frac{MPK_{j,t+1} + P_{j,t+1}}{P_{j,t}}$$

where $MPK_{j,t+1}$ is the marginal productivity of type-$j$ capital. Under the assumption of competitive equilibrium, the marginal product of capital should equal across different sectors. However, in the economy with uncertainty, net rates of return from investment to intangible capital and tangible capital are not necessarily equalized within each firm.

The one-period risk-free interest rate, $r^f_t$ equals to the reciprocal of the date $t$ conditional expectation of stochastic discount factor $m_{t,t+1}$:

$$r^f_t = \left( \mathbf{E}_t \left[ m_{t,t+1} \right] \right)^{-1}$$

The return of the market portfolio is the weighted average of the return on both types of capital

$$r^{t+1}_t = \frac{\hat{P}_{m,t} K_{m,t} r_{m,t+1} + \hat{P}_{u,t} K_{u,t} r_{u,t+1}}{P_{m,t} K_{m,t} + P_{u,t} K_{u,t}}$$

$$= \frac{K_{m,t} q_{m,t} r_{m,t+1} + K_{u,t} q_{u,t} r_{u,t+1}}{K_{m,t} q_{m,t} + K_{u,t} q_{u,t}}$$

Similarly, returns on the capital in each sector are given by

$$r^{c,t+1}_t = \frac{\eta_{m,t} K_{m,t} q_{m,t} r_{m,t+1} + \eta_{u,t} K_{u,t} q_{u,t} r_{u,t+1}}{\eta_{m,t} K_{m,t} q_{m,t} + \eta_{u,t} K_{u,t} q_{u,t}}$$
\[
\tau_{t+1}^x = \frac{(1 - \eta_{m,t}) K_{m,t} q_{m,t} r_{m,t+1} + (1 - \eta_{u,t}) K_{u,t} q_{u,t} r_{u,t+1}}{(1 - \eta_{m,t}) K_{m,t} q_{m,t} + (1 - \eta_{u,t}) K_{u,t} q_{u,t}}
\]

Hence the cross-section variation in the stock returns comes from the variation in the relative importance of the intangible capital as well as the difference between the rates of return from investment on intangible capital and physical capital.

3 Bayesian Estimation of the Model

Following An and Schorfheide (2007), Bayesian estimation procedure for non-linearly approximated DSGE models is employed for our empirical analysis. This requires series of computationally intensive procedures, which is explained in the following subsection. In terms of the methodology our paper is closely related to Binsbergen, Fernández-Villaverde, Kojien, and Rubio-Ramírez (2012) who investigate a DSGE model with recursive preferences and solve the model by the third order approximation and estimate the model by maximum likelihood constructed with particle filter. Besides the fact that Bayesian estimation procedure is used, our paper delivers the estimates of unobservable time series, e.g., Tobin’s \( q \), physical and intangible capital, via the particle smoother, which is the main methodological contribution of our paper.

3.1 Methodology

3.1.1 Solution Method and State-Space Representation

Until recently, models with recursive preferences are considered empirically intractable, or very complicated to say the least. Since the log-linearization of the model with recursive preferences will degenerate the approximated model to that of the time-separable expected utility specification, computationally expensive solution method has been required for an analysis of full scale DSGE models. Early empirical works include Tallarini (2000), Bansal and Yaron (2004) and Hansen, Heaton, and Li (2008), with limited model specifications. Cañára, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) show that perturbation method is generally applicable to bring this class of models to the data. As pointed out in An (2013), equilibrium conditions of the model can be directly fed into Dynare for an empirical analysis, with a bit of tweak. Noting that the certainty equivalent of future utility, \( W_t \), captures all the distinctive properties of recursive preferences and that it shows up only in the stochastic discount factor, writing \( W_t^{1-\gamma} = E_t[V_{t+1}^{1-\gamma}] \) and
adding it to Dynare code will numerically solve any order of approximation via perturbation method. The second-order approximated model can be represented as a nonlinear state space model

\begin{align*}
y_t &= f(s_t) + u_t \\
x_t &= h(x_{t-1}, z_t) \\
z_t &= \Gamma z_{t-1} + \epsilon_t
\end{align*}  

where \( y_t \) contains observables and \( s_t = [x_t, z_t]' \) is vector of state variables consisting of endogenous state variables (\( x_t \)) and exogenous shock processes (\( z_t \)). Generally, function \( f \) in (14) is specified as a selection (or loading) matrix that maps state variables into observables, and function \( h \) in (15) is a quadratic function in both arguments. That is, equations (15) and (16) make a nonlinear transition equation where innovations to the exogenous processes enter as a quadratic form. This hinders the popular Kalman filter from being exploited in evaluation of the likelihood.

Our model is equipped with two structural shocks, economy-wide aggregate productivity shock and the investment-specific shock. Hence we choose the aggregate consumption growth rate (\( CGR_t \)) and the investment growth rate (\( XGR_t \)) as our main observables. To exploit the information from the stock market, we also include the excess return (\( EXRET_t \)) as observable. That is, a set of measurement equations is given as

\begin{align*}
CGR_t &= 100 \times \left( \tilde{c}_t - c_{t-1} + g^c + \tilde{z}_t + \alpha \tilde{x}_t \right) + u^c_t \\
XGR_t &= 100 \times \left( \tilde{x}_{m,t} - \tilde{x}_{m,t-1} + g^x + \tilde{z}_t + \tilde{\xi}_t \right) + u^x_t \\
EXRET_t &= 100 \times \left( \exp(\tilde{r}_t) - \exp(\tilde{r}_t^f) \right) + u^r_t
\end{align*}

where \( \tilde{x}_t \) represents the log deviation of the variable \( x_t \) from its steady state and \( u^c_t, u^x_t, \) and \( u^r_t \) are measurement errors with zero means.

### 3.1.2 Likelihood Evaluation and Particle Filter

According to our presentation, the main difficulty with nonlinear state space in evaluating the likelihood is that the transition equation is non-linear with non-Gaussian innovations. Generally, filters utilize the following recursion to process the information contained in newly arrived observations:

\[
p(s_t|y_{1:t}) = \frac{p(y_t|s_t)}{p(y_t|y_{1:(t-1)})} \int p(s_t|s_{t-1}) p(s_{t-1}|y_{1:(t-1)}) \ ds_{t-1}
\]
where \( y_{t:r} = \{ y_t, y_{t+1}, \ldots, y_r \} \). The filtered state at \( t-1 \), \( p(s_{t-1}|y_{1:t-1}) \), is updated to period-\( t \) filtered state with the help of densities from transition and measurement equations, \( p(s_t|s_{t-1}) \) and \( p(y_t|s_t) \), respectively. In a standard linear Gaussian state-space model, we can show analytically that all the filtered states are normally distributed and it is enough to track only the means and variances of the filtered state variables at each time period. But for a nonlinear state space, as in our representation of the second-order approximated model, the transition density is non-Gaussian, and hence the normal recursion of Kalman filter breaks down. Instead, particle filter employs computationally heavy Monte Carlo integration to process the recursion and the filtered state is approximated by random samples from \( p(s_t|y_{1:t}) \). Finally, the likelihood is evaluated based on the forecasting error decomposition with the following marginal contribution to the likelihood:

\[
p(y_t|y_{1:(t-1)}) = \int p(y_t|s_t) \left( \int p(s_t|s_{t-1}) p(s_{t-1}|y_{1:(t-1)}) \, ds_{t-1} \right) \, ds_t,
\]

\[
\approx \frac{1}{MN} \sum_{k=1}^{M} \sum_{i=1}^{N} \left[ p(y_t|s_t^{(k)}) \int p(s_t^{(i)}|s_{t-1}^{(i)}) \, ds_{t-1}^{(i)} \right]
\]

