# New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models*

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**Abstract**

This paper features entropy-based restrictions on SDFs, and its correlated multiplicative components, to evaluate models under the setting that stochastic discount factors (SDFs) jointly price a vector of returns. Specifically, our entropy bound on the square of the SDFs is intended to capture the time-variation in the conditional volatility of the log SDF, as well as non-normalities. Each entropy bound can be inferred from the mean and the variance-covariance matrix of a vector of asset returns. Extending extant treatments, we develop entropy codependence measures, and our bounds generalize to multi-period SDFs. Our approach offers ways to improve model performance.



KEY WORDS: Entropy, stochastic discount factors, permanent component, lower entropy bounds, entropy codependence, asset pricing models, eigenfunction problem.

JEL CLASSIFICATION CODES: C51, C52, G12.

# 1. Introduction

The quest for well-performing stochastic discount factors (hereby SDFs) has dominated the agenda in asset pricing. Despite substantial progress, identifying the desirable properties of the SDFs and the embedded permanent and transitory components, in addition to their link to economic fundamentals, remains an unresolved issue. The search is ongoing, as can be inferred from the treatments in Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Bakshi and Chabi-Yo (2012), Hansen (2012), Backus, Chernov, and Zin (2014), and Christensen (2014).

Our approach lies within the tradition of examining the SDFs, together with their correlated permanent and transitory components (e.g., Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)), and we propose new entropy restrictions to evaluate asset pricing models. In the vein of Hansen and Jagannathan (1991) and Bakshi and Chabi-Yo (2012), among others, our entropy representations are cast in a framework in which the SDF correctly prices finitely many asset returns (note that when an SDF correctly prices a portfolio, it is not tantamount to correctly pricing each of the assets constituting the portfolio). More specifically, our entropy bound representations rely on a framework that exploits the return properties of the risk-free bond, the long-term discount bound, and a vector of risky assets.

We offer several theoretical results. First, we generalize and extend the Alvarez and Jermann (2005) entropy bound on the permanent component of SDFs, as well as the Backus, Chernov, and Zin (2014) entropy bound on the SDFs. The new entropy bounds are parameterized in terms of both a vector of expected returns and a variance covariance matrix of returns, and they have no analytical analogs. Because the new bounds incorporate information about the joint dynamics of multiple asset returns, the bounds are quantitatively tighter. Such a feature allows for a better discrimination among asset pricing models.

Second, and equally pertinent from economic perspectives, we develop a new entropy measure based on the square of the SDF, and the square of the permanent component of the SDF. We establish that such performance measures are suitable for characterizing departures from lognormality and for capturing time-variation in the conditional volatility of the log SDF. We further show that our bounds are distinct from the bounds derived in Hansen and Jagannathan (1991) and Bakshi and Chabi-Yo (2012), who focus on the variance of the SDFs and the variance of the permanent component of SDFs, respectively.

Third, motivated by Hansen (2012), we develop new entropy codependence measures between the permanent and the transitory components of the SDF. Our characterizations highlight a feature essential to all models; in particular, we show that a viable SDF should admit a positive codependence between the permanent and transitory components of the SDF. We analytically show that the sign of codependence can be inferred from Treasury bond data. Our codependence measures give insights into the properties of the SDFs that cannot be obtained by applying the performance measures in Hansen and Jagannathan (1991), Alvarez and Jermann (2005), Bakshi and Chabi-Yo (2012), and Backus, Chernov, and Zin (2014).

Fourth, we provide analytical expressions for entropy bounds in a multi-period investment setting. One possible benefit of these bounds is that they offer flexibility in benchmarking the models to asset return data measured over both short and long horizons.

Directly relevant to our characterizations are the following questions: Why should one care about the entropy bound on the permanent component of the SDFs? If, as asserted in Alvarez and Jermann (2005, Proposition 1) and Hansen and Scheinkman (2009, Proposition 7.2), the permanent component can be uniquely identified from the multiplicative decomposition of the SDF, what do we additionally learn by studying the restrictions that the permanent and transitory component impose on asset market data? First, isolating the desirable properties of the permanent and the transitory components could move us closer to a better understanding of viable SDFs. Second, entropy bounds on the permanent component of the SDFs could help to "look under the hood of a model," and uncover potential inconsistencies of a model with short-run and long-run implications of SDFs (Hansen (2012, page 913)). Third, it is of interest to correctly model the sign of the dependence between the permanent and the transitory components, which, we argue, is necessary for capturing key aspects of asset markets.

We illustrate the usefulness of our bounds in the context of three (state-of-the-art) asset pricing models: (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity (as presented in Backus, Chernov, and Zin (2014)). We analytically solve the eigenfunction problem and derive the permanent and transitory components of the SDF of each model. Our framework provides new perspectives on the performance of these models and their ability to fit asset market quantities.

When the bounds are constructed based on the premise that the SDF correctly prices the risk-free bond, the long-term discount bond, and finitely many risky assets, our implementation reveals that each model produces insufficient entropy to satisfy the lower bound on both the permanent component of the SDF and the SDF itself. A block bootstrap-based procedure provides statistical support for our conclusions.

The entropy bound on the square of the SDF enables a crucial dimension of model assessment. Specifically, the difference habit and the recursive utility with stochastic variance models are rejected. These models can explain only about half of the lower bound estimated from returns data. However, the recursive utility with constant jump intensity model generates entropy that is substantially higher than the lower entropy bound implied from the data. In our search for a possible explanation, we find that this model's success can be traced to jump parameterizations of consumption growth that also produce unrealistic distributional higher-moments of the SDF.

We also show that the recursive utility with jump intensity model struggles to match properties of bond returns, as gauged by its lack of consistency with the transitory component of the SDF. Moreover, each model appears to be inconsistent with entropy-based measures of codependence between the permanent and the transitory components of the SDF.

Our work belongs to a branch of asset pricing that explores the relevance of entropy bounds to distinguish among models. We show that our entropy measure on the square of the SDF is related to the expected return of a security that pays the SDF (Theorem 1). We further derive bounds on entropy of the square of the SDFs and the permanent component of the SDFs (Theorem 2), and we feature general entropy bounds on the SDFs and the permanent component of the SDFs (Theorems 3 and 4).

Our bounds are aimed at complementing the approaches in Alvarez and Jermann (2005), Bakshi and Chabi-Yo (2012), and Backus, Chernov, and Zin (2014). In the manner of Hansen and Jagannathan (1991, 1997), our formalizations strive to understand model attributes, but our thrust is on the SDFs in conjunction with their correlated permanent and transitory components. Moreover, our approach inherits the model-free flavor of Hansen and Jagannathan (1991); we propose a codependence measure (Theorem 5), and we develop a multi-period extension (Theorem 6). The entropy bounds are tractable, convey rich economic interpretations, can encapsulate data considerations that transcend model calibrations, and our framework can incorporate statistical concerns in model assessment.

## 2. Correlated multiplicative decomposition of SDFs and motivating entropy

Let $m_{t,t+1}$ represent the stochastic discount factor between date $t$ and $t+1$. Our objective is to propose new entropy measures to evaluate asset pricing models when $m_{t,t+1}$ is required to price many (distinct) asset returns. Hansen and Jagannathan (1991, equation (3)) emphasize that omitting returns can weaken the implications for $m_{t,t+1}$.

### *2.1. Attributes of the multiplicative decomposition of SDFs*

As a starting point, we employ a result in Alvarez and Jermann (2005, Proposition 1, page 1983) and Hansen and Scheinkman (2009, page 200), who establish that $m_{t,t+1}$ admits a multiplicative decomposition:

$$m_{t,t+1} = m^P_{t,t+1} m^T_{t,t+1} \qquad \text{with} \qquad E\left[m^P_{t,t+1}\right] = 1 \qquad \text{and} \qquad m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1}, \tag{1}$$

where $m^P_{t,t+1}$ ($m^T_{t,t+1}$) is the permanent (transitory) component of $m_{t,t+1}$, $R_{t,t+1,\infty}$ is the gross return of an infinite-maturity discount bond, and $E[.]$ is the unconditional expectation.

Alvarez and Jermann (2005, Proposition 1) show that the $m^P_{t,t+1}$ component of the SDF is unique when $m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1}$. It is the case of uniquely identified $m^P_{t,t+1}$ that is of economic interest.

The components $m^P_{t,t+1}$ and $m^T_{t,t+1}$ can be correlated, and, if they exist, can be obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009, Corollary 6.1). In the context of parameterized asset pricing models, both Hansen (2012) and Christensen (2014, Section 3) show that an appropriately solved eigenfunction problem will ensure a unique $m^P_{t,t+1}$.

### 2.2. *Motivating the entropy of $m^2_{t,t+1}$ and $\left(m^P_{t,t+1}\right)^2$ in asset pricing tests*

To assess the merits of an asset pricing model, Alvarez and Jermann (2005, page 1985) propose using the entropy of $m^P_{t,t+1}$, defined below (the entropy $L[m_{t,t+1}]$ is similarly defined):

$$L[m^P_{t,t+1}] \;=\; \log(E[m^P_{t,t+1}]) \;-\; E[\log(m^P_{t,t+1})] \;=\; -E[\log(m^P_{t,t+1})], \qquad (\text{since } E[m^P_{t,t+1}]=1) \tag{2}$$

Alvarez and Jermann show that for some distributions, $L[m^P_{t,t+1}]$ completely characterizes the distribution of $\log(m^P_{t,t+1})$. For example, if $m^P_{t,t+1}$ is lognormally distributed, we must have $\exp(E[\log(m^P_{t,t+1})] + \frac{1}{2}Var[\log(m^P_{t,t+1})]) = 1$. Hence, $L[m^P_{t,t+1}] = -E[\log(m^P_{t,t+1})] = \frac{1}{2}Var[\log(m^P_{t,t+1})]$, and it is only the variance (or equivalently the mean in this setting) of $\log(m^P_{t,t+1})$ that matters for asset pricing. To generate higher entropy, modeling approaches often incorporate non-normalities in $m^P_{t,t+1}$ and $m_{t,t+1}$, as noted also in Backus, Chernov, and Zin (2014), which could also help to achieve consistency with asset market data.

#### 2.2.1. *New entropy measure and its economic interpretation*

One aim of this paper is to motivate an alternative entropy-based measure, specifically $L[(m^P_{t,t+1})^2]$ (or $L[m^2_{t,t+1}]$), as a metric for evaluating asset pricing models, in conjunction with $L[m^P_{t,t+1}]$. In an analogy to equation (2), we consider

$$L[m^2_{t,t+1}] = \log(E[m^2_{t,t+1}]) - E[\log(m^2_{t,t+1})], \qquad \text{and,} \tag{3}$$

$$L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)]. \tag{4}$$

Our measure $L[m^2_{t,t+1}]$ is related to *Jensen's gap*, defined as, $J\{u\} \equiv E[f\{u\}] - f\{E[u]\} \geq 0$ applied to the convex function $f\{u\} = -\log(u^2)$. In contrast, the variance measure used in Hansen and Jagannathan (1991) is related to Jensen's gap applied to the convex function $f\{u\} = u^2$.

There is an important economic interpretation associated with $L[m_{t,t+1}^2]$. In particular, $L[m_{t,t+1}^2]$ encodes information about the expected return of a fundamental security, namely, the security that entitles the investor a payoff of $m_{t,t+1}$. The return of this security is $r_{t,t+1}^{\text{SDF}} \equiv \frac{m_{t,t+1}}{E_t[m_{t,t+1}^2]} - 1$. Any variable that comoves with $r_{t,t+1}^{\text{SDF}}$ is a potential factor (e.g., Merton (1973)). We now prove.

**Theorem 1** *The expected return on a security that pays the SDF is related to $L[m_{t,t+1}^2]$ as follows:*

$$0 \;<\; E[\log(R_t^f)] \;-\; E[\log(1+r_{t,t+1}^{\text{SDF}})] \;\leq\; L[m_{t,t+1}^2], \tag{5}$$

*where $R_t^f$ is the gross return of the risk-free bond.*

**Proof**: See Appendix A. ■

The square of the entropy of $m_{t,t+1}^2$ is the maximum excess log return on a security that pays the SDF. There is a counterpart interpretation for $L[(m_{t,t+1}^P)^2]$. The security that pays $m_{t,t+1}^P/m_{t,t+1}^T$ has a time $t$ price of $E_t[(m_{t,t+1}^P)^2]$, and return $r_{t,t+1}^{\text{PSDF}} \equiv \frac{m_{t,t+1}^P/m_{t,t+1}^T}{E_t[(m_{t,t+1}^P)^2]} - 1$. Then it can be shown that $E[\log(R_{t,t+1,\infty})] - E[\log(1+r_{t,t+1}^{\text{PSDF}})] \leq L[(m_{t,t+1}^P)^2]$.

### *2.2.2. Insights from example economies*

While developing the implications of this new entropy measure, our analysis centers around two key issues. First, what do we miss when the entropy measure $L[m_{t,t+1}^P]$ (or $L[m_{t,t+1}]$) is employed to assess the consistency of $m_{t,t+1}^P$ (or $m_{t,t+1}$) of an asset pricing model with observed asset prices? Second, what do we gain when $L[(m_{t,t+1}^P)^2]$ (or $L[m_{t,t+1}^2]$) is applied to asset pricing problems? Our framework is also pertinent to understanding how one could use observed asset prices to learn about dependence in $m_{t,t+n}^P$ (or $m_{t,t+n}$) over any generic investment horizon $n$.

The next example first showcases an environment where $L[(m_{t,t+1}^P)^2]$ has no role beyond $L[m_{t,t+1}^P]$. Steps leading to most of the results that follow are shown in Online Appendix I.

**Example 1** Let the dynamics of the permanent and transitory components be given by (Alvarez and Jermann (2001, page 9); see also Campbell (1986, equation (3))):

$$\log\left(m_{t,t+1}^{P}\right) = -\frac{1}{2}\sigma_P^2 + \varepsilon_{t+1}^{P} \quad \text{and} \quad \log\left(m_{t,t+1}^{T}\right) = \log(\beta) + \alpha_0\,\varepsilon_{t+1}^{T} + \sum_{i=1}^{\infty}(\alpha_i - \alpha_{i-1})\,\varepsilon_{t+1-i}^{T}, \tag{6}$$

where the two shocks are normally distributed and homoskedastic, i.e., $\varepsilon_{t+1}^{P} \sim \mathcal{N}(0,\sigma_P^2)$ and $\varepsilon_{t+1}^{T} \sim \mathcal{N}(0,\sigma_T^2)$, with constant correlation. Then

$$L[(m_{t,t+1}^{P})^2] \;=\; 4L[m_{t,t+1}^{P}] \qquad \text{where} \qquad L[m_{t,t+1}^{P}] \;=\; \frac{\sigma_P^2}{2}. \tag{7}$$

Equation (7) shows that when the conditional volatility of $\log(m_{t,t+1}^{P})$ is time-invariant, the two entropy measures $L[m_{t,t+1}^{P}]$ and $L[(m_{t,t+1}^{P})^2]$ contain identical information, i.e., $L[(m_{t,t+1}^{P})^2] - 4L[m_{t,t+1}^{P}] = 0$. ♣

To address possible advantages of $L[(m_{t,t+1}^{P})^2]$ over $L[m_{t,t+1}^{P}]$ from the vantage point of asset pricing, we apply the definition of $L[u]$ to random variables $u^2$ and $u$, and arrive at the following result:

$$L[(m_{t,t+1}^{P})^2] \;-\; 4L\left[m_{t,t+1}^{P}\right] \;=\; \log(E\left[(m_{t,t+1}^{P})^2\right]) \;+\; E[\log((m_{t,t+1}^{P})^2)]. \tag{8}$$

Equation (8) indicates that the departure between $L[(m_{t,t+1}^{P})^2]$ and $4L[m_{t,t+1}^{P}]$ can be attributed to the time-variation in the conditional volatility of $\log(m_{t,t+1}^{P})$. The following example puts this notion on a solid footing.

