Generalized Deferred Acceptance Auctions with Multiple Relinquishment Options for Spectrum Reallocation

By EIICHIRO KAZUMORI*

This paper studies the design of reverse auctions in the US incentive auctions where TV broadcasters in the UHF band offer to relinquish the usage right or to relocate to another band to increase the spectrum for mobile communication uses. The state of the art is the deferred-acceptance auctions by Paul Milgrom and Ilya Segal (2014) that determine allocations based on scores and are strategy-proof when sellers are single-minded. But when sellers have multiple relinquishment options and are restricted to be single-minded, there would be possibilities of strategic considerations about which option the seller should bid. Nevertheless just allowing multiple bids is not possible since a seller switching one option to another may violate interference constraints. To resolve this issue, this paper proposes generalized deferred acceptance auctions with the supplementary phase where sellers make multiple offers, the buyer does not need to recalculate interference constraints, and are strategy-proof. This auction can be regarded as a generalization of 'ladder auctions' of Jon Levin, Paul Milgrom, and Ilya Segal (2012) for multiple offers. These results show generalities and robustness of incentive auctions design described in FCC NPRM 12-118, Paul Milgrom, Larry Ausubel, Jon Levin, and Ilya Segal (2012), and Paul Milgrom and Ilya Segal (2014).

There have been rapid progresses in the mobile broadband technology that allows for any time, anywhere access of high-bandwidth applications, contents, and communications. The first wireless Internet access, which became available in 1991, offered the speed up to 237Kbits/second (GSM Edge) as part of the second generation (2G) technology. Then, the next 3G technology provided up to 16 Mbit/second (UMTS TDD). Now the current fourth (4G) generations technology delivers up to 100–300Mbits/second (LTE Advanced).

These progresses in mobile broadband technology have led to wide adoption of mobile communication devices that rely on extensive uses of spectrum. In 2012, there are 172 million smartphone subscribers and more than 29% of adults own tablet computers or eReaders, up from 3% in 2009 in the United States (Mary Meeker (2012)). Furthermore, the demands are expected to increase dramatically in the near future. The average amount of traffic per smartphone in 2012 is 342 MB

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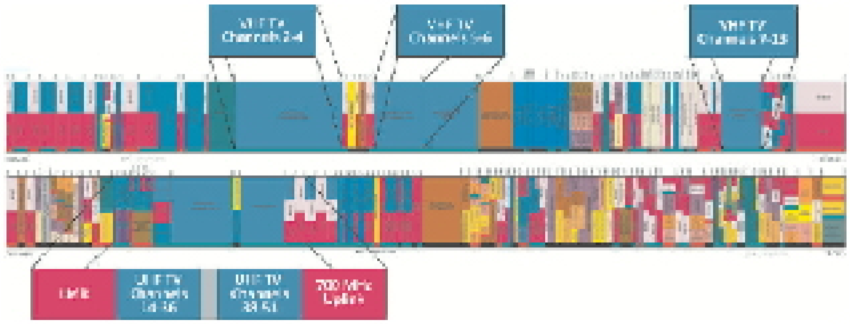

Figure 1. : The Current Allocation of Spectrum to TV Stations

per month, up from 189 MB per month in 2011. Wireless data traffic in the US has spiked 104% over the past year. Global mobile traffic is projected to grow 13 folds between 2012 and 2017, reaching 11.2 exabytes per month (Cisco (2013)).

Given the importance of mobile communications in modern life and economic growth, reallocation of spectrum from current licensees TV broadcasters to mobile uses has become a public policy agenda[1]. Currently, there are 8402 broadcasters operating in 3 VHF (very high frequency) bands and 2 UHF (ultra high frequency) bands[2], and each broadcaster has been assigned to a 6MHZ block of spectrum covering a specific geographic area (Figure **??**). Given the technological characteristics of spectrum of the UHF bands, the FCC focuses on reallocations of spectrum in the UHF band[3]. FCC (2013a) describes the transition plan of the UHF 600MHZ band allocation from the current one that allocates most of the current spectrum to TV broadcasters to the new one that increases the allocation to mobile operators.

The Congress authorized FCC to conduct incentive auctions in the Spectrum Act in February 2012[4]. Incentive auctions are voluntary, market-based means of repurposing spectrum by encouraging licensees to relinquish spectrum usage rights in exchange for a share of the proceeds from an auction of new licenses to use the repurposed spectrum. Its economic significances are that incentive auctions provide opportunities for reallocating of spectrum from lower-value uses to high-value uses through the market-based mechanism[5].

[1]See Susan Athey, Emilio Calvano, and Joshua Gans (2010) for an analysis of consumers switching from tranditional news outlet to the Internet.

[2]See FCC (2013b). VHF is the ITU-designated range of radio frequency electromagnetic waves from 30 MHz to 300 MHz. UHF designates the radio frequency range between 300 MHz and 3 GHz (3,000 MHz).

[3]The spectrum in the UHF 600MHZ band has a further reach and better penetration through buildings than higher frequencies. The 600MHZ band is adjacent to the 700MHZ band where mobile operators are building LTE 4G data networks after auctions in 2008.

[4]Incentive auctions are based on the National Broadband Plan announed by FCC in March 2010. The FCC issued the Broadband Television Spectrum Incentive Auctions NPRM (FCC (2012b)) to develop a rulemaking record. FCC (2013a) summarizes the proposed incentive auction process.

[5]Eval Kwerel (2013) and Harold Feld, Evan Kwerel, John Marsh, and Preston Padden (2013). For its economic impact, the Congressional Budget Office (2011) notes that the budgetary impact of spectrum auctions, mainly from the incentive auctions, is

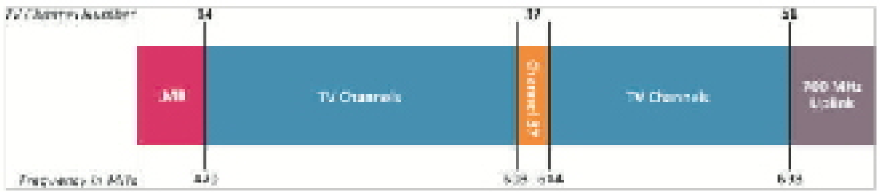

Figure 2. : The Current Allocation of 600MHZ Band.

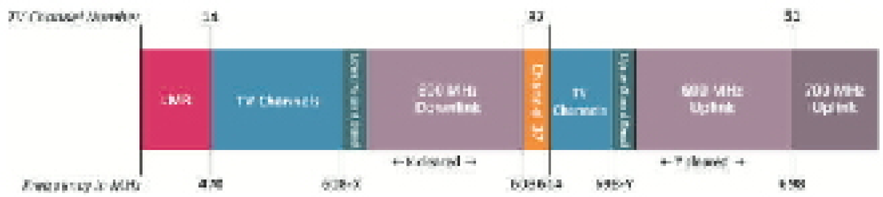

Figure 3. : The Reallocation of 600MHZ Band.

The fundamental structure of the incentive auctions process was proposed by the FCC NPRM (FCC (2012a))-Milgrom, Ausubel, Levin, and Segal (2012) and it consists of 3 parts[6]. The first step is Reverse Auctions: TV stations submit bids to voluntarily relinquish spectrum rights to FCC in exchange for payments. Then Repacking is a reorganization of the broadcast television bands where television stations that remain on-the-air will occupy a smaller portion of the UHF band. Then the FCC can reconfigure a portion of the band into contiguous blocks of spectrum for flexible uses. Then the Forward Auctions identify the prices that potential users of repurposed spectrum would pay for the new licenses.

