Bank Runs, Fire-Sales, and Equity Injections

Roberto Robatto,* University of Wisconsin-Madison

December 30, 2015

PRELIMINARY AND INCOMPLETE

Abstract
I present a general equilibrium model of banking in which financial crises can be due to panics (i.e., coordination failures that give rise to multiple equilibria) or fundamentals. Crises are systemic, in the sense that many banks are insolvent and subject to runs. A financial crisis is also characterized by a flight to safety-liquidity, and by sales of assets by banks at a depressed price, that is, a fire-sale. Fire-sales arise despite all agents have the same ability in managing assets (contrary to the common assumption of having “nonspecialists” who have less expertise in managing assets) and despite no liquidation cost is paid when an asset is sold.

Motivated by the large intervention of the US government in October 2008, I use the model to study equity injections into banks. Large equity injections eliminate a banking crisis. However, if the policy intervention is not large enough, it amplifies the flight to safety-liquidity and (preliminary results show that it also) reduces welfare.

---

*E-mail: robatto@wisc.edu. I am grateful to Briana Chang, Dean Corbae, and participants to seminars at the University of Wisconsin-Madison for comments.
1 Introduction

During the 2008 Columbus-day weekend, at a critical moment during the 2008 financial crisis, the US government announced the largest ever US intervention in the financial sector. The CEOs of the biggest US financial institutions were called for a meeting and persuaded to accept equity injections into their banks. The interventions in October 2008 and in the following months contributed to the recapitalization of the US banking system, and by early 2009 the acute phase of the crisis was over.

According to an empirical analysis by Veronesi and Zingales (2010), the intervention in October 2008 had a net benefit on the economy of $86-$109 billion, but costed taxpayers an estimate of $21-$44 billion.

A rationale for such large intervention targeting even institutions that looked solvent and well-capitalized (such as JPMorgan Chase at the time) is often provided by a comparison with other episodes of financial crises. Looking at Japan, Hoshi and Kashyap (2010) draw two lessons about banks’ recapitalization. First, they argue that the rescue package must be large, because acute banking crises are typically large events. Moreover, a well-functioning financial sector is crucial for growth, thus a full recapitalization of banks is required to avoid spillovers on the real economy. Second, they point out that banks might refuse equity assistance for fear of admitting larger losses than had previously disclosed.

This paper takes a complementary approach by analyzing these issues in a novel theoretical model of banking crises. Depending on parameter values, there exists either a good equilibrium only, or a bad equilibrium only, or multiple equilibria. Thus, the model encompasses both crises driven by fundamentals and crises driven by panics, in the sense of coordination failure. The paper provides two results related to the recapitalization of banks. First, small equity injections are not only unable to resolve a crisis, but they even reduce welfare.1 Second, the model provides a framework in which the need of a government intervention at the height of a crisis and the concern that banks might refuse equity injection can be rationalized.

A ultimate objective, although outside the scope of this paper, is to analyze the trade-off between the benefits of ex-post equity injection and the moral hazard that this intervention might create ex-ante. The framework that I develop in this paper is particularly important

---

1So far, I have shown the reduction of welfare only using numerical examples. However, since the bad equilibrium can be easily solved in closed form, the result can be generalized and a welfare analysis can be undertaken more formally; I plan to do that in the next revision of the paper.
because of non-monotonic effects of equity injections. On the one hand, a large intervention is ex-post beneficial, but ex-ante costly if it creates moral hazard. On the other hand, a commitment to a smaller interventions in the event of a crisis reduces moral hazard ex-ante, but might deteriorate a crisis ex-post.

While the model uses a structure of preferences similar to the one used by Diamond and Dybvig (1983) and shared by most of the banking literature, some of the assumptions about technology and endowments are crucially different. As a result, the channel that gives rise to multiple equilibria is also different from Diamond and Dybvig (1983), and is instead related to a strategic complementarity in the decision to hold liquidity. The main features of the model and some of the results are related to a companion paper, Robatto (2015), although the model in Robatto (2015) is a monetary version of the one presented here and is thus used to analyze monetary policy during financial crises.2

In standard models of bank runs, the investment technology has a fixed return and can be scaled up or down at no cost. In my model, I take the opposite assumption: the investment technology is represented by capital in fixed supply.3 If depositors believe that their own bank might be subject to a run, they fly to liquidity in order to self-insure against future liquidity needs.4 The flight to liquidity has two effects. First, depositors reduce the demand for capital in order to increase their demand for liquidity; but due to the fixed supply of capital, the price of capital drops in equilibrium. Second, the flight to liquidity reduces the resources intermediated by banks, thereby forcing banks to sell capital at a depressed price, a fire-sale. Fire-sales require banks to have assets in the first place (otherwise they have nothing to sell); this is where the other departure from standard models of runs has a role. In the model, banks have some endowment in the initial period, including liabilities (pre-existing deposits), contrary to typical models in which banks have either no endowments or only endowment of assets. Finally, the sale of assets by banks at a depress price is responsible for the insolvency of the weakest banks in the economy, which are then subject to runs, confirming the initial belief of depositors.