In a naive version of the particle filter which is employed in our analysis, \( M = 1 \), that is, only one \( s_t^{(k)} \) is simulated for each \( s_{t-1}^{(i)} \). Before we proceed, we need to note that there is a trade-off between the number of particles, \( N \), and the standard deviation of the measurement error. When the measurement error is very small, \( p(y_t|s_t^{(k)}) \) becomes negligible for most of the particles and eventually the swarm of particles will not properly represent the density it approximates.

### 3.1.3 Bayesian Inference and Particle Smoother

Functions and matrices of the system in the state-space representation of the approximated model, e.g. \( f, h, \) and \( \Gamma \) in (14)–(16), are functions for the parameter vector

\[
\Theta = \left( \rho, \gamma, \theta, \alpha, \alpha^e_m, \alpha^x_m, \delta_m, \delta_u, \phi_m, \phi_u, g^e, g^x, \beta, \rho_z, \rho_\xi, \sigma_z, \sigma_\xi \right)
\]

Bayesian approach treats a parameter as an unobservable random variable and the belief over the parameter distribution is updated as more observations arrive. The prior belief is modeled as \( p(\Theta) \) and updated to a posterior belief via the Bayes rule:

\[
p(\Theta|y_{1:T}) = \frac{p(y_{1:T}|\Theta)p(\Theta)}{P(\Theta)}
\]
Once the likelihood, \( p(y_{1:T}|\Theta) \), is evaluated by the particle filter, Markov-chain Monte Carlo (MCMC) method can be used to draw from the posterior distribution. We use the random-walk Metropolis algorithm as the posterior simulator.

Given the parameter vector, we can consider estimating unobservable time series using either filter or smoother. This is possible if we estimate the state vector \( s_t \), because relevant measurement equations can be specified to map the state vector into a time series variable of our interest. With Bayesian estimation, however, the filtered state may not be appropriate to generate the desired time series. Any statistic from the posterior parameter distribution, e.g. the posterior mean or even a random draw from the posterior, reflects the information from the current observation, \( p(\Theta|y_{1:T}) \). Hence, the information contained in the posterior statistic is not consistent with the filtered state, \( p(s_t|y_{1:t}) \). In this paper, we use the forward-filtering-backward-sampling algorithm to estimate the smoother, \( p(s_t|y_{1:T}) \). More details on particle filter and smoother can be found in Doucet and Johansen (2011).

### 3.2 Estimation Results

#### 3.2.1 Prior Specification

There are 17 model parameters in \( \Theta \). First, we calibrate some parameter values that is less controversial in the literature. The capital share in the production function \( \alpha \) is 0.36, which is assumed to be same in the two sectors. The capital share of physical capital, \( \alpha_c^e = 0.23 \) and \( \alpha_x^e = 0.20 \), are chosen to match the average ratio of nominal private fixed nonresidential investment to nominal nondurable goods and services consumption, which is 19\% in US economy. The annual depreciation rate of physical capital, \( \delta_m \), is fixed at 10\% based on the NIPA data of capital stocks and investment of private fixed nonresidential assets, which is same as that of Hall (2001).

Table 1 shows the prior specification of parameters to be estimated. The inverse of elasticity of intertemporal substitution is debatable whether its value is above or below 1. Accordingly, the prior mean of \( \rho \) is centered at 1. The risk aversion parameter is a key parameter to generate high return on risky asset. For an agent with high risk aversion, higher compensation is required to persuade him to substitute into another risky state. There is not much of consensus on the value of this parameter, and we choose both prior mean and standard deviation high values, 100 and 50, respectively. The labor supply parameter \( \theta \) governs the elasticity of substitution between consumption and leisure and is chosen to be 2.75 such that average working hour is 25\%. The depreciation rate of
intangible capital is centered at 10% annual rate which is the same as that of physical capital. The adjustment cost associated with the installation of new investment goods is governed by $\phi_m$ and $\phi_u$, which is the elasticity of investment with respect to Tobin’s $q$. For physical capital, we set the prior mean at 0.3 and standard deviation at 0.15, which roughly covers Abel (1980)’s estimates. The adjustment cost parameter for intangible capital is not available in the literature, so we set a relatively diffuse prior with high mean. Parameters $g^x$ and $g^c$ are closely related to the investment growth rate and consumption growth rate, so priors are centered near the historical averages. We should note, however, that the model is approximated to the second order and the certainty equivalence does not hold. Therefore, the temporal average can be away from the steady state values depending on the size of shocks. Instead of the discount factor, $\beta$, we set up the prior for the steady state real interest rate. Persistence parameters and standard deviations of shock processes are set identical.

3.2.2 Data

We use data on both macroeconomic variables and stock market value to estimate the parameters and value of intangible capital. See Data Appendix for details. The growth rates $CGR_t$ and $XGR_t$ that are used for estimation are constructed as the percentage term from log-difference of real values of each variables for the period from 1952Q1 to 2011Q4. Excess return is the difference between the return on the value-weighted NYSE portfolio and that on the 3 month T-bill. All the values are calculated at the quarterly rate.

3.2.3 Posterior Estimates

We draw from the posterior distribution via the random-walk Metropolis algorithm using Dynare with customized codes. With the swarm of 5,000 particles, 100,000 draws are generated and the first 40,000 draws are discarded. The standard deviations of measurement errors are fixed at 50% of those of observables.

Table 2 reports the posterior means and 90% credible interval in comparison with prior means. The posterior mean of the inverse EIS, $\rho$, is 0.955 which is less than 1 but the 90% credible interval covers the unity. There is a big information gain in the risk aversion. The 90% credible interval of $\gamma$ is between 53.6 and 81 and this is consistent with the literature. Binsbergen, Fernández-Villaverde, Kojien, and Rubio-Ramírez (2012) reports values between 40 and 80 depending on the specification. The depreciation of intangible
capital has posterior with mean of 0.025 which is not different from physical capital. The capital adjustment cost parameter for physical capital, $\phi_m$, coincides with the prior, but $\phi_u$ for intangible capital is estimated quite high. Its prior mean is 1 and the posterior mean is 2.234 which means that the likelihood brings in strong signal for high adjustment cost. Posterior 90% credible interval for the steady state values of investment growth, $g^x$, is outside of the prior mean which implies the bias correction from the second order approximation plays some role in updating the belief. Investment shock persistence, $\rho_\xi$, is estimated very high, 0.919, with tight credible interval.