The Focus of the Paper: The Design of Reverse Auctions. In this paper, we focus on the design of reverse auctions because of 2 reasons. First, active participation by TV stations into the reverse auctions is crucial to achieve the mandate of reclaiming large amounts of spectrum and raising revenues. Second, TV stations going off the air or moving to the VHF band will provide spectrum for whitespace uses[7] that are important for experimentation in new technologies, community broadband services for underserved areas, scientific uses (radio astronomy), and emerging IEEE 802.11AF white-fi technologies[8].

around $23.3 billion.

[6]See Evan Kwerel and William Falker (1985) and Evan Kwerel and John Williams (2002) for pioneering proposals for market-based mechanisms. See Preston McAfee, John McMillan, and Simon Wilkie (2009) for account of FCC auctions. FCC (2013a-c) discusses recent developments in the incentive auctions rulemaking.

[7]See Paul Milgrom, Jon Levin, and Assaf Eilat. (2011).

[8]Furthermore, this question of designing reverse auction mechanisms can be potentially applied to the other, related issue of relocation and sharing of the government spectrum initiated by the 2010 Presidential Memorandum “Unleashing the Wireless Broadband

The state of the art of the design of reverse auctions is deferred acceptance auctions proposed by Milgrom and Segal (2014). In reverse auctions, a station that does not relinquish spectrum will be repacked according to the 600MHZ band plan. In the repacking assignment, a station cannot interfere with other stations[9]. In other words, repacking of TV stations is possible only if the assignment does not create interferences among stations[10]. Since each station is different in its location and coverage area, checking feasibility of assignment requires checking interference constraints for each pair of stations, and is a computationally hard problem[11]. Thus auction rules based on a full optimization subject to interference constraints may well be too complicated for practical purposes. Thus Milgrom and Segal (2014) consider auctions that assign a scalar value score to each offer and determine allocation and payments based on the score.

Specifically, in deferred acceptance auctions, each seller submits a single offer. At each round, an offer is evaluated with its scoring function. The buyer deletes the offer with the highest score. Then the set of frozen sellers (i.e., the set of stations that are assigned to the band) will be updated and offers are re-evaluated. Auctions end when only offers with zero score remain. The payment rule is determined by the threshold price that is the maximum amount that the seller can offer to sell. When bidders are single-minded, the deferred acceptance auctions are strategy-proof, can be implemented in descending clock auctions, nearly-optimal[12], group strategy-proofness [13], and equivalent to pay-as-bid auctions[14].

As a practical implementation, Milgrom, Levin, and Segal (2013) propose "ladder auctions" where the buyer starts with assigning stations to go off the air, Lo-VHF, Hi-VHF, and the UHF band as the price clock goes down and that a seller cannot go back from the higher band to the lower band.

The Research Question: Allowing Multiple Relinquishment Options. The Milgrom and Segal (2014) analysis assumes single-minded bidders. But the Spectrum Act 6403(a)(2) stipulates that bidders shall have multiple relinquishment options such as going off the air, moving to the VHF band, and channel sharing[15]. Then the research question of the paper is to define a generalization of deferred acceptance to allow multiple relinquishment options, establish its strategy-proofness, and construct equivalent descending clock auctions.

Revolution" that directed the National Telecommunications and Information Administration (NTIA) within the Department of Commerce to collaborate with FCC to develop a plan and a timetable to make 500 megahertz (MHz) of federally and nonfederally allocated spectrum available for wireless broadband use in the next 10 years.

[9]Specifically, given an engineering characteristics of each station such as locations and the power of transmitter, the TV software determines the service area where the signals from the station is higher than some threshold levels. This contour will determine the separation distance. Figure 4 provides an example of 2 TV stations in the New York area. The interference constraints can apply for TV stations and TV stations and mobile services (interservice interference constraints).

[10]In addition, market variations can lead to interservice constraints between TV stations and mobile communication services.

[11]See FCC (2013d, 2014a,b) and Robert Weller (2013).

[12]That is, the buyer divides the geographic markets into cliques (metropolitan areas) that do not intersect with each other. If there are few cross-clique interference constraints, then the deferred-acceptance heuristic auctions can be designed to be almost efficient.

[13]That is, for any group deviation, there is at least one seller who does not strictly gain from the group deviation.

[14]That is, the pay-as-bid auctions are dominance solvable and its Nash equilibrium of the auction game is the equivalent to the one in the deferred-acceptance heuristic auctions.

[15]Specifically the Act 6403 (2) says the following: "Eligible Relinquishments—A relinquishment of usage rights for purposes of paragraph (1) shall include the following: (A) Relinquishing all usage rights with respect to a particular television channel without receiving in return any usage rights with respect to another television channel. (B) Relinquishing all usage rights with respect to an ultra high frequency television channel in return for receiving usage rights with respect to a very high frequency television channel. (C) Relinquishing usage rights in order to share a television channel with another licensee."

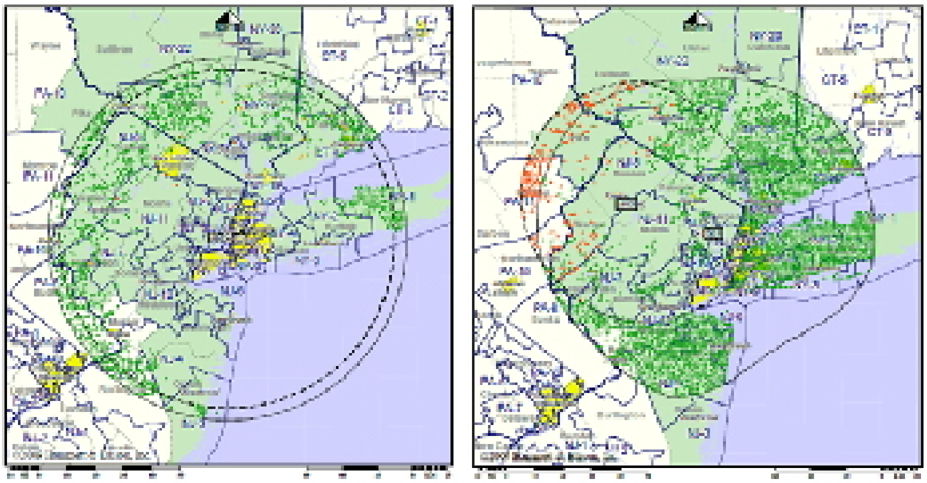

Figure 4. : An Example of 2 TV Stations Operating in the New York Market (WABC-TV in Channel 7 and WMBC-TV in Channel 18)

There are 3 reasons that we consider this extension. First, this extension can reduce strategic manipulations regarding the choice of auctions that the seller is wiling to bid. When the seller can make only one offer, the seller has to choose which auctions to bid. And the seller's choice of auctions depends on the payment the seller will get. Then the payment depends on the choice of auctions by other bidders. Therefore, the seller's choice of auctions can depend on other sellers' choices and sincere bidding may not realize. Allowing multiple offers can resolve this question.

Second, this extension can improve efficiency of reallocation. When the seller's choice depends on other sellers' choice, it can be that the seller may offer to relinquish through an option that may not be efficient only because there may not be enough competitions in this auction. Then, allowing multiple offers can improve efficiency.