**Example 2** Suppose an eigenfunction problem yields $\log(m_{t,t+1}^{P}) \sim \mathcal{N}(\mu_t,\sigma_t^2)$. Then

$$L[(m_{t,t+1}^{P})^2] = \log\left(E\left[e^{\sigma_t^2}\right]\right) \;+\; E[\sigma_t^2] \qquad \text{and} \qquad L[m_{t,t+1}^{P}] = \frac{1}{2}E[\sigma_t^2]. \tag{9}$$

Using the Taylor expansion of $e^{\sigma_t^2}$ around $\sigma_t^2 = E\left[\sigma_t^2\right]$, we observe that

$$L[(m_{t,t+1}^{P})^2] \;-\; 4L[m_{t,t+1}^{P}] \;=\; \log\left(1 + \sum_{j=2}^{\infty}\frac{1}{j!}E\left[(\sigma_t^2 - E[\sigma_t^2])^j\right]\right). \tag{10}$$

The information embedded in the distribution of $\sigma_t^2$ differentiates $L[(m_{t,t+1}^{P})^2]$ from $L[m_{t,t+1}^{P}]$. In general,

$L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ contain distinct information relevant to distinguishing asset pricing models. ♣

We further note that $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ do not coincide because $L[m^P_{t,t+1}] > 0$; hence, $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ reflect distinct entropies. More generally, $L[(m^P_{t,t+1})^2]$ subsumes $L[m^P_{t,t+1}]$.

The entropy measure $L[(m^P_{t,t+1})^2]$ offers flexibility in detecting non-normalities in $\log(m^P_{t,t+1})$. From a Taylor expansion of $\exp\left(\log((m^P_{t,t+1})^2)\right)$ around $E[\log(m^P_{t,t+1})]$, we note that equation (4) implies:

$$L[(m^P_{t,t+1})^2] = \log\left(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!}\kappa_j\right), \qquad \text{where} \qquad \kappa_j \equiv E[(\log(m^P_{t,t+1}) - E[\log(m^P_{t,t+1})])^j] \tag{11}$$

is the $j$th central moment of $\log(m^P_{t,t+1})$. The normality of $\log(m^P_{t,t+1})$ imposes two restrictions: first, that $\kappa_j = 0$ for $j \geq 3$, and, second, that $L[m^P_{t,t+1}] = -E[\log(m^P_{t,t+1})] = \frac{1}{2}Var[\log(m^P_{t,t+1})]$. Therefore, under the normality of $\log(m^P_{t,t+1})$,

$$L[(m^P_{t,t+1})^2] \;=\; \log\left(1 \;+\; 2Var\left[\log\left(m^P_{t,t+1}\right)\right]\right) \;=\; \log\left(1 - 4E\left[\log\left(m^P_{t,t+1}\right)\right]\right) \;\approx\; 4L[m^P_{t,t+1}]. \tag{12}$$

Thus, $L[(m^P_{t,t+1})^2]$ may be construed as capturing the departure of $\log(m^P_{t,t+1})$ from normality.

In a similar vein, $L[m^2_{t,t+1}]$ captures asymmetries in $\log(m_{t,t+1})$. We note that $L[m^2_{t,t+1}] = \log(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!}E(\log(m_{t,t+1}) - E[\log(m_{t,t+1})])^j)$, and therefore $L[m^2_{t,t+1}] = 4L[m_{t,t+1}]$ under the normality of $\log(m_{t,t+1})$. In Colacito, Ghysels, and Meng (2013, equation (11)), the $\log(m_{t,t+1})$ is not normal and $L[m^2_{t,t+1}] \neq 4L[m_{t,t+1}]$, illustrating that $L[m^2_{t,t+1}]$ could be a suitable candidate for evaluating asset pricing models under deviations from lognormality.

There is also an exact relation between $L[m^P_{t,t+1}]$, $L[(m^P_{t,t+1})^2]$ and $Var[m^P_{t,t+1}]$, illustrating that asset pricing models might satisfy restrictions on $L[m^P_{t,t+1}]$ and not on $L[(m^P_{t,t+1})^2]$. It may be verified that

$$L[(m^P_{t,t+1})^2] \;-\; 2L[m^P_{t,t+1}] \;=\; \log\left(1 \;+\; Var\left[m^P_{t,t+1}\right]\right) \;\approx\; Var\left[m^P_{t,t+1}\right]. \tag{13}$$

When $m^P_{t,t+1}$ is the permanent component with the lowest variance, $Var[m^P_{t,t+1}]$ corresponds to the *minimum* variance of $m^P_{t,t+1}$ in Bakshi and Chabi-Yo (2012, equation (6)).

The following example further synthesizes the various elements of our analysis.

**Example 3** Suppose the SDF is governed by (Backus, Foresi, and Telmer (2001, equation (19))),

$$\log(m_{t,t+1}) = -\delta - \gamma z_t - \lambda z_t^{\frac{1}{2}} \varepsilon_{t+1}, \quad z_{t+1} = (1-\varphi)\theta + \varphi z_t + \sigma z_t^{\frac{1}{2}} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \quad (14)$$

for a state variable $z_t$. We solve the eigenfunction problem to derive (see the Online Appendix II):

$$m^T_{t,t+1} = \exp\left(-\delta + \xi(1-\varphi)\theta + \xi(z_t - z_{t+1})\right), \qquad m^P_{t,t+1} = \exp\left((-\xi - \gamma + \xi\varphi)z_t + (\xi\sigma - \lambda)\, z_t^{\frac{1}{2}} \varepsilon_{t+1}\right), \quad (15)$$

where $\xi \equiv \frac{-(\varphi-1-\lambda\sigma)-\sqrt{(\varphi-1-\lambda\sigma)^2-2\sigma^2\left(\frac{1}{2}\lambda^2-\gamma\right)}}{\sigma^2}$ and $\gamma \equiv \frac{1}{2}\left(1+\lambda^2\right)$. It can be further shown that

$$L[(m^P_{t,t+1})^2] - 2L[m^P_{t,t+1}] = \log\left(E\left[e^{\left(2(-\xi-\gamma+\xi\varphi)\, +\, 2(\xi\sigma-\lambda)^2\right)z_t}\right]\right). \quad (16)$$

The stochastic nature of $z_t$ can introduce a wedge between $L[(m^P_{t,t+1})^2]$ and $L[m^P_{t,t+1}]$. ♣

In summary, our analytical links highlight that if one is interested in using entropy to learn about properties of the SDFs, one may need to use $L[(m^P_{t,t+1})^2]$ ($L[m^2_{t,t+1}]$) in conjunction with $L[m^P_{t,t+1}]$ ($L[m_{t,t+1}]$). An essential distinguishing trait of the entropy measure $L[(m^P_{t,t+1})^2]$ is its ability to more effectively cope with the effect of time-varying volatility and distributional non-normalities in $\log(m^P_{t,t+1})$.

## 3. Entropy bounds when the SDF correctly prices finitely many returns

This section features four theoretical results when the SDF is required to correctly price finitely many returns. First, we develop the bounds on $L[m^2_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$. Second, we present the bound on $L[m_{t,t+1}]$ and then a bound on $L[m^P_{t,t+1}]$. Next, we develop restrictions that are based on the entropy codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$. In so doing, we also illustrate the advantages of the bounds on $L[m^2_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ in assessing asset pricing models versus the bounds on $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$.

To proceed with the development of our entropy bounds, consider a set $\mathbb{S}$ of SDFs that correctly prices a finite number of returns:

$$\mathbb{S} = \{m_{t,t+1} > 0 : E_t\left[m_{t,t+1}\right] = q_t,\ \ E\left[m_{t,t+1}R_{t,t+1,\infty}\right] = 1,\ \text{ and }\ E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}\}, \tag{17}$$

where $\mathbf{1}$ is a vector column of ones. Moreover, $\mathbf{R}_{t,t+1}$ is an $N \times 1$ vector of gross returns that excludes the risk-free bond and the infinite-maturity discount bond.

We further postulate that some SDFs that belong to the set $\mathbb{S}$ can be decomposed into permanent and transitory components:

$$\mathbb{S}_P = \left\{m_{t,t+1} \in \mathbb{S} : m_{t,t+1} = m^P_{t,t+1}\, m^T_{t,t+1},\ \text{ and }\ m^T_{t,t+1} = \left(R_{t,t+1,\infty}\right)^{-1}\right\}. \tag{18}$$

Hence, this paper focuses on the class of SDFs that can be decomposed uniquely into a permanent and a transitory component, that is, which admit $m^T_{t,t+1} = \left(R_{t,t+1,\infty}\right)^{-1}$.

Equally important, equation (17) requires the SDF to correctly price each of the $N+2$ distinct assets. Therefore, due to the pricing of additional risky assets, set $\mathbb{S}$ is considerably smaller than its counterparts based on pricing three assets, for example, a risk-free bond, a long-term bond, and a generic portfolio of risky assets, as in Alvarez and Jermann (2005). Overall, the formulations in equations (17)–(18) allow us to develop entropy bounds that are based on the return properties of $N+2$ assets and, hence, offer considerable generality and may be sharper. It is known from Hansen and Jagannathan (1991, page 230) that excluding the full pricing information in the theoretical analysis can weaken the implications for $m_{t,t+1}$.

We further note that Backus, Chernov, and Zin (2014, equation (2), page 56) consider a set of SDFs that correctly prices a single asset return, or a single return based on a generic portfolio of assets, that is, $S^* = \{m_{t,t+1} > 0 : E_t\left[m_{t,t+1}\right] = q_t, E[m_{t,t+1}(\sum_{i=1}^{I} \varpi_i R^i_{t,t+1})] = 1\}$, for some predetermined weight $\varpi_i$. Clearly, the set $\mathbb{S}$ is considerably smaller than $S^*$ because $\mathbb{S} \subset S^*$. In their implementation, Backus, Chernov, and Zin (2014, Column 2 of Table I) present the mean excess log returns on equity portfolios, equity options, currencies, and nominal bonds, which also coincide with a lower bound on $L[m_{t,t+1}]$ (i.e., the right-hand side of expression for $I(1)$ in Backus, Chernov, and Zin (2014, equation (5))).

By considering a set of SDFs that simultaneously prices a number of risky assets, our approach aims to extend Backus, Chernov, and Zin (2014) and deliver entropy bounds that exploit the information contained in the vector of returns $\mathbf{R}_{t,t+1}$, more specifically, the variance-covariance matrix and the vector of average returns. The necessity of developing entropy bounds that rely on an SDF that correctly prices multiple risky assets simultaneously has also been highlighted in Christensen (2014, footnote 10). In one extreme, our implementation could accommodate entropy bounds that are based on the ability of SDFs to correctly price all available individual assets in the economy (in the flavor of Ang, Liu, and Schwarz (2013)).

Guided by our discussions, we first derive the bounds on the entropies $L[m_{t,t+1}^2]$ and $L[(m_{t,t+1}^P)^2]$.

### *3.1. Characterizing the bounds on the entropies $L[m_{t,t+1}^2]$ and $L[(m_{t,t+1}^P)^2]$*

**Theorem 2** *The following bounds are germane to asset pricing models:*

*(a) The entropy of $m_{t,t+1}^2$ satisfies:*

$$L[m_{t,t+1}^2] \geq 2\left(E\left[\log\left(\frac{(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])'\Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])}\mathbf{R}_{t,t+1}\right)\right] - \log\left((E[q_t])^{-1}\right)\right) + \log\left(1+(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])'\Sigma^{-1}(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])/(E[q_t])^2\right), \tag{19}$$

*where $\Sigma$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$.*

*(b) The entropy of $(m_{t,t+1}^P)^2$ satisfies*

$$L[(m_{t,t+1}^P)^2] \geq 2\left(E\left[\log\left(\frac{(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])'\Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])}\mathbf{R}_{t,t+1}\right)\right] - E[\log(R_{t,t+1,\infty})]\right) + \log\left(1+(\mathbf{1}-E[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}])'\Sigma_P^{-1}(\mathbf{1}-E[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}])\right), \tag{20}$$

*where $\Sigma_P$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$.*

**Proof**: See Appendix B. ■

The entropy bounds stipulated in equations (19) and (20) summarize properties of the distribution of

$m_{t,t+1}$ and $m^P_{t,t+1}$ and, hence, contain information that could help to gauge asset pricing models. Moreover, the lower bounds presented in the right-hand side of equations (19) and (20) in Theorem 2 are computable from the time-series of asset returns and discount bonds. In addition, our bounds are model-free.

One may interpret the lower bound on $L[m^2_{t,t+1}]$ in equation (19) of Theorem 2 as having two economically meaningful components. The first term surrogates an excess rate of return, whereas the second term is proportional to a Sharpe ratio-related component. Both equations (19) and (20) offer the result that the entropies $L[m^2_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ are determined by the vector of mean returns and a quadratic form of mean and variance of the vector of returns.

### *3.2. Bound on the entropy $L[m_{t,t+1}]$ and comparison with Backus, Chernov, and Zin (2014)*

**Theorem 3** *The entropy of $m_{t,t+1}$ satisfies:*

$$L[m_{t,t+1}] \;\geq\; E\left[\log\left(\frac{\left(\Sigma^{-1}\left(\mathbf{1}-E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)\right)'\mathbf{R}_{t,t+1}}{\mathbf{1}'\Sigma^{-1}\left(\mathbf{1}-E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)}\right)\right] \;-\; \log\left(\left(E\left[q_t\right]\right)^{-1}\right), \tag{21}$$

*where $\Sigma$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$.*

**Proof**: See Appendix C. ■

Importantly, our bound on $L[m_{t,t+1}]$ in equation (21) extends the Backus, Chernov, and Zin (2014, equation (5)) bound when the SDF correctly prices many risky assets. To elaborate, consider augmenting the stock market (or any benchmark portfolio) with a risky security, that is, $\mathbf{R}_{t,t+1} \equiv [R^m_{t,t+1}, R^i_{t,t+1}]$. It is shown in the Online Appendix III that the entropy bound in equation (21) becomes

$$L[m_{t,t+1}] \;\geq\; \underbrace{E\left[\log\left(R^m_{t,t+1}\right)\right] - \log\left(\left(E\left[q_t\right]\right)^{-1}\right)}_{\text{BCZ entropy bound}} \;+\; \underbrace{E\left[\log\left(1 \;+\; b_0\frac{(R^i_{t,t+1}-R^m_{t,t+1})}{R^m_{t,t+1}}\right)\right]}_{\text{positive incremental contribution}}. \tag{22}$$

The constant $b_0$ depends on the first and second moment of $\mathbf{R}_{t,t+1}$ and is presented in equation (I8) of Online Appendix III. Thus, increasing the dimensionality of $\mathbf{R}_{t,t+1}$ beyond a single asset (or benchmark portfolio) $R^m_{t,t+1}$ could be expected to lead to a more stringent lower bound on $L[m_{t,t+1}]$.

