Bilateral Trading in Divisible Double Auctions*

Songzi Du Haoxiang Zhu

March 2, 2014

Preliminary and Incomplete. Comments Welcome

Abstract

We study bilateral trading between two bidders in a divisible double auction. The bidders (1) submit demand schedules, (2) have interdependent and linearly decreasing marginal values, and (3) can be asymmetric. Existing models of divisible double auctions typically require at least three bidders for the existence of linear equilibria. In this paper, we characterize a family of nonlinear ex post equilibria with two bidders, implicitly given by a solution to an algebraic equation. We show that the equilibrium amount of trading is strictly less than that in the ex post efficient allocation. If marginal values do not decrease with quantity, we solve the family of ex post equilibria in closed form. Our theory of bilateral trading differs from the bargaining literature and can serve as a tractable building block to model dynamic trading in decentralized markets.

Keywords: divisible double auctions, bilateral trading, bargaining, ex post equilibrium
JEL Codes: D44, D82, G14

*First draft: November 2011. Du: Simon Fraser University, Department of Economics, 8888 University Drive, Burnaby, B.C. Canada, V5A 1S6. songzid@sfu.ca. Zhu: MIT Sloan School of Management, 100 Main Street E62-623, Cambridge, MA 02142. zhuh@mit.edu.

1 Introduction

Trading with demand schedules, in the form of double auctions, is common in many financial and commodity markets. A large literature is devoted to characterizing the trading behavior in this mechanism as well as the associated price discovery and allocative efficiency (see, for example, Kyle (1989), Vives (2011), Rostek and Weretka (2012), and Du and Zhu (2013), among others). In a typical model of divisible double auctions, bidders simultaneously submit linear demand schedules (i.e., a set of limit orders, or price-quantity pairs), and trading occurs at the market-clearing price. A limitation of these models is that the existence of linear equilibria requires at least three bidders. While the $n \geq 3$ assumption is relatively innocuous for centralized markets, it is restrictive for decentralized, over-the-counter (OTC) markets, where trades are conducted bilaterally.

In this short paper, we aim to fill this gap by studying the previously unexplored case of two bidders (bilateral trading) in double auctions. Each bidder in our model receives a one-dimensional private signal about the asset and values the asset at a weighted average of his and the other bidder's signals. That is, values are interdependent. In addition, the bidder's marginal value for owning the asset declines linearly in quantity. Moreover, the bidders can be asymmetric, in the sense that their values can have different weights on each other's signals, and that their marginal values can decline at different rates.

We characterize a family of non-linear equilibria in this model. In an equilibrium, each bidder's demand schedule is implicitly given by a solution to a non-linear algebraic equation. We show that these equilibria always lead to a trading quantity that is strictly lower (in absolute values) than the ex post efficient quantity. This is consistent with the "demand reduction" property commonly seen in multi-unit auctions (see, for example, Ausubel, Cramton, Pycia, Rostek, and Weretka (2011)). The equilibria that we characterize are ex post equilibria; that is, the equilibrium strategies remain optimal even if each bidder would observe the private information of the other bidder. In the special case of constant marginal values, we obtain a bidder's equilibrium demand schedule in closed form: it is simply a constant multiple of a power function of the difference between the bidder's signal and the price, where the exponent is decreasing in the weight a bidder assigns on his own signal.

Besides the literature on divisible auctions, our model is also related to the bilateral bargaining literature. For example, the literature pioneered by Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989) studies double auction of an indivisible object with two bidders. Although the indivisibility of the asset in their models is suitable for markets like real estates and art, our model of divisible auctions better characterizes the markets for financial securities, derivatives, and commodities.

Our equilibria are directly applicable to a sequential bargaining model in which the two parties go back and forth proposing prices and quantities that they are willing to trade, until one side accepts the terms proposed by the other. Suppose that the two parties both have private signals and interdependent values as in our model, and that they have no time discounting during the bargaining process. The final outcome from the sequential proposals of prices and quantities is equivalent to the outcome from a static double auction in which the two parties submit demand schedules and trade at the market-clearing price; consequently, our ex post equilibria become sequential equilibria in this sequential bargaining model. The application of our equilibria to sequential bargaining is distinct from and complementary to the sequential bargaining literature pioneered by Rubinstein (1982) for which time discounting plays an important role.[1] An advantage of our approach is the robustness of our equilibria (due to their ex post optimality) to the precise ordering of proposals and counter-proposals in the sequential bargaining.

In addition to characterizing a single trade between two parties, our model can also serve as an alternative building block for the analysis of dynamic trading in OTC markets. Whereas dynamic search models by Duffie, Garleanu, and Pedersen (2005) and extensions assume private values and symmetric information between the two parties of a trade, our model allows interdependent values and asymmetric information. Related, Duffie, Malamud, and Manso (2009) study how dispersed information regarding a common value percolates in a large OTC markets. They take as given that the two parties exchange their information upon the pairwise meeting, but abstract away from the mechanism or incentive associated with the information exchange.[2] Our model of divisible auction provides a strategic microfoundation for such information exchange. For this application, a particular advantage of our equilibria is that they are ex post optimal, which makes the model more tractable and more robust to the private information of counterparties.

[1] Ausubel, Cramton, and Deneckere (2002) summarize the literature on sequential bargaining with private information. Many sequential bargaining papers focus on the case of one-sided private information; Deneckere and Liang (2006) is the closest paper from this literature to us, as they study one-sided private information and interdependent values. We are not aware of any paper that studies sequential bargaining with two-sided private information and interdependent values.

[2] In an extension, Duffie, Malamud, and Manso (2013) consider an indivisible double auction in each pairwise meeting.

2 Model

There are $n = 2$ players, whom we call "bidders," trading a divisible asset. Each bidder i observes a private signal, $s_i \in [\underline{s}, \overline{s}] \subset \mathbb{R}$, about the value of the asset. We use j to denote the bidder other than i. Bidder i's value for owning the asset is:

$$v_i = \alpha_i s_i + (1 - \alpha_i) s_j, \tag{1}$$

where $\alpha_1 \in (0, 1]$ and $\alpha_2 \in (0, 1]$ are commonly known constants that capture the level of interdependence in bidders' valuations. We assume that $\alpha_1 + \alpha_2 > 1$.