Third, allowing multiple offers provides more flexibility for TV stations. It may increase TV stations participation in the reverse auctions, which implies more availability of spectrum. This is especially important since the active participation of TV stations is crucial to the success of the incentive auctions[16]

But allowing multiple offers to sellers is not trivial because of interference constraints since switching to a band can create interferences with stations that are already assigned to this band[17].

[16]For example, an FCC commissioner Pai states as follows (Pai (2014)):

> We need to redouble our outreach efforts to broadcasters. If there is not sufficient broadcaster participation in the reverse auction, the incentive auction will fail. It is as simple as that. (...) We also need to let broadcasters know that participating in the reverse auction doesn't necessarily mean exiting the industry—the channel-sharing and UHF-to-VHF options allow broadcasters to stay on the air while getting new funding that can strengthen their operations.

[17]I thank Ilya Segal for helpful conversations on this issue.

EXAMPLE 1: *Consider a station in the UHF band that is contemplating on going off the air or moving to the VHF band. Suppose the buyer runs deferred acceptance auctions simultaneously both for the UHF band and for the VHF band. Suppose further that a station can make only one offer, but that a station is allowed to switch from the going off the air auction to the moving to the VHF auction during the course of the auctions depending on the history of the auctions.*

Now consider a station has an offer of going off the air rejected. It implies that the station will be repacked to the UHF band. Since the offer of going off the air is rejected, the station would very well consider switching its offer to the moving to the VHF auction. In standard simultaneous ascending auctions, this switching will not cause any problems. But in reverse auctions with interference constraints, switching to the VHF band will require feasibility checking with stations currently assigned to the VHF band. If the station interferes with any of stations currently assigned to the VHF band, then switching may not be possible even if the station offers to move to the VHF band with a very low offer. These interference problem may create a strategic manipulation.

Generalized Deferred Acceptance Auctions with the Supplementary Phase. In this paper, we consider a generalization of Milgrom-Segal deferred acceptance auctions, called generalized deferred acceptance auctions with the supplementary phase. In this auction, a seller makes multiple offers for multiple options. An offer is scored according to the interference with frozen TV stations and other existing spectrum uses (medical telemetry, radio astronomy, and wireless microphone, and community broadband services). The threshold price of an offer is the maximum offer that the seller can make to sell through the option. The seller sells through the option that provides the seller the highest profit.

In these generalized deferred acceptance auctions, a seller does not need to switch across options in the auction phase since the seller can make multiple offers for multiple options from the beginning. Furthermore, the buyer does not need to adjust the interference constraints since a seller does not need to switch. Finally, the final allocation satisfies interference constraints.

EXAMPLE 2: *(Continued) In the setting of the previous example, in generalized deferred acceptance auctions with the supplementary phase, the station can make offers in both going off the air auctions and moving to the VHF band auction at the same time in the first auction phase. In supplementary phase, the seller is assigned to the auction that will provide the seller the maximum profit. Thus a seller does not switch across auctions in the process.*

This auctions have three desirable properties. First, when sellers have multiple sell options but a unit supply, deferred acceptance auctions are strategy-proof. With the seller-optimal allocation rule and the threshold pricing, sellers do not have incentives to manipulate the payment through offers for other options. Second, generalized deferred acceptance auctions can be implemented as a two-stage mechanism. The first stage is clock auctions for each relinquishment option. The second stage determines the final assignment. Finally, the generalized deferred acceptance auctions with the supplementary phase inherit the properties of deferred acceptance auctions, can realize high efficiency levels, robust to group deviation, and equivalent outcomes with pay-as-bid auctions. These extensions are possible because of the structure that sellers have only unit supply although they can make multiple offers.

The generalized deferred acceptance auctions with the supplementary phase can be considered as a generalization of ladder auctions of Paul Milgrom, Jon Levin, and Ilya Segal (2013) where a seller is allowed to bid in multiple auctions and sell through the option that provides the seller the highest profit.

When each bidder is committed to one option, the ladder auctions will be strategy-proof for bidders. But if we consider a larger game where a bidder begins with choosing an option and then bids in the auctions, the choice of a relinquishment option can involve strategic consideration. For example, if there are many stations who wish to move to the VHF band, then, a station has to offer a very low price to move to the VHF band. Then it might be that the station may wish to go off the air with a higher price. Providing the supplementary phase and allowing multiple offer options will resolve this problem.

When each bidder is allowed to make multiple offers, the original ladder auctions can still have strategic considerations since a seller is not sure how aggressively the seller should bid in the first auction without knowing the information of the second auction. Providing the supplementary phase will resolve this problem and restore strategy-proofness since the seller is guaranteed to make the maximum profit[18].

There are 2 contributions of the paper. First, theoretically, this paper shows that the deferred acceptance auctions can be extended to the practically important economic environment that allows multiple relinquishment options for a seller. It extends the scope of the applicability of Milgrom-Segal (2014) deferred acceptance auctions. A general lesson is that the stability and the deferred acceptance idea are central ideas in market design that can be applicable not just to the matching problem but also to the auction problems[19].

Second, practically, in terms of the FCC incentive auctions setting, the paper shows that the deferred acceptance auctions of Paul Milgrom and Ilya Segal (2014) can be generalized to accommodate the multiple seller relinquishment options. This generalized deferred acceptance auctions can be implemented as “ladder auctions” where sellers can make multiple offers and sell through the option that will provide the seller the highest profit. This will reduce strategic manipulation regarding the choice of auctions at pre-auctions and/or auction stage.

The organization of the paper is as follows. Section 1 describes the reverse auctions economic environments. Section 2 discusses our approach. Section 3 defines generalized deferred acceptance auctions. Section 4 establishes its strategy-proofness. Section 5 studies descending clock auctions. Section 6 studies near-optimality. Section 7 examines weakly group strategy-proofness. Section 8 considers equivalence to simultaneous pay-as-bid auctions. The Online Appendix will discuss the

[18] Furthermore, the choice of simultaneous versus sequential clearing was examined at the design of first FCC auctions (see FCC (1993, 1994a-e) and Evan Kwerel and Greg Rosston (1999)). The FCC initially proposed sequential ascending bid auctions in the NPRM (FCC (1993)). But researchers, including Paul Milgrom, Preston McAfee (1993a,b), and Robert Wilson, argued that simultaneous clearings have advantages in maximizing the amount of information available to bidders, permitting bidders to pursue efficient backup strategies as more information became available during an auction, likely to be perceived as fair, and being favored by the experimental studies (Charles Plott (1997)).

[19] Indeed, Milgrom and Segal (2014) write: “In this paper, we extend that success to new class of auction design problems, connect DA auctions to encompass clock auctions, and reaffirm that the deferred acceptance idea as the basis for many of the most successful mechanisms of market design.” This paper develops this research agenda further by showing that the DA auctions can be extended to more general economic environment. See also Kazumori (2014). I am grateful for a reviewer for pointing out the importance of emphasizing the general theoretical contributions of the paper.

case where the buyer has the buying target and also describes the software implementations.

I. Modelling the Reverse Auctions Economic Environment.

The model of this paper is a generalization of the Milgrom-Segal (2014) model that allows multiple relinquishment options for potential sellers.

The buyer. There is a single buyer in this economic environment. The FCC incentive auctions NPRM Paragraph 37 states that the FCC identifies a set of bidders that would voluntarily relinquish spectrum usage rights and the compensation that each would receive.