Table 3 reports the moments of aggregate macro variables calculated from the data and the simulated series based on the posterior distribution. ‘Posterior Mean’ column shows the simulation result at the posterior mean of parameters. Then we proceed with randomly sampled 6,000 posterior draws (10% of the total posterior parameter draws). The model is simulated without the measurement errors at each posterior draw, and the simulation statistics such as mean, standard deviation, and autocorrelation are stacked. The posterior median and 90% credible interval are calculated based on this simulation result across posterior parameter draws. We can see that means of the quantity variables are all within the corresponding credible intervals. Comparison on the standard deviation shows a different result. The output growth is much less volatile in the model with the credible interval between 0.601 and 0.803 against 1.005 in the data. The consumption growth in the model is more volatile, [0.655, 0.865] against 0.530. The investment growth is again less volatile in the model, [0.864, 1.445] against 2.532. The model does not look explaining the data at all, but a special attention should be paid to interpret this result. For the particle filter to run, we have introduced fairly large measurement errors, but we ignored them in the simulation. It is clear that means and autocorrelations will not be affected by measurement errors. Since the measurement error is specified being orthogonal to other part of the model, standard deviations from the simulation can be adjusted by adding those of measurement errors. As a result, adjusted standard errors for the consumption and investment growth are [0.92, 1.13] and [2.13, 2.71], respectively. With the adjustment the standard deviation of investment growth matches the data. But the result becomes uglier for the consumption growth. The volatility of consumption growth is even higher than that of output growth, which is often found in the model with recursive preferences (Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012)). The volatility of the output growth is difficult to be accurately adjusted since it is not used as an observable for the estimation while the output is defined as weighted combination of consumption and investment in the model. Autocorrelations in the model are consistently higher than those in the data.

Table 4 presents the statistics of the sample and posterior distribution of risk free
rate, market equity return and excess return of market index. The mean rate of return of market index matches that of data, but risk free rate is too high, [0.579, 1.286] in the model against 0.318 in the data. The standard deviation of equity return is low while the standard deviation of risk free matches with that of the data. Again, the standard deviation is needed to be adjusted. For excess return which is used as an observable for the estimation the volatility becomes much closer with the adjustment, [6.03, 7.15] against 8.477.

4 The Nature of Intangible Capital

In this section, we study the nature of intangible capital estimated from the data based on the model proposed in Section 2. Table 2 presents the prior and posterior mean and 90% credible interval of the parameter estimates. The capital adjustment cost technology is governed by parameter $\phi_m$ and $\phi_u$, which is the elasticity of investments with respect to Tobin’s $q$. The estimates of $\phi_m$ for physical capital that Abel (1980) and Eberly (1997) got in somewhat different models range from 0.27 to 0.52 and from 0.37 to 1.06, respectively. Our estimate of this parameter with posterior mean of 0.29 lies within the range reported in the literature. However, the estimates of adjustment costs for intangible capital is much lower with posterior mean of 2.234, which implies that intangible capital is much more responsive to variation in the Tobin’s $q$. Figure 5 shows that the estimates of the physical capital is more volatile than intangible capital and more volatile than the data. This is in contrast with the estimates of Hall (2001) which is replicated in Figure 6. In Hall (2001)'s model, intangible capital and physical capital are perfect substitutes of each other. Hence the difference between the total capital stock imputed from the stock market value (solid line for small and dashed line for large capital adjustment costs) and the measured physical capital (dotted line) captures the quantity of intangible capital. The implied quantity of intangible capital is much more volatile and negative in 1980s for fast adjustment case. However, this would imply that the risk or uncertainty of intangible capital would be higher than that of physical capital and the return on investment from intangible capital should also be higher.

Table 5 presents the statistics of the posterior distribution of rates of return and market value of firms relative to the book value of physical capital. The mean and standard deviation of return on the physical capital is much higher than that of intangible capital. Furthermore, return of the intangible-capital-intensive sector (sector $x$), which is the investment producing sector has lower mean and standard deviation as compared with that of tangible-capital-intensive sector. This is consistent with the point of view of
Hansen, Heaton, and Li (2005) that the empirical evidence of Fama and French (1995) on cross-section variation of stock returns based on book-to-market ratio suggesting possibly important differences in the risk exposure of technologies that feature different mixes of tangible and intangible capital.

In this model, the uncertainty associated with intangible capital is much less than that of physical capital. The left panel of Figure 7 depicts the estimated time series of rates of return on physical capital and intangible capital, and the difference between rates of return with 90% credible interval depicted by the shaded area. The return on intangible capital varies much less than that of physical capital, and the difference drives the difference between return on consumption and investment sectors which is depicted in the right panel of Figure 7. However, the implied excess return of consumption sector over investment sector is only 0.6% per year which is less than the observed value premium of 6% per year.

McGrattan and Prescott (2000), Atkeson and Kehoe (2002) and McGrattan and Prescott (2010) measure the value of intangible capital or technological capital in the neoclassical growth model without uncertainty by assuming the equality of return on different types of capital. Kapicka (2012) study the importance of technological capital in a two-country model of McGrattan and Prescott (2009) and find that technological capital is about one-third of tangible capital in the US economy. Hall (2001) and Laitner and Stolyarov (2003) infer the value of intangible capital in an economy with uncertainty, and find that intangible capital has became an increasingly important component of corporate value in the US economy. As showed in Table 5, the market value of firm to book value of physical capital ($q^*$) is well above one, which implies that intangible capital is an important component of assets owned by the firms in US. Figure 8 depicts posterior median of $q^*$ and $q$ with 90% credible interval as the shaded area. The difference between estimates of $q^*$ and $q$ is large and stable, which implies that the intangible capital is an important component of corporate value during the postwar period. However, for the period before 1970s the error bands are very large and the accuracy of estimates is not very high due to the limitation of the particle smoother as discussed in Section 3. As the model estimates of intangible capital is very stable, the estimates of $q^*$ does not matches the large variation in the observed $q^*$ of nonfarm and nonfinancial business. This is consistent with the Li (2009)'s results, which are obtained using first-order approximation and Kalman filter in a DSGE model with CRRA utility. If the value of the intangible

---

4McGrattan and Prescott (2009) introduce the concept of technological capital in the neoclassical growth model, and in our paper we don't differentiate technological capital and intangible capital and use them interchangeably.
capital is the main focus of the measurement, then first-order approximation and linear filter is sufficient. However, if the focus is the risk and uncertainty of intangible capital, then higher order approximation is necessary.