---

2 There are also some other important differences. In Robatto (2015), deposits are denominated in nominal terms and the bad equilibrium is associated with deflation instead of fire-sales. Thus, that model relies on debt-deflation. Debt-deflation arises from the fact that deposits are denominated in nominal term, therefore deflation increases their real value. Since deposits are liabilities for banks, deflation contributes to distress in the banking sector. None of these effects is present in this paper.

3 This is an extreme form of technological illiquidity, whose importance is emphasized by e.g. Brunnermeier and Sannikov (2014).

4 In a bad equilibrium, depositors form beliefs about the probability of a bank run. Despite a similarity with Goldstein and Pauzner (2005), the results are different. The assumptions about technology and endowment in Goldstein and Pauzner (2005) build on Diamond and Dybvig (1983), while I depart from those assumption.
The model has three periods, \( t = 0, 1, 2 \), and the preference shocks (early vs late consumers) is realized at \( t = 1 \). The bad equilibrium requires a timing in which the flight to liquidity, fire-sales, and insolvencies happen at \( t = 0 \), while runs happen at \( t = 1 \). Thus, to make sure that depositors do not run on insolvent banks at \( t = 0 \), I also assume temporary asymmetric information about the balance sheet of banks. At \( t = 0 \), asymmetric information about the balance sheet of banks prevent depositors from telling apart solvent and insolvent banks. At \( t = 1 \) information about the balance sheet of banks is exogenously revealed, and insolvent banks are finally subject to runs.

Whether or not multiple equilibria exists depend on the initial endowment of banks. If the initial endowment of banks is large, only a good equilibrium exists. If the initial endowment is low, only a bad equilibrium exists. For intermediate values, multiple equilibria exists.

Notably, fire-sales arise in the model despite all agents have the same ability at managing assets (contrary to the common assumption of having “nonspecialists” who have less expertise in managing the asset, as in the financial fire-sales theory described by Shleifer and Vishny (2011) and widely used in macroeconomics and finance) and despite no liquidation cost is paid when an asset is sold (as it is typically assumed in models of bank runs). Thus, another contribution of the paper besides the analysis of equity injections is to provide a deeper foundation for a theory of financial assets fire-sales.\(^5\)

2 Model

The economy is populated by a unit mass of banks indexed by \( b \in \mathbb{B} \equiv [0, 1] \), a double continuum of households indexed by \( h \in \mathbb{H} = [0, 1] \times [0, 1] \), and a government. Superscripts \( h \) and \( b \) refer to household \( h \) and bank \( b \).

There are three time periods denoted by \( t \in \{0, 1, 2\} \).

2.1 Preferences

Household \( h \in \mathbb{H} \) enjoys utility from consumption \( C^h_1 \) and \( C^h_2 \) at period \( t = 1 \) and \( t = 2 \):

\[
\mathbb{E} \left[ \tilde{u} \left( C^h_1 \right) \right] + \beta C^h_2
\]

\(^5\)See also Guerrieri and Shimer (2014) for another theory of endogenous fire-sales.
where \(0 < \beta < 1\). The functional form of \(\tilde{u}(\cdot)\) depends on the realization of a preference shock, at the beginning of \(t = 1\):

\[
\tilde{u}(\cdot) = \begin{cases} 
\bar{u}(\cdot) & \text{(impatient) with probability } \kappa \\
u(\cdot) & \text{(patient) with probability } 1 - \kappa 
\end{cases}
\]  

where \(\tilde{u}(\cdot)\) and \(u(\cdot)\) are defined by:

\[
\tilde{u}(C) = \begin{cases} 
\theta C & \text{if } C < \overline{C} \\
\theta \overline{C} + (C - \overline{C}) & \text{if } C \geq \overline{C}
\end{cases}, \theta > 1
\]  

\[
u(C) = C
\]  

The preference shock is private information of household \(h\), is i.i.d. across households, and the law of large numbers holds for each subset of \(\mathbb{H}\) with a continuum of households.

Each bank is run by a banker with linear utility in consumption \(C_b^2\) at time \(t = 2\).

### 2.2 Technology

There is a fixed supply \(K\) of capital in the economy.

At \(t = 0\), each unit of capital produces \(A\) units of output. After production, capital and output can be traded in a Walrasian market, and the price of capital is \(Q_0\) (the price of output is normalized at one). Additionally, each period, there is a storage technology (each unit of output stored at \(t\) is available at \(t + 1\), with no net return).

At \(t = 1\), there is no market. Households have only access to their own bank (as described later). Output produced at \(t = 0\) is the source of consumption good at \(t = 1\).

At \(t = 2\), each unit of capital produces \(R\) units of consumption good.

### 2.3 Endowments

At \(t = 0\), each household \(h\) is endowed with \(K_{-1}^h\) units of capital, and \(D_{-1}^h\) deposits at a bank \(b(h)\).\(^6\) The endowment of deposits \(D_{-1}^h\) is denominated in consumption goods, i.e., it is a promise by bank \(b(h)\) to pay \(D_{-1}^h\) units of consumption goods on demand.

\(^6\)Formally, \(b(h)\) is an exogenous function that maps the set of households \(\mathbb{H}\) into the set of banks \(\mathbb{B}\), so that each household has initial deposits at one bank.
Each bank $b \in B$ is endowed with $K^b_{-1}$ units of capital and with an obligation to pay $D^b_{-1}$ units of money to a unit continuum of households denoted by $H(b) \subset H$, as represented in Figure 1.