Potential Sellers. Let $i = 1, ..., I$ denote potential sellers. Let N be the set of potential sellers. The NPRM paragraph 73 states that full power and class A stations are eligible to participate in the reverse auctions. We consider these sellers in the UHF band[20].

Unit Supply. We assume that each seller has a unit of goods that the seller can choose to relinquish. The NPRM paragraph 90 states that a winning reverse auction bidder relinquishes all usage rights for that channel and would retain no further rights with regard to the channel.

Relinquishment Options. Let $j = 1, ..., J$ be a bid option through which a seller can relinquish the goods. The NPRM paragraph 90 states that the reverse auction shall include 3 relinquishment options for participants. The first option is going off the air. In this case, a winning bidder in the reverse auctions relinquishes all the usage rights with respect to its pre-incentive auctions television channel. The second option is switching to the VHF band. A station contributes a UHF channel in exchange for auction proceeds and the promise of a VHF channel. The station can retain 6MHZ, can still multicast, but will have further interferences in the coverage area. The third option is channel sharing. A station shares the single transmitter and the antenna. Each station is licensed a portion of 6 MHZ channel and retains primary FCC licensee.

Private Values. Let $c_{i,j}$ be the private cost and expenses that seller i incur from selling a good through option j. In FCC reverse auctions, $c_{i,1}$ is the cost of a station from going off the air by forgoing all future revenues from its operation as a TV broadcaster. $c_{i,2}$ refers to the cost of moving to a lower VHF band with some additional adjustment costs due to a limited range of VHF digital signals compared with UHF digital signals. Finally, $c_{i,3}$ involves the cost from moving to a different band and continuing operations while sharing the spectrum with another station.

Preferences. We assume that sellers' preferences are quasi-linear and risk-neutral. Let p_{ij} be the compensation that seller i receives from selling through the jth option. Then seller i has payoff of $p_{ij} - c_{ij}$ when the seller sells the good through the jth option and 0 otherwise.

Interference Constraints. A crucial consideration in the reverse auctions is interferences among TV stations. Harmful interference can occur when two services use the same or adjacent frequencies in the same geographic area. When television stations operate on the same or adjacent channels, television stations are often required to operate in geographically separate areas to avoid interference. Specifically, for the UHF band, rejecting an offer and reassigning the station in the UHF band can create interferences with existing stations in the new UHF band. For the VHF band, accepting an offer so that a station can move to the VHF band can create interferences with existing stations in the VHF band. For both UHF and VHF band, assignment can create interferences with

[20] The decision problem of sellers in the VHF band can be analyzed in a similar manner.

TV whitespace uses (wireless medical telemetry, radio astronomy, wireless microphone, community broadband services, IEEE 802.11AF).

Milgrom and Segal (2014) assume that sellers are single-minded. That is, a seller is assumed to be interested in selling through only one option. Thus we can interpret the Milgrom and Segal model as a case $J = 1$.

II. Generalizing Deferred Acceptance Auctions.

This section discusses our approach of extending deferred acceptance auctions to allow multiple relinquishment options.

Let us begin by reviewing the Milgrom-Segal deferred acceptance auction procedure. It proceeds in the following 3 steps: in the first bid collection process, each seller submits one offer. In the second assignment step, sellers' offers are evaluated with the scoring function. The buyer rejects the offers with the highest score. The scores are updated with the updated set of frozen sellers. The auctions end when all offers have zero scores and these offers are accepted. Finally, the payment for the winner is the (personalized) threshold price.

Then the research question is to generalize the deferred acceptance auctions to the reverse auctions environment with multiple relinquishment options while preserving strategy-proofness, feasibility, and low computational costs.

Let us begin by recalling the idea of the threshold pricing rule. In auctions with the threshold pricing, a seller sells the good if and only if the profit from selling the good is positive when the seller makes truthtelling offers. Thus, threshold auctions can be interpreted as the buyer makes a take-it-or-leave-it offer for each seller with the threshold prices, and the seller with the positive profit will accept the offer.

EXAMPLE 3: *Suppose that the buyer has 1 unit to buy. There are 3 potential sellers with costs 10, 5, and 4. Suppose that these sellers offer truthfully. Then the threshold prices for these sellers are 4, 4, and 5. Then the payoff for these sellers are -6, -1, and 1. Thus the allocation rule is interpreted as allocating to the seller whose provisional profit, that is, (the threshold price) - (the cost), is positive. With the threshold pricing, for all offer profile such that threshold price is greater than the cost, it is a best response to offer truthfully. Also for all offer profiles such that the threshold price is less than cost, it is a best reply to offer truthfully and not to sell the good. Consequently, truthful offers are a dominant strategy. This reformulation extends to the case where the buyer has multiple units of the goods to procure. Suppose that the buyer is now interested in buying 2 units. Then the threshold pricing is 5, 10, and 10. So the payoffs are -5, 5, and 6. Thus buying from the seller with positive payoff leads to an efficient and truthful equilibrium outcome.*

We now extend the idea to allow multiple relinquishment options for a seller. Suppose a seller has multiple options through which the seller can relinquish a unit of goods. Then, one generalization of the previous approach is that the seller makes offers for multiple options, the buyer calculates the threshold price for each seller for each option, and the buyer makes a take-it-or-leave-it offer for each seller and for each option.

EXAMPLE 4: *The buyer wishes to procure goods. Each station has 2 options (going off the air or moving to the VHF band) to relinquish a unit of goods. Suppose there are 3 potential sellers.*

Costs	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Station 1	*10*	*8*
Station 2	*7*	*2*
Station 3	*4*	*3*

Suppose that the interference constraints are such that (1) the buyer can accommodate at most 2 stations in the new UHF band plan. That is, the buyer can buy 1 unit from this option and (2) the buyer can accommodate at most 1 station in the VHF band[21]*.*

Suppose that sellers make truthful offers. Then the threshold prices are:

Threshold Prices	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Seller 1	*4*	*2*
Seller 2	*4*	*3*
Station 3	*7*	*2*

Then the profits (=threshold prices - costs) are given by

Threshold Prices-Costs	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Station 1	*-6*	*-6*
Station 2	*-3*	*1*
Station 3	*3*	*-1*

The assignment is: seller 1 stays in the UHF band, seller 2 moves to the VHF band, and seller 3 goes off the air.

In the previous example, the parameter values are such that each seller has at most 1 relinquishment option that provides a positive profit. But it is possible that a seller has low costs for both options and that the seller can have positive payoffs from selling through multiple options. But the seller cannot sell more than one unit of good. In this case, the buyer conducts a supplementary phase. Allowing a seller to sell through the option with the highest profit will provide the seller the

[21]This assumption is a simplification of the interference constraints. In reality, the number of stations that a UHF band plan can accomodate depends on the assignments of stations in the UHF band. To explain this point, consider some city with 3 TV stations in the UHF band. Station A is located in the center of the city. Station B is located in the south of the city. Station C is located in the north of the city. The FCC plans to reduce the amount of spectrum available to TV stations in the new UHF band plan so that mobile operators can have more spectrum. In this new UHF band, if Station A is assigned to the new UHF band, station B and C have interferences. That is, these two stations need to go off the air or to move to the VHF band. But it may very well be that Station B and C can coexist since B is in the south and C is in the north,. In this case, one station, Station A needs to go off the air or move to the VHF band. As this example shows, the number of stations that need to go off the air or move to another band, is endogenous and cannot be determined ex ante.

incentive to report the cost truthfully.