The dashed line in Figure 8 depicts the observed \( q^* \), that is, ratio of market value of firms to the replacement costs of physical capital owned by nonfarm and nonfinancial business in US economy from 1970Q1 to 2011Q4. The estimates of \( q^* \), depicted by the solid line, matches the peak value of observed \( q^* \) in early 2000s. However, the unmatched large swings in the observed \( q^* \) in 1970s and 1990s suggests that capital adjustment induced by a wave of Schumpeter (1912) creative destruction might be an important factor that drives the variation in the observed \( q^* \) before 1980s.

5 Conclusion

We estimate the value of intangible capital in a dynamic stochastic general equilibrium model that features capital adjustment costs, investment-specific technological progress and recursive utility. We use second-order approximation method to solve the model and estimate the model using Bayesian method with the help of particle filter. Data from US economy in the postwar period imply that corporations indeed have formed large amounts of intangible capital as Hall (2001) found, but the implied investment in intangible capital is much smoother than that of physical capital, and the return on investment in intangible capital is much less risky, which implies that intangible-capital intensive firms have a lower average return. This is consistent with empirical evidence found by Fama and French (1995) and others that the average return of firms with low book to market equity ratio (growth firms) is lower than that of the firms with high book-to-market ratio (value firms). Market value of firms measures the value of all the capital owned by the firm, including intangible capital, while the book value of the firm only contains replacement cost of physical capital, as intangible capital is expensed in the accounting book. Hence intangible-capital-intensive firms are more likely to be growth firms and have lower book-to-market ratio.

However, we find that the estimates of market value of firms is relative high with large error bands before 1970s. It is still a challenge for the model to generate a less-than-one ratio of market value of the firm to the book value of the physical capital. Our results suggest that the arguments of Chun, Kim, Morck, and Yeung (2008) make sense that a wave of Schumpeter (1912)'s creative destruction affecting a wide range of U.S. corporations and making the existing physical capital and intangible capital obsolete. Measurement of
capital in a general equilibrium model that incorporates a wave of "creative destruction" is left for future research.
References


<table>
<thead>
<tr>
<th>Name</th>
<th>Density</th>
<th>Mean</th>
<th>S.D.</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Gamma</td>
<td>1</td>
<td>0.1</td>
<td>inverse EIS</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Gamma</td>
<td>100</td>
<td>50</td>
<td>risk aversion</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Gamma</td>
<td>2.75</td>
<td>0.5</td>
<td>labor elasticity</td>
</tr>
<tr>
<td>$\delta_u$</td>
<td>Beta</td>
<td>0.025</td>
<td>0.005</td>
<td>depreciation u-capital</td>
</tr>
<tr>
<td>$\phi_m$</td>
<td>Gamma</td>
<td>0.3</td>
<td>0.15</td>
<td>adjustment cost m-capital</td>
</tr>
<tr>
<td>$\phi_u$</td>
<td>Gamma</td>
<td>1</td>
<td>0.5</td>
<td>adjustment cost u-capital</td>
</tr>
<tr>
<td>$g^x_\ast$</td>
<td>Normal</td>
<td>0.9</td>
<td>0.2</td>
<td>x-sector growth rate</td>
</tr>
<tr>
<td>$g^c_\ast$</td>
<td>Normal</td>
<td>0.8</td>
<td>0.2</td>
<td>c-sector growth rate</td>
</tr>
<tr>
<td>$r^\ast$</td>
<td>Normal</td>
<td>0.35</td>
<td>0.1</td>
<td>riskfree rate</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.1</td>
<td>shock persistence</td>
</tr>
<tr>
<td>$\rho_\xi$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>InvGamma</td>
<td>0.007</td>
<td>Inf</td>
<td>shock volatility</td>
</tr>
<tr>
<td>$\sigma_\xi$</td>
<td>InvGamma</td>
<td>0.007</td>
<td>Inf</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td>0.36</td>
<td></td>
<td>total capital share</td>
</tr>
<tr>
<td>$\alpha_m^x$</td>
<td></td>
<td>0.23</td>
<td></td>
<td>c-sector m-capital share</td>
</tr>
<tr>
<td>$\alpha_m^x$</td>
<td></td>
<td>0.20</td>
<td></td>
<td>x-sector m-capital share</td>
</tr>
<tr>
<td>$\delta_m$</td>
<td></td>
<td>0.025</td>
<td></td>
<td>depreciation m-capital</td>
</tr>
<tr>
<td>Prior</td>
<td>Posterior</td>
<td>90% Interval</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>-----------</td>
<td>--------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>Mean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>1</td>
<td>0.955</td>
<td>[0.900, 1.015]</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>100</td>
<td>66.697</td>
<td>[53.61, 81.00]</td>
<td></td>
</tr>
<tr>
<td>$\theta$</td>
<td>2.75</td>
<td>4.505</td>
<td>[4.120, 4.900]</td>
<td></td>
</tr>
<tr>
<td>$\delta_u$</td>
<td>0.025</td>
<td>0.025</td>
<td>[0.023, 0.028]</td>
<td></td>
</tr>
<tr>
<td>$\phi_m$</td>
<td>0.3</td>
<td>0.290</td>
<td>[0.276, 0.308]</td>
<td></td>
</tr>
<tr>
<td>$\phi_u$</td>
<td>1</td>
<td>2.234</td>
<td>[2.060, 2.404]</td>
<td></td>
</tr>
<tr>
<td>$g^{x*}$</td>
<td>0.9</td>
<td>1.062</td>
<td>[0.933, 1.174]</td>
<td></td>
</tr>
<tr>
<td>$g^{c*}$</td>
<td>0.8</td>
<td>0.804</td>
<td>[0.702, 0.925]</td>
<td></td>
</tr>
<tr>
<td>$r^*$</td>
<td>0.35</td>
<td>0.317</td>
<td>[0.261, 0.367]</td>
<td></td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>0.5</td>
<td>0.567</td>
<td>[0.494, 0.632]</td>
<td></td>
</tr>
<tr>
<td>$\rho_\xi$</td>
<td>0.5</td>
<td>0.919</td>
<td>[0.912, 0.929]</td>
<td></td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.007</td>
<td>0.005</td>
<td>[0.002, 0.008]</td>
<td></td>
</tr>
<tr>
<td>$\sigma_\xi$</td>
<td>0.007</td>
<td>0.007</td>
<td>[0.002, 0.013]</td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Moments of Quantity Variables