Banks are heterogeneous in their endowment of capital:

$$K^b_{-1} = \begin{cases} K^L & \text{for a fraction } \alpha \text{ of banks} \\ K^H & \text{for a fraction } 1 - \alpha \text{ of banks} \end{cases}, \quad K^H > K^L > 0$$

and, at $t = 0$, they have private information about their own endowment of capital.

**Assumption 2.1. (Private information about banks’ capital endowment)** At $t = 0$, the endowment of capital $K^b_{-1}$ of each bank is private information of the respective bank; other agents in the economy only know:

$$Pr \left( K^b_{-1} = K^L \right) = \alpha$$

$$Pr \left( K^b_{-1} = K^H \right) = 1 - \alpha;$$

at $t = 1$, the value of $K^b_{-1}$ of each bank becomes common knowledge and the informational asymmetry is resolved.\(^7\)

For simplicity, there is no heterogeneity in the endowment of deposits across banks, and there is no heterogeneity in endowment across households.\(^8\)

Endowments satisfy $\int K^h_{-1} dh + \int K^b_{-1} db = K$ and $\int D^h_{-1} dh = \int D^b_{-1} db.$
2.4 Budgets

Initial net worth. Given the price of capital $Q_0$, the balance sheet of a bank $b$ at the beginning of $t = 0$ is:

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of capital</td>
<td>Value of deposits</td>
</tr>
<tr>
<td>$= K^b_{-1}Q_0$</td>
<td>$= D^b_{-1}$</td>
</tr>
<tr>
<td>Output</td>
<td>Net worth</td>
</tr>
<tr>
<td>$= AK^b_{-1}$</td>
<td>$= N^b_0$</td>
</tr>
</tbody>
</table>

where net worth is the difference between the value of assets and the value of deposits:

$$N^b_0 = K^b_{-1} (A + Q_0) - D^b_{-1}. \quad (5)$$

If $N^b_0 \geq 0$, the bank is solvent (the value of its assets is larger than deposits $D^b_{-1}$). If $N^b_0 < 0$, the bank is insolvent (the value of its assets is less than deposits $D^b_{-1}$). Note that a bank with negative net worth can be active in equilibrium because of asymmetric information about its holdings of capital.

Using a similar logic, let $N^h_0$ be the net worth (or wealth) of household $h$:

$$N^h_0 = K^h_{-1} (A + Q_0) + D^h_{-1}. \quad (6)$$

The difference between the net worth of banks $N^b_0$ and the net worth of households $N^h_0$ is that deposits are liabilities for banks, but they are assets for households. Since banks and households takes the price $Q_0$ as given, the net worth $N^b_0$ and $N^h_0$ summarize, respectively, the value of bank $b$’s endowment and household $h$’s endowment.

$t = 0$: trading and deposits. At $t = 0$, household split their net worth $N^h_0$ across storage $S^h_0$, deposits, and capital:

$$S^h_0 + D^h_0 + Q_0 K^h_0 \leq N^h_0. \quad (7)$$

subject to the non-negativity constraints $S^h_0 \geq 0$, $D^h_0 \geq 0$, and $K^h_0 \geq 0$. While there is perfect competition in the banking sector, I impose the restriction that households can hold
deposits at no more than one bank.\footnote{Assumption 2.2 can be justified by costs of maintaining banking relationships. Formally, the cost would be zero if household $h$ holds deposits at one bank, and infinite if household $h$ holds deposits at two or more banks. Assumption 2.2 can be relaxed, but it is crucial that households cannot hold deposits at a large number of banks.}

**Assumption 2.2. (One bank per household)** Each household $h \in \mathbb{H}$ can hold deposits (at most) at one bank.

Without loss of generality, I assume that households keep banking with the same bank $b (h)$ with which they had initial deposits $D_h^h$.\footnote{Perfect competition in the banking sector implies that, in equilibrium, households will be indifferent between using any bank.}

At $t = 0$, bank $b$ can store goods $S^b_0$, and hold capital $K^b_0$. The bank finances itself with its net worth $N^b_0$ and with deposits $D^b_0$:

$$\underbrace{S^b_0}_{\text{storage}} + \underbrace{K^b_0 Q_0}_{\text{capital}} \leq \underbrace{N^b_0}_{\text{net worth}} + \underbrace{D^b_0}_{\text{deposits}} \tag{8}$$

subject to the non-negativity constraints $S^b_0 \geq 0$, $K^b_0 \geq 0$, and $D^b_0 \geq 0$. Moreover, $K^b_0$ is also private information of bank $b$ (otherwise, if $K^b_0$, $S^b_0$, and $D^b_0$ were all observable, it would be possible to figure out the value of net worth $N^b_0$ and of endowment $K^{b - 1}_1$).