We further assume that bidder i's marginal value for owning the asset decreases linearly in quantity at a commonly known rate $\lambda_i \geq 0$. Thus, if bidder i acquires quantity q_i at the price p, bidder i has the ex post utility:

$$U_i(q_i, p; v_i) = v_i q_i - \frac{\lambda_i}{2}(q_i)^2 - p q_i. \tag{2}$$

By construction, if $q_i = 0$, then $U_i = 0$.

The trading mechanism is an one-shot divisible double auction. We use $x_i(\,\cdot\,; s_i)$, where $x_i(\,\cdot\,; s_i) : [\underline{s}, \overline{s}] \to \mathbb{R}$, to denote the demand schedule that bidder i submits conditional on his signal s_i. The demand schedule $x_i(\,\cdot\,; s_i)$ specifies that bidder i wishes to buy a quantity $x_i(p; s_i)$ of the asset at the price p when $x_i(p; s_i)$ is positive, and that bidder i wishes to sell a quantity $-x_i(p; s_i)$ of the asset at the price p when $x_i(p; s_i)$ is negative.

Given the submitted demand schedules $(x_1(\,\cdot\,; s_1), x_2(\,\cdot\,; s_2))$, the auctioneer (a human or a computer algorithm) determines the transaction price $p^* \equiv p^*(s_1, s_2)$ from the market-clearing condition

$$x_1(p^*; s_1) + x_2(p^*; s_2) = 0. \tag{3}$$

After p^* is determined, bidder i is allocated the quantity $x_i(p^*; s_i)$ of the asset and pays $x_i(p^*; s_i)p^*$. If no market-clearing price exists, there is no trade, and each bidder gets a utility of zero.[3]

We make no assumption about the distribution of (s_1, s_2). Therefore, the solution concept that we use is *ex post equilibrium*. In an ex post equilibrium, each bidder has no regret—he would not deviate from his strategy even if he would learn the signal of the other bidder.

Definition 1. An **ex post equilibrium** is a profile of strategies (x_1, x_2) such that for every

[3] If multiple market-clearing prices exist, we can pick one arbitrarily.

profile of signals $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$, every bidder i has no incentive to deviate from x_i. That is, for any alternative strategy $\tilde{x}_i$ of bidder i,

$$U_i(x_i(p^*; s_i), p^*; v_i) \geq U_i(\tilde{x}_i(\tilde{p}; s_i), \tilde{p}; v_i),$$

where v_i is given by (1), p^* is the market-clearing price given x_i and x_j, and $\tilde{p}$ is the market-clearing price given $\tilde{x}_i$ and x_j, $j \neq i$.

Note that in an ex post equilibrium, a bidder can guarantee a non-negative ex post utility, since he can earn zero utility by submitting a demand schedule that does not clear the market (and hence trading zero quantity).

3 Characterize a Family of Ex Post Equilibria

We first define the sign function:

$$\operatorname{sign}(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}. \tag{4}$$

Proposition 1. *Suppose that $1 < \alpha_1 + \alpha_2 < 2$. Let C be any positive constant such that*

$$C \geq \frac{(\overline{s} - \underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2} \left(\frac{\lambda_2 \left(1 - \frac{\alpha_1}{2}\right) + \lambda_1 \frac{\alpha_2}{2}}{\alpha_1 + \alpha_2 - 1} \right)^{\alpha_1+\alpha_2-1}, \text{ and} \tag{5}$$

$$C \geq \frac{(\overline{s} - \underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1} \left(\frac{\lambda_1 \left(1 - \frac{\alpha_2}{2}\right) + \lambda_2 \frac{\alpha_1}{2}}{\alpha_1 + \alpha_2 - 1} \right)^{\alpha_1+\alpha_2-1}. \tag{6}$$

Then, there exists a family (parameterized by C) of ex post equilibria in which:

$$x_i(p; s_i) = y_i(|s_i - p|) \cdot \operatorname{sign}(s_i - p), \ i \in \{1, 2\}, \tag{7}$$

where, for $z_1, z_2 \in [0, \overline{s} - \underline{s}]$, $y_1(z_1)$ and $y_2(z_2)$ are the smaller solutions to

$$(2 - \alpha_1 - \alpha_2) z_1 = C \alpha_2 y_1(z_1)^{\alpha_1+\alpha_2-1} - \left(\lambda_2 \left(1 - \frac{\alpha_1}{2}\right) + \lambda_1 \frac{\alpha_2}{2} \right) y_1(z_1), \tag{8}$$

$$(2 - \alpha_1 - \alpha_2) z_2 = C \alpha_1 y_2(z_2)^{\alpha_1+\alpha_2-1} - \left(\lambda_1 \left(1 - \frac{\alpha_2}{2}\right) + \lambda_2 \frac{\alpha_1}{2} \right) y_2(z_2). \tag{9}$$

The equilibrium price $p^* = p^*(s_1, s_2)$ is in between s_1 and s_2, and satisfies

$$p^* = \frac{\alpha_1 s_1 + \alpha_2 s_2}{\alpha_1 + \alpha_2} + \frac{\alpha_1 \lambda_2 - \alpha_2 \lambda_1}{2(\alpha_1 + \alpha_2)} x_1(p^*; s_1) \tag{10}$$