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Variable</th>
<th>Data Mean</th>
<th>Data Median</th>
<th>Data 90% Interval</th>
<th>Posterior Mean</th>
<th>Posterior Median</th>
<th>Posterior 90% Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Output Growth</td>
<td>0.740</td>
<td>0.764</td>
<td>0.782</td>
<td>0.387</td>
<td>1.150</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Consumption Growth</td>
<td>0.821</td>
<td>0.756</td>
<td>0.789</td>
<td>0.403</td>
<td>1.183</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Investment Growth</td>
<td>0.862</td>
<td>1.496</td>
<td>1.039</td>
<td>0.163</td>
<td>1.932</td>
<td></td>
</tr>
<tr>
<td>S.D.</td>
<td>Output Growth</td>
<td>1.005</td>
<td>0.692</td>
<td>0.694</td>
<td>0.601</td>
<td>0.803</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Consumption Growth</td>
<td>0.530</td>
<td>0.763</td>
<td>0.749</td>
<td>0.655</td>
<td>0.865</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Investment Growth</td>
<td>2.532</td>
<td>1.076</td>
<td>1.135</td>
<td>0.864</td>
<td>1.445</td>
<td></td>
</tr>
<tr>
<td>Autocorr.</td>
<td>Output Growth</td>
<td>0.400</td>
<td>0.780</td>
<td>0.779</td>
<td>0.698</td>
<td>0.855</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Consumption Growth</td>
<td>0.489</td>
<td>0.741</td>
<td>0.755</td>
<td>0.661</td>
<td>0.834</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Investment Growth</td>
<td>0.509</td>
<td>0.907</td>
<td>0.925</td>
<td>0.881</td>
<td>0.960</td>
<td></td>
</tr>
<tr>
<td>Cross Corr.</td>
<td>Consumption Growth</td>
<td>0.625</td>
<td>0.963</td>
<td>0.978</td>
<td>0.955</td>
<td>0.996</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Investment Growth</td>
<td>0.670</td>
<td>0.524</td>
<td>0.569</td>
<td>0.403</td>
<td>0.726</td>
<td></td>
</tr>
</tbody>
</table>
Table 4: Moments of Financial Variables I

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Variable</th>
<th>Data Mean</th>
<th>Data Median</th>
<th>Posterior Mean</th>
<th>Posterior Median</th>
<th>Posterior 90% Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Risk Free Rate</td>
<td>0.318</td>
<td>1.007</td>
<td>0.937</td>
<td>0.579</td>
<td>1.286</td>
</tr>
<tr>
<td></td>
<td>Equity Return</td>
<td>2.014</td>
<td>1.966</td>
<td>1.982</td>
<td>1.375</td>
<td>2.633</td>
</tr>
<tr>
<td></td>
<td>Excess Return</td>
<td>1.697</td>
<td>0.959</td>
<td>1.053</td>
<td>0.752</td>
<td>1.378</td>
</tr>
</tbody>
</table>

Table 5: Moments of Financial Variables II

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Variable</th>
<th>Mean</th>
<th>Median</th>
<th>Posterior 90% Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>$r_m$</td>
<td>2.762</td>
<td>2.785</td>
<td>[1.933, 3.727]</td>
</tr>
<tr>
<td></td>
<td>$r_u$</td>
<td>0.532</td>
<td>0.540</td>
<td>[0.263, 0.809]</td>
</tr>
<tr>
<td></td>
<td>$r_c$</td>
<td>2.023</td>
<td>2.039</td>
<td>[1.391, 2.683]</td>
</tr>
<tr>
<td></td>
<td>$r_x$</td>
<td>1.849</td>
<td>1.865</td>
<td>[1.293, 2.477]</td>
</tr>
<tr>
<td></td>
<td>$q$</td>
<td>5.344</td>
<td>1.069</td>
<td>[0.701, 1.645]</td>
</tr>
<tr>
<td></td>
<td>$q^*$</td>
<td>2.982</td>
<td>2.100</td>
<td>[1.449, 3.215]</td>
</tr>
<tr>
<td></td>
<td>$q^{cs}$</td>
<td>2.878</td>
<td>2.008</td>
<td>[1.371, 3.109]</td>
</tr>
<tr>
<td></td>
<td>$q^{x*}$</td>
<td>3.286</td>
<td>2.360</td>
<td>[1.671, 3.544]</td>
</tr>
<tr>
<td>S.D.</td>
<td>$r_m$</td>
<td>4.691</td>
<td>4.734</td>
<td>[4.181, 5.343]</td>
</tr>
<tr>
<td></td>
<td>$r_u$</td>
<td>1.444</td>
<td>1.403</td>
<td>[1.250, 1.543]</td>
</tr>
<tr>
<td></td>
<td>$r_c$</td>
<td>3.168</td>
<td>2.671</td>
<td>[2.196, 3.218]</td>
</tr>
<tr>
<td></td>
<td>$r_x$</td>
<td>2.751</td>
<td>2.232</td>
<td>[1.669, 2.770]</td>
</tr>
<tr>
<td></td>
<td>$q$</td>
<td>0.278</td>
<td>0.264</td>
<td>[0.066, 0.536]</td>
</tr>
<tr>
<td></td>
<td>$q^*$</td>
<td>0.491</td>
<td>0.451</td>
<td>[0.110, 0.997]</td>
</tr>
<tr>
<td></td>
<td>$q^{cs}$</td>
<td>0.486</td>
<td>0.446</td>
<td>[0.109, 0.985]</td>
</tr>
<tr>
<td></td>
<td>$q^{x*}$</td>
<td>0.510</td>
<td>0.469</td>
<td>[0.115, 1.042]</td>
</tr>
</tbody>
</table>
Figure 1: Price and Quantity of Private Nonresidential Fixed Investment
Figure 2: Average Weekly Working Hours and Real Wage
Figure 3: Observed Tobin’s $q$

Note: Ratio of market value of the securities nonresidential to physical capital at replacement costs of nonfarm and nonfinancial corporations
Figure 4: Market Value and Book Value of Asset and Equity
Figure 5: Estimates of Physical and Intangible Capital
Figure 6: Comparison of Estimates of the Quantity of Capital in Hall (2001)
Figure 7: Rates of Return of Investments

Note: Posterior median of smoothed rates of returns of investments on physical and intangible capital and their difference with 90% credible interval (left panels); Posterior median of smoothed rates of returns of investments on consumption and investment sector and their difference with 90% credible interval (right panels)
Figure 8: Market Value of Firm to the Replacement Costs of Capital

Note: The dashed line depicts the market value of firms to the replacement cost of physical capital owned by nonfarm and nonfinancial business in US economy from 1947Q1-2012Q4. The dotted line depicts the posterior median of market value of firms to the replacement costs of physical capital and intangible capital, and the solid line depicts the posterior median of market value of firms to the replacement costs of physical capital. The 90% credible interval of posterior distribution is the shaded area.
Appendix

A Solving the Two-Sector Model with Intangible Capital

This section provides the first-order conditions that characterize the equilibrium of the two-sector model with intangible capital in Section 2. To solve the social planner’s problem, we first need to transform the nonstationary economy into a stationary system. The transformation involves dividing all the nonstationary variables by their stochastic trends as following

\[ c_t = \frac{C_t}{\Psi_t}, \quad k_{j,t} = \frac{K_{j,t}}{\Psi_t^c}, \quad x_{j,t} = \frac{X_{j,t}}{\Psi_t^x}, \quad u_t = \frac{U_t}{\Psi_t^c}, \quad v_t = \frac{V_t}{\Psi_t^c}, \quad w_t = \frac{W_t}{\Psi_t^c} \]

for \( j = m, u \). The labor inputs \( n_t^c \) and \( n_t^x \) as well as the capital share \( \eta_{m,t} \) and \( \eta_{u,t} \) are stationary and have no trend.