We are often asked to clarify the sense in which our bound extends the Backus, Chernov, and Zin (2014) paper to multiple assets. While their equation (5) is derived for an arbitrary return that satisfies the Euler equation, this return could be associated with any portfolio of traded assets. However, note that when the Euler equation is satisfied for a portfolio, it does not imply that the SDF correctly prices each individual return in the portfolio (granted that individual mispricings could cancel out). In contrast, when an SDF correctly prices individual returns, the linearity of the pricing rule ensures correct pricing of a portfolio (with non-random weights). It turns out that using the framework of equation (17) exerts a non-trivial effect on the magnitude of the entropy bound, which could reveal novel insights into the properties of $m_{t,t+1}$. The lesson is that the set of SDFs under consideration bear considerably in evaluating asset pricing models.

Completing our arguments, when there is a single risky asset, the gross return vector $\mathbf{R}_{t,t+1}$ reduces to a single return $R^m_{t,t+1}$. In this case, our lower bound on $L[m_{t,t+1}]$ depends only on excess log returns:

$$E[\log\left(R^m_{t,t+1}\right)] \;-\; \log\left(1/E\left[q_t\right]\right) \qquad \text{versus} \qquad E[\log\left(R^m_{t,t+1}\right)] \;-\; E[\log(R^f_t)] \tag{23}$$

in Backus, Chernov, and Zin (2014). Since $R^f_t = 1/q_t$, Jensen's inequality implies $E[\log\left(q_t\right)] \leq \log\left(E\left[q_t\right]\right)$.

The general bound on $L[m_{t,t+1}]$ in equation (21) can be employed in conjunction with the bound on $L[m^2_{t,t+1}]$ to evaluate whether SDF properties are consistent with observed asset prices.

### *3.3. Bound on the entropy $L[m^P_{t,t+1}]$ and comparison with Alvarez and Jermann (2005)*

**Theorem 4** *The entropy of $m^P_{t,t+1}$ satisfies:*

$$L[m^P_{t,t+1}] \;\geq\; E\left[\log\left(\frac{\left(\Sigma^{-1}\left(\mathbf{1}-E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)\right)'\mathbf{R}_{t,t+1}}{\mathbf{1}'\Sigma^{-1}\left(\mathbf{1}-E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)}\right)\right] \;-\; E\left[\log\left(R_{t,t+1,\infty}\right)\right]. \tag{24}$$

**Proof**: See Appendix D. ■

When $\mathbf{R}_{t,t+1}$ specializes to a single risky asset, our lower entropy bound on $L[m^P_{t,t+1}]$ in equation (24) specializes to the one in Alvarez and Jermann (2005, equation (4)), that is, $E[\log(R^m_{t,t+1})] - E[\log(R_{t,t+1,\infty})]$. We will show that our bound on $L[m^P_{t,t+1}]$ which relies on the return properties of risk-free bond, the long-term discount bond, and $N$ risky assets, is considerably more stringent.

We argue that the general bound on $L[m^P_{t,t+1}]$ in equation (24) can be employed in conjunction with the bound on $L[(m^P_{t,t+1})^2]$ in equation (20) to evaluate the consistency of $m^P_{t,t+1}$ with observed asset prices.

### *3.4. Sharpness of our entropy bounds*

Apart from theoretical arguments, how sharp are our generalized bounds compared to the single-asset (or a benchmark portfolio) based bound on $L[m_{t,t+1}]$ in Backus, Chernov, and Zin (2014, Column 2 of Table I) and the corresponding bound on $L[m^P_{t,t+1}]$ in Alvarez and Jermann (2005)?

To address this question, Table 1 reports our lower bounds on $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$ and the associated bootstrap $p$-values. In doing so, we consider several $N$ (the dimensionality of $\mathbf{R}_{t,t+1}$) and draw two conclusions from our computations in Table 1:

- First, our bounds on $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$ are quantitatively sharper with $N > 1$, implying greater hurdles on pricing models (e.g., compare bounds in Panel V versus those in Panels I through IV);
- Second, the bounds obtained with a portfolio are far less stringent than the corresponding bounds that hinge on the SDFs correctly pricing each of the assets comprising the portfolio. This can be seen by comparing the bound displayed in row (c) versus (i) and between row (d) versus (j).

Similar to Hansen and Jagannathan (1991) variance bound, we have derived entropy bounds that are related to the entropy of an arbitrary portfolio of assets, which are all correctly priced by the SDF. We will show that considering $L[m^2_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ can further help to discern across models.

### 3.5. Further clarifying discussions

Recognize that the lower bound on $L[(m^P_{t,t+1})^2]$ in equation (20) is distinct from the lower bound on $Var[m^P_{t,t+1}]$ in Bakshi and Chabi-Yo (2012, equation (6)). Analogously, the lower bound on $Var[m_{t,t+1}]$, that is, the Hansen and Jagannathan (1991, equation (12)) bound, and our lower bound on $L[(m_{t,t+1})^2]$ in equation (19), constitute distinctly relevant metrics for evaluating asset pricing models. Moreover, Ghosh, Julliard, and Taylor (2012, Section II.1) construct entropy bounds when the SDF can be factorized into observable and model-specific unobservable components. Our entropy bounds on the SDF are distinct from their bounds, allow correlated multiplicative components, and can be directly inferred from the returns data.

The analysis in Backus, Chernov, and Zin (2014, Section A.2) points to the distinct nature of the lower bound on $Var[m_{t,t+1}]$ versus the lower bound on $L[m_{t,t+1}]$ (see also a discussion in Alvarez and Jermann (2005, page 1985)). As noted in equation (13), the entropy of $m^P_{t,t+1}$, the entropy of $(m^P_{t,t+1})^2$, and the variance of $m^P_{t,t+1}$ are related by the expression: $\exp(L[(m^P_{t,t+1})^2] - 2L[m^P_{t,t+1}]) - 1 = Var[m^P_{t,t+1}]$. Such a relation implies that it may be possible for a model to satisfy the bound on $Var[m^P_{t,t+1}]$, but not the bound on $L[(m^P_{t,t+1})^2]$, and vice versa. Rearranging, appreciate that $L[(m^P_{t,t+1})^2 = \log[Var(m^P_{t,t+1}) + 1] + 2L[m^P_{t,t+1}] \geq \log[Var(m^P_{t,t+1}) + 1] + 2E[\log(\frac{\left(\Sigma^{-1}(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])\right)'\mathbf{R}_{t,t+1}}{\mathbf{1}'\Sigma^{-1}(\mathbf{1}-E[q_t]E[\mathbf{R}_{t,t+1}])})] - 2E\left[\log\left(R_{t,t+1,\infty}\right)\right]$.

Liu (2012, Proposition 1 and Collorary 1) derives an upper bound on $E[m^\delta_{t,t+1}]$ when $\delta \in [0,1]$, and a lower bound on $E[m^\delta_{t,t+1}]$ when $\delta < 0$, where $\delta$ is expressed in terms of the risk aversion parameter $\gamma \equiv \frac{1}{1-\delta}$. Moreover, our results in Theorem 2 through Theorem 4 can be contrasted to the single-return-based bound on the generalized entropy function in Liu (2012, equations (11) and (12)). In addition, our entropy bound on $L[m^2_{t,t+1}]$ offers a distinction to the bounds considered in Snow (1991, equations (7) and (12)). Specifically, our bounds are easier to implement and do not involve solving an optimization problem.

### 3.6. Restrictions on the transitory component of the SDFs

Our analysis can be adapted to develop restrictions on the entropy of $(m^T_{t,t+1})^2$ and $m^T_{t,t+1}$. Intuitively, both $L[m^T_{t,t+1}]$ and $L[(m^T_{t,t+1})^2]$ capture the departure of $R_{t,t+1,\infty}$ from lognormality. Moreover, when there is no time-variation in the conditional variance of $\log(R_{t,t+1,\infty})$, we obtain equivalence of the type $L[(m^T_{t,t+1})^2] = 4L[m^T_{t,t+1}]$. Absent distributional assumptions, the general restrictions are

$$L[(m^T_{t,t+1})^2] \;=\; \log(E\left[1/R^2_{t,t+1,\infty}\right]) \;+\; 2E[\log(R_{t,t+1,\infty})] \quad \text{and} \tag{25}$$

$$L[m^T_{t,t+1}] \;=\; \log(E\left[1/R_{t,t+1,\infty}\right]) \;+\; E[\log(R_{t,t+1,\infty})]. \tag{26}$$

Restrictions (25) and (26) inherit the model-free attribute of the entropy bounds on $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$. Given a proxy for $R_{t,t+1,\infty}$, the quantities on the right-hand side of (25)–(26) are computable. These quantitative restrictions can be helpful in investigating whether a pricing model is aligned with the properties of the transitory component of the SDFs, as reflected in the return time-series of long-term discount bonds.

### *3.7. Restrictions on entropy-based codependence, motivated by Hansen (2012)*

Inspired by a treatment in Hansen (2012, Section 4.3), we develop two additional results in the context of the permanent and transitory components of the SDF. First, we note that

$$\underbrace{L\left[m^P_{t,t+1}m^T_{t,t+1}\right] \;-\; L\left[m^P_{t,t+1}\right] \;-\; L\left[m^T_{t,t+1}\right]}_{\text{Intrinsic to an asset pricing model}} \;=\; \underbrace{\log\left(E\left[q_t\right]\right) \;-\; \log\left(E\left[1/R_{t,t+1,\infty}\right]\right)}_{\text{Can be recovered from bond data}}, \tag{27}$$

where recognizing that the left-hand side of equation (27) simplifies to $\log(E[m^P_{t,t+1}m^T_{t,t+1}]) - \log(E[m^P_{t,t+1}]) - \log(E[m^T_{t,t+1}])$ by virtue of the definition of entropy, while the right-hand side of equation (27) can be inferred from bond data. Second, we develop an upper bound on the entropy-based codependence between $(m^P_{t,t+1})^2$ and $(m^T_{t,t+1})^2$ and state it as a formal result.

**Theorem 5** *The following upper bound on the entropy-based codependence measure is true:*

$$0 \;\leq\; L[\left(m^P_{t,t+1}\right)^2(m^T_{t,t+1})^2] - L\left[\,(m^P_{t,t+1})^2\right] - L[(m^T_{t,t+1})^2] \;\leq\; \log\left(1+\frac{\mathbf{y}'\Sigma\mathbf{y}}{\left(E\left[q_t\right]\right)^2}\right), \tag{28}$$

*where* $\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)$ *and* $\Sigma$ *is the variance-covariance matrix of* $\mathbf{R}_{t,t+1}$.

**Proof:** See Appendix E. ■

The two codependence measures capture fundamentally different information embedded in an asset pricing model. Specifically, the restriction in equation (27) traces codependence exclusively to bond prices, while inequality in equation (28) of Theorem 5 traces codependence predominantly to the mean and variance-covariance matrix of a generic set of risky asset returns.

# 4. Asset pricing models

Our goal is to learn about the properties of $m^P_{t,t+1}$ and $m^T_{t,t+1}$, and their consistency with the entropy restrictions and entropy codependence measures. We focus on three asset pricing models: (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity. Our analysis can be expanded to consider other asset pricing models. We complement the analysis in Backus, Chernov, and Zin (2014) by solving the eigenfunction problem and by studying the implications of entropy codependence measures (for ease of exposition, we also closely follow their model notation).

## *4.1. Difference habit model*

In the difference habit model (e.g., Campbell and Cochrane (1999)), the SDF is

$$m_{t,t+1} \;=\; \beta\, g_{t+1}^{\rho-1}\,(s_{t+1}/s_t)^{\rho-1}\,, \tag{29}$$

where $g_{t+1}$ is consumption growth, $\beta$ is the time discount parameter, and $1-\rho$ is the coefficient of relative risk aversion. Define $s_t \equiv 1-\exp(z_t)$ and $z_t \equiv \log(h_t)-\log(c_t)$, where $s_t$ is the surplus ratio corresponding to $z_t$, and the habit $h_{t+1}$ is known at time $t$. The laws of motion for $h_t$ and $g_t$ are

$$\log(h_{t+1}) = \log(h) + \eta\,[B]\,\log(c_t) \qquad \text{and} \qquad \log(g_{t+1}) = \log(g) + \gamma[B]\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}, \tag{30}$$

where $B$ is the lag operator, such that $B\{s_{t+1}\} = s_t$, with backshift operators $\gamma[B] = \sum_{j=0}^{\infty} \gamma_j B^j$ and $\eta[B] = \sum_{j=0}^{\infty} \eta_j B^j$. Moreover, $\upsilon$ denotes the constant variance of $\log(g_t)$, and $\omega_{gt+1}$ is i.i.d. standard normal variable.

Loglinear approximation of $\log(s_t)$, in conjunction with the laws of motion in equation (30), leads to the surplus ratio dynamics:

$$\log(s_{t+1}) - \log(s_t) = \left(\frac{s-1}{s}\right)(\eta[B]B - 1)\log(g_{t+1}). \tag{31}$$

Completing model description, we define the state variable $x_t = (\gamma[B] - \gamma_0)\upsilon^{\frac{1}{2}}\omega_{gt+1}$, which governs the dynamics of the log consumption growth:

$$x_t = \gamma_1 \upsilon^{\frac{1}{2}} \omega_{gt} + \varphi_g x_{t-1} \qquad \text{with} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{32}$$

Solving the eigenfunction problem (as formalized in equations (H1) and (H2)) results in the following:

**Proposition 1** *For the SDF of the habit model specified in equation (29), the permanent component is:*

$$m_{t,t+1}^{P} = \exp\left(-D_1 + D_2 x_{t-1} + D_3 x_t + D_4 x_{t+1}\right), \tag{33}$$

*where the dynamics of $x_t$ is displayed in equation (32) and the coefficients $D_1$ through $D_4$ are defined in equations (J15) through (J18) of Online Appendix IV.*

**Proof**: See the steps in Online Appendix IV. ■

We employ equation (33) of Proposition 1 to compute the left-hand side of the bound expressions (19)-(20) of Theorem 2. Asset pricing models that accommodate habit have shown promise in matching salient attributes of the asset market data, including the equity premium, procyclicality of stock prices, counter-cyclicality of stock volatility, and return predictability at long-horizons (e.g., see, among others, Bekaert and Engstrom (2012), Chapman (1998), Chan and Kogan (2002), and Santos and Veronesi (2010)).

### 4.2. *Recursive utility models*

The two recursive utility models that we consider are adopted from Backus, Chernov, and Zin (2014):

$$U_t = \left[(1-\beta)\, c_t^{\rho} \;+\; \beta\left(\mu_t\left[U_{t+1}\right]\right)^{\rho}\right]^{\frac{1}{\rho}}, \tag{34}$$

with certainty equivalent function $\mu_t\left[U_{t+1}\right] = \left(E_t\left[U_{t+1}^{\alpha}\right]\right)^{\frac{1}{\alpha}}$. Moreover, $\rho$ is the time preference parameter, $1/(\rho-1)$ is the intertemporal elasticity of substitution, and $1-\alpha$ is the coefficient of relative risk aversion.