To clarify the notation and the timing, note that household $h$ starts with endowment $D_h^h$ of deposits, and bank $b$ starts with endowment $D^b_1$ of deposits. The choice of deposits $D^h_0$ taken by household $h$ at $t = 1$ is a decision regarding rolling over preexisting deposits $D_{-1}^h$ (fully or partially) and/or increasing deposits. For instance, if $D^h_0 = D^h_{-1}$, then the dollar value of household $h$’s bank account remains constant. For bank $b$, the difference $D^b_0 - D^b_{-1}$ is the net issuance of deposits. If $D^b_0 > D^b_{-1}$, bank $b$ increases its deposits and thus receives new resources from households. Otherwise, bank $b$ reduces its amount of preexisting deposits and must pay back some resources to households.\footnote{To describe precisely the interaction between banks and depositors, I must specify what happens if many preexisting deposits are not rolled over at $t = 0$ and the bank does not have enough resources to repay them, i.e., the bank does not have enough endowment. If such circumstance occurs, the bank is shut down immediately and depositors get pro-rata repayments.}

$t = 1$: withdrawals and consumption  \quad$ At $t = 1$, households learn the realization of their own preference shock that determines the functional form of $\tilde{u} (\cdot)$. They then decide
withdrawals \( W^h_1 \) from their own bank subject to a sequential service constraint,\(^{12}\) and then consumption \( C^h_1 \) subject to a cash-in-advance constraint.

Banks use the goods \( S^b_0 \) that they stored at \( t = 0 \) to pay out withdrawals. Banks do not make any economic decisions at \( t = 1 \); the amount withdrawn by depositors of bank \( b \) is:

\[
W^b_1 = \int_{\mathcal{H}(b)} W^h_1 dh.
\]

Withdrawals \( W^h_1 \) are limited by the feasibility constraint \( W^h_1 \leq S^b_0 \), because at \( t = 1 \) there is no market in which banks can sell capital in exchange for consumption good.

In the event of large withdrawals from a bank, the bank might not have enough goods to serve all households. Formally, let \( l^h_1 \in \{0, +\infty\} \) be a limit on withdrawals determined by the position in the line. Then, withdrawals \( W^h_1 \) are constrained by:

\[
0 \leq W^h_1 \leq \begin{cases} 
D^h_0 & \text{if } l^h_1 = +\infty \text{ (no run, or among those first in line in a run)} \\
0 & \text{if } l^h_1 = 0 \text{ (among those last in line in a run)}
\end{cases}
\]

or, more compactly, \( 0 \leq W^h_1 \leq \min \{D^h_0, l^h_1\} \). If household \( h \) is served when the bank is out of money, then \( l^h_1 = 0 \) and thus \( W^h_1 = 0 \). If household \( h \) is served when the bank still has money, then \( l^h_1 = +\infty \) and \( 0 \leq W^h_1 \leq D^h_0 \). Allowing for partial or full suspension of convertibility does not affect the results.

Bank \( b \) is subject to a run if the limit on withdrawals is \( l^h_1 = 0 \) for some depositors of bank \( b \). If bank \( b \) is subject to a run, the bank is liquidated at \( t = 2 \). In the event of liquidation, assets that the bank has at \( t = 2 \) are used to repay deposits not withdrawn. If the value of assets is insufficient, depositors are repaid pro-rata; if the value of assets is greater than the value of deposits not withdrawn, banks can use the difference for consumption \( C^b_2 \).

Consumption of households cannot exceed the sum of storage \( S^h_0 \) at \( t = 0 \), and of withdrawals \( W^h_1 \) at \( t = 1 \):

\[
C^h_1 \leq S^h_0 + W^h_1.
\]

\(^{12}\)The sequential service constraint is imposed as a physical constraint as in Wallace (1988), rather than as a restriction on contracts.
At \( t = 2 \), households are entitled to receive a return \( 1 + r_2^D \) on deposits that are not withdrawn at \( t = 1 \). I refer to \( r_2^D \) as the \textit{promised return on deposits}. However, banks might not have enough resources to pay the promised return \( r_2^D \). Define \( r_b^2 \leq r_2^D \) to be the \textit{actual return on deposits} of bank \( b \). Note that \( r_b^2 \) can be lower than the promised return; if that is the case, then the quantity \( 1 + r_b^2 \) has the interpretation of recovery rate.

Before describing consumption of households and banks, it is useful to define the return on capital \( 1 + r_2^K \):

\[
1 + r_2^K = \frac{R}{Q_0}
\]

Consumption of households at \( t = 2 \) is:

\[
C^h_2 = K^h_0 R + (D^h_0 - W^h_1) (1 + r_2^{b(h)}) + (S^h_0 + W^h_1 - C^h_1)
\]

The sources of consumption goods at \( t = 2 \) are: capital \( K^h_0 \) bought at \( t = 0 \) and transformed into consumption goods at rate \( R \); deposits not withdrawn \( D^h_0 - W^h_1 \) plus the actual return \( r_2^{b(h)} \); and stored goods \( S^h_0 + W^h_1 - C^h_1 \) (in equilibrium, storage will be zero).

Similarly, consumption of banks at \( t = 2 \) is:

\[
C^b_2 = \max \left\{ 0; \frac{K^b_0 R}{D^b_0 - W^b_1} + (S^b_0 - W^b_1) - (D^b_0 - W^b_1) (1 + r_2^D) \right\}
\]

If the value of assets minus liabilities is negative, the bank is insolvent and \( C^b_2 \) is bounded below at zero.

---

13 For simplicity, I assume that no return is paid on deposit withdrawn at \( t = 1 \). This contractual arrangement is (weakly) optimal because households have linear utility from consumption at \( t = 2 \).