The social planner’s problem in the transformed economy can be written as

\[
v_t = \max \left\{ (1 - \beta)u_t^{1-\rho} + \beta w_t^{1-\rho} \right\}^{\frac{1}{1-\rho}} + \sum_{j=m,u} \lambda_{j,t} \left[ (1 - \delta_j)\tilde{k}_{j,t} + \Phi_j \left( \frac{x_{j,t}}{k_{j,t}} \right) \tilde{k}_{j,t} - k_{j,t} \right] + \mu_t \left[ \left( \eta_{m,t}\tilde{k}_{m,t} \right)^{\alpha_m} \left( \eta_{u,t}\tilde{k}_{u,t} \right)^{\alpha_u} (n_t^c)^{1-\alpha} - c_t \right] + \mu_t^x \left[ \left( (1 - \eta_{m,t})\tilde{k}_{m,t} \right)^{\alpha_m} \left( (1 - \eta_{u,t})\tilde{k}_{u,t} \right)^{\alpha_u} (n_t^x)^{1-\alpha} - x_{m,t} - x_{u,t} \right]
\]

where \( \lambda_{m,t}, \lambda_{u,t}, \mu_t^c \) and \( \mu_t^x \) are Lagarangian multipliers and \( \tilde{k}_{j,t} \) is defined for \( j = m, u \) as following,

\[ \tilde{k}_{j,t} = k_{j,t-1} \exp \left( -\Delta \log \Psi_t^x \right) \]

and

\[
u_t = c_t\left( 1 - n_t^c - n_t^x \right)\theta \\
w_t = \left( \mathbb{E}_t \left[ v_{t+1}^{1-\gamma} \exp \left( (1 - \gamma)\Delta \log \Psi_{t+1}^c \right) \right] \right)^{\frac{1}{1-\gamma}}
\]
A.1 First-order Conditions

The necessary conditions for the maximization problem of the social planner are nonlinear functions of the transformed variables as following,

\[ c_t : (1 - \beta)\nu_t u_t^{1-\rho} (c_t)^{-1} = \mu_t^c \quad (17a) \]
\[ n_t^c : -\theta(1 - \beta)\nu_t u_t^{1-\rho} (1 - n_t^c - n_t^x)^{-1} = (1 - \alpha)\mu_c^c (n_t^c)^{-1} \quad (17b) \]
\[ n_t^x : -\theta(1 - \beta)\nu_t u_t^{1-\rho} (1 - n_t^c - n_t^x)^{-1} = (1 - \alpha)\mu_t^x (x_{m,t} + x_{u,t}) (n_t^x)^{-1} \quad (17c) \]
\[ x_{j,t} : \lambda_{j,t} \Phi'_j \left( \frac{x_{j,t}}{k_{j,t}} \right) = \mu_t^x \quad \text{for } j = m, u \quad (17d) \]
\[ \eta_{j,t} : \mu_t^c \alpha_c^c \frac{c_t}{\eta_{j,t}} = \mu_t^x \alpha_x^x \frac{x_{m,t} + x_{u,t}}{1 - \eta_{j,t}} \quad \text{for } j = m, u \quad (17e) \]
\[ k_{j,t} : \beta\nu_t w_t^{-\rho} \frac{\partial w_t}{\partial k_{j,t}} = \lambda_{j,t} \quad \text{for } j = m, u \quad (17f) \]

where we have from the Envelope Theorem, for \( j = m, u \)

\[ \frac{\partial w_t}{\partial k_{j,t}} = E_t \left[ \frac{\partial v_{t+1}}{\partial k_{j,t+1}} \left( \frac{v_{t+1}}{w_t} \right)^{-\gamma} \exp \left( (1 - \gamma) \Delta \log \Psi_{t+1}^c - \Delta \log \Psi_{t+1}^x \right) \right] \quad (18) \]
\[ \frac{\partial v_t}{\partial k_{j,t}} = \lambda_{j,t} \left[ (1 - \delta_j) + \Phi_{j,t} - \Phi'_{j,t} \frac{x_{j,t}}{k_{j,t}} \right] + \frac{\mu_t^c \alpha_c^c c_t + \mu_t^x \alpha_x^x (x_{m,t} + x_{u,t})}{k_{j,t}} \quad (19) \]

where \( \Phi_{j,t} = \Phi_j \left( \frac{x_{j,t}}{k_{j,t}} \right) \) and \( \Phi'_{j,t} = \Phi'_j \left( \frac{x_{j,t}}{k_{j,t}} \right) \).

A.2 Euler Equations

From the equilibrium condition of production sector, the transformed wage rate at time \( t \), \( y_t \), equals the atemporal marginal rate of substitution between consumption and labor, from equation (17a) and (17b), we have

\[ y_t = -\frac{\partial v_t}{\partial n_t^c} / \frac{\partial v_t}{\partial c_t} = \frac{\theta c_t}{1 - n_t} \]

Then we can derive the Euler equation from the first-order conditions (17)-(19),

\[ E_t \left[ m_{t,t+1} r_{j,t+1} \right] = 1 \quad \text{for } j = m, u \]
where $m_{t,t+1}$ is the stochastic discount factor defined as

$$m_{t,t+1} = \beta \exp \left( -\rho \Delta \log \Psi^c_{t+1} \right) \left( \frac{c_{t+1}}{c_t} \right)^{-\rho} \left( \frac{1 - n_{t+1}}{1 - n_t} \right)^{\theta(1-\rho)} \left( \frac{\tilde{v}_{t+1}}{w_t} \right)^{\rho-\gamma}$$

(20)

where $\tilde{v}_t = v_t \exp(\Delta \log \Psi^c_t)$ and $r_{j,t}$ is the investment return of type $j$ capital, for $j = m, u$

$$r_{j,t} = \left\{ \frac{\alpha_j^c c_t + \alpha_j^x p^x_t (x_{m,t} + x_{u,t})}{\tilde{p}_{j,t-1} k_{j,t}} + \frac{\tilde{p}_{j,t}}{\tilde{p}_{j,t-1}} \left( 1 - \delta_j + \Phi_{j,t} - \Phi'_{j,t} \frac{x_{j,t}}{k_{j,t}} \right) \right\}$$

$$\times \exp \left( \Delta \log \Psi^c_t - \Delta \log \Psi^x_t \right)$$

where

$$\tilde{p}_t^x = \frac{p_t^x}{\Phi'_{j,t}}$$

is the period-$t$ price of investment goods relative to consumption in the transformed economy and

$$\tilde{p}_{j,t} = \frac{p_t^x}{\Phi'_{j,t}}$$

is the period-$t$ price of capital installed at beginning of period $t + 1$, for $j = m, u$.