EXAMPLE 5: *(Continued). In a previous example, suppose that the cost structure is*

Costs	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Station1	*10*	*8*
Station 2	*7*	*3*
Station 3	*4*	*2*

In this example, seller 3 has low costs for both options. Then the threshold prices are:

Threshold prices	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Seller 1	*4*	*2*
Seller 2	*4*	*2*
Seller 3	*7*	*3*

Then the profits are:

Threshold prices - costs	*Relinquishment by Going Off the Air*	*Relinquishment by Moving to the VHF Band*
Station 1	*-6*	*-6*
Station 2	*-3*	*-1*
Station 3	*3*	*1*

Then the seller 3 has a largest profit from going off the air. Thus the final assignments are that seller 1 and 2 stay in the UHF band, and seller 3 goes off the air.

We now need to check whether the assignment after the supplementary phase satisfies interference constraints. Suppose a station has both bids rejected. The station is repacked to the UHF band. Its feasibility is checked at the time the bid for going off the air was rejected. Thus this assignment satisfies the interference constraints. If a station has going off the air bid accepted and that the bid for the VHF band rejected, then the station will go off the air since this option will provide a higher profit. This assignment does not violate the interference constraints in the VHF band. Suppose that a station has a going off the air bid rejected and the bid for the VHF band accepted. In this case, the station will move to the VHF band since it will provide a higher profit. Removing the station from the UHF band will not create new interferences. Also the interferences with stations at the VHF band are already checked at the auction phase. Finally, suppose that a station has both offers accepted. Then the station will sell through the option with the highest profit. If the station turns out to be going off the air, removing stations from the VHF band will not create new interferences at the VHF band. If the station turns out to be or moving to the VHF band, the interference with stations at the VHF band is already checked at the auction phase.

III. Generalized Deferred Acceptance Auctions with the Supplementary Phase.

In generalized deferred acceptance auctions, the buyer will collect bids from sellers for each relinquishment option, calculate a set of sellers that would relinquish the good or move to other bands based on their offers and feasibility constraints, and determine the payment that each seller would receive. In case where there are any seller selected to sell through multiple options, the seller will pick the option that would provide the seller the highest profit.

A. The Bid Collection Procedure.

Generalized deferred acceptance auctions are a single round sealed bid procedure. Each seller, for each relinquishment option, offers the payment that the seller is willing to accept in exchange for relinquishing the goods through the option. Formally, let an offer made by seller $i = 1, ..., n$ for the jth option be $b_{i,j}$. Let the set of possible bids by seller i for option j be $B_{i,j}$. We assume these sets are finite and actually the same across all bidders and options: there exists B such that $B = B_{i,j}$ for all i and j. The profile of bids for the jth option is $b_j = (b_{1,j}, ..., b_{n,j})$.

B. The Auction Phase.

The auction phase determines which bids are acceptable and which bids are rejected. In deferred acceptance auctions, a bid is evaluated with the scoring function that is a function of other bids for that option. In the assignment process, let A_j be the set of active bidders in the auction for the jth option. Let $N \backslash A_j$ be the set of "frozen" bidders in the auction for the jth option. In the going-off-the-air auction, $N \backslash A_j$ is the set of bidders whose bids are rejected and repacked at the UHF band. In the auctions for moving to the VHF band, $N \backslash A_j$ is the set of bidders whose bids are accepted and assigned to the VHF band.

Then the scoring function of seller i for the jth option is $s_i^{A_j} : B_{i,j} \times B_{N \backslash A_j} \rightarrow \mathbf{R}_+$ that is nondecreasing in the first argument. Here the scoring function for seller i for the jth option depends only on the bids for the jth option. In the auctions for going off the air, the scoring function checks interferences with already repacked stations at the UHF band. If there are interferences with existing UHF stations, then the score would be zero, the offer would be accepted, and the station would go off the air. Also in the auction for moving to the VHF band, the scoring function checks interferences with stations already assigned to the VHF band. If there are interferences with existing VHF stations, then the score would be high, the offer would not be accepted, and the station cannot be assigned to the VHF band.

AUCTIONS FOR GOING OFF THE AIR.

The auctions for "the going off the air" option are the deferred acceptance auctions of Paul Milgrom and Ilya Segal (2014). Let this option be $j = 1$. The bids for this option is $b_1 = (b_{1,1}, ..., b_{n,1})$. The scoring function for seller i is $s_i^{A_1}(b_{i,1}, b_{N \backslash A_1,1})$ that depends on seller i's offer for the 1st option and bids for the jth option by sellers already repacked to the UHF band.

The auction at stage t works as follows. Let $A_{1,t} \subseteq N$ be the set of active bidders for option 1 at stage t. Then, let $A_{1,t+1} = A_{1,t} \setminus \arg\max_{i \in A_{1,t}} s_i^{A_{1,t}}(b_{i,1}, b_{N \setminus A_{1,t},1})$. That is, the buyer removes the seller with the highest score. That seller (the station) is repacked at the UHF band.

The auction stops at the first stage t' such that $s_i^{A_{1,t'}}(b_{i,1}, b_{N \setminus A_{1,t'}}) = 0$ for all $i \in A_{1,t'}$. That is, all remaining stations have zero scores and cannot be repacked. Then the set of provisional winner from the first stage auctions for option 1 is $\alpha_1(b_1) = \{i \in N : s_i^{A_{1,t'}}(b_{i,1}, b_{N \setminus A_{1,t'},1}) = 0\}$. Here a "winner" refers to selling the usage right and going off the air. The threshold price for bidder i is $p_{i,1}(b_1) = \sup\{b_{i,1} : i \in \alpha_1(b'_{i,1}, b_{-i,1})\}$. That is, the maximum price that bidder i can offer to sell through the 1st option given the offers by other sellers.

AUCTIONS FOR "MOVING TO THE VHF BAND"

In the auctions for offers for moving to the VHF band, the buyer needs to check interference constraints with the station to the VHF band. Then the auctions start with accepting offers with low scores and stop when it is no longer possible to accept stations because of interference constraints. Specifically, the auctions for the VHF band consider bids $b_2 = (b_{1,2}, ..., b_{n,2})$ for the option $j = 2$. Let $s_i^{A_2}(b_{i,2}, b_{N \setminus A_2,2})$ be the score of bidder i for option $j = 2$ when the set of active bidders is $A_2 \subseteq N$.

The operation at stage t is as follows. Let $A_{2,t} \subseteq N$ be the set of active bidders for option 2 at stage t. Then, let $A_{2,t+1} = A_{2,t} \setminus \arg\min_{i \in A_{1,t}} s_i^{A_{2,t}}(b_{i,2}, b_{N \setminus A_{2,t},2})$. That is, the buyer removes the seller with the lowest score. That seller (the station) will be assigned to the VHF band.

The auction stops at the first stage $t"$ where $s_i^{A_{2,t"}}(b_{i,2}, b_{N \setminus A_{2,t"},2}) > 0$ for all $i \in A_{2,t"}$. That is, the auctions end when all remaining bidders have interferences with the already assigned stations $N \setminus A_{2,t"}$ so that they cannot be assigned to the VHF band.

The set of provisional winners from the auction for moving to the VHF band is $\alpha_2(b_2) = \{i \in N : s_i^{A_{2,t"}}(b_{i,1}, b_{N \setminus A_{2,t"},2}) = 0\}$. That is, a winner is a seller (a station) whose offers are so low that they can be assigned to the VHF band. The threshold price for bidder i is $p_{i,2}(b_2) = \sup\{b_{i,2} : i \in \alpha_2(b'_{i,2}, b_{-i,2})\}$.