With backshift operators characterized by $\nu\left[B\right] = \sum_{j=0}^{\infty} \nu_j B^j$ and $\psi\left[B\right] = \sum_{j=0}^{\infty} \psi_j B^j$, the state-variables in this model obey the dynamics:

$$\log(g_t) = \log(g) + \gamma[B]\, \upsilon_{t-1}^{1/2}\omega_{gt} + \psi\left[B\right] z_{gt} - \psi\left[1\right] h\theta, \qquad h_t = h + \eta\left[B\right]\omega_{ht}, \tag{35}$$

$$\upsilon_t = \upsilon \;+\; \nu\left[B\right]\omega_{\upsilon t}, \quad z_{gt}|j \sim \mathcal{N}\left(j\theta, j\delta^2\right), \quad P\left[j\right] = \exp\left(-h_{t-1}\right)\frac{(h_{t-1})^j}{j!}, \tag{36}$$

where $\omega_{gt}$, $z_{gt}$, and $\omega_{ht}$ are standard normal random variables, independent of each other and across time. The jump component $z_{gt}$ is a Poisson mixture of normals: conditional on the number of jumps $j$, $z_{gt}$ is normal, with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date $t$ is $e^{h_{t-1}} h_{t-1}^{j}/j!$, and the jump intensity, $h_{t-1}$, is the mean of $j$.

*A. Recursive utility model with stochastic variance*. Set $h = 0$, $\eta\left[B\right] = 0$, $\psi\left[B\right] = 0$ in equations (35) and (36). For tractability, we consider the evolution of the transformed variable:

$$x_t \;=\; \varphi_g\, x_{t-1} \;+\; \gamma_1\, \upsilon_{t-1}^{1/2}\, \omega_{gt}. \tag{37}$$

Now we state the following proposition.

**Proposition 2** *For the SDF of the recursive utility model with stochastic variance, the permanent compo-*

*nent is:*

$$m^P_{t,t+1} \quad = \quad \exp\left(H_6 + (H_2 - \tau_0)\, x_t \; + \; (H_3 + \tau_0)\, x_{t+1} + (H_4 - \tau_1)\, \upsilon_t + (H_5 + \tau_1)\, \upsilon_{t+1}\right), \tag{38}$$

*where the coefficients $H_2$ through $H_6$, $\tau_0$, and $\tau_1$ are presented in Online Appendix V.*

Proof: See the steps in Online Appendix V. ■

*B. Recursive utility model with constant jump intensity*: In equations (35) and (36), set $\nu[B] = 0$. We obtain the following result.

**Proposition 3** *For the SDF of the recursive utility model with constant jump intensity, the permanent component is:*

$$m^P_{t,t+1} \quad = \quad \exp\left(G_9 - G_8\, h_t + (G_5 + \varsigma_1)\, z_{gt+1} + (G_6 + \varsigma_2\gamma_1)\, \upsilon^{\frac{1}{2}}\, \omega_{gt+1} + (G_7 + \varsigma_0\eta_0)\, \omega_{ht+1}\right), \tag{39}$$

*where the coefficients $G_5$ through $G_9$, $\varsigma_0$ through $\varsigma_3$, and $\eta_0$ are presented in Online Appendix V.*

**Proof**: See the steps in Online Appendix V. ■

Models that incorporate recursive preferences in conjunction with stochastic variance or jumps in the consumption growth dynamics have proved successful in explaining asset pricing quantities. Notable applications include, among others, Epstein and Zin (1991), Bansal and Yaron (2004), Campbell and Vuolteenaho (2004), Hansen, Heaton, and Li (2008), Martin (2013), Wachter (2013), and Zhou and Zhu (2009). Wachter (2013) emphasizes that her model can reconcile the size of the equity premium, the behavior of equity volatility, and the return predictability of Treasury bonds, pointing to a possible link between seemingly disparate phenomena from equity and bond markets.

# 5. Analyzing asset pricing models

Our benchmark for assessing whether a model produces sufficient entropy are the bounds in Theorems 2 through 4, which are are based on the SDF correctly pricing each of the $N+2$ asset returns.

Moreover, we consider a block bootstrap procedure to judge whether a model statistically meets our data-based lower entropy bounds. We then juxtapose our analysis with new entropy-based measures of codependence, which are motivated by a discussion in Hansen (2012). Our contention is that performance metrics based on (i) the bounds on the stochastic discount factors and their permanent and transitory components, and (ii) the bound on entropy codependence could furnish new perspectives on how to specify well-performing stochastic discount factors that admit a correlated multiplicative decomposition.

## *5.1. There is empirical rationale for considering the entropy of $L[m_{t,t+1}^2]$ and $L[(m_{t,t+1}^P)^2]$*

Pertinent to our empirical inquiry is first the question: How meaningful are our entropy bounds on $m_{t,t+1}^2$ and $(m_{t,t+1}^P)^2$? To answer this question, we need to show that entropy $L[m_{t,t+1}^2]$ (or $L[(m_{t,t+1}^P)^2]$) contains information beyond that which is contained in entropy $L[m_{t,t+1}]$ (or $L[m_{t,t+1}^P]$).

Note that in a setting where $m_{t,t+1}$ is lognormally distributed with no time-variation in the conditional volatility of $\log(m_{t,t+1})$, one obtains the restriction: $L[m_{t,t+1}^2] - 4L[m_{t,t+1}] = 0$ (see also our motivating Example 1 that correspondingly highlights $L[(m_{t,t+1}^P)^2] - 4L[m_{t,t+1}^P] = 0$). One implication of this restriction is that the lower bound on $L[m_{t,t+1}^2]$ is proportional to the lower bound on $L[m_{t,t+1}]$, which is amenable to validation from the returns data.

Guided by this reasoning, we combine the right-hand sides of equation (19) and equation (21) (and isomorphically equation (20) and equation (24) for $m_{t,t+1}^P$) and consider the quantities:

$$\Pi_m \equiv \frac{2(E[\log(\mathbf{a}'\mathbf{R}_{t+1})] - \log\left((E[q_t])^{-1}\right)) + \log(1 + \mathbf{y}'\Sigma^{-1}\mathbf{y}/(E[q_t])^2)}{4(E[\log(\mathbf{a}'\mathbf{R}_{t+1})] - \log((E[q_t])^{-1}))} - 1 \quad \text{and} \tag{40}$$

$$\Pi_{m^P} \equiv \frac{2(E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})]) + \log(1 + \mathbf{y}_P'\Sigma_P\mathbf{y}_P)}{4(E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})])} - 1, \tag{41}$$

where, for brevity, we set $\mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}$, $\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right] E\left[\mathbf{R}_{t,t+1}\right]\right)$, and $\mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right)$. The hypothesis $\Pi_m = 0$ amounts to testing whether $L[m_{t,t+1}^2]$ and $L[m_{t,t+1}]$ impound the same information.

Table 2 provides a point estimate of $\Pi_m$, and also $\Pi_{m^P}$, for three sets of $\mathbf{R}_{t,t+1}$, and a bootstrap $p$-value that tests whether $\Pi_m = 0$ versus $\Pi_m \neq 0$. Our empirical analysis elicits the observation that the hypothesis of $\Pi_m = 0$ is rejected, whereby the data-based lower bound on $L[m_{t,t+1}^2]$ can depart from its $4L[m_{t,t+1}]$ counterparts by as much as 56.17%. The reported $p$-values are based on a block bootstrap, with a block size of 20, with 50,000 replications from the data. The hypothesis of $\Pi_{m^P} = 0$ is also rejected.

Our evidence provides some rationale for considering $L[m_{t,t+1}^2]$ and $L[(m_{t,t+1}^P)^2]$ in assessing asset pricing models.

### 5.2. *Implementation and calculation of model-based entropies*

How do the models under consideration fare when viewed from the perspective of our data-based entropy bounds? Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows the calibration procedure in Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). The corresponding model parameterizations are displayed in our Table Appendix-I, which indicates that each model reasonably calibrates to consumption growth data.

Aided by the analytical representations of $m_{t,t+1}^P$ derived in our Propositions 1 through 3, we generate the paths for $m_{t,t+1}^P$, along with those of $m_{t,t+1}$, over 966 months corresponding to our returns data over 1931:07 to 2011:12. The paths are based on the model parameters in Table Appendix-I and shocks driving the fundamentals (e.g., $\omega_{\upsilon t}$ and $\omega_{gt}$ for the RU-SV). Then we obtain the sample averages of the series $\{(m_{t,t+1}^P)^2, m_{t,t+1}^P, m_{t,t+1}^2, m_{t,t+1} : t = 1, \ldots 966\}$, and accordingly compute the entropies $L[(m_{t,t+1}^P)^2]$, $L[m_{t,t+1}^P]$, $L[m_{t,t+1}^2]$, and $L[m_{t,t+1}]$.

Next, we draw 50,000 paths for the shocks driving a model and, hence, obtain 50,000 paths for $m_{t,t+1}^P$

and $m_{t,t+1}$. Panels A and B of Table 3 report the entropies across the models, obtained by averaging the entropies over the 50,000 replications. The $p$-values, shown in square brackets, represent the proportion of replications for which the model-based entropy measure exceeds the corresponding lower bound obtained from the returns data in 50,000 replications of a simulation over 966 months.

### 5.3. *Model assessment based on the bound on $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$*

The next question to ask is: how successful are the three models in generating $L[m_{t,t+1}]$ that is consistent with the data? Panel A of Table 3 reveals an $L[m_{t,t+1}]$ of 0.0196, 0.0217, and 0.0190, respectively, for the DH, RU-SV, and RU-CJI models. Based on our data-based performance measure, that is, the lower bound on $L[m_{t,t+1}]$, displayed on the right-hand side of equation (21), computed based on SET B, all the models are rejected at the 5% level (as seen by the bootstrap $p$-values).

This implication from our generalized bound, calculated using the return properties of the risk-free bond, the long-term discount bond, the equity market, and the 25 portfolios sorted by size and momentum, differ from a finding in Backus, Chernov, and Zin (2014). Specifically, the data-based lower bounds in Backus, Chernov, and Zin (2014, Table 1) are generally of an order lower than the average conditional entropy $E[L_t[m_{t,t+1}]]$ obtained from asset pricing models. In particular, all of the 11 $E[L_t[m_{t,t+1}]]$ in Backus, Chernov, and Zin (2014, Tables II through IV) exceed the lower bound inferred from the data on the S&P 500 index.

How does one explain this discrepancy? We note that the magnitude of the lower bound on $L[m_{t,t+1}]$ in the calculations of Backus, Chernov, and Zin (2014, Table 1, row S&P 500) is 0.0040, whereas it is 0.0367, based on our lower bound and SET B. It bears emphasizing that a single-asset based lower bound on $L[m_{t,t+1}]$ may provide an insufficient hurdle in evaluating the merits of an asset pricing model. When the entropy calculations exploit the information in the distribution of the return vector $\mathbf{R}_{t,t+1}$, it imposes stronger implications for $m_{t,t+1}$.

We are now prompted to ask: Are the properties of $m^P_{t,t+1}$ implicit in the models consistent with the entropy bound $L[m^P_{t,t+1}]$? We find that the $L[m^P_{t,t+1}]$ produced by the DH, RU-SV, and RU-CJI models are 0.0203, 0.0237, and 0.0197, respectively, while the data-based lower bound on $L[m^P_{t,t+1}]$ is 0.0348 (see Panel B of Table 3). The reported $p$-values indicate that all the three models are rejected at the 5% level; namely, the models generate insufficient entropy $L[m^P_{t,t+1}]$. In essence, the bounds on both $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$ agree in suggesting that the three models are misspecified.

### *5.4. Model assessment based on the bound on $L[m^2_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ yields additional insights*

Elaborating further, we now argue that considering the entropy $L[m^2_{t,t+1}]$ (or $L[(m^P_{t,t+1})^2]$) in the model assessment can provide an important contrast to our findings based on the entropy $L[m_{t,t+1}]$ (or $L[m^P_{t,t+1}]$).

One prominent result is that the entropy $L[(m^P_{t,t+1})^2]$ of the RU-CJI model is about 15-fold higher than the other two models that do not incorporate the random jump feature in the dynamics of the consumption growth. For example, the DH, RU-SV, and RU-CJI models generate $L[(m^P_{t,t+1})^2]$ of 0.0811, 0.095, and 1.4858, respectively (see the entries in Panel B of Table 3).

We further note that since the lower bound restriction implied from asset prices is 0.1851, the DH and RU-SV models are rejected at the 5% level. However, the RU-CJI model with constant jump intensity cannot be rejected at the 5% level, which is a point of departure based on the entropy $L[m^P_{t,t+1}]$.

Accordingly, one key question emerges: Why does the RU-CJI fails to explain features of $m^P_{t,t+1}$, as reflected in asset prices when $L[m^P_{t,t+1}]$-based performance measure is used, while the model is successful in explaining features of $m^P_{t,t+1}$, as reflected in asset prices when $L[(m^P_{t,t+1})^2]$-based performance measure is used? To investigate a source of model outperformance, we note that the entropy measure $L[(m^P_{t,t+1})^2]$ is substantially more sensitive to tail asymmetries and tail size of the $m^P_{t,t+1}$ distribution as opposed to the entropy measure $L[m^P_{t,t+1}]$.

Taking such a trait of entropies into consideration, we report the moments of $m_{t,t+1}$ and $m^P_{t,t+1}$ for each of the models in Panel C and Panel D of Table 3. The unexpected finding is that the RU-CJI model embeds excessive levels of skewness and kurtosis of $m^P_{t,t+1}$, while generating variance that is almost 90 times its DH and RU-SV model counterparts. Our contention is that the inordinate levels of the higher-order moments of $m^P_{t,t+1}$ ($m_{t,t+1}$) give rise to the reported $L[(m^P_{t,t+1})^2]$ ($L[m^2_{t,t+1}]$) of 1.4858 (1.4331) for the RU-CJI model.

How should one interpret a model, such as the RU-CJI, that calibrates well to the first-moment, the second-moment and the autocorrelation of consumption growth, but does not produce finite central moments for the distribution of both $m^P_{t,t+1}$ and $m_{t,t+1}$. This result arises because a convex transform of a random variable, which is here poisson-distributed, increases the skewness to the right (see van Zwet (1966, page 10, Theorem 2.2.1)). To see this analytically, we can invoke the density of the poisson random variable to show that $E_t[(m_{t,t+1})^k] = E_t\left[e^{k\log(m_{t,t+1})}\right] = E_t\left[E_t\left[e^{k\log(m_{t,t+1})}|j\right]\right] = e^{G[k]}E_t\left[e^{H[k]j}\right]$, for some constants $G[k]$ and $H[k]$. Note that $e^{H[k]j}$ is a convex transformation of the poisson variable $J$, and for certain parameterizations, does not admit finite higher-moments of $m_{t,t+1}$. The inordinate amounts of skewness and kurtosis do not appear to be a reasonable depiction of valuation operators, which are likely to be characterized by exponential, rather than power, tails.