Proof. See Section A.1. □

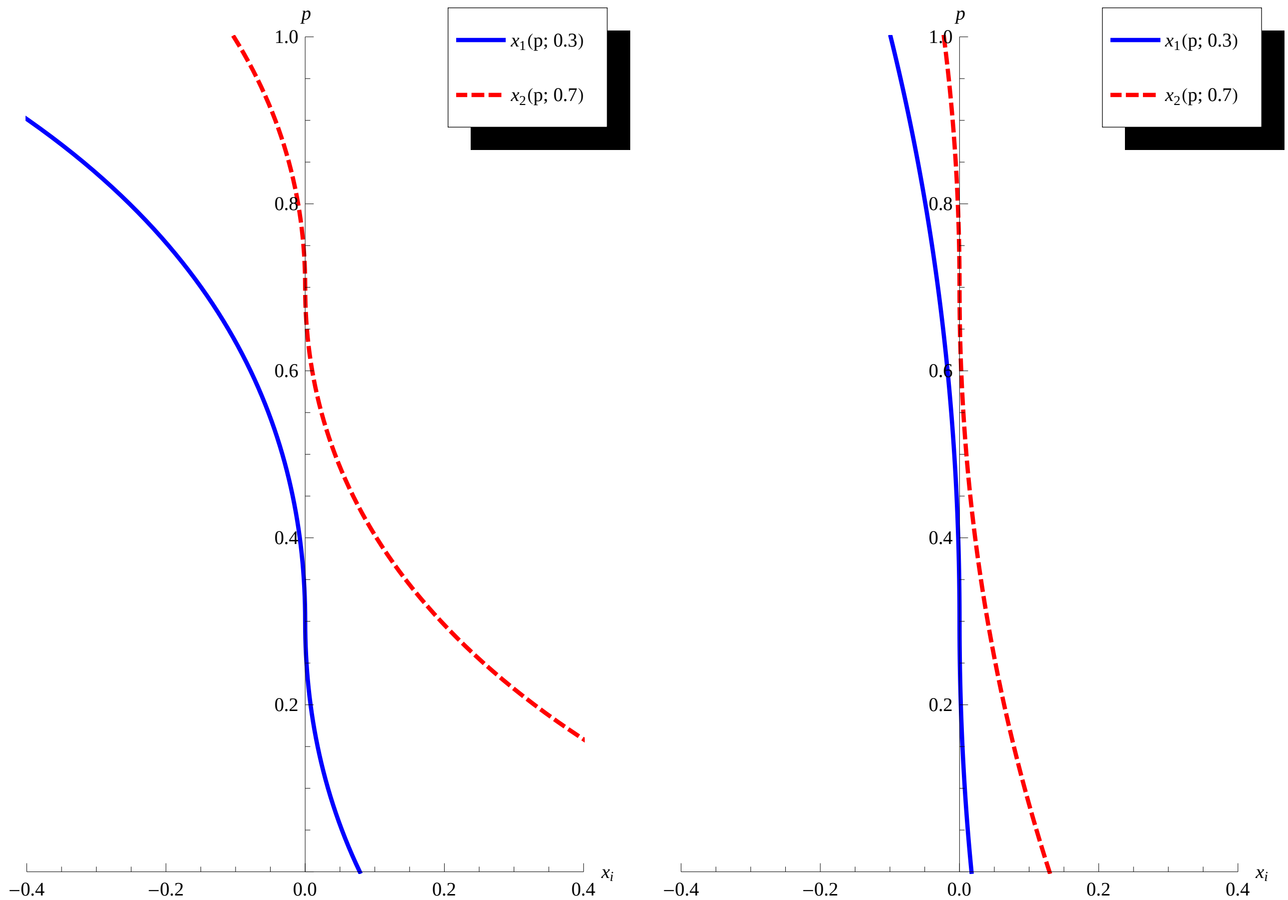

Figure 1: Equilibria from Proposition 1 with $\alpha_1 = 0.7, \alpha_2 = 0.8, \lambda_1 = 0.1, \lambda_2 = 0.2, \underline{s} = 0, \overline{s} = 1, s_1 = 0.3$ and $s_2 = 0.7$. The equilibrium on the left has $C = 0.729$ and the equilibrium on the right has $C = 1.458$.

Figure 1 demonstrates two equilibria of Proposition 1. The primitive parameters are $\alpha_1 = 0.7, \alpha_2 = 0.8, \lambda_1 = 0.1, \lambda_2 = 0.2, \underline{s} = 0$, and $\overline{s} = 1$. The realized signals are $s_1 = 0.3$ and $s_2 = 0.7$. On the left-hand plot, we show the equilibrium with $C = 0.729$, which, subject to conditions (5)–(6), is the equilibrium that maximizes trading volume (hence the most efficient equilibrium). The equilibrium price is $p^* = 0.5126$, bidder 1 gets $x_1(p^*; s_1) = -0.037$, and bidder 2 gets $x_2(p^*; s_2) = 0.037$. On the right-hand plot, we show the equilibrium with $C = 1.458$. The equilibrium price is $p^* = 0.513$, bidder 1 gets $x_1(p^*; s_1) = -0.009$, and bidder 2 gets $x_2(p^*; s_2) = 0.009$.

While the full proof of Proposition 1 is provided in Section A.1, we briefly discuss its intuition. The conditions (5)–(6) guarantee that the algebraic equations (8)–(9) have solutions. For example, the right-hand side of Equation (8), rewritten as

$$f_1(y_1) \equiv C\alpha_2 y_1^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1 \frac{\alpha_2}{2}\right) y_1, \tag{11}$$

is clearly a concave function of y_1. Condition (5) ensures that the maximum of $f_1(y_1)$ is above $(2 - \alpha_1 - \alpha_2)(\overline{s} - \underline{s})$. Hence, by the Intermediate Value Theorem, a solution exists.

Moreover, whenever the inequalities (5)–(6) strict, there always exist two solutions $y_1(z_1)$: one before $f_1(y_1)$ obtains its maximum and the other after. Between the two, we select the former. It is easy to see that the smaller solution $y_1(z_1)$ is increasing in z_1 because $f_1(y_1)$ is increasing in y_1 before it obtains its maximum. This means that bidder i's demand $x_i(p; s_i)$ is decreasing in p, by (7). The other solution implies an upward-sloping demand schedule and should be discarded.