The above auction for moving to the VHF band can be implemented as deferred acceptance by taking the negative (or the inverse) of the scoring function defined above.

C. The Supplementary Phase.

The above auction phases make provisional assignments of stations for each band based on bids for each band. The auction phase does not take into account of the quantity constraint that each seller can supply only one unit of good. Then the supplementary phase determines the final allocation.

Let $\alpha_j(b_j)$, $j = 1, ..., J$ be the set of provisional winners for each option. If $\alpha_j(b_j) \cap \alpha_{j'}(b_{j'}) = \phi$ for all j and j', then, there are no overlaps of winners across auctions. In this case, the provisional winners are the final winners. Otherwise, there exists some sellers who are winners in multiple auctions. Let i' be a such seller and let $j_1, .., j_k$ be options such that $i' \in \alpha_{j_1}(b_{j_1}), ..., i' \in \alpha_{j_k}(b_{j_k})$. Then, seller i' is assigned to the option j^* such that $j^* \in \arg\max_{j' \in \{j_1,...,j_k\}}(p_{i',j'} - b_{i',j'})$. That is,

j^* is the option that provides the highest profit for seller i'. By construction, $\max_{j' \in \{j_1,\ldots,j_k\}}(p_{i',j'} - b_{i',j'}) \geq 0$ and the seller has nonnegative profits.

IV. Strategy-Proofness of Generalized Deferred acceptance Auctions with the Supplementary Phase.

In the generalized deferred acceptance auctions, sellers bid for multiple options to sell. Since the buyer would buy through the options that would provide the highest payoff for a seller and a seller's offer cannot affect the threshold price, the seller's best response is to make offers truthfully.

PROPOSITION 1: *In the reverse auctions economic environment, generalized deferred acceptance auctions with the supplementary phase are strategy-proof.*

The result provides a generalization of the strategy-proofness result of deferred acceptance auctions with single-minded bidders to the environment with bidders who have multiple offers and have unit supply.

PROOF:

Let us consider seller i's offer for the jth option. We would like to show that player i prefers to offer truthfully against for every other sellers' offers.

Consider the case where seller i has the highest provisional profits with the jth option among all options when the seller i bid truthfully. In this case, seller i sells from truthtelling and obtains the positive profit. Even if seller i increases the offer, it will not affect the outcome. If the seller reduces the offer, since seller i may not be able to sell through the jth option anymore, the profit may decrease.

When seller i does not have the highest provisional profits with the jth option with truthtelling, seller i does not sell from the jth option with truthtelling. Lowering an offer may lead to an outcome of selling through the jth option that will reduce the payoff. If the seller increases the offer, the outcome will not change.

Consequently, truthtelling is a best response for seller i even in this case. □

V. Implementations in Simultaneous Descending Clock Auctions.

The previous section presented simultaneous in a single round sealed bid procedure. Incentive Auctions NPRM paragraph 39 states that the second option for reverse auctions is a multiple round or dynamic process in which bidders indicate the willingness to accept lower payment in exchange for relinquishing rights. This section provides such a dynamic implementation of the generalized deferred acceptance auctions. The dynamic mechanism is a combination of simultaneous descending clock auctions and the supplementary phase.

A. Simultaneous Descending Clock Auctions with the Supplementary Phase.

The first stage can be considered as running descending clock auctions of Paul Milgrom and Ilya Segal (2014) for each option.

Let $A_{j,t} \subset N$ be the set of active bidders for auction j at stage t. The profile of set of active bidders for all auctions is $A_t = (A_{1,t}, ..., A_{J,t})$. Of course the set of active bidders for auction j can be different from the set of active bidders for auction j' for $j \neq j'$.

The history of auction j at stage t is $A^t_j = (A^1_j, ..., A^t_j)$. It is the sets of active bidders for auction j up to stage t. The history of auctions $j = 1, ..., J$ is $A^t = (A^t_1, ..., A^t_J)$. The set of all histories for auction j is denoted by H_j.

The price clock for the jth auction is a mapping $p_j : H_j \to \mathbf{R}^N$ with the property that, for each seller i, $\underbrace{p_{i,j}}_{\text{personalized price clock for seller } i \text{ in auction } j}(A^t_j) \leq p_{i,j}(A^t_j)$. That is, the price clock is decreasing over time.

Then the simultaneous descending clock auctions proceed as follows: At the first stage, we have $A^1_j = N$ for all j. That is, in the initial stage, the set of active bidders is the whole set of bidders. At each period t, for each auction j, given history A^t_j, each seller i is offered $p_{i,j}(A^t_j)$. When $p_{i,j}(A^t_j) < p_{i,j}(A^{t-1}_j)$, seller i can decide whether to stay in the auction or exit (and repacked). For each auction j, let $E_{j,t} \subseteq A_{j,t}$ be the set of bidders who choose to exit. Then the set of active bidders is $A^{t+1}_j = A_{j,t} \backslash E_{j,t}$. Auction j ends at stage t when $p_{i,j}(A^t_j) = p_{i,j}(A^{t-1}_j)$ for all $i \in A^t_j$, that is, when the price clock stops. Then the active bidders $i \in A^t_j$ will be provisional sellers with price $p_{i,j}(A^t_j)$[22].

When there are sellers who are selected as a provisional seller in multiple auctions. the supplementary phase will assign such sellers to sell through the auction with that provides the highest profit to the seller.

B. *Equivalence with the Simultaneous Descending Clock Auctions.*

This subsection shows that the above simultaneous descending auctions with the supplementary phase can implement the generalized deferred acceptance auctions by defining correspondences between the bid scoring rule and the price reduction rule.

As defined by Milgrom and Segal (2014), the cutoff strategy is a strategy $b_{i,j}$ such that bidder i exits from the auction if $p_{i,j}(A^t) < b_{i,j}$. Then we have

PROPOSITION 2: *For any generalized deferred acceptance auction with a finite set of bids, there exists an equivalent simultaneous clock auctions with cutoff strategies.*

The result implies that the generalized deferred acceptance auctions can be implemented with descending clock auctions, as in the original deferred acceptance auctions.

PROOF:

For each auction j, as in Milgrom and Segal, construct a price clock that reduces the prices for sellers with the highest score by the minimum price increments while keeping the price clock fixed for other sellers. Then, for every history, the next set of sellers to quit is the set of bidders with the maximum score. Thus the outcome of the auction phase in deferred-acceptance heuristic auctions

[22]We will impose activity rules that a seller who has dropped out an auction cannot come back to that auction. I thank Kevin-Leyton Brown for comments.

and the outcome simultaneous descending clock auctions are equivalent. Then the supplementary phase will lead to the equivalent outcome of the whole auction process. □

VI. Near-Optimality of Generalized Deferred Acceptance Auctions

Deferred acceptance auctions determine allocation based on scores. The algorithm does not fully optimize the allocation subject to interference constraints because of its computational burden. Then, it is significant to understand the efficiency properties of the algorithm.

Milgrom and Segal (2014) use the fact that the interference graph among stations is known before the auction. Furthermore, these stations can be partitioned into cliques (“metropolitan areas"). Then, Paul Milgrom and Ilya Segal (2014) show that there exists a lower bound on the number of stations such that removing more than this number of stations at each clique will make any assignment, especially an efficient assignment, feasible without violating interference constraints. That is, it is possible to construct a scoring rule and deferred acceptance auctions, that is based on the prior knowledge of that interference graph, such that the resulting allocation assigns the stations to the channels in the most efficient way subject to these constraints.