How general are our conclusions with respect to the RU-CJI model? Specifically, are there model combinations that produce reasonable higher-order moments of $m^P_{t,t+1}$, calibrate well to consumption growth data, and yet deliver high entropies? To probe this issue, we vary the jump distribution parameters ($\theta$, $\delta$, $h$) of the consumption growth dynamics (see equation (36)), and report the results in Table Appendix-II. The takeaway message is that jump parameterizations (among the 27 parameter combinations) that yield reasonable levels of skewness and kurtosis of $m^P_{t,t+1}$ do not appear to produce enough entropies to satisfy the lower bound on $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$.

### *5.5. Models also fail to satisfy the data-based restrictions on* $L[(m^T_{t,t+1})^2]$

Next we examine the entropy of $(m^T_{t,t+1})^2$ and $m^T_{t,t+1}$, which enables us to further challenge models by assessing their ability to fit certain aspects of the Treasury market data.

Two features of our findings are worth emphasizing in Table 4. First, all the models fail to produce a transitory component of the SDFs that are consistent with return properties of the long-term discount bond.

Second, the RU-CJI model is worse than the other two models when performance is assessed based on the transitory component. Specifically, the jump parametrization of the consumption growth dynamics lead to implausible entropies of the transitory component of the RU-CJI model.

In sum, our exercises suggest that asset pricing models need to do a better job of satisfying the entropy-based performance measures of both the permanent and the transitory components of the SDFs. Thus, by identifying the dimensions where pricing models may be lacking, our bounds could reveal a greater appreciation of the desirable properties of $m_{t,t+1}$.

### *5.6. Models fail to satisfy the data-based upper bound restrictions on codependence*

How deft are the models in matching entropy-based codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$? Table 5 shows that the DH, RU-SV, and RU-CJI models are not able to reproduce the magnitude and the sign of the dependence measures obtained from asset prices.

Although the observed asset prices indicate a positive dependence between the permanent and transitory components of the SDF, all three models suggest a negative codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$. Therefore, these models are not properly aligned with codependence imputed from asset market data.

Importantly, the entropy codependence measures could shed light onto the properties of the $m_{t,t+1}$ that one cannot get by applying the variance bounds in Hansen and Jagannathan (1991) or the entropy bounds in Alvarez and Jermann (2005) and Backus, Chernov, and Zin (2014). The relevance of the entropy codependence measure in model assessment provides an additional motivation for decomposing the stochastic discount factors into permanent and transitory components.

### *5.7. Summary and empirical implications*

In sum, for the set of parameter values in Table Appendix-I, the asset pricing models we investigate are not able to generate entropies $L[m_{t,t+1}]$ and $L[m^P_{t,t+1}]$ that surpass the data-based lower bounds, and hence unable to describe the features of the SDF, as reflected in the asset returns.

While the RU-CJI model does meet the lower entropy bound $L[m_{t,t+1}^2]$, the model success is achieved at the expense of implausible central moments of the $m_{t,t+1}$ distribution. Each asset pricing model also appears inconsistent with the data on long-term bond returns and with our entropy-based codependence measures inferred from the returns data.

## 6. Generalizing the entropy bounds to alternative investment horizons

The objective is to generalize the highlighted set of entropy bounds to the case when returns are measured over more than a single-period. We are guided by Hansen (2012), who emphasizes the need to study the behavior of long-term entropy of SDFs. This problem entails imposing additional restrictions on the dynamic link between the permanent and transitory components over an $n$-period investment horizon.

Consider the $n$-period SDF, $m_{t,t+n}$, defined as:

$$m_{t,t+n} = \prod_{j=1}^{n} m_{t+j-1,t+j}, \qquad \text{where } m_{t+j-1,t+j} \text{ is the SDF from } t+j-1 \text{ to } t+j. \tag{42}$$

We postulate that the $n$-period SDF can be decomposed as

$$m_{t,t+n} = m_{t,t+n}^{P} m_{t,t+n}^{T}, \quad \text{where} \quad m_{t,t+n}^{P} = \prod_{j=1}^{n} m_{t+j-1,t+j}^{P} \quad \text{and} \quad m_{t,t+n}^{T} = \prod_{j=1}^{n} m_{t+j-1,t+j}^{T}, \tag{43}$$

with $m_{t+j-1,t+j}^{T} = 1/R_{t+j-1,t+j,\infty}$, $E[m_{t+j-1,t+j}^{P}] = 1$, and $R_{t+j-1,t+j,\infty}$ is the gross return from holding a discount bond with infinite-maturity from time $t+j-1$ to $t+j$.

Now consider the sets that correctly price each of the $N+2$ assets over $n$-periods:

$$\mathbb{S}^{(n)} = \left\{ m_{t,t+n} > 0 : E_t\left[m_{t,t+n}\right] = q_t^{(n)},\ E\left[m_{t,t+n} R_{t,t+n,\infty}\right] = 1, \text{ and } E\left[m_{t,t+n} \mathbf{R}_{t,t+n}\right] = \mathbf{1} \right\} \quad \text{and} \tag{44}$$

$$\mathbb{S}_P^{(n)} = \left\{ m_{t,t+n} \in \mathbb{S}^{(n)} \colon m_{t,t+n} = m_{t,t+n}^{P} m_{t,t+n}^{T}, \ \text{ and } \ m_{t,t+n}^{T} = \left(R_{t,t+n,\infty}\right)^{-1} \right\}, \tag{45}$$

where $\mathbf{R}_{t,t+n}$ is a vector column of risky asset returns and $q_t^{(n)}$ is the price of an $n$-period discount bond.

Each component of $\mathbf{R}_{t,t+n}$ is of the form $R_{t,t+n} = \prod_{j=1}^{n} R_{t+j-1,t+j}$, where $R_{t+j-1,t+j}$ is the return of the risky asset from $t+j-1$ to $t+j$. Moreover, $R_{t,t+n,\infty} = \prod_{j=1}^{n} R_{t+j-1,t+j,\infty}$.

Let $\Sigma^{(n)}$ be the variance-covariance matrix of $\mathbf{R}_{t,t+n}$ and $\Sigma_P^{(n)}$ be the variance-covariance of $\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}$. Our main characterization is presented next.

**Theorem 6** *The following entropy bounds are applicable to n-period stochastic discount factors:*

*(a) The entropy of $m^2_{t,t+n}$ satisfies:*

$$L[m^2_{t,t+n}] \geq 2\left(E\left[\log\left(\frac{\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)'\left(\Sigma^{(n)}\right)^{-1}}{\mathbf{1}'\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)}\mathbf{R}_{t,t+n}\right)\right]-\log\left((E[q_t^{(n)}])^{-1}\right)\right)$$
$$+\log\left(1+\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)'\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)\left(E[q_t^{(n)}]\right)^{-2}\right). \quad (46)$$

*(b) The entropy of $(m^P_{t,t+n})^2$ satisfies*

$$L[(m^P_{t,t+n})^2] \quad \geq \quad 2\left(E\left[\log\left(\frac{\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)'\left(\Sigma^{(n)}\right)^{-1}}{\mathbf{1}'\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E[q_t^{(n)}]E\left[\mathbf{R}_{t,t+n}\right]\right)}\mathbf{R}_{t,t+n}\right)\right] - E\left[\log\left(R_{t,t+n,\infty}\right)\right]\right)$$
$$+\log\left(1+\left(\mathbf{1}-E\left[\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}\right]\right)'\left(\Sigma_P^{(n)}\right)^{-1}\left(\mathbf{1}-E\left[\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}\right]\right)\right). \quad (47)$$

*(c) The entropy of $m_{t,t+n}$ satisfies:*

$$L[m_{t,t+n}] \geq E\left[\log\left(\frac{\left(\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E\left[q_t^{(n)}\right]E\left[\mathbf{R}_{t,t+n}\right]\right)\right)'\mathbf{R}_{t,t+n}}{\mathbf{1}'\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E\left[q_t^{(n)}\right]E\left[\mathbf{R}_{t,t+n}\right]\right)}\right)\right]-\log\left(\left(E\left[q_t^{(n)}\right]\right)^{-1}\right). \quad (48)$$

*(d) The entropy of $m^P_{t,t+n}$ satisfies:*

$$L[m^P_{t,t+n}] \;\geq\; E\left[\log\left(\frac{\left(\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E\left[q_t^{(n)}\right]E\left[\mathbf{R}_{t,t+n}\right]\right)\right)'\mathbf{R}_{t,t+n}}{\mathbf{1}'\left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1}-E\left[q_t^{(n)}\right]E\left[\mathbf{R}_{t,t+n}\right]\right)}\right)\right] \;-\; E\left[\log\left(R_{t,t+n,\infty}\right)\right]. \quad (49)$$

**Proof:** See Appendix F. ■

The entropy bounds derived in Theorem 6 reflect information about the dynamics of asset returns and Treasury bond returns. Our entropy restrictions on $m^{P}_{t,t+n}$ and $m_{t,t+n}$ can be used to evaluate consistency of asset pricing models with observed prices over any investment horizon.

Other forms of codependence could be clarified in a multi-period setting, whereby

$$L[m^{P}_{t,t+n}m^{T}_{t,t+n}] - L[m^{P}_{t,t+n}] - L[m^{T}_{t,t+n}] = \log(E[q_t^{(n)}]) - \log\left(E\left[1/R_{t,t+n,\infty}\right]\right). \tag{50}$$

Elaborating further, the dependence between $m_{t,t+k}$ and $m_{t+k,t+n}$ can be expressed in terms of the Treasury term structure quantities as:

$$L[m_{t,t+k}m_{t+k,t+n}] - L\left[m_{t,t+k}\right] - L[m_{t+k,t+n}] = \log(E[q_t^{(n)}]) - \log(E[q_t^{(k)}]) - \log(E[q_{t+k}^{(n-k)}]). \tag{51}$$

Overall, the restrictions over the $n$-periods could enrich our understanding of the codependence between the permanent and the transitory components of the SDF and help to build models that are more adept at mimicking asset pricing quantities over alternative investment horizons.

## 7. Conclusions

A central problem in finance is the specification of the stochastic discount factor. We study this problem by providing new asset pricing restrictions that are based on the entropy of the square of the stochastic discount factor. Our entropy measure is suitable for capturing the conditional volatility and non-normalities in the log stochastic discount factor. The entropy restrictions we develop are based on the ability of the stochastic discount factor to *jointly* price the risk-free bond, the long-term discount bond, and a set of risky assets. We also present new entropy codependence measures to assess asset pricing models.

Our bounds framework hinges on understanding the permanent and transitory components of the stochastic discount factors and are in the tradition of Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Bakshi and Chabi-Yo (2012), and Hansen (2012). Key to our analysis are the expressions for the permanent and the transitory components of the stochastic discount factor, which we obtain by solving the eigenfunction problem. We ask whether the entropy of the permanent and the transitory components of the stochastic discount factor from a model are sufficient to meet the corresponding lower bounds.

There are a number of implications of our entropy framework for asset pricing models. First, our evaluation reveals that the difference habit model, the recursive utility model with stochastic variance, and the recursive utility model with constant jump intensity generally fail to satisfy the posited bounds on the permanent and the transitory components of the stochastic discount factors. Second, while the recursive utility model with constant jump intensity meets the lower bound on the square of the permanent component, we attribute the model success to unrealistic higher-order moments associated with the parametrization of the stochastic discount factor. Finally, these models are incompatible with the entropy codependency restrictions inferred from the returns data.

We also extend our framework to bounds that are valid for stochastic discount factors over alternative investment horizons. Borovicka, Hansen, Hendricks, and Scheinkman (2011) have advocated looking at risk and valuation dynamics over different investment horizons.

With some modifications, our framework could be expanded to analyze other asset pricing models, including generalized recursive smooth ambiguity utility (as in Ju and Miao (2012)) and generalized disappointment aversion (as in Routledge and Zin (2010)). One could also refine our bounds framework to incorporate conditioning information to further learn about the properties of the stochastic discount factors and the dynamics of the permanent and transitory components.

The push to attain well-specified stochastic discount factors has applications that transcend stock, bond, commodity, currency, and options valuation.

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## Appendix A: Proof of Theorem 1

The security that pays the SDF is a hedging asset and will have a negative expected return. The gross return $1+r^{\mathrm{SDF}}_{t,t+1} = \frac{m_{t,t+1}}{E_t[m^2_{t,t+1}]}$ satisfies the Euler equation with $E_t[m_{t,t+1}(1+r^{\mathrm{SDF}}_{t,t+1})] = 1$. Hence, taking logs of the expression for $1+r^{\mathrm{SDF}}_{t,t+1}$, and adding and subtracting $\log(m^2_{t,t+1})$, we obtain:

$$\log(1+r^{\mathrm{SDF}}_{t,t+1}) = \log(m_{t,t+1}) - \log(E_t\left[m^2_{t,t+1}\right]) + \log(m^2_{t,t+1}) - 2\log(m_{t,t+1}). \tag{A1}$$

Taking expectations on both sides of equation (A1):

$$E_t[\log(1+r^{\mathrm{SDF}}_{t,t+1})] + E_t[\log(m_{t,t+1})] \quad = \quad \overbrace{E_t[\log(m^2_{t,t+1})] - \log(E_t[m^2_{t,t+1}])}^{-L_t[m^2_{t,t+1}]} \tag{A2}$$

It then follows that,

$$L_t[m^2_{t,t+1}] + \log(E_t[m_{t,t+1}]) \quad = \quad -E_t[\log(1+r^{\mathrm{SDF}}_{t,t+1})] + \underbrace{\log(E_t[m_{t,t+1}]) - E_t[\log(m_{t,t+1})]}_{L_t[m_{t,t+1}]} \tag{A3}$$

$$> \quad -E_t[\log(1+r^{\mathrm{SDF}}_{t,t+1})]. \tag{A4}$$

Rearranging, we can derive the expression

$$L_t[m^2_{t,t+1}] \ > \ -E_t[\log(1+r^{\mathrm{SDF}}_{t,t+1})] - \log(E_t[m_{t,t+1}]). \tag{A5}$$

Since the gross return of the risk-free bond satisfies $R^f_t\, E_t\left[m_{t,t+1}\right] = 1$, we obtain

$$\log(R^f_t) - E_t[\log(1+r^{\mathrm{SDF}}_{t,t+1})] \ < \ L_t[m^2_{t,t+1}]. \tag{A6}$$

Taking unconditional expectations on both sides of equation (A6):

$$E[\log(R^f_t)] - E[\log(1+r^{\mathrm{SDF}}_{t,t+1})] \ < \ E[L_t[m^2_{t,t+1}]]. \tag{A7}$$

Now invoke the following relation:

$$E[L_t[m_{t,t+1}^2]] \;\leq\; L[m_{t,t+1}^2] \qquad \text{since} \qquad L[m_{t,t+1}^2] = E[L_t[m_{t,t+1}^2]] \;+\; L[E_t[m_{t,t+1}^2]]. \tag{A8}$$

Therefore, $E[\log(R_t^f)] - E[\log(1 + r_{t,t+1}^{\text{SDF}})] \;\leq\; L[m_{t,t+1}^2]$. Our measure is tied to the maximum expected log return on a security that pays the SDF and we derive a lower bound on $L[m_{t,t+1}^2]$.