In these equilibria, each bidder i buys $y_i(s_i - p)$ units of asset if the price p is below his signal s_i; he sells $y_i(p - s_i)$ units if p is above s_i. The constant C represents the aggressiveness of the bidding strategy; the smaller is C, the larger is $y_i(z_i)$, and hence the more aggressive the bidders bid at each price.[4] The most aggressive equilibrium is also the most efficient one, because it maximizes the amount of trading while obtaining non-negative utilities for both bidders. On the other hand, as C tends to infinity, $y_i(z_i)$ tends to zero, and hence the amount of trading in equilibrium tends to zero. Among this family of ex post equilibria, the most efficient one, which corresponds to the smallest C satisfying conditions (5)–(6), is a natural candidate for equilibrium selection.

The equilibrium price in (10) reveals to each bidder the signal of the other bidder. Since in the double auction each bidder can condition his quantity on the price, each bidder effectively conditions his trading on the signal of the other bidder. We construct our ex post equilibria by writing each bidder's value in terms of the equilibrium price and the other bidder's signal; hence each bidder's ex post optimality condition can be written purely in terms of the equilibrium price and the other bidder's demand schedule. This ex post optimality condition is a first order condition, and hence an ordinary differential equation;

[4]See Equations (8) and (9): as C gets larger, $y_i(z_i)$ must become smaller since the left-hand sides of (8) and (9) are not changing.

its solution is thus the demand schedule that the other bidder uses in an ex post equilibrium.

3.1 Demand Reduction and Efficiency

In this subsection we work under the condition that $\lambda_1 + \lambda_2 > 0$.

The ex post efficient allocation is given by:

$$\max_{q_1 \in \mathbb{R}} \; v_1 q_1 - \frac{\lambda_1}{2}(q_1)^2 + v_2(-q_1) - \frac{\lambda_2}{2}(-q_1)^2, \tag{12}$$

where $q_2 = -q_1$. Let us denote the efficient allocation by (q_1^e, q_2^e), where q_1^e solves the above maximization problem, and $q_2^e = -q_1^e$. We have

$$v_1 - \lambda_1 q_1^e - (v_2 - \lambda_2 q_2^e) = 0, \tag{13}$$

that is,

$$q_1^e = \frac{v_1 - v_2}{\lambda_1 + \lambda_2} = \frac{(\alpha_1 + \alpha_2 - 1)(s_1 - s_2)}{\lambda_1 + \lambda_2}. \tag{14}$$

Let us denote

$$q^e \equiv |q_1^e| = \frac{(\alpha_1 + \alpha_2 - 1)|s_1 - s_2|}{\lambda_1 + \lambda_2}, \tag{15}$$

which is the amount of trading (in absolute value) in the efficient allocation. Note that we have suppressed the dependence of q^e on (s_1, s_2) for notational simplicity.

Let $q^*(C) \equiv |x_1(p^*; s_1)|$ be the amount of trading (in absolute value) in an ex post equilibrium (x_1, x_2) from Proposition 1, where the constant C satisfies Conditions (5) and (6). Let us also define

$$f(y) \equiv C(\alpha_1 + \alpha_2) y^{\alpha_1 + \alpha_2 - 1} - (\lambda_1 + \lambda_2) y. \tag{16}$$

In Section A.1 we show that $y = q^*(C)$ is the smaller solution to

$$f(y) = (2 - \alpha_1 - \alpha_2)|s_1 - s_2|, \tag{17}$$

before $f(y)$ reaches its maximum.[5]

Proposition 2. *Suppose that $\lambda_1 + \lambda_2 > 0$. For every signal profile (s_1, s_2), the amount of trading in every ex post equilibrium of Proposition 1 is strictly less than that in the ex post*

[5] It is straightforward to show that given Conditions (5) and (6), there always exist two solutions to (17), one before and one after $f(y)$ reaches the maximum.

efficient allocation. That is, $q^*(C) < q^e$ for every C satisfying (5) and (6) and for every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$.

Proof. See Section A.2. □

3.2 Special Cases

In this subsection we consider a few special cases of the equilibria of Proposition 1.

3.2.1 Constant Marginal Values

In the special case that $\lambda_1 = \lambda_2 = 0$, we obtain explicit closed-form solutions.

Corollary 1. *Suppose that $\alpha_1 + \alpha_2 > 1$ and $\lambda_1 = \lambda_2 = 0$. There exists a family of ex post equilibria in which:*

$$x_i(p; s_i) = C|\alpha_i(s_i - p)|^{\frac{1}{\alpha_1+\alpha_2-1}} \cdot \operatorname{sign}(s_i - p),\ i \in \{1, 2\}, \tag{18}$$

where C is any positive constant, and the equilibrium price is independent of C and is given by

$$p^*(s_1, s_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2} s_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} s_2. \tag{19}$$

Corollary 1 shows that if $\lambda_1 = \lambda_2 = 0$, the equilibrium price $p^*(s_1, s_2)$ tilts toward the signal of the bidder who assigns a larger weight on his private signal.

3.2.2 Symmetric Bidders

Corollary 2. *Suppose that $\alpha_1 = \alpha_2 \equiv \alpha \in (1/2, 1)$ and $\lambda_1 = \lambda_2 \equiv \lambda$. There exists a family of ex post equilibria in which:*

$$x_i(p; s_i) = y(|s_i - p|) \cdot \operatorname{sign}(s_i - p),\ i \in \{1, 2\}, \tag{20}$$

where $y(z)$ is the smaller solution to

$$2(1 - \alpha)z = Cy(z)^{2\alpha-1} - \lambda y(z), \tag{21}$$

for $z \in [0, \overline{s} - \underline{s}]$, and C is any positive constant such that

$$C \geq (\overline{s} - \underline{s})^{2-2\alpha} \left(\frac{\lambda}{2\alpha - 1}\right)^{2\alpha-1}. \tag{22}$$

The equilibrium price is independent of C and is given by

$$p^*(s_1, s_2) = \frac{s_1 + s_2}{2}. \tag{23}$$

3.2.3 Common Value

Corollary 3. *Suppose that $\lambda_1 + \lambda_2 > 0$. As $\alpha_1 + \alpha_2$ tends to 1, every ex post equilibrium (x_1, x_2) from Proposition 1 implies zero trading:*

$$\lim_{\alpha_1+\alpha_2 \to 1} x_i(p; s_i) = 0, \tag{24}$$

for every $i \in \{1, 2\}, s_i \in [\underline{s}, \overline{s}], p \in [\underline{s}, \overline{s}]$, and C that satisfies Conditions (5) and (6).