14 \( r_2^D \) is an endogenous return that is taken as given by both banks and households. The results are unchanged if I allow each bank to post a bank-specific return.
3 Equilibrium

**Actual return on deposits.** The actual return on deposits $r^b_2$ is defined by:

$$1 + r^b_2 = \begin{cases} 
1 + r^D_2 \text{ (promised return)} & \text{if } C^b_2 > 0 \\
\frac{K^b_0 R}{D^b_0 - W^b_1} & \text{if } C^b_2 = 0
\end{cases} \tag{12}$$

If the assets of bank $b$ at $t = 2$ are greater than its liabilities and thus $C^b_2 > 0$, the actual return on deposits $r^b_2$ is equal to the promised return $r^D_2$. Otherwise, $r^b_2$ is the return paid using the assets of banks at $t = 2$, $K^b_0 R$ (capital transformed into consumption good). The resources $K^b_0 R$ are split across deposits not withdrawn $D^b_0 - W^b_1$.

**Limit on withdrawal** Recall that the limit on withdrawal $l^h_1$ can take two values, $l^h_1 \in \{0, +\infty\}$. If all depositors of bank $b$ attempt to withdraw money at night, only a fraction of depositors can withdraw at night, i.e., $l^h_1 = +\infty$ for a subset of depositors of bank $b$. In the (relevant) case in which all depositors of bank $b$ choose the same value of deposits:

$$\Pr(l^h_1 = +\infty) = \frac{S^b_{0(h)}}{D^b_{0(h)}} \quad \text{and} \quad \Pr(l^h_1 = 0) = 1 - \frac{S^b_{0(h)}}{D^b_{0(h)}} \tag{13}$$

3.1 Beliefs

At $t = 0$, households form beliefs about the limit on withdrawals $l^h_1$ (that is, beliefs about the chance of being last in line in a run) and beliefs about the actual return on deposits $r^b_2$. Let $\Pr^h\left(l^h_1, r^b_2(h)\right)$ be the belief of household $h$. In equilibrium, I require such belief to be rational, that is, to be equal to the realized probability distribution over $l^h_1$ and $r^b_2(h)$ in the economy, denoted as $\Pr\left(l^h_1, r^b_2(h)\right)$. The equilibrium requirement can thus be stated as:

$$\Pr^h\left(l^h_1, r^b_2(h)\right) = \Pr\left(l^h_1, r^b_2(h)\right). \tag{14}$$

The probability distribution $\Pr\left(l^h_1, r^b_2(h)\right)$ can be obtained by combining (12) and (13) with the choices taken by banks in equilibrium.

---

15Alternatively, if $C^b_2 < 0$, the actual return $r^b_2$ can be computed using equation (11) by setting consumption equal to zero, storage equal to zero, by replacing $r^D_2$ with $r^b_2$, and then by solving for $r^b_2$. 
3.2 Market clearing conditions

The market clearing conditions are as follows.

\[
\int_B K^b_0 db + \int_H K^h_0 dh = K. \quad (15)
\]

Output and storage:

\[
\int_B S^b_0 db + \int_H S^h_0 dh = A K \quad (16)
\]

Deposits:

\[
\int_B D^b_0 db = \int_H D^h_0 dh. \quad (17)
\]

3.3 Equilibrium definition

Since I force all banks to offer the same promised return on deposits \( r^D_2 \), I impose a pooling equilibrium in the banking market, similar to Akerlof (1970). The results are unchanged if I allow each bank to post a bank-specific promised return on deposits. In this case, the equilibrium that arises is still a pooling one because bad banks want to imitate good banks to survive as long as possible.

The next definition formalizes the equilibrium concept.

**Definition 3.1.** An equilibrium is a collection of:

- price of capital \( Q_0 \) and promised return on deposits \( r^D_2 \);
- for each households \( h \in H \), beliefs \( Pr^h(\cdot) \); choices \( S^h_0, D^h_0, K^h_0, W^h_1, C^h_1, C^h_2 \); and limit on withdrawals \( l^h_1 \in \{0, +\infty\} \);
- for each bank \( b \in B \), choices \( D^b_0, S^b_0, K^b_0, C^b_2 \); and actual return on deposits \( r^b \);

such that:

- households maximize utility (1) subject to (7), (9), and (10);
- households’ beliefs are rational, i.e., equation (14) holds for all \( h \in H \);
- banks maximize consumption at \( t = 2 \), equation (11), subject to (8);
- actual return on deposits are defined by Equation (12), and limits on withdrawals satisfy: \(^{16}\)

\[ l^h_1 = 0 \text{ for depositor } h \text{ of bank } b \Rightarrow \int_{\mathbb{H}(b)} W^{h'}_1 \left( l^{h'}_1 = +\infty \right) dh' > S^b_0; \]

\(^{16}\)This condition says that if a household faces a limit on withdrawals \( l^h_1 = 0 \), then withdrawals chosen by depositors of bank \( b (h) \), if the limit on withdrawals were \(+\infty\) for all of them, would be greater than the amount of goods \( S^b_0 \) stored at \( t = 0 \) by bank \( b \).
(market clearing) the market clearing conditions hold.