The Milgrom-Segal (2014) result concerns the deferred acceptance auctions where a seller can only make one offer. In contrast this paper considers its generalization to allow sellers to make multiple offers for multiple relinquishment options, Thus it would be significant to understand whether these efficiency properties extend to generalized deferred acceptance auctions.

The generalized deferred acceptance auctions with the supplementary phase proceed in 2 steps. In the first step, the algorithm matches stations to channels in each of the UHF and VHF band separately. In the second step, the algorithm adjusts the allocation to accommodate the quantity constraints that each seller can sell only one usage right. Then the first step of analysis is to consider assignments of UHF stations to the UHF band and also to the VHF band separately and derive their efficiency bounds. Then in the second step we modify the scoring function assignment of stations in a way that will be improving efficiency of allocations.

Recall that N is the set of UHF stations. For each station s, s', let $\{Z(s, s')\}_{s,s' \in N}$ be the interference matrix where $Z(s, s') = 1$ implies that two stations are geographically so close that they cannot be assigned to the same channel. Let n_{UHF} and n_{VHF} be the number of channels in the UHF and VHF bands, respectively.

Let $\{1, ..., M\}$ be a collection of cliques (metropolitan areas). An interpretation is that each station belongs to one of these metropolitan areas. We assume that these metropolitan areas are common to both stations in the UHF band and the VHF band since these locations are driven by economic considerations. Let $m : N \to \{1, ..., M\}$ be a partition of UHF stations into the metropolitan areas. We assume that for any two stations s and s' that are in the same metropolitan areas, $Z(s, s') = 1$ so that they cannot be assigned to the same channel.

Assignment of UHF stations to the UHF band. This corresponds to the “Going Off the Air Auctions”. Let $S \subset N$ be a collection of stations in the UHF band. Let $c : S \to \{1, ..., n_{UHF}\}$ be an assignment of stations into the UHF channels.

Then an idea is that if we remove sufficiently many stations from each metropolitan area (i.e. these stations will go off the air), then the interference constraints among stations will no longer be binding. Specifically, let

$$F^k_{mn,UHF} = \{S \subset N_{UHF} : \text{ for each station } j, |S \cap m^{-1}(j)| < n_{UHF} - k\}.$$

That is, $F^k_{mn,UHF}$ is a set of stations in the UHF band where at each metropolitan area j, there are less than $n_{UHF} - k$ stations.

Then the Milgrom-Segal theorem says that there exists a lower bound k_{UHF} such that, for each $k \geq k_{UHF}$, there exists a one-to-one assignment between stations in $F^k_{mn,UHF}$ and UHF channels in all metropolitan areas that are feasible. Furthermore, there exists scoring functions and the deferred acceptance auctions algorithm such that, for any set of bids, the algorithm assigns at least $n - d_{mn,UHF}(Z)$ most valuable stations in each partition $m^{-1}(j)$ and that the efficiencies are at least $1 - d_{mn,UHF}(Z)/n$.

Assignments of UHF stations to the VHF band. Similarly, consider the assignment of UHF stations into the VHF band that corresponds to the Moving to the VHF band auctions. As in the above case, we can construct a lower bound k_{VHF} such that, for each $k > k_{VHF}$, there exists a one-to-one assignment between stations in $F^k_{mn,VHF}$ and VHF channels, and a deferred acceptance auctions algorithm such that the algorithm assigns at least $n - d_{mn,VHF}(Z)$ most valuable stations in each partition $m^{-1}(j)$ and that the efficiencies are at least $1 - d_{mn,VHF}(Z)/n$.

Since the generalized deferred acceptance auctions can improve the allocation over each auctions with an appropriate choice of scoring functions, it assigns at least $n - \max\{d_{mn,UHF}(Z), d_{mn,VHF}(Z)\}$ most valuable stations in each partition and the efficiencies are at least $1 - \max\{d_{mn,UHF}(Z), d_{mn,VHF}(Z)\}/n$.

In the supplementary phase, a seller's choice among channels depends on profits that a seller obtains from each auction. But we can still construct scoring functions that will essentially restrict a seller's choice to the one that improves efficiency. (For example, if it is more desirable from a viewpoint of a buyer to assign a station to going off the air rather than the VHF band, then the buyer can develop a scoring function that would artificially create an interference with the stations in the VHF band that will prevent the stations to be assigned to the VHF band.) Thus we have,

PROPOSITION 3: *There exists a generalized deferred acceptance auction algorithm such that, for any set of bids, assigning at least the* $n - \max\{d_{mn,UHF}(Z), d_{mn,VHF}(Z)\}$ *stations in each metropolitan area and the efficiencies are at least* $1 - \max\{d_{mn,UHF}(Z), d_{mn,VHF}(Z)\}/n$.

This result implies that a near-optimality property of deferred acceptance auctions of Milgrom and Segal (2014) generalizes to generalized deferred acceptance auctions. This generalization is possible due to the fact that a seller has only unit supply although they can make offers for multiple relinquishment options.

VII. Weakly Group Strategy-Proofness.

Milgrom and Segal (2014) further consider robustness of deferred acceptance auctions to group deviations. Milgrom and Segal (2014) employ a notion of *weak group strategy-proofness* which says, for every profile of values c, for every set of players $S \subset N$, and for every strategy profile σ_S of these players, at least one bidder in S has a weakly higher payoff from the profile of truthful bids c than from the strategy profile of players in a set S players according to the strategy profile σ_S and other players behaving truthfully. That is, for any group deviation, there is at least one seller who does not strictly gain from the group deviation. Milgrom and Segal (2014) show that the deferred acceptance auctions are weakly group strategy proof. Indeed, in generalized deferred acceptance auctions, the seller's profit is the maximum of the profits from each auction. Thus, when a seller has a deviation from one auction, a seller prefers to use this deviation in that auction when the seller chooses the bidding strategy in overall auctions.

PROPOSITION 4: *Generalized deferred acceptance auctions with the supplementary phase is weakly group strategy-proof.*

VIII. Equivalence to Simultaneous Pay-as-Bid Auctions.

The previous sections consider generalized deferred acceptance auctions with the supplementary phase and study their properties. An alternative method is pay-as-bid auctions. Paul Milgrom and Ilya Segal (2014) compare these two mechanisms and derive the equivalence between these two auction mechanisms: the pay-as-bid auctions are dominance solvable and a Nash equilibrium of the auction game is the equivalent to the one in the deferred acceptance auctions. The intuition developed by Paul Milgrom and Ilya Segal (2014) is that an iterated deletion of weakly dominated strategies is closely related to the deletion of least favorable offers in the deferred acceptance algorithm. These equivalence properties imply that deferred acceptance auctions encompass another important class of auction mechanisms.

Then a question is whether these equivalence properties extend to generalized deferred acceptance auctions with the supplementary phase. An intuition is that these auctions are a combination of deferred acceptance auctions and the supplementary phase where the seller chooses the most favorable outcome. Thus we can extend the process of iterated deletion of weakly dominated strategies to the two-stage mechanism.