3.3 Private Values

Private values correspond to $\alpha_1 = \alpha_2 = 1$. Strictly speaking, private values are not covered by Proposition 1, but one can easily obtain ex post equilibria using the same line of arguments as in Proposition 1.

Corollary 4. *Suppose that $\alpha_1 = \alpha_2 = 1$. Let C_1 and C_2 be positive constants satisfying*

$$C_1 - C_2 = \frac{\lambda_1 - \lambda_2}{2}, \tag{25}$$

and

$$C_i \geq \frac{\lambda_1 + \lambda_2}{2} \left(\log \frac{2(\overline{s} - \underline{s})}{\lambda_1 + \lambda_2} + 1 \right), i \in \{1, 2\}. \tag{26}$$

There exists a family (parameterized by C_1 and C_2) of ex post equilibria in which:

$$x_i(p; s_i) = y_i(|s_i - p|) \cdot \operatorname{sign}(s_i - p),\ i \in \{1, 2\}, \tag{27}$$

where, for $z_i \in [0, \overline{s} - \underline{s}]$, $y_i(z_i)$ is the smaller solution to

$$C_i y_i(z_i) - \frac{\lambda_1 + \lambda_2}{2} y_i(z_i) \log(y_i(z_i)) = z_i. \tag{28}$$

The equilibrium price $p^* = p^*(s_1, s_2)$ is in between s_1 and s_2, and satisfies

$$p^* = \frac{s_1 + s_2}{2} + \frac{\lambda_2 - \lambda_1}{4} x_1(p^*; s_1). \tag{29}$$

Proof. See Section A.3. ☐

Appendix

A Proofs

A.1 Proof of Proposition 1

Step 1: writing the first order conditions as differential equations.

Let (x_1, x_2) be an ex post equilibrium. The first-order conditions for the ex post optimality are: for every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$, $i \in \{1, 2\}$ and $j \neq i$,

$$\begin{aligned}
&-x_i(p^*; s_i) - (\alpha_i s_i + (1-\alpha_i)s_j - p^* - \lambda_i x_i(p^*; s_i)) \left(-\frac{\partial x_j}{\partial p}(p^*; s_j)\right) = 0, \\
&x_1(p^*; s_1) + x_2(p^*; s_2) = 0,
\end{aligned} \tag{30}$$

where $p^* \equiv p^*(s_1, s_2)$ is the market-clearing price.

We first conjecture that the market-clearing price satisfies

$$s_1 - p^* + \mu x_1(p^*; s_1) = -\frac{\alpha_2}{\alpha_1}(s_2 - p^*), \tag{31}$$

where μ is a constant to be determined in Step 2.

Given the conjecture in (31), we have

$$\begin{aligned}
v_1 - p^* - \lambda_1 x_1(p^*; s_1) &= \alpha_1(s_1 - p^* + \mu x_1(p^*; s_1)) + (1-\alpha_1)(s_2 - p^*) - (\lambda_1 + \alpha_1 \mu) x_1(p^*; s_1) \\
&= \alpha_2(p^* - s_2) + (1-\alpha_1)(s_2 - p^*) - (\lambda_1 + \alpha_1 \mu) x_1(p^*; s_1) \\
&= (\alpha_1 + \alpha_2 - 1)(p^* - s_2) + (\lambda_1 + \alpha_1 \mu) x_2(p^*; s_2),
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
v_2 - p^* - \lambda_2 x_2(p^*; s_2) &= \alpha_2(s_2 - p^*) + (1-\alpha_2)(s_1 - p^*) - \lambda_2 x_2(p^*; s_2) \\
&= \alpha_1(p^* - s_1 - \mu x_1(p^*; s_1)) + (1-\alpha_2)(s_1 - p^*) - \lambda_2 x_2(p^*; s_2) \\
&= (\alpha_1 + \alpha_2 - 1)(p^* - s_1) + (\lambda_2 - \alpha_1 \mu) x_1(p^*; s_1).
\end{aligned} \tag{33}$$

Using Equations (32) and (33), we can rewrite the first order condition of bidder i in (30) as a differential equation that involves only bidder j, $j \neq i$:

$$x_1(p^*; s_1) = ((\alpha_1 + \alpha_2 - 1)(p^* - s_1) + (\lambda_2 - \alpha_1 \mu) x_1(p^*; s_1)) \frac{\partial x_1}{\partial p}(p^*; s_1), \tag{34}$$

$$x_2(p^*; s_1) = ((\alpha_1 + \alpha_2 - 1)(p^* - s_2) + (\lambda_1 + \alpha_1 \mu) x_2(p^*; s_1)) \frac{\partial x_2}{\partial p}(p^*; s_2) \tag{35}$$

To solve Equations (34) and (35), we first solve a simpler system:

$$y(z) = (\eta z - \lambda y(z)) y'(z), \quad y(0) = 0, \quad y'(z) > 0 \text{ for } z > 0. \tag{36}$$

After solving (36), we obtain a solution to (34) and (35) by setting

$$x_i(p; s_i) = y(|s_i - p|) \operatorname{sign}(s_i - p). \tag{37}$$

Lemma 1. *Suppose that $0 < \eta < 1$ and $\lambda > 0$. The differential equation*

$$y(z) = (\eta z - \lambda y(z)) y'(z) \tag{38}$$

is solved by the implicit solution to:

$$(1 - \eta) z = C y(z)^\eta - \lambda y(z), \tag{39}$$

where C is a positive constant. If

$$C \geq \left(\frac{\lambda}{\eta}\right)^\eta (\overline{s} - \underline{s})^{1-\eta}, \tag{40}$$

we can select $y(z)$ that solves (39) such that $y(0) = 0$, $y(z) > 0$, $y'(z) > 0$ and $y''(z) > 0$ for every $z \in (0, \overline{s} - \underline{s}]$.