Completing the picture, observe that $E_t[m_{t,t+1}^2] > (E_t\,[m_{t,t+1}])^2$ (because $\text{Var}\,(m_{t,t+1}) > 0$). Hence,

$$E_t[1 + r_{t,t+1}^{\text{SDF}}] = \frac{E_t\,[m_{t,t+1}]}{E_t[m_{t,t+1}^2]} < \frac{E_t\,[m_{t,t+1}]}{(E_t\,[m_{t,t+1}])^2} = \frac{1}{E_t\,[m_{t,t+1}]}. \tag{A9}$$

This implies

$$\log\left(E_t\,\left[1 + r_{t,t+1}^{\text{SDF}}\right]\right) < \overbrace{\log\left(\frac{1}{E_t[m_{t,t+1}]}\right)}^{\log(R_t^f)}. \tag{A10}$$

By an application of Jensen's inequality:

$$E_t[\log(1 + r_{t,t+1}^{\text{SDF}})] < \log\left(E_t\,\left[1 + r_{t,t+1}^{\text{SDF}}\right]\right) < \log(R_t^f). \tag{A11}$$

It then follows that

$$\log(R_t^f) - E_t[\log(1 + r_{t,t+1}^{\text{SDF}})] > 0. \qquad \text{Therefore,} \qquad E[\log(R_t^f)] - E[\log(1 + r_{t,t+1}^{\text{SDF}})] > 0. \tag{A12}$$

This is what we intended to show. ■

## Appendix B: Proof of Theorem 2

We adopt the following notation to streamline equation presentation and the steps of the proof:

$$\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\,[q_t]\,E\,[\mathbf{R}_{t,t+1}]\right), \qquad \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\,[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}]\right), \qquad \text{and} \qquad \mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}, \tag{B1}$$

where $\Sigma$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$, and $\Sigma_P$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$. We assume that $\mathbf{a}'\mathbf{R}_{t,t+1}$ is strictly positive. Further define,

$$\mathrm{er}_R \equiv E\left[\log\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)\right] - \log\left((E[q_t])^{-1}\right), \tag{B2}$$

$$\mathrm{er}_\infty \equiv E\left[\log\left(R_{t,t+1,\infty}\right)\right] - \log\left((E[q_t])^{-1}\right). \tag{B3}$$

**Proof of the entropy bound on $m^2_{t,t+1}$ in equation (19)**. By the definition of entropy: $L[m^2] = \log(E[m^2]) - E[\log(m^2)]$. Then

$$\begin{aligned}
L\left[m^2_{t,t+1}\right] &= \log\left(E\left[m^2_{t,t+1}\right]\right) - 2\log\left(E\left[q_t\right]\right) + 2L\left[m_{t,t+1}\right], \\
&= \log\left(1 + \frac{E\left[m^2_{t,t+1}\right] - \left(E\left[q_t\right]\right)^2}{\left(E\left[q_t\right]\right)^2}\right) + 2L\left[m_{t,t+1}\right], \\
&= \log\left(1 + \frac{Var\left[m_{t,t+1}\right]}{\left(E\left[q_t\right]\right)^2}\right) + 2L\left[m_{t,t+1}\right], \\
&\geq \log\left(1 + \frac{Var\left[m_{t,t+1}\right]}{\left(E\left[q_t\right]\right)^2}\right) + 2\,\mathrm{er}_R \qquad \left(\text{since } L\left[m_{t,t+1}\right] \geq \mathrm{er}_R;\ \text{see } (C4)\right).
\end{aligned} \tag{B4}$$

Because $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}$ and setting $q_t = E\left[m_{t,t+1}\right]$, it follows that

$$E\left[m_{t,t+1}\left(\mathbf{R}_{t,t+1} - E\left(\mathbf{R}_{t,t+1}\right)\right)\right] = \left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right). \tag{B5}$$

Multiplying equation (B5) by $\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)'\Sigma^{-1}$ yields

$$\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)'\Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right) = E\left[m_{t,t+1}\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)'\Sigma^{-1}\left(\mathbf{R}_{t,t+1} - E\left[\mathbf{R}_{t,t+1}\right]\right)\right]. \tag{B6}$$

Applying the Cauchy Schwartz inequality to the right-hand side of equation (B6), it can be shown that

$$
\begin{aligned}
Var\left[m_{t,t+1}\right] &\geq \left(\mathbf{1}-\left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)^{'}\Sigma^{-1}\left(\mathbf{1}-\left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right), \\
&\geq \left(\mathbf{1}-\left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)^{'}\Sigma^{-1}\Sigma\Sigma^{-1}\left(\mathbf{1}-\left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right), \\
&\geq \mathbf{y}'\Sigma\mathbf{y}. \qquad \text{(B7)}
\end{aligned}
$$

Combining the expressions in equations (B4) and (B7), we obtain the bound on $L[m^2_{t,t+1}]$ in equation (19) of Theorem 2. ■

**Proof of the entropy bound on $(m^P_{t,t+1})^2$ in equation (20) of Theorem 2**. We write

$$
\begin{aligned}
L[(m^P_{t,t+1})^2] &= \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)], \\
&= \log(E[(m^P_{t,t+1})^2]) - 2E[\log(m^P_{t,t+1})], \\
&= \log(E[(m^P_{t,t+1})^2]) + 2L\left[m^P_{t,t+1}\right], \\
&= \log\left(1+Var\left[m^P_{t,t+1}\right]\right) + 2L[m^P_{t,t+1}]. \qquad \text{(B8)}
\end{aligned}
$$

We show in equation (D4) that $L[m^P_{t,t+1}] \geq \text{er}_R - \text{er}_\infty$ (the complete expressions for $\text{er}_R$ and $\text{er}_\infty$ are in equations (B2) and (B3), respectively). Therefore, we deduce that

$$
L[\left(m^P_{t,t+1}\right)^2] \geq \log\left(1+Var\left[m^P_{t,t+1}\right]\right) + 2\left(\text{er}_R - \text{er}_\infty\right). \qquad \text{(B9)}
$$

Since $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = E\left[m^P_{t,t+1}\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right] = \mathbf{1}$, we then obtain:

$$
E\left[m^P_{t,t+1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right] = \mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]. \qquad \text{(B10)}
$$

Multiplying each side of equation (B10) by $\left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)^{'}\Sigma_P^{-1}$, we get

$$
\left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)^{'}\Sigma_P^{-1}\left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right) = E\left[m^P_{t,t+1}\left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)^{'}\Sigma_P^{-1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right]. \qquad \text{(B11)}
$$

Applying the Cauchy Schwartz inequality to the right-hand side of equation (B11), we note that

$$\begin{aligned} Var\left[m_{t,t+1}^{P}\right] &\geq \left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)'\Sigma_P^{-1}\left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right), \\ &\geq \left(\mathbf{1}-E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)'\left(\Sigma_P^{-1}\right)'\Sigma_P\Sigma_P^{-1}\left(\mathbf{1}-E\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right)\right), \\ &\geq \mathbf{y}_P'\Sigma_P\mathbf{y}_P. \qquad \text{(where noting } \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1}-E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right)\text{)}. \qquad \text{(B12)} \end{aligned}$$

Inserting the bound derived in equation (B12) into equation (B9) leads to the bound in equation (20) of Theorem 2. ■

## Appendix C: Proof of Theorem 3

**Generalizing the Backus, Chernov, and Zin (2014) entropy bound on $m_{t,t+1}$ to many risky assets**. Recognizing from equation (B1) that $\mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}$,

$$\begin{aligned} E\left[\log\left(m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1}\right)\right] &\leq \log\left(E\left[m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{a}'E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{a}'\mathbf{1}\right) = \log(1), \\ &\leq 0. \end{aligned} \tag{C1}$$

From equation (C1) and noting that $\log(m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1}) = \log(m_{t,t+1}) + \log(\mathbf{a}'\mathbf{R}_{t,t+1})$, we deduce that

$$E\left[\log\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)\right] \leq -E\left[\log\left(m_{t,t+1}\right)\right]. \tag{C2}$$

Adding $\log\left(E\left[m_{t,t+1}\right]\right)$ to both sides of equation (C2) yields

$$\log\left(E\left[m_{t,t+1}\right]\right) + E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] \leq \log(E[m_{t,t+1}]) - E\left[\log\left(m_{t,t+1}\right)\right] = L\left[m_{t,t+1}\right]. \tag{C3}$$

Since $q_t = E_t\left[m_{t,t+1}\right]$, equation (C3) simplifies to

$$L\left[m_{t,t+1}\right] \geq E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] \; - \; \log\left(1/E\left[q_t\right]\right). \tag{C4}$$

Our equation (C4) generalizes Backus, Chernov, and Zin (2014) when the bounds incorporate more than a single risky asset, specifically the set of assets outlined in equation (17). This was the final step in the proof of Theorem 3. ■

## Appendix D: Proof of Theorem 4

**Generalizing the Alvarez and Jermann (2005) entropy bound on $m^P_{t,t+1}$ to many risky assets**. Consider an SDF $m^P_{t,t+1} \in \mathbb{S}_P$. We note that

$$E\left[\log\left(m_{t,t+1}\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)\right] = E\left[\log\left(m^P_{t,t+1}\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right]. \tag{D1}$$

Invoking Jensen's inequality, we have

$$\begin{aligned} E\left[\log\left(m^P_{t,t+1}\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right] &= E\left[\log\left(m_{t,t+1}\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)\right], \\ &\leq \log\left(\mathbf{a}^{'}E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{a}^{'}\mathbf{1}\right), \\ &\leq 0. \end{aligned} \tag{D2}$$

From equation (D2), we deduce

$$E\left[\log\left(\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right] \leq -E[\log\left(m^P_{t,t+1}\right)] = L[m^P_{t,t+1}]. \tag{D3}$$

Hence,

$$L\left[m^P_{t,t+1}\right] \geq E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})],$$
$$\geq E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] - \log\left(1/E\left[q_t\right]\right) - \left(E[\log(R_{t,t+1,\infty})] - \log\left(1/E\left[q_t\right]\right)\right). \quad \text{(D4)}$$

This bound generalizes Alvarez and Jermann (2005) to $N+2$ assets. ■

## Appendix E: Proof of Theorem 5 on codependence

To streamline expressions, we write our measure of codependence as:

$$D_b \equiv L[(m^P_{t,t+1})^2(m^T_{t,t+1})^2] - L[(m^P_{t,t+1})^2] - L[(m^T_{t,t+1})^2], \quad \text{(E1)}$$
$$= \log(E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2]) - \log(E[(m^P_{t,t+1})^2]) - \log(E[(m^T_{t,t+1})^2]). \quad \text{(E2)}$$

From the expression in equation (E1),

$$\log(E[(m^P_{t,t+1})^2]) + \log(E[(m^T_{t,t+1})^2]) = \log(E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2]) - D_b. \quad \text{(E3)}$$

From the Cauchy Schwartz inequality, we have

$$\left(E\left[m^P_{t,t+1}m^T_{t,t+1}\right]\right)^2 \leq E[(m^P_{t,t+1})^2]\,E[(m^T_{t,t+1})^2]. \quad \text{(E4)}$$

Taking the log of the expression in equation (E1) gives

$$\log(\left(E[m^P_{t,t+1}m^T_{t,t+1}]\right)^2) \leq \log(E[\left(m^P_{t,t+1}\right)^2]) + \log(E[\left(m^T_{t,t+1}\right)^2]). \quad \text{(E5)}$$

Replacing equation (E3) in the expression (E5) yields

$$\log((E[q_t])^2) \leq \log(E[\left(m^P_{t,t+1}\right)^2(m^T_{t,t+1})^2]) - D_b, \qquad (\text{since } q_t = E_t\left[m^P_{t,t+1}m^T_{t,t+1}\right]) \quad \text{(E6)}$$

and

$$D_b \; + \; \log((E\,[q_t])^2) \; \leq \; \log\left(E[(m^P_{t,t+1})^2\left(m^T_{t,t+1}\right)^2]\right). \tag{E7}$$

Since $Var[m_{t,t+1}] = E[m^2_{t,t+1}] - (E[m_{t,t+1}])^2 = E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2] - (E\,[q_t])^2$, one could write equation (E7) as:

$$D_b + \log((E\,[q_t])^2) \leq \log(Var\,[m_{t,t+1}] + (E\,[q_t])^2). \tag{E8}$$

From the proof of Theorem 2, we have shown that $\mathbf{y}^{'}\Sigma\mathbf{y} \; \leq \; Var\,[m_{t,t+1}]$. Therefore,

$$\log\left(\mathbf{y}^{'}\Sigma\mathbf{y} \; + \; (E\,[q_t])^2\right) \; \leq \; \log\left(Var\,[m_{t,t+1}] + (E\,[q_t])^2\right). \tag{E9}$$

Because $\mathbf{y}^{'}\Sigma\mathbf{y}$ is the highest lower bound on $Var\,[m_{t,t+1}]$, it follows that $\log(\mathbf{y}^{'}\Sigma\mathbf{y} + (E\,[q_t])^2)$ is the highest of the lower bounds on $\log(Var\,[m_{t,t+1}] + (E\,[q_t])^2)$. Therefore, any other lower bound on the quantity $\log(Var\,[m_{t,t+1}] + (E\,[q_t])^2)$ must be lower than $\log(\mathbf{y}^{'}\Sigma\mathbf{y} + (E\,[q_t])^2)$. As a result,

$$D_b + \log\left(\left(E\left[q^1_t\right]\right)^2\right) \; \leq \; \log\left(\mathbf{y}^{'}\Sigma\mathbf{y} + (E\,[q_t])^2\right), \tag{E10}$$

which implies

$$D_b \; \leq \; \log\left(1 + \frac{\mathbf{y}^{'}\Sigma\mathbf{y}}{(E\,[q_t])^2}\right). \tag{E11}$$

To establish the positivity of the codependence measure $D_b$, we note that

$$D_b \quad = \quad \log\left(\frac{E\left[\left(m^P_{t,t+1}\right)^2\left(m^T_{t,t+1}\right)^2\right]}{E\left[\left(m^P_{t,t+1}\right)^2\right]E\left[\left(m^T_{t,t+1}\right)^2\right]}\right), \tag{E12}$$

$$\quad = \quad \log\left(\frac{Cov\left[\left(m^P_{t,t+1}\right)^2,\left(m^T_{t,t+1}\right)^2\right]}{E\left[\left(m^P_{t,t+1}\right)^2\right]E\left[\left(m^T_{t,t+1}\right)^2\right]} + 1\right). \tag{E13}$$

Since $Cov[m^P_{t,t+1}, m^T_{t,t+1}] = E\,[q_t] - E\,[1/R_{t,t+1,\infty}] > 0$, we have $Cov[(m^P_{t,t+1})^2, (m^T_{t,t+1})^2] > 0$. Therefore, $D_b > \log(1) = 0$. The proof of Theorem 5 is complete. ■