I focus on symmetric equilibria in which banks with the same net worth make the same choices, in particular for deposits ($N^b_0 = N^{b'}_0 \Rightarrow D^b_0 = D^{b'}_0$ for $b, b' \in B$).

**Restriction on parameters.** I impose the following restriction on $R$ and $C_{17}$:

$$R = \frac{1}{1-\beta} A, \quad C = \frac{\overline{A}K}{\kappa}. \quad (18)$$

As a result of the assumption about $C$, optimality requires that all consumption goods at $t = 1$ is consumed by impatient households. To see this, note that consumption good at $t = 1$ is $\overline{A}K$, thus if all impatient households consume the same amount, feasibility implies that each of them can consume at most $\frac{\overline{A}K}{\kappa}$, i.e., at most $\overline{C}$. Since marginal utility is higher for impatient up to $\overline{C}$, the result follows.

## 4 Good equilibrium

The next Proposition describe the good equilibrium of the model, the proof is provided in the Appendix.

**Proposition 4.1.** If and only if:

$$K^b_{-1} \left( \frac{A}{1-\beta} \right) - D^b_{-1} \geq 0 \quad (19)$$

for all banks $b \in B$, then a good equilibrium exists. The good equilibrium is characterized by:

- **price of capital**
  $$Q_0 = \frac{\beta}{1-\beta} A \equiv Q^* \quad (20)$$

- **$t = 0$, households:** $S^h_0 = 0$, $D^h_0 = D^*$, and $K^h_0 = \frac{N^h_0 - D^*}{Q_0}$, where $D^* = \frac{\overline{A}K}{\kappa}$;
- **$t = 0$, banks:** $D^b_0 = D^*$, $S^b_0 = \kappa D^*$, and $K^b_0 = \frac{1}{Q_0} [N^b_0 + (1 - \kappa) D^*]$;
- **$t = 1$:** $L^h_1 = +\infty$ for all $h$; $W^h_1 = D^*$ if $\tilde{u} (\cdot) = \bar{u} (\cdot)$ (impatient) and $W^h_1 = 0$ if $\tilde{u} (\cdot) = u (\cdot)$ (patient); $C^h_1 = D^* = \overline{C}$ if impatient, and $C^h_1 = 0$ if patient;

---

17The restriction about $R$ can be motivated by an infinite-horizon formulation in which capital can be carried to the next period.
\[ t = 2: \]

- *return on capital* \( 1 + r^K_2 = \frac{1}{\beta} \equiv 1 + r^*; \)
- *promised return on deposits* \( 1 + r^D_2 = \frac{1}{\beta} \) and *actual return on deposits* \( 1 + r^b_2 = 1 + r^D_2 = \frac{1}{\beta} \) for all \( b \).

Given the equilibrium price of capital, condition (19) guarantees that the net worth of all banks in the economy is positive. Thus, in the good equilibrium, all banks are solvent and households expect no runs. Therefore, they hold deposits \( D^* > 0 \) and no money because banks pool the liquidity risk of households, allowing impatient households to withdraw money at \( t = 1 \) and offering a return on deposits not withdrawn.

Patient households are (weakly) better off by postponing consumption to \( t = 2 \), because the marginal utility at \( t = 1 \) is one, the (discounted) marginal utility at \( t = 2 \) is \( \beta \), and capital gives a return of \( 1/\beta \). Impatient households consume \( \bar{C} \), therefore the marginal utility of any additional unit of consumption is also one. Thus, impatient households are similarly willing to postpone any additional consumption to \( t = 2 \).

Banks store a fraction \( \kappa \) of deposits (in order to serve withdrawals by impatient households at \( t = 1 \)), while the remainder fraction \( 1 - \kappa \) is invested in capital. Since the return on capital is \( 1 + r^K_2 = 1/\beta \), each dollar of deposit yields a gross return \( (1 - \kappa) (1 + r^K_2) = (1 - \kappa) / \beta \) at \( t = 2 \). As the bank has to pay a gross return \( 1 + r^D_2 = 1/\beta \) on the fraction \( 1 - \kappa \) of deposits not withdrawn, banks’ profits for each dollar of deposits are \( (1 - \kappa) \left[ (1 + r^K_2) - (1 + r^D_2) \right] = 0 \). Thus, banks are indifferent about the quantity of deposits \( D^b_0 \) that they hold. Note that if \( r^K_2 > r^D_2 \) or \( r^K_2 < r^D_2 \), banks would make, respectively, positive or negative profits on each dollar of deposits, and thus would choose, respectively, \( D^b_0 = +\infty \) or \( D^b_0 = 0 \). But this outcome would violate the market clearing condition for deposits, thus in equilibrium \( r^K_2 = r^D_2 \) must hold.

5 Bad equilibria without policy intervention

**Restriction on endowments.** I restrict the endowment of banks to depends on the parameters of the model, as if the initial conditions where generated by an infinite-horizon economy in which a good equilibrium is realized before time \( t = 0 \). To govern the heterogeneity in the endowment of capital of banks, I introduce two parameters, \( X^L \) and \( X^H \), satisfying \( X^L < X^H \).
Assumption 5.1. The endowment of banks satisfy:

\[
K^L = \bar{K} (1 + X^L) \frac{1 - \kappa (1 - \beta)}{\kappa \beta} \\
K^H = \bar{K} (1 + X^H) \frac{1 - \kappa (1 - \beta)}{\kappa \beta} \\
D^b_{-1} = AK \frac{1 - \kappa}{\kappa} \frac{1}{\beta} \text{ for all } b \in \mathbb{B}.
\]

Before turning to the analysis of bad equilibria, the next Corollary derives some implications of Assumption 5.1 for the existence of the good equilibrium.