Proof of Lemma 1. Suppose that $y(z)$ satisfies (39). We first show that it must also satisfy

(38). Differentiate (39) with respect to z gives:

$$\begin{aligned}1-\eta &= (C\eta y(z)^{\eta-1}-\lambda)y'(z)\\ &= (Cy(z)^{\eta-1}-\lambda)y'(z) - C(1-\eta)y(z)^{\eta-1}y'(z)\\ &= \frac{(1-\eta)z}{y(z)}y'(z) - C(1-\eta)y(z)^{\eta-1}y'(z)\\ &= \frac{(1-\eta)z - C(1-\eta)y(z)^{\eta}}{y(z)}y'(z),\end{aligned}$$

i.e.,

$$\begin{aligned}1 &= \frac{z - Cy(z)^{\eta}}{y(z)}y'(z)\\ &= \frac{z-((1-\eta)z+\lambda y(z))}{y(z)}y'(z),\end{aligned}$$

which is exactly Equation (38).

Let

$$f(y) = Cy^{\eta} - \lambda y. \tag{41}$$

The function f is clearly strictly concave and obtains its maximum at

$$y^* = \left(\frac{C\eta}{\lambda}\right)^{\frac{1}{1-\eta}}. \tag{42}$$

We choose $C > 0$ so that

$$f(y^*) = C\left(\frac{C\eta}{\lambda}\right)^{\frac{\eta}{1-\eta}} - \lambda\left(\frac{C\eta}{\lambda}\right)^{\frac{1}{1-\eta}} = C^{\frac{1}{1-\eta}}\left(\frac{\eta}{\lambda}\right)^{\frac{\eta}{1-\eta}}(1-\eta) \geq (1-\eta)(\overline{s}-\underline{s}) \tag{43}$$

which is equivalent to (40). Given this choice of C, for every $z \in [0, \overline{s}-\underline{s}]$, by the Intermediate Value Theorem there is a unique $y(z) \in [0, y^*]$ that solves $f(y(z)) = (1-\eta)z$. Since $f'(y(z)) > 0$ for $y(z) \in (0, y^*)$, we have $y'(z) = \frac{1-\eta}{f'(y(z))} > 0$ for $z \in (0, \overline{s}-\underline{s}]$.

Finally, we differentiate both sides of (38) to obtain:

$$y'(z) = (\eta z - \lambda y(z))y''(z) + (\eta - \lambda y'(z))y'(z), \tag{44}$$

i.e.,

$$(\eta z - \lambda y(z))y''(z) = (1-\eta)y'(z) + \lambda y'(z)^2. \tag{45}$$

Since $y'(z) > 0$ and $\eta z - \lambda y(z) = \frac{y(z)}{y'(z)} > 0$, we conclude that $y''(z) > 0$ for $z > 0$.

□

Step 2: deriving the equilibrium strategy.

Given Lemma 1, we let $y_1(z_1)$ and $y_2(z_2)$, where $z_1, z_2 \geq 0$, be implicitly defined by

$$(2-\alpha_1-\alpha_2)z_1 = C_1 y_1(z_1)^{\alpha_1+\alpha_2-1} - (\lambda_2 - \alpha_1\mu)y_1(z_1), \tag{46}$$

$$(2-\alpha_1-\alpha_2)z_2 = C_2 y_2(z_2)^{\alpha_1+\alpha_2-1} - (\lambda_1 + \alpha_1\mu)y_2(z_2), \tag{47}$$

and let

$$x_1(p; s_1) = y_1(|s_1 - p|)\,\mathrm{sign}(s_1 - p), \tag{48}$$

$$x_2(p; s_2) = y_2(|s_2 - p|)\,\mathrm{sign}(s_2 - p). \tag{49}$$

Clearly, (46) is equivalent to

$$(2-\alpha_1-\alpha_2)(z_1 + \mu y_1(z_1)) = C_1 y_1(z_1)^{\alpha_1+\alpha_2-1} - (\lambda_2 - \alpha_1\mu - (2-\alpha_1-\alpha_2)\mu)y_1(z_1), \tag{50}$$

so to satisfy conjecture (31), we let

$$C_1 = \alpha_2 C, \quad C_2 = \alpha_1 C, \tag{51}$$

for a constant $C > 0$, and

$$\lambda_2 - \alpha_1\mu - (2-\alpha_1-\alpha_2)\mu = \frac{\alpha_2}{\alpha_1}(\lambda_1 + \alpha_1\mu), \tag{52}$$

i.e.,

$$\mu = \frac{1}{2}\left(\lambda_2 - \frac{\alpha_2}{\alpha_1}\lambda_1\right). \tag{53}$$

Substituting (51) and (53) into (46) and (47) gives:

$$(2-\alpha_1-\alpha_2)z_1 = C\alpha_2 y_1(z_1)^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}\right)y_1(z_1), \tag{54}$$

$$(2-\alpha_1-\alpha_2)z_2 = C\alpha_1 y_2(z_2)^{\alpha_1+\alpha_2-1} - \left(\lambda_1\left(1 - \frac{\alpha_2}{2}\right) + \lambda_2\frac{\alpha_1}{2}\right)y_2(z_2). \tag{55}$$

If a market-clearing price p^* exists, we have $\mathrm{sign}(s_1 - p^*) = -\,\mathrm{sign}(s_2 - p^*)$ and $y_1(|s_1 -$

$p^*|) = y_2(|s_2 - p^*|)$. Thus, Equations (51) and (52) ensure that from (47) and (50) we have

$$|s_1 - p^*| + \mu y_1(|s_1 - p^*|) = \frac{\alpha_2}{\alpha_1}|s_2 - p^*|, \tag{56}$$

or equivalently

$$(s_1 - p^*) + \mu x_1(p^*; s_1) = -\frac{\alpha_2}{\alpha_1}(s_2 - p^*), \tag{57}$$

which is exactly our conjecture in (31).