### A.3 Equilibrium

Given the initial capital stock $k_{m,0}$ and $k_{u,0}$, and shock $\xi_0$ and $z_0$, the equilibrium of the transformed economy is characterized by the following system of equations:

\begin{align*}
\theta(n_t - n^{x}_t) &= (1 - \alpha)(1 - n_t) \\
p^x_t (x_{m,t} + x_{u,t}) &= \frac{n^{x}_t}{c_t} \\
p^x_t (x_{m,t} + x_{u,t}) &= \frac{\alpha_j^x (1 - \eta_j,t)}{c_t} \\
E_t \left[ m_{t,t+1} r_{j,t+1} \right] &= 1 \\
\tilde{v}_t &= \exp(-\rho \Delta \log \Psi^c_t) \left( \frac{\tilde{v}_t}{w_{t-1}} \right)^{\rho-\chi} \\
q_{j,t} \Phi'_{j,t} &= 1 \\
p_{j,t} &= q_{j,t} \Phi'_{j,t} \left( 1 - \delta_j + \Phi_{j,t} - \Phi'_{j,t} \frac{x_{j,t}}{k_{j,t}} \right) \\
r_{j,t} &= \frac{1}{q_{j,t-1} p^x_{t-1}} \left[ p_{t,j} + \frac{\alpha_j^c c_t + \alpha_j^x p^x_t (x_{m,t} + x_{u,t})}{k_{j,t}} \right] \exp \left( \Delta \log \Psi^c_t - \Delta \log \Psi^x_t \right)
\end{align*}

(21a) \hspace{1cm} (21b) \hspace{1cm} (21c) \hspace{1cm} (21d) \hspace{1cm} (21e) \hspace{1cm} (21f) \hspace{1cm} (21g) \hspace{1cm} (21h)
\[ v_t^{1-\rho} = (1 - \beta) (c_t (1 - n_t)^{\theta})^{1-\rho} + \beta w_t^{1-\rho} \]  
(21i)

\[ w_t^{1-\gamma} = \mathbf{E}_t [v_{t+1}^{1-\gamma}] \]  
(21j)

\[ k_{j,t} = (1 - \delta_j) \tilde{k}_{j,t} + \Phi_{j,t} \tilde{k}_{j,t} \]  
(21k)

\[ c_t = \left( \eta_{m,t} \tilde{k}_{m,t} \right)^{\alpha_m} \left( \eta_{u,t} \tilde{k}_{u,t} \right)^{\alpha_u} (n_t - n_x t)^{1-\alpha} \]  
(21l)

\[ x_{m,t} + x_{u,t} = \left( (1 - \eta_{m,t}) \tilde{k}_{m,t} \right)^{\alpha_m} \left( (1 - \eta_{u,t}) \tilde{k}_{u,t} \right)^{\alpha_u} (n_t - n_x t)^{1-\alpha} \]  
(21m)

\[ \tilde{k}_{j,t} = k_{j,t-1} \exp \left( -\Delta \log \Psi_t \right) \]  
(21n)

\[ \tilde{v}_t = v_t \exp (\Delta \log \Psi_t) \]  
(21o)

\[ \Delta \log \Psi_t^x = g^x + \log z_t + \log \xi_t \]  
(21p)

\[ \Delta \log \Psi_t^c = g^c + \log z_t + \alpha \log \xi_t \]  
(21q)

\[ \log z_t = \rho_z \log z_{t-1} + \varepsilon_{z,t} \]  
(21r)

\[ \log \xi_t = \rho_\xi \log \xi_{t-1} + \varepsilon_{\xi,t} \]  
(21s)

The value function and the continuation value are included in the equilibrium conditions. It can be easily checked that the stochastic discount factor of this model given in (21e), \( m_t \), is turned into the standard one if the preference is the expected utility with CRRA (\( \rho = \gamma \)). Note that the marginal adjustment of capital is equal to zero, that is, \( \Phi_{j,t} = \Phi_{j,t} \cdot x_{j,t} / \tilde{k}_{j,t} \) for a model without adjustment cost (\( \Phi_j(x) = x \)). In such a model, the Tobin’s \( q_{j,t} \) in (21f) is constant at one.

### A.4 Steady State

We solve the nonlinear system (21a)-(21s) by second-order approximation around the steady state using DYNARE, then use particle filter to estimate the time series of the unobserved capital stocks and investments in intangible from the observed variables.

Before we proceed to obtain the nonstochastic steady state of the detrended economy, we need to specify the functional forms of the capital adjustment cost. Following Jermann (1998), the capital adjustment function is defined as

\[ \Phi_j(x) = \frac{a_{j1}}{1 - \frac{1}{\phi_j}} x^{1 - \frac{1}{\phi_j}} + a_{j2} \]

for \( \phi_j \geq 0 \). Parameters \( a_{j1} \) and \( a_{j2} \) are chosen so that the nonstochastic steady state along the growth path is invariant to the specification of \( \phi_j \). From the steady state relationship
we have
\[
\frac{1}{q_j} = \Phi_j'\left(\frac{x_j}{k_j}\right) = a_{j1}\left(\frac{x_j}{k_j}\right)^{-\frac{1}{\sigma_j}}
\]
\[
e^{g^x} - 1 + \delta_j = \Phi_j\left(\frac{x_j}{k_j}\right) = a_{j1}\left(\frac{\phi_j}{\phi_j - 1}\right)\left(\frac{x_j}{k_j}\right)^{1-\frac{1}{\sigma_j}} + a_{j2}
\]

We further assume that the steady state investment-capital ratio is given as
\[
\frac{x_j}{k_j} = \exp(g^x) - 1 + \delta_j
\]

which requires the steady state value of \(q_j\) to be a unity. Therefore, for steady state values to be independent of \(\phi_j\) we should have
\[
a_{j1} = \left(\exp(g^x) - 1 + \delta_j\right)^{\frac{1}{\sigma_j}}, \quad a_{j2} = \frac{\exp(g^x) - 1 + \delta_j}{1 - \phi_j}
\]

Hence, we have at the nonstochastic steady state,
\[
\Phi_j\left(\frac{x_j}{k_j}\right) = \frac{x_j}{k_j}, \quad \Phi_j'\left(\frac{x_j}{k_j}\right) = 1 \quad (22)
\]