Corollary 5.2. Assume Assumption 5.1 holds. Then a good equilibrium (described by Proposition 4.1) exists if and only if \( X^L \geq 1 \).

Bad equilibrium. The next Proposition presents the bad equilibrium, and the following paragraphs explain the results.\(^\text{18}\)

Proposition 5.3. Under some parameter restrictions, a bad equilibrium exists, and is characterized by:

- **price of capital:**
  \[ Q_0 = A \frac{\beta}{1 - \beta} \frac{1}{1 + \kappa (\theta - 1)} < Q^* \]  
  (21)
  where \( Q^* \) is defined by Equation (20);
- **\( t = 0 \), households’ choices:**
  \[ D^h_0 < D^* \]
  \[ S^h_0 \in (0, AK) \]
- **\( t = 0 \), banks’ choices:**
  - net worth \( N^b_0 \geq 0 \) (solvent bank) if \( K^b_{-1} = K^H \), and \( N^b_0 < 0 \) (insolvent bank) if \( K^b_{-1} = K^L \);
  - \( D^b_0 = D^h_0 \), \( M^b_0 = \kappa D^h_0 \), and \( K^b_0 = \frac{1}{Q_0} \left[ N^b_0 + (1 - \kappa) D^h_0 \right] \);
- **\( t = 1 \):**

\(^{18}\) All the prices and quantities of the bad equilibrium can be easily expressed as function of parameters, although some of them are not reported here.
• for all depositors of solvent banks: \( L_1^h = +\infty \), and

\[
W_1^h = \begin{cases} 
D_0^h & \text{if } \tilde{u} (\cdot) = \bar{u} (\cdot) \text{ (patient)} \\
0 & \text{if } \tilde{u} (\cdot) = u (\cdot) \text{ (impatient)}
\end{cases},
\]

\[
C_1^h = \begin{cases} 
D_0^h + S_0^h & \text{if } \tilde{u} (\cdot) = \bar{u} (\cdot) \text{ (patient)} \\
S_0^h & \text{if } \tilde{u} (\cdot) = u (\cdot) \text{ (impatient)}
\end{cases};
\]

• for a fraction \( \kappa \) of depositors of insolvent banks: \( L_1^h = +\infty \), \( W_1^h = D_0^h \), \( C_1^h = D_0^h + S_0^h \);

• for a fraction \( 1 - \kappa \) of depositors of insolvent banks: \( L_1^h = 0 \), \( W_1^h = 0 \), and \( C_1^h = S_0^h \).

• \( t = 2 \):
  • return on capital \( 1 + r_2^K = \frac{1}{\beta} + \frac{\kappa (\theta - 1)}{\beta} > 1 + r^* \);
  • promised return on deposits \( 1 + r_2^D = 1 + r_2^K \);
  • actual return on deposits: if \( N_0^b \geq 0 \) (solvent bank) then \( 1 + r_2^b \left( N_2^b \geq 0 \right) = 1 + r_2^D \); if \( N_0^b < 0 \) (insolvent bank) then \( r_2^b < 0 \).

At \( t = 0 \), households believe that banks with low endowment of capital, \( K^L \), will be subject to runs. However, due to asymmetric information, households do not know whether their own bank has high or low holdings of capital, and thus they assign positive probability to a run on their own bank. Therefore, at \( t = 0 \), households flight to safe liquid assets (storage) choosing \( S_0^h > 0 \) and \( D_0^h < D^* \) in order to self-insure against the risk of being among those last in line during a run. As a result, an increase in storage implies a reduction in the demand for capital, resulting in a reduction in the price of capital \( Q_0 \). Due to the low \( Q_0 \), a bank \( b \) with endowment \( K_{-1}^b = K^L \) is insolvent, that is, \( N_0^b < 0 \).

I claim that banks with endowment \( K^L \) will not be able to pay the promised return on deposits \( r_2^D \) at \( t = 2 \) for two reasons. First, these banks are insolvent at \( t = 0 \), i.e., \( N_0^b < 0 \), as discussed before. Second, the promised return on deposits \( r_2^D \) is equal to the return on capital \( r_2^K \), thus banks make zero profits on deposits (the logic is the same one discussed after Proposition 5.3). Thus, insolvency at \( t = 0 \) and no profits on deposits implies that these banks will not have enough resources at \( t = 2 \) and will pay an actual return on deposits \( r_2^b < r_2^D \).

At \( t = 1 \), impatient households want to withdraw money from their own banks in order to finance consumption expenditure. Moreover, depending on the value of the actual return
on deposits $r^b_2$, even patient households might prefer to withdraw. If $r^b_2 < 0$, the actual return on deposits is negative, and thus a patient household prefers to run, otherwise she will incur losses on deposits not withdrawn if she waits until $t = 2$.\footnote{If $r^b_2 = -1$, it means that deposits not withdrawn loose their entire value, thus $r^b_2$ is bounded below by $-1$.}

Note that the initial belief, at $t = 0$, that some bank in the economy would be subject to run, is then verified at $t = 1$, confirming that this is indeed an equilibrium.