The following lemma gives conditions that guarantee the existence of market-clearing price.

Lemma 2. *Suppose that*

$$C \geq \frac{(\overline{s} - \underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2}\left(\frac{\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}}{\alpha_1 + \alpha_2 - 1}\right)^{\alpha_1+\alpha_2-1}, \textit{ and} \tag{58}$$

$$C \geq \frac{(\overline{s} - \underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1}\left(\frac{\lambda_1\left(1 - \frac{\alpha_2}{2}\right) + \lambda_2\frac{\alpha_1}{2}}{\alpha_1 + \alpha_2 - 1}\right)^{\alpha_1+\alpha_2-1}. \tag{59}$$

Then for every profile $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$*, there exists a unique* $p^* \in [\underline{s}, \overline{s}]$ *that satisfies* $x_1(p^*; s_1) + x_2(p^*; s_2) = 0$*.*

Proof. By Lemma 1, conditions (58) and (59) give $y_1 : [0, \overline{s} - \underline{s}] \to [0, \infty)$ and $y_2 : [0, \overline{s} - \underline{s}] \to [0, \infty)$, respectively, that are strictly increasing and convex.

Without loss of generality, suppose that $s_1 < s_2$. There exists a minimum $\bar{y} > 0$ that solves: [6]

$$(2 - \alpha_1 - \alpha_2)(s_2 - s_1) = C(\alpha_1 + \alpha_2)\bar{y}^{\alpha_1+\alpha_2-1} - (\lambda_1 + \lambda_2)\,\bar{y}. \tag{60}$$

Let z_1 satisfies

$$(2 - \alpha_1 - \alpha_2)z_1 = C\alpha_2\bar{y}^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}\right)\bar{y}, \tag{61}$$

[6] By construction, we have

$$(2 - \alpha_1 - \alpha_2)(s_2 - s_1) = C\alpha_2 y^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}\right)y$$

when $y = y_1(s_2 - s_1)$, and

$$(2 - \alpha_1 - \alpha_2)(s_2 - s_1) = C\alpha_1 y^{\alpha_1+\alpha_2-1} - \left(\lambda_1\left(1 - \frac{\alpha_2}{2}\right) + \lambda_2\frac{\alpha_1}{2}\right)y$$

when $y = y_2(s_2 - s_1)$. Hence, by the Intermediate Value Theorem, there exists a $\bar{y} \leq \min(y_1(s_2 - s_1), y_2(s_2 - s_1))$ that satisfies Equation (60).

and let z_2 satisfies

$$(2 - \alpha_1 - \alpha_2) z_2 = C\alpha_1 \bar{y}^{\alpha_1+\alpha_2-1} - \left(\lambda_1 \left(1 - \frac{\alpha_2}{2}\right) + \lambda_2 \frac{\alpha_1}{2}\right) \bar{y}. \tag{62}$$

Clearly, we have $z_1 > 0$, $z_2 > 0$ and $z_1 + z_2 = s_2 - s_1$. Let $p^* = s_1 + z_1$. Then we have $z_1 = p^* - s_1$, $z_2 = s_2 - p^*$, and $y_1(z_1) = y_2(z_2) = \bar{y}$, i.e., $x_1(p^*; s_1) = x_2(p^*; s_2)$.

Finally, the uniqueness of p^* follows from the fact that both $x_1(p; s_1)$ and $x_2(p; s_2)$ are strictly decreasing in p. □

Step 3: verifying ex post optimality.

Finally, we directly verify the ex post optimality of (x_1, x_2). Let

$$\Pi_i(p) = (v_i - p)(-x_j(p; s_j)) - \frac{\lambda_i}{2}(-x_j(p; s_j))^2, \tag{63}$$

for $i \in \{1, 2\}$ and $j \neq i$. We will show that

$$\Pi_i(p^*) \geq \Pi_i(p), \tag{64}$$

for every $p \in [\underline{s}, \overline{s}]$ and every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$.

Without loss of generality, fix $i = 1$ and $s_2 < s_1$. By construction, we have $s_2 < p^* < s_1$, $x_1(p^*; s_1) = -x_2(p^*; s_2) > 0$. Since $x_1(p^*; s_1) > 0$, the first order condition (30) implies that

$$v_1 - p^* - \lambda_1 x_1(p^*; s_1) = v_1 - p^* + \lambda_1 x_2(p^*; s_2) > 0. \tag{65}$$

Let $\bar{p} > p^*$ be such that

$$v_1 - \bar{p} + \lambda_1 x_2(\bar{p}; s_2) = 0. \tag{66}$$

We note that

$$\Pi_1'(p) = (v_1 - p + \lambda_1 x_2(p; s_2)) \left(-\frac{\partial x_2}{\partial p}(p; s_2)\right) + x_2(p; s_2) < 0 \tag{67}$$

for $p > \bar{p}$.

We have

$$\Pi_1(p^*) = \int_0^{x_1(p^*; s_1)} (v_1 - p^* - \lambda_1 q)\, dq > 0. \tag{68}$$

On the other hand, when $p \leq s_2$, we have $x_2(p; s_2) \geq 0$, hence $\Pi_1(p) \leq 0$. Thus, $\Pi_1(p)$ cannot be maximized by $p \in [\underline{s}, s_2]$.

For $p \in (s_2, \overline{s}]$, we have $x_2(p; s_2) = -y_2(p - s_2)$, and hence:

$$\begin{aligned}\Pi_1'(p) =&(v_1 - p - \lambda_1 y_2(p - s_2))y_2'(p - s_2) - y_2(p - s_2) \\ =&(v_i - p - \lambda_1 y_2(p - s_2))y_2'(p - s_j) \\ &- ((\alpha_1 + \alpha_2 - 1)(p - s_2) - (\lambda_1 + \alpha_1 \mu)y_2(p - s_2))y_2'(p - s_2) \\ =&(v_i - p - (\alpha_1 + \alpha_2 - 1)(p - s_2) + \alpha_1 \mu y_2(p - s_2))y_2'(p - s_2)\end{aligned} \tag{69}$$

where the second line follows by the differential equation in (35) and (36). Since $y'(p-s_2) > 0$ for $p > s_2$, $\Pi_1'(p) = 0$ for $p > s_2$ if and only if

$$v_i - p - (\alpha_1 + \alpha_2 - 1)(p - s_2) + \alpha_1 \mu y_2(p - s_2) = 0 \tag{70}$$

for $p > s_2$.