Given the functional form of \(\Phi_j\), the steady state of detrended economy is
\[
z = \xi = 1
\]
\[
m = \beta \exp(-\rho g^c)
\]
\[
r^f = m^{-1}
\]
\[
r = r_m = r_u = r_f
\]
\[
q = q_m = q_u = 1
\]
\[
p_m = 1 - \delta_m
\]
\[
p_u = 1 - \delta_u
\]
\[
\tilde{k_j} = \frac{\alpha_j^{c_n} + \alpha_j^r - \alpha_j^c}{r \exp(g^x - g^c) - 1 + \delta_j}
\]
\[
\frac{x_j}{x_m + x_u} = \frac{\exp(g^x) - 1 + \delta_j}{r \exp(g^x - g^c) - 1 + \delta_j}
\]
\[
A_j = \frac{\exp(g^x) - 1 + \delta_j}{r \exp(g^x - g^c) - 1 + \delta_j}
\]
\[
\begin{align*}
\frac{n^x}{n} &= \frac{\alpha^c_m A_m + (\alpha - \alpha^c_m) A_u}{1 + (\alpha^c_m - \alpha^x_m)(A_m - A_u)} \\
\left\{ \frac{n^x}{n} = \alpha A \quad \text{if we assume } \delta_m = \delta_u = \delta \text{ and hence } A_m = A_u = A \right\} \\
\\
n &= \left[ 1 + \frac{\theta}{1 - \alpha} \left( 1 - n^x \right) \right]^{-1} \\
n^x &= n \times \frac{n^x}{n} \\
\eta_j &= \left( 1 + \frac{\alpha_j^c}{\alpha_j^x} \cdot \frac{n^x}{n} \right)^{-1} \\
x_m + x_u &= \left[ \left( 1 - \eta_m \right) \frac{\tilde{k}_m}{y^x} \right]^{\alpha^c_m} \left( 1 - \eta_u \right) \frac{\tilde{k}_u}{y^x} \left( n^x \right)^{1-\alpha} \\
\\
k_j &= (x_m + x_u) \times \frac{\tilde{k}_j}{x_m + x_u} \exp(g^x) \\
x_j &= (x_m + x_u) \times \frac{x_j}{x_m + x_u} \\
c &= (\eta_m k_m)^{\alpha^c_m} (\eta_u k_u)^{\alpha - \alpha^c_m} \left( n - n^x \right)^{1-\alpha} \exp(-\alpha g^x) \\
p^x &= \frac{(x_m + x_u)(n - n^x)}{c n^x} \\
p_m &= p^x \times \frac{p_m}{p_x} \\
p_u &= p^x \times \frac{p_u}{p_x} \\
v &= \left[ \frac{1}{1 - \beta} \exp \left( 1 - \rho g^c \right) \right]^{\frac{1}{1 - \beta}} c(1 - n)^\theta \\
w &= v \exp \left( g^c \right)
\end{align*}
\]
B Data Appendix

B.1 Macroeconomic data

- Consumption ($C$): Quarterly seasonally adjusted nominal Personal Consumption Expenditure of nondurable goods and services from NIPA Table 1.1.2 Line 4 and Line 5.

- Investment in physical capital ($I_m$): Quarterly seasonally adjusted nominal Fixed Nonresidential Investment from NIPA Table 1.1.5, Line 9.

- GDP ($Y$): Quarterly seasonally adjusted nominal GDP from NIPA Table 1.1.5 Line 1.

- Price Deflator for Consumption ($P_c$): We first download quarterly seasonally adjusted Implicit Price Deflators ($2005 = 100$) for nondurable goods and services from NIPA Table 1.1.9, Line 4 and Line 5, then construct the price deflator for consumption as weighted average of implicit price deflator of personal nondurable consumption and implicit price deflator of personal services consumption. See Hansen, Heaton and Li (2005) for details.

- Relative Price of Investment ($P^e$): We first download quarterly seasonally adjusted Implicit Price Deflators ($2005 = 100$) for Fixed Nonresidential Investment from NIPA Table 1.1.9, Line 9, then compute the relative price as the ratio of implicit price deflator of investment to the price deflator of consumption ($P_c$).

- Labor Income: Quarterly seasonally adjusted Compensation of Employees from NIPA Table 1.12, Line 2.

- Hours: We follow King et al. (1988) to construct hours series as monthly total employment multiplied by average weekly hours worked divided by the civilian non-institutional population 16 years and older. Data on total employment, average weekly hours and civilian non-institutional population 16 years and older is downloaded from website of Bureau of Labor Statistics.

B.2 Stock Market Data

- We follow Hall (2001) to construct the market value of US nonfarm nonfinancial corporation using data from Flow of Funds Accounts maintained by Federal Reserve Board. See Hall (2001) online Data Appendix for details.
• We download the quarterly return on market index of NYSE, AMEX and NASDAQ stocks from CRSP maintained by WRDS, University of Pennsylvania

• We use quarterly return on 30-day Treasury Bills as proxy for the risk free rate.

We get real market value, real return on market index and real risk free rate by deflating the nominal value by the growth rate of price deflator of consumption.
C Particle Smoother

The smoother can be applicable to estimate the state variables using the information available in future period, \( p(s_t|y_{1:T}) \). The joint posterior distribution of the state variable, \( p(s_{1:T}|y_{1:T}) \), would be sufficient for this purpose since it can be marginalized into a smoothered distribution of the state at any period. With the Markov property of the state space, the joint posterior distribution of the state variable can be decomposed as

\[
p(s_{1:T}|y_{1:T}) = p(s_{T}|y_{1:T}) \prod_{t=1}^{T-1} p(s_t|s_{t+1}, y_{1:t})
\]

Therefore, a procedure for the particle smoother can be implemented using (23):

1. **Forward filtering**: Run the particle filter (with \( N \) particles) to approximate posterior distributions of the state variable, \( \{p(s_t|y_{1:t})\}_{t=1}^{T} \).

2. **Initialize at the terminal period**: Sample randomly \( S^{(k)}_{T} \) from the approximated \( p(s_{T}|y_{1:T}) \).

3. **Backward sampling**: Sample recursively \( S^{(k)}_{t} \sim p \left( s_{t}|S^{(k)}_{t+1}, y_{1:t} \right) \) for \( t = T-1, T-2, \ldots, 1 \). However, direct sampling from this conditional distribution is not possible. Instead, note that

\[
p \left( s_{t}|S^{(k)}_{t+1}, y_{1:t} \right) = \frac{p \left( S^{(k)}_{t+1}|s_{t} \right) p(s_{t}|y_{1:t})}{\int p \left( S^{(k)}_{t+1}|s_{t} \right) p(s_{t}|y_{1:t}) \, ds_{t}}
\]

which suggests the following sampling scheme:

(a) Evaluate \( w^{(i)}_{t} = p \left( S^{(i)}_{t+1}|S^{(i)}_{t} \right) \) for \( i = 1, \ldots, N \), where \( S^{(i)}_{t} \)'s are particles that approximate \( p(s_{t}|y_{1:t}) \).

(b) Sample \( S^{(k)}_{t} \) from the approximated \( p(s_{t}|y_{1:t}) \) with probability \( \frac{w^{(k)}_{t}}{\sum_{i=1}^{N} w^{(i)}_{t}} \).

When we repeat (a) and (b) for \( t = T-1, T-2, \ldots, 1 \), we obtain one draw from \( p(s_{1:T}|y_{1:T}) \).

4. Repeat 2 and 3 for \( k = 1, \ldots, K \). Eventually we obtain \( \left\{ S^{(k)}_{1:T} \right\}_{k=1}^{K} \) that approximates \( p(s_{1:T}|y_{1:T}) \).