After withdrawals, households choose consumption. Impatient households consume all the goods they have because they have marginal utility of consumption, $\theta > 1$ (in the bad equilibrium, their consumption is less than $\bar{C}$). Patient households prefer to consume their goods as well. This last result derives by comparing, for a patient household, the marginal utility of consuming one unit of good at $t = 1$ with the discounted marginal utility of consuming it at $t = 2$:

\[
\frac{1}{\text{marginal utility at } t=1} > \beta \frac{1}{\text{marginal utility at } t=2}.
\]

Note that the result is unchanged even if the marginal utility of patient households at $t = 1$ is between $\beta$ and one.

Under some parameter restriction, the holdings of capital of banks decrease, as stated by the next result.

\textbf{Proposition 5.4.} Under some restriction on endowments, in the bad equilibrium described by Proposition 5.3:

\[K^b_0 < K^b_1 \text{ for all } b \in \mathbb{B}, \quad K^h_0 > K^h_1 \text{ for all } h \in \mathbb{H}.\]

Thus, the bad equilibrium described by Proposition 5.3 exhibits sales of capital by banks at a depressed real price, that is, fire-sales. Capital sold by banks is bought by households. Since households are buying capital at very convenient terms, fire-sales imply a transfer of wealth from banks to households. This transfer of wealth is a crucial element in causing the insolvency of the banks with low endowment of capital.

Finally (although the restriction on parameters are not reported in the statement of Proposition 5.3), multiple equilibria exists for a subset of the parameter space. In other regions of the parameter space, there exists either a good equilibrium only, or a bad equilibrium only.
6 Policy analysis: equity injection

I model an equity injection at $t = 0$ as an increase of the endowment of capital of banks. The government is able to transfer endowment of capital from households to banks at $t = 0$. I assume that the government is subject to the same asymmetric information as private agents and thus does not observe the endowment of capital of banks. Therefore, the government is not able to distinguish between solvent and insolvent banks. Thus, the government transfers endowment from households to all banks in the economy.

However, in this model, a transfer of endowment from households to all banks is equivalent, from an equilibrium perspective, to increasing only the endowment of capital of banks that are insolvent in the bad equilibrium (that is, increasing $K^L$)\(^{20}\) combined with a transfer of consumption, at $t = 2$, from households to solvent banks (that is, banks with endowment $K^H$). This result follows from the fact that transferring capital from an household to a solvent bank has no effect on equilibrium prices and beliefs. Solvent banks pay anyway the promised return on deposits $r^b_2$, thus transferring some endowment to them has only the effect of increasing their consumption at $t = 2$.

Thus, the effect of the equity injection are equivalent to the effect of a change in $K^L$, which is analyzed by the next Proposition.

**Proposition 6.1.** Assume a bad equilibrium exists. Then:

$$\frac{dQ_0}{dK^L} = 0, \quad \frac{dK^b_1|_{K^b_1 = K^L}}{dK^L} = 0$$

$$\frac{dD^b_0}{dK^L} < 0, \quad \frac{dS^h_0}{dK^L} > 0.$$

In the bad equilibrium, a small equity injection financed by taxing households does not affect prices, while it exacerbates the flight to liquidity by reducing deposits and increasing storage by households. Thus, at $t = 1$, the consumption of patient agents increase and the consumption of impatient agents decreases (this results follows from the fact that patient agents consume all the goods they have, which are now higher due to the increase in storage, and from the fact that aggregate consumption is constant). As a result, welfare decreases, because the consumption of patient agents, who have high marginal utility,
If instead the increase of $K^L$ is large enough, the bad equilibrium is eliminated. Intuitively, if all banks have a large endowment, their net worth at $t = 0$ remains positive even if the price of capital were to drop. Thus, there is no flight to liquidity and only the good equilibrium exists.

**Proposition 6.2.** There exists a threshold $\hat{K} < \bar{K}$ such that, if the endowment of all banks satisfy $K_b \geq \hat{K}$, the good equilibrium exists and the bad equilibrium does not exist.

### 7 Conclusions

I have presented a new framework to analyze bank runs, fire-sales, and equity injections. The paper has two main contributions. First, on the theoretical side, fire-sales arise despite all agents have the same ability in managing assets (contrary to the common assumption of having “nonspecialists” who have less expertise in managing assets) and despite no liquidation cost is paid when an asset is sold. Second, on the policy side, small equity injections amplify the flight to liquidity and reduce welfare (the latter result is derived using numerical examples, but I plan to generalize and prove it formally). The results have thus implications for the analysis of both ex-post policy interventions that aims at mitigating the effects of financial crises but might create moral hazard, and ex-ante policies that aims at reducing the possibilities of such crises.

### References


\footnote{So far, I have shown the reduction of welfare only using numerical examples. However, since the bad equilibrium can be easily solved in closed form, the result can be generalized and a welfare analysis can be undertaken more formally; I plan to do that in the next revision of the paper.}