We distinguish between two cases:

1. When $\mu \leq 0$, the left-hand side of (70) is strictly decreasing in p, since by Lemma 1 $y(p - s_2)$ is strictly increasing in p. Thus, Equation (70) has only one solution: $p = p^*$ (by the construction in Step 1 and 2, we have $\Pi_1'(p^*) = 0$).

2. When $\mu > 0$, the left-hand side of (70) is strictly convex in p, since by Lemma 1 $y(p - s_2)$ is strictly convex in p. Thus, Equation (70) has at most two solutions (one of the solutions is $p = p^*$). However, we know that for any $p > \bar{p}$, the left-hand side of the (70) is negative (see Equation (67)). Therefore, $p = p^*$ is the only solution to (70).

Therefore, Equation (70) has only one solution on $(s_2, \overline{s}]$: $p = p^*$. This implies that $\Pi_1'(p) = 0$ has only one solution on $(s_2, \overline{s}]$: $p = p^*$. Since the maximum point of $\Pi_i(p)$ over $[\underline{s}, \overline{s}]$ cannot be in $[\underline{s}, s_2]$ or in $[\bar{p}, \overline{s}]$, it must be in $(s_2, \bar{p})$ and satisfies $\Pi_i'(p) = 0$. We thus conclude that $p = p^*$ maximizes $\Pi_i(p)$ over all $p \in [\underline{s}, \overline{s}]$.

A.2 Proof of Proposition 2

We show that $f(q^e) > (2 - \alpha_1 - \alpha_2)|s_1 - s_2|$, where f is defined in Equation (16). Since $y = q^*(C)$ is the smaller solution to $f(y) = (2 - \alpha_1 - \alpha_2)|s_1 - s_2|$, we must have $q^*(C) < q^e$.

Clearly, $f(q^e) > (2 - \alpha_1 - \alpha_2)|s_1 - s_2|$ is equivalent to:

$$C > \frac{|s_1 - s_2|^{2-\alpha_1-\alpha_2}}{\alpha_1 + \alpha_2} \left(\frac{\lambda_1 + \lambda_2}{\alpha_1 + \alpha_2 - 1} \right)^{\alpha_1+\alpha_2-1}. \tag{71}$$

Let us define:

$$\mathcal{C}_1 \equiv \frac{(\overline{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2}\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \tag{72}$$

$$\mathcal{C}_2 \equiv \frac{(\overline{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1}\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \tag{73}$$

$$\mathcal{C} \equiv \frac{(\overline{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1+\alpha_2}\left(\frac{\lambda_1+\lambda_2}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}. \tag{74}$$

We claim that $\max(\mathcal{C}_1,\mathcal{C}_2) > \mathcal{C}$. For the sake of contradiction, suppose $\max(\mathcal{C}_1,\mathcal{C}_2) \le \mathcal{C}$; this implies:

$$\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le \frac{\alpha_2}{\alpha_1+\alpha_2}$$
$$\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le \frac{\alpha_1}{\alpha_1+\alpha_2},$$

which implies

$$\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} + \left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le 1,$$

which is clearly false given $0 < \alpha_1+\alpha_2-1 < 1$.

Hence Conditions (5) and (6), which state that $C \ge \max(\mathcal{C}_1,\mathcal{C}_2)$, imply that $C > \mathcal{C}$, which implies (71).

A.3 Proof of Corollary 4

The proof of Corollary 4 follows the exact same steps as that of Proposition 1; the only difference is solving the differential equation (c.f. Lemma 1):

$$(z-\lambda y(z))y'(z) = y(z), \tag{75}$$

whose solution is given by the implicit equation

$$Cy(z) - \lambda y(z)\log(y(z)) = z, \tag{76}$$

where C is a constant.

References

Ausubel, L., P. Cramton, and R. Deneckere (2002): "Bargaining with Incomplete Information," in *Handbook of Game Theory*, ed. by R. J. Aumann and S. Hart, Elsevier, chap. 50.

Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2011): "Demand Reduction, Inefficiency and Revenues in Multi-Unit Auctions," Working paper.

Chatterjee, K. and W. Samuelson (1983): "Bargaining under Incomplete Information," *Operations Research*, 31, 835–851.

Deneckere, R. and M.-Y. Liang (2006): "Bargaining with Interdependent Values," *Econometrica*, 74, 13091364.

Du, S. and H. Zhu (2013): "Dynamic Ex Post Equilibrium, Welfare, and Optimal Trading Frequency in Double Auctions," Working paper.

Duffie, D., N. Garleanu, and L. Pedersen (2005): "Over-the-Counter Markets," *Econometrica*, 73, 1815–1847.

Duffie, D., S. Malamud, and G. Manso (2009): "Information Percolation with Equilibrium Search Dynamics," *Econometrica*, 77, 1513–1574.

——— (2013): "Information Percolation in Segmented Markets," Working paper.

Kyle, A. S. (1989): "Informed Speculation with Imperfect Competition," *Review of Economic Studies*, 56, 317–355.

Rostek, M. and M. Weretka (2012): "Price Inference in Small Markets," *Econometrica*, 80, 687–711.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97109.

Satterthwaite, M. A. and S. R. Williams (1989): "Bilateral Trade with the Sealed Bid k-Double Auction: Existence and Efficiency," *Journal of Economic Theory*, 48, 107–133.

Vives, X. (2011): "Strategic Supply Function Competition with Private Information," *Econometrica*, 79, 1919–1966.