We consider a following version of simultaneous pay-as-bid auctions with the supplementary phase. Each seller makes offers for multiple auctions. For each auction, offers with the lowest scores are accepted subject to quantity and interference constraints and the winner pays its own bid in contrast to the threshold prices in deferred acceptance auctions. If a seller wins multiple auctions, the seller picks the auction that provides the highest profit for the seller. This auction generalizes the pay-as-bid auctions considered in Paul Milgrom and Ilya Segal (2014) by allowing sellers to make multiple offers while the seller can sell only one unit of supply.

Recall $j = 1, ..., J$ denote relinquishment options. Let $b_j = (b_{1,j}, ..., b_{n,j})$ be a profile of bids for the jth option by sellers $1, ..., n$. Let us first consider auctions for the jth option. Let $p_i(b_{-i,j}) = \max\{b'_{i,j} \in B : i \in \alpha(b'_{i,j}, b_{-i,j})\}$ be the threshold price for player i given $b_{-i,j}$. That is, $p_i(b_{-i,j})$ is the maximum amount that i can offer to sell through the jth option. Then,

PROPOSITION 5: *In simultaneous pay-as-bid auctions with the supplementary phase, in a complete information environment, for each player i, for each option j, bidding $b_{i,j} = \max\{c^{+}_{i,j}, p_i(b_{-i,j})\}$ forms a Nash equilibrium.*

That is, in simultaneous pay-as-bid auctions with the supplementary phase, it is a Nash equilibrium strategy to bid according to the Milgrom-Segal equilibrium strategy in each auction. It is because the seller has a unit supply and the seller obtains the maximum payoff from auctions in the supplementary phase.

PROOF:

In a complete information environment, when other players bid according to the bidding strategy $b_{i,j}$, then, a player wins the auction according to $b_{i,j}$ if and only if the profit from the auction is positive. That is, bidding $b_{i,j}$ maximizes the profit from each auction given that other players follow the bidding strategy $b_{i,j}$. Since that a seller's payoff from the whole auction is the maximum profits from all auctions and a bid in one auction does not affect the outcome of other auctions, $b_{i,j}$ is still the optimal response from the viewpoint of the whole auctions. □

We now apply the notion of iterated deletion of weakly dominated strategies to our setting. In the definition of Milgrom and Segal (2014), an auction is *dominance-solvable* in state c if under full information, there exists a payoff profile that is the unique outcome of iterated deletion of weakly dominated strategies, regardless of the order of elimination. Here Milgrom and Segal consider elimination in single deferred acceptance auctions and we now consider multiple deferred acceptance auctions with the supplementary phase.

We also extend the notion of nonbossyness. That is, for each station $i \in N$ and for each option j and for each bid profile b_j, $\alpha_j(b'_{i,j}, b_{-i,j}) \cap \{i\} = \alpha_j(b_{i,j}, b_{-i,j}) \cap \{i\}$ implies that $\alpha_j(b'_{i,j}, b_{-i,j}) = \alpha_j(b_{i,j}, b_{-i,j})$. That is, a seller who is winning cannot affect others' allocations for any option without changing own allocation (i.e. without affecting the threshold prices). We now develop the equivalence result.

PROPOSITION 6: *Consider a simultaneous paid-as-bid auction with the supplementary phase with a monotonic, non-bossy assignment rule. Say that a value profile c is generic if for each* i, $c_i \in [0, v_i]\backslash B_i$. *(1). The auction is pure-strategy dominance-solvable for all generic value profiles if and only if can be implemented via a generalized deferred acceptance auction with the supplementary phase. (2). In this case, for every generic value profile, the unique payoff profile surviving an iterated deletion of dominated strategies is also the unique (pure or mixed) Nash equilibrium payoff profile in undominated strategies. (3). In this case, one strategy profile that survives an iterated deletion of dominated strategies and is a Nash equilibrium in undominated strategies is the one described in Proposition 5.*

Proposition 6 implies that the generalized deferred acceptance auctions are equivalent to corresponding pay-as-bid auctions and that they encompass pay-as-bid auctions, as in the original deferred acceptance auctions.

PROOF:

(Part 1 If part) The proof of the if part proceeds in 2 steps. In Step 1, from deferred-acceptance heuristic auctions, by Proposition 2, we can construct multiple descending clock auctions with the threshold pricing rule with the equivalent outcome. Furthermore, as noted in Paul Milgrom and Ilya Segal (2014), the clock auctions can be implemented by a pay-your-bid pricing rule. In Step 2, we construct a process of an iterated deletion of weakly dominated strategies from each descending clock auctions. For each auction, we apply the process constructed in Milgrom and Segal (2014): at the beginning, the bid with the highest score will be weakly dominated and deleted. It will update the scores and the threshold prices for each bid. Then the bid with the highest updated score will be weakly dominated and deleted. The process ends when all remaining bids have zero scores and are winners. Then, for each seller who wins multiple auctions, every bid other than the bid that will provide the highest profit for the seller will be deleted. Then the final outcome is indeed dominance-solvable.

(Part 1 Only if part) The proof proceeds in 2 parts. In Part 1, since the game is dominance solvable, there exists a process of iterated deletion of weakly dominated strategies. Then, we can construct a sequence of scores and descending clock auctions that assign the highest score to the bid that will be deleted in the first round, second round, ... and in the supplementary phase. Then in Part 2, from these scoring function and descending clock auctions, by Proposition 2, we can construct simultaneous deferred-acceptance heuristic auctions with the supplementary phase.

(Part 2 and 3.) Consider a undominated mixed strategy Nash equilibrium. We proceed in 2 steps. In the first step, by the application of the Milgrom and Segal (2014) argument, the mixed strategy equilibrium has to put probability 1 on a single bid and the bid must be on $b_{i,j}$. Step 2 consider the supplementary phase. Since the profit from the whole auction is the maximum of payoffs from auctions, bidding strategy $b_{i,j}$ is not strictly dominated.

The proof of Part 3 follows from the above results.

IX. Conclusion.

This paper studies the design of reverse auctions in the US incentive auction process. The state of the art is the generalized deferred acceptance auctions of Paul Milgrom and Ilya Segal (2014). We consider its extensions to allow stations to make multiple offers. A significance of this research question is that allowing multiple options can increase efficiency, revenue, and encourage participation in auctions. A main difficulty is that just running simultaneous auctions is not feasible since seller switching across auctions will create interferences with stations already assigned to the band.

There are 4 results in the paper. We propose generalized deferred acceptance auctions with the supplementary phase. Since a seller can make multiple offers in multiple auctions, the seller does not switch across auctions in the middle. First, the auctions are strategy-proof because of the threshold pricing and also because the seller can maximize the profits from auctions. Second, the auctions can be implementable. in descending clock auctions with the supplementary phase by developing a correspondence between the scoring function and the descending price clock. Third, auctions are nearly-optimal, weakly group strategy-proof, and equivalent to pay-as-bid auctions. Finally, the generalized deferred acceptance auctions inherit favorable structural properties of the original deferred acceptance auctions. It is because of the environment specific properties that a seller has a unit supply.

The contribution of the paper is the strengthening of the results of Paul Milgrom and Ilya Segal (2014) by resolving the conflict between switching and interference constraints. This generalization can be useful in related reallocation and sharing of the government spectrum and other complex allocation problems. A future research agenda is to validate these theoretical results by conducting experiments among human bidders[23][24].

[23]Belch and Kazumori (2013a, b) develop the software prototype. Kazumori (2005, 10) conduct experimental studies of package auctions.

[24]A long-term research and policy for spectrum allocation will include dynamic spectrum allocation and open/hierachical access

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