## Appendix F: Proof of Theorem 6

**Proof of the $n$-period bounds for the SDF:** We have $L[m_{t,t+n}] = \log(E[m_{t,t+n}]) - E[\log(m_{t,t+n})]$ and $q_t^{(n)} = E_t[m_{t,t+n}]$. Before proceeding, we define

$$\mathbf{y}^{(n)} = \left(\Sigma^{(n)}\right)^{-1}\left(\mathbf{1} - E[q_t^{(n)}]\,E[\mathbf{R}_{t,t+n}]\right) \qquad \text{and} \qquad \mathbf{a}^{(n)} \equiv \mathbf{y}^{(n)} / \left(\mathbf{1}'\mathbf{y}^{(n)}\right), \tag{F1}$$

where $\Sigma^{(n)}$ is the variance-covariance matrix of $\mathbf{R}_{t,t+n}$. Assume $(\mathbf{a}^{(n)})'\mathbf{R}_{t,t+n} > 0$. Using Jensen's inequality, we have

$$\begin{aligned} E\left[\log\left(m_{t,t+n}\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] &\leq \log\left(E\left[m_{t,t+n}\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right]\right), \\ &\leq \log\left(\mathbf{a}^{(n)'}E[m_{t,t+n}\mathbf{R}_{t,t+n}]\right), \\ &\leq \log\left(\mathbf{a}^{(n)'}\mathbf{1}\right), \\ &\leq 0, \end{aligned} \tag{F2}$$

and

$$E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] \leq -E[\log(m_{t,t+n})]. \tag{F3}$$

Adding $\log(E[m_{t,t+n}])$ to both sides of equation (F3) yields

$$\log(E[m_{t,t+n}]) + E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] \leq \log(E[m_{t,t+n}]) - E[\log(m_{t,t+n})] = L[m_{t,t+n}], \tag{F4}$$

and

$$\begin{aligned} L[m_{t,t+n}] &\geq E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] + \log\left(E\left[q_t^{(n)}\right]\right), \\ &\geq E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(n)}\right]\right), \\ &\geq E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] + \log\left(E\left[q_t^{(1)}\right]\right) - \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(n)}\right]\right), \\ &\geq E\left[\log\left(\mathbf{a}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - n\log\left(1/E\left[q_t^{(1)}\right]\right) - \left(n\log\left(E\left[q_t^{(1)}\right]\right) - \log\left(E\left[q_t^{(n)}\right]\right)\right). \end{aligned} \tag{F5}$$

Upon simplification, we get the bound in equation (46). Now note the relation, $L\left[m_{t,t+n}^2\right] = \log\left(E\left[m_{t,t+n}^2\right]\right) - E\left[\log\left(m_{t,t+n}^2\right)\right]$. We expand $L\left[m_{t,t+n}^2\right]$ to

$$\begin{aligned} L\left[m_{t,t+n}^2\right] &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2E\left[\log\left(m_{t,t+n}\right)\right], \\ &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[m_{t,t+n}\right]\right) + 2\left(\log\left(E\left[m_{t,t+n}\right]\right) - E\left[\log\left(m_{t,t+n}\right)\right]\right), \\ &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[q_t^{(n)}\right]\right) + 2L\left[m_{t,t+1}\right]. \end{aligned} \tag{F6}$$

Recall that

$$\begin{aligned} L[m_{t,t+n}^2] &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2E\left[\log\left(m_{t,t+n}\right)\right], \\ &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[q_t^{(n)}\right]\right) + 2L\left[m_{t,t+n}\right], \\ &= \log\left(\frac{E\left[m_{t,t+n}^2\right]}{\left(E\left[q_t^{(n)}\right]\right)^2}\right) + 2L\left[m_{t,t+n}\right]. \end{aligned} \tag{F7}$$

Using an application of the Cauchy Schwartz inequality, the result in equation (47) follows. The corresponding bounds on $L[m_{t,t+n}^P]$ and $L[(m_{t,t+n}^P)^2]$ obey a similar construction, and the details are available from the authors. ∎

## Online Appendix I: Proofs of the results in Section 2.2

**Proof of equation (7) in Example 1**. We apply the definition of entropy to $(m^P_{t,t+1})^2$, i.e., $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)]$. Under our assumption, $\log(m^P_{t,t+1})$ is normally distributed with mean $-\frac{1}{2}\sigma^2_P$ and variance $\sigma^2_P$. Therefore,

$$L[m^P_{t,t+1}] = -E\left[\log\left(m^P_{t,t+1}\right)\right] = \frac{1}{2}\sigma^2_P, \tag{G1}$$

and,

$$L[(m^P_{t,t+1})^2] = \log\left(\exp\left(-\frac{2}{2}\sigma^2_P + \frac{4}{2}\sigma^2_P\right)\right) \; - \; 2\left(-\frac{1}{2}\sigma^2_P\right) = 2\sigma^2_P. \tag{G2}$$

Using equations (G1) and (G2), we see that $L[(m^P_{t,t+1})^2] = 4L[m^P_{t,t+1}]$. ■

**Proof of equation (8)**. Using the definition of entropy,

$$\begin{aligned} L[(m^P_{t,t+1})^2] - 4L[m^P_{t,t+1}] &= \log(E[\left(m^P_{t,t+1}\right)^2]) - 2E[\log\left(m^P_{t,t+1}\right)] + 4E[\log(m^P_{t,t+1})], && \text{(G3)} \\ &= \log(E[\left(m^P_{t,t+1}\right)^2]) + E[\log\left(((m^P_{t,t+1})^2\right)]. && \text{(G4)} \end{aligned}$$

Thus, we have the desired expression. ■

**Proof of equation (9) in Example 2**. Combining our assumption that $\log(m^P_{t,t+1})$ follows $\mathcal{N}\left(\mu_t, \sigma^2_t\right)$ with the fact that $E_t[m^P_{t,t+1}] = 1$, we note

$$e^{\mu_t + \frac{1}{2}\sigma^2_t} = 1, \;\text{ which implies }\; \mu_t = -\frac{1}{2}\sigma^2_t. \;\text{ Hence, }\; L[m^P_{t,t+1}] = \frac{1}{2}E[\sigma^2_t]. \tag{G5}$$

Next, we evaluate $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)]$ in two steps. Since $\log(m^P_{t,t+1})$ follows $\mathcal{N}\left(\mu_t, \sigma^2_t\right)$, we obtain

$$E[(m^P_{t,t+1})^2] = E[E_t[\left(m^P_{t,t+1}\right)^2]] = E[e^{2\mu_t + 2\sigma^2_t}] = E[e^{\sigma^2_t}] \qquad (\text{since } \mu_t = -\frac{1}{2}\sigma^2_t). \tag{G6}$$

With the above results, we note that $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)] = \log(E[e^{\sigma^2_t}]) -$

$2\mu_t = \log(E[e^{\sigma_t^2}]) + E[\sigma_t^2].$ ■

**Proof of equation (10) in Example 2**. Observe that

$$L[(m_{t,t+1}^P)^2] - 4L[m_{t,t+1}^P] = \log(E[e^{\sigma_t^2}]) + E[\sigma_t^2] - 4\left(\frac{1}{2}E\left[\sigma_t^2\right]\right) = \log\left(\frac{E\left[e^{\sigma_t^2}\right]}{\exp\left(E\left[\sigma_t^2\right]\right)}\right). \tag{G7}$$

The Taylor expansion of $e^{\sigma_t^2}$ around $\sigma_t^2 = E\left[\sigma_t^2\right]$ yields

$$e^{\sigma_t^2} = e^{\left(E\left[\sigma_t^2\right]\right)} + \sum_{j=1}^{\infty} \frac{1}{j!}\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j e^{\left(E\left[\sigma_t^2\right]\right)}, \tag{G8}$$

which implies

$$\frac{E\left[e^{\sigma_t^2}\right]}{e^{\left(E\left[\sigma_t^2\right]\right)}} = 1 + \sum_{j=1}^{\infty} \frac{1}{j!}E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right] = 1 + \sum_{j=2}^{\infty} \frac{1}{j!}E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right]. \tag{G9}$$

Therefore,

$$L[(m_{t,t+1}^P)^2] - 4L[m_{t,t+1}^P] = \log\left(\frac{E\left[e^{\sigma_t^2}\right]}{e^{\left(E\left[\sigma_t^2\right]\right)}}\right) = \log\left(1 + \sum_{j=2}^{\infty} \frac{1}{j!}E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right]\right). \tag{G10}$$

This ends the proof of equation (10). ■

**Proof of equation (11)**. We observe that

$$\exp\left(\log((m_{t,t+1}^P)^2)\right) = \exp\left(2\log\left(m_{t,t+1}^P\right)\right). \tag{G11}$$

The Taylor expansion series of $\exp\left(\log((m_{t,t+1}^P)^2)\right)$ around $E[\log(m_{t,t+1}^P)]$ produces

$$\exp\left(2\log\left(m_{t,t+1}^P\right)\right) = \exp\left(2E\left[\log\left(m_{t,t+1}^P\right)\right]\right) + \sum_{j=1}^{\infty} \frac{2^j}{j!}\left(\log\left(m_{t,t+1}^P\right) - E\left[\log\left(m_{t,t+1}^P\right)\right]\right)^j. \tag{G12}$$

We apply the expectation operator to (G12) and get

$$E\left[\exp\left(2\log\left(m_{t,t+1}^{P}\right)\right)\right] = \exp\left(2E\left[\log\left(m_{t,t+1}^{P}\right)\right]\right) + \sum_{j=1}^{\infty}\frac{2^{j}}{j!}\kappa_{j}\exp\left(2E\left[\log\left(m_{t,t+1}^{P}\right)\right]\right), \tag{G13}$$

with $\kappa_j = E\left[\left(\log\left(m_{t,t+1}^{P}\right) - E\left[\log\left(m_{t,t+1}^{P}\right)\right]\right)^{j}\right]$. Next, we apply the log function to (G13) and get

$$\log\left(E\left[\exp\left(2\log\left(m_{t,t+1}^{P}\right)\right)\right]\right) = \log\left(\exp\left(2E\left[\log\left(m_{t,t+1}^{P}\right)\right]\right)\right) + \log\left(1+\sum_{j=1}^{\infty}\frac{2^{j}}{j!}\kappa_{j}\right), \tag{G14}$$

and

$$\log\left(E\left[\exp\left(2\log\left(m_{t,t+1}^{P}\right)\right)\right]\right) - 2E\left[\log\left(m_{t,t+1}^{P}\right)\right] = \log\left(1+\sum_{j=1}^{\infty}\frac{2^{j}}{j!}\kappa_{j}\right). \tag{G15}$$

Expression (G15) is equivalent to

$$L[\left(m_{t,t+1}^{P}\right)^{2}] = \log(E[(m_{t,t+1}^{P})^{2}]) - E[\log((m_{t,t+1}^{P})^{2})] = \log\left(1+\sum_{j=1}^{\infty}\frac{2^{j}}{j!}\kappa_{j}\right). \tag{G16}$$

This completes our description of the steps. ■

**Proof of equation (12)**. Under the normality of $\log(m_{t,t+1}^{P})$, we get

$$\begin{aligned} L[(m_{t,t+1}^{P})^{2}] &= \log\left(1+2Var\left[\log\left(m_{t,t+1}^{P}\right)\right]\right) \\ &= \log\left(1-4E\left[\log\left(m_{t,t+1}^{P}\right)\right]\right) \quad (\text{since } E\left[\log\left(m_{t,t+1}^{P}\right)\right] = -\frac{1}{2}Var\left[\log\left(m_{t,t+1}^{P}\right)\right]) \\ &= -4E\left[\log\left(m_{t,t+1}^{P}\right)\right] = 4L[m_{t,t+1}^{P}], \end{aligned} \tag{G17}$$

as desired. ■

**Proof of equation (13)**. Observe that

$$
\begin{aligned}
L[(m^P_{t,t+1})^2] - 2L[m^P_{t,t+1}] &= \log(E[(m^P_{t,t+1})^2]) - 2E[\log(m^P_{t,t+1})] - 2\log(E[m^P_{t,t+1}]) + 2E\left[\log\left(m^P_{t,t+1}\right)\right] \\
&= \log\left(E\left[\left(m^P_{t,t+1}\right)^2\right]\right) \\
&= \log\left(E[(m^P_{t,t+1})^2] - (E[m^P_{t,t+1}])^2 + 1\right) \\
&= \log\left(1 + Var\left[m^P_{t,t+1}\right]\right) \approx Var\left[m^P_{t,t+1}\right], \qquad \text{(G18)}
\end{aligned}
$$

which is what we present in the main body of the paper. ■

## Online Appendix II: Analytical solution for the eigenfunction problem in Example 3

Consider the eigenfunction problem for the dynamics of the SDF in equation (14):

$$E_t\left[m_{t,t+1}\, e_{t+1}\right] = \zeta\, e_t, \text{ where } \zeta \text{ is the eigenvalue and } e_{t+1} \text{ is the eigenfunction.} \qquad \text{(H1)}$$

Accordingly, the permanent and transitory components of the SDF are

$$m^P_{t,t+1} = m_{t,t+1}\left(\frac{e_{t+1}}{\zeta e_t}\right) \qquad \text{and} \qquad m^T_{t,t+1} = \frac{\zeta e_t}{e_{t+1}}. \qquad \text{(H2)}$$

We conjecture that the eigenfunction $e_{t+1}$ takes the form $e_{t+1} = \exp\left(\xi\, z_{t+1}\right)$. Consider the expression

$$
\begin{aligned}
\log(m_{t,t+1}) + \log(e_{t+1}) - \log(e_t) &= -\delta - \gamma z_t - \lambda z_t^{\frac{1}{2}}\varepsilon_{t+1} + \xi z_{t+1} - \xi z_t, \\
&= -\delta + \xi\,(1-\varphi)\,\theta + (-\gamma + \xi\varphi - \xi)\, z_t + (-\lambda + \xi\sigma)\, z_t^{\frac{1}{2}}\varepsilon_{t+1}, \qquad \text{(H3)}
\end{aligned}
$$

and, thus,

$$E_t\left[m_{t,t+1}\frac{e_{t+1}}{e_t}\right] = \exp\left(-\delta + \xi\,(1-\varphi)\,\theta + \left(-\gamma + \xi\varphi - \xi + \frac{1}{2}\,(-\lambda + \xi\sigma)^2\right) z_t\right). \qquad \text{(H4)}$$

Therefore,

$$\log(\zeta) = -\delta + \xi(1-\varphi)\theta \qquad \text{and} \qquad -\gamma + \xi\varphi - \xi + \frac{1}{2}(-\lambda + \xi\sigma)^2 = 0. \tag{H5}$$

It may be seen that the second expression in equation (H5) is amenable to the simplification:

$$\frac{1}{2}\lambda^2 - \gamma + \xi(\varphi - 1 - \lambda\sigma) + \frac{1}{2}\xi^2\sigma^2 = 0. \tag{H6}$$

To be consistent with Backus, Foresi, and Telmer (2001, Section II.B), we must have $\gamma = \frac{1}{2}\left(1+\lambda^2\right)$. Let $\Delta = (\varphi - 1 - \lambda\sigma)^2 - 2\sigma^2\left(\frac{1}{2}\lambda^2 - \gamma\right) > 0$. Following Hansen and Scheinkman (2009), we select the solution associated with the negative root. Consequently, we choose

$$\xi = \frac{-\xi(\varphi - 1 - \lambda\sigma) - \sqrt{\Delta}}{\sigma^2}. \tag{H7}$$

The transitory component of the SDF is $m^T_{t,t+1} = \exp\left(-\delta + \xi(1-\varphi)\theta + \xi(z_t - z_{t+1})\right)$. Hence, the log permanent component of the SDF is $\log(m^P_{t,t+1}) = \log(m_{t,t+1}) - \log(m^T_{t,t+1})$, which delivers equation (15). The entropies in (14) follow by exploiting the conditional expectation. ■

## Online Appendix III: Expression for our bound in our Theorem 3 when *N*=2

Consider the two-asset specialization with market portfolio and another asset with gross return $R^i_{t,t+1}$: $\mathbf{R}_{t,t+1} = [R^m_{t,t+1}, R^i_{t,t+1}]$. Therefore,

$$\Sigma = \begin{bmatrix} Var[R^m_{t,t+1}] & Cov(R^m_{t,t+1}, R^i_{t,t+1}) \\ Cov(R^m_{t,t+1}, R^i_{t,t+1}) & Var[R^i_{t,t+1}] \end{bmatrix} \equiv \begin{bmatrix} \sigma^2_M & \beta_i\sigma^2_M \\ \beta_i\sigma^2_M & \sigma^2_i \end{bmatrix}, \tag{I1}$$

and

$$\Sigma^{-1} = \frac{1}{(\sigma^2_M\sigma^2_i - \beta^2_i\sigma^4_M)} \begin{bmatrix} \sigma^2_i & -\beta_i\sigma^2_M \\ -\beta_i\sigma^2_M & \sigma^2_M. \end{bmatrix}. \tag{I2}$$

Then $\mathbf{y} = \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)$ is

$$\mathbf{y} = \frac{1}{\left(\sigma_M^2\sigma_i^2 - \beta_i^2\sigma_M^4\right)}\begin{bmatrix} \sigma_i^2 & -\beta_i\sigma_M^2 \\ -\beta_i\sigma_M^2 & \sigma_M^2 \end{bmatrix}\left(\begin{bmatrix} 1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right] \\ 1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right] \end{bmatrix}\right), \tag{I3}$$

$$= \frac{1}{\left(\sigma_M^2\sigma_i^2 - \beta_i^2\sigma_M^4\right)}\begin{pmatrix} \sigma_i^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) - \beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right) \\ -\beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) + \sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right) \end{pmatrix}. \tag{I4}$$

Further manipulation yields

$$\mathbf{1}'\mathbf{y} = \frac{\left(\sigma_i^2 - \beta_i\sigma_M^2\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) + \sigma_M^2\left(1 - \beta_i\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)}{\left(\sigma_i^2 - \beta_i^2\sigma_M^2\right)\sigma_M^2}. \tag{I5}$$

Therefore,

$$\begin{aligned} & \frac{1}{\mathbf{1}'\mathbf{y}}\left(\mathbf{y}'\mathbf{R}_{t+1}\right) \\ = \; & \frac{1}{\mathbf{1}'\mathbf{y}}\frac{1}{\left(\sigma_M^2\sigma_i^2 - \beta_i^2\sigma_M^4\right)}\left\{\begin{array}{c} \sigma_i^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right)R_{t,t+1}^m - \beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)R_{t,t+1}^m \\ -\beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)R_{t,t+1}^i + \sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)R_{t,t+1}^i \end{array}\right\} \\ = \; & a_0 R_{t,t+1}^m + b_0 R_{t,t+1}^i, \end{aligned} \tag{I6}$$

where $a_0 + b_0 = 1$ and setting

$$a_0 \equiv \frac{\sigma_i^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) - \beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)}{\left(\sigma_i^2 - \beta_i\sigma_M^2\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) + \sigma_M^2\left(1 - \beta_i\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)}, \tag{I7}$$

$$b_0 \equiv \frac{\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right) - \beta_i\sigma_M^2\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right)}{\left(\sigma_i^2 - \beta_i\sigma_M^2\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^m\right]\right) + \sigma_M^2\left(1 - \beta_i\right)\left(1 - E\left[q_t\right]E\left[R_{t,t+1}^i\right]\right)}. \tag{I8}$$

The final expression for the entropy bound $L[m_{t,t+1}]$ can therefore be written as:

$$\begin{aligned} L[m_{t,t+1}] & \geq E\left[\log\left(\frac{1}{\mathbf{1}'\mathbf{y}}\left(\mathbf{y}'\mathbf{R}_{t+1}\right)\right)\right] - \log\left(\left(E\left[q_t\right]\right)^{-1}\right), \\ & = E\left[\log\left(a_0 R_{t,t+1}^m + b_0 R_{t,t+1}^i\right)\right] - \log\left(\left(E\left[q_t\right]\right)^{-1}\right). \end{aligned} \tag{I9}$$

Upon rearranging equation (I9), we obtain the expression in equation (22). ■

**Online Appendix IV: Details of the difference habit model in Proposition 1**

For the law of motions of the habit and consumption growth in equation (30), we define the backshift operators $\eta[B]$ and $\gamma[B]$:

$$\eta[B] = \sum_{j=0}^{\infty} \eta_j B^j \qquad \text{and} \qquad \gamma[B] = \sum_{j=0}^{\infty} \gamma_j B^j, \tag{J1}$$

with $\eta_0 = 1 - \varphi_h$ and $\eta_{j+1} = \varphi_h \eta_j$, $j \geq 0$, and $\gamma_0 = 1$. Invoking a log linear approximation of $\log(s_t)$,

$$\log(m_{t,t+1}) = D_0 + (\rho - 1)\frac{1}{s}\left(1 - (1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}, \tag{J2}$$

$$\text{where } D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h} - 1\right)\log(g).$$

Using a log linear approximation $\log(s_t) \approx 1 + \frac{(s-1)}{s} z_t$, the dynamics of the surplus consumption ratio is

$$\log(s_{t+1}) - \log(s_t) = \frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g_{t+1}). \tag{J3}$$

Therefore, we may write the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g) \\ &\quad + (\rho-1)\frac{1}{s}\left(1-(1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}.\end{aligned} \tag{J4}$$

To solve for the permanent and transitory components of the SDF, we write the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g) + (\rho-1)\frac{1}{s}x_t \\ &\quad - (\rho-1)\frac{1}{s}(1-s)\eta[B]Bx_t - (\rho-1)\frac{1}{s}(1-s)\eta[B]B\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ &\quad + (\rho-1)\frac{1}{s}\upsilon^{\frac{1}{2}}\omega_{gt+1},\end{aligned} \tag{J5}$$

where

$$x_t = (\gamma[B] - \gamma_0)\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}, \qquad \text{implying} \qquad x_{t+1} - \varphi_g\, x_t = \gamma_1 \upsilon^{1/2}\,\omega_{gt+1}. \tag{J6}$$

We simplify the log SDF as

$$\begin{aligned} \log(m_{t,t+1}) &= D_0 + (\rho-1)\,\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right)x_t + (\rho-1)\,\frac{1}{s\gamma_1}x_{t+1} \\ &\quad + (\rho-1)\,\frac{1}{s}\,(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta\,[B]\,x_{t-1} - (\rho-1)\,\frac{1}{s}\,(1-s)\,\frac{1}{\gamma_1}\eta\,[B]\,x_t. \end{aligned} \tag{J7}$$

We conjecture that the eigenfunction $e_{t+1}$ corresponding to the general problem in equations (H1) and (H2) is of the form:

$$\log(e_{t+1}) = \delta\,[B]\,x_{t+1}, \qquad \text{where} \qquad \delta\,[B] = \sum_{j=0}^{\infty} \delta_j B^j \qquad \text{with} \qquad \delta_0 = 1. \tag{J8}$$

To verify the solution, we expand to the following:

$$\begin{aligned} \log(m_{t,t+1}) + \log\left(\frac{e_{t+1}}{e_t}\right) &= D_0 + (\rho-1)\,\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right)x_t + (\rho-1)\,\frac{1}{s}\,(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta\,[B]\,x_{t-1} \\ &\quad - \left((\rho-1)\,\frac{1}{s}\,(1-s)\,\frac{1}{\gamma_1}\eta\,[B] + \delta\,[B]\right)x_t + (\delta\,[B] - \delta_0)\,x_{t+1} \\ &\quad + \left((\rho-1)\,\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)x_{t+1}. \end{aligned} \tag{J9}$$

Upon simplifying the expectation involving the eigenfunction problem, we derive $\zeta$ as

$$\begin{aligned} \log(\zeta) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\,\frac{(s-1)}{s}\,(\eta\,[B]\,B - 1)\log(g) + \frac{1}{2}\left((\rho-1)\,\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)^2 \gamma_1^2\upsilon \\ &\quad + \left((\rho-1)\,\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right) + \left((\rho-1)\,\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)\varphi_g\right)x_t + (\rho-1)\,\frac{1}{s}\,(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta\,[B]\,x_{t-1} \\ &\quad + \left(-(\rho-1)\,\frac{1}{s}\,(1-s)\,\frac{1}{\gamma_1}\eta\,[B] - \delta\,[B]\right)x_t + (\delta\,[B] - \delta_0)\,x_{t+1}. \end{aligned} \tag{J10}$$

Using the identification approach, we deduce

$$\log(\zeta) = D_0 + \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1} + \delta_0\right)^2 \gamma_1^2 \upsilon \tag{J11}$$

and

$$\begin{aligned}
\delta_1 &= -\left((\rho-1)\frac{1}{s} + \delta_0\varphi_g\right) - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_0 - \delta_0\right), \\
\delta_{j+1} &= -(\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta_{j-1} - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_j - \delta_j\right) \text{ for } j \geq 1.
\end{aligned} \tag{J12}$$

Exploiting the solution to the eigenfunction function, we derive the transitory component of the SDF as

$$m_{t,t+1}^T = \exp\left(D_0 + D_1 + D_5\,(x_t - x_{t+1})\right). \tag{J13}$$

Equation (J13) implies the permanent component in equation (33) of Proposition 1, where

$$D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h} - 1\right)\log(g), \tag{J14}$$

$$D_1 = \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1} + \delta_0\right)^2 \gamma_1^2 \upsilon, \tag{J15}$$

$$D_2 = (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B], \tag{J16}$$

$$D_3 = -\delta[B] - (\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] + (\rho-1)\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right), \tag{J17}$$

$$D_4 = (\rho-1)(s\gamma_1)^{-1} + \delta[B], \quad \text{and} \tag{J18}$$

$$D_5 = \delta[B]. \tag{J19}$$

This ends the proof. ■

### Online Appendix V: Details of the recursive utility models in Propositions 2 and 3

Based on equations (34) and (36), we note that $\omega_{gt}$, $z_{gt}$, and $\omega_{ht}$ are standard normal random variables, independent of each other and across time. The jump component $z_{gt}$ is a poisson mixture of normals:

conditional on the number of jumps $j$, $z_{gt}$ is normal with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date $t+1$ is $e^{h_t} h_t^j / j!$ expands to

$$\begin{aligned} m_{t,t+1} &= \exp\left(\chi_0 + a_g[B]\,\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + a_z[B]\,z_{gt+1} + a_\upsilon[B]\,\omega_{\upsilon t+1} + a_h[B]\,\omega_{ht+1}\right), \quad &(\text{K1}) \\ \chi_0 &= \log(\beta) + (\rho - 1)\log(g) \\ &\quad - (\alpha - \rho)(D\upsilon - Jh) - (\alpha - \rho)(\alpha/2)\left((Db_1\nu[b_1])^2 + (Jb_1\eta[b_1])^2\right), \quad &(\text{K2}) \end{aligned}$$

where $a_g[B]$, $a_z[B]$, $a_\upsilon[B]$, and $a_h[B]$ are backshift operators defined as follows:

$$a_g[B] = (\rho - 1)\gamma[B] + (\alpha - \rho)\gamma[b_1], \qquad a_z[B] = (\rho - 1)\psi[B] + (\alpha - \rho)\psi[b_1], \tag{K3}$$

$$a_\upsilon[B] = (\alpha - \rho)\,D\,(b_1\nu[b_1] - \nu[B]B), \qquad a_h[B] = (\alpha - \rho)\,J\,(b_1\eta[b_1] - \eta[B]B), \tag{K4}$$

$$D = (\alpha/2)(\gamma[b_1])^2, \quad \text{and} \qquad J = \left(\frac{e^{\alpha\psi[b_1]\theta + (\alpha\psi[b_1]\delta)^2} - 1}{\alpha}\right). \tag{K5}$$

The functions $\eta[b_1]$, $\nu[b_1]$, and $\gamma[b_1]$ are polynomial functions of $b_1$:

$$\eta[b_1] = \sum_{j=0}^{\infty} b_1^j \eta_j, \qquad \gamma[b_1] = \sum_{j=0}^{\infty} b_1^j \gamma_j, \qquad \nu[b_1] = \sum_{j=0}^{\infty} b_1^j \nu_j, \qquad \psi[b_1] = \sum_{j=0}^{\infty} b_1^j \psi_j, \tag{K6}$$

with $\gamma_0 = 1$, where

$$\sum_{j=1}^{\infty} \gamma_j < \infty, \qquad \sum_{j=1}^{\infty} \eta_j < \infty, \qquad \sum_{j=1}^{\infty} \nu_j < \infty, \qquad \sum_{j=1}^{\infty} \psi_j < \infty, \tag{K7}$$

and

$$\nu[B] = \sum_{j=0}^{\infty} \nu_j B^j \qquad \text{and} \qquad \psi[B] = \sum_{j=0}^{\infty} \psi_j B^j. \tag{K8}$$

**A. Recursive utility with stochastic variance:** The SDF is a special case of (K1) with $h = 0$, $\eta[B] = 0$,

$J = 0$. The SDF takes the following form:

$$m_{t,t+1} = \exp\left(\begin{array}{c} H_0 + (\rho-1)\gamma[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + (\alpha-\rho)\gamma[b_1]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} \\ +(\alpha-\rho)Db_1\nu[b_1]\omega_{\upsilon t+1} - (\alpha-\rho)D\nu[B]B\omega_{\upsilon t+1} \end{array}\right),$$

with

$$H_0 = \log(\beta) + (\rho-1)\log g - (\alpha-\rho)(D\upsilon) - (\alpha-\rho)(\alpha/2)\left((Db_1\nu[b_1])^2\right). \tag{K9}$$

Now, define

$$x_t = (\gamma[B]-\gamma_0)\upsilon_t^{\frac{1}{2}}\omega_{gt+1}. \tag{K10}$$

The state variable $x_t$ dynamics is:

$$x_t = \varphi_g x_{t-1} + \gamma_1\upsilon_{t-1}^{\frac{1}{2}}\omega_{gt}, \qquad \text{with} \qquad \gamma_j = \varphi_g\gamma_{j-1} \text{ for } j \geq 2 \qquad \text{and} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{K11}$$

It can also be shown that the dynamics of the state variable $\upsilon_t$ is

$$\upsilon_t - \upsilon = \varphi_\upsilon(\upsilon_{t-1}-\upsilon) + \nu_0\omega_{\upsilon t}, \qquad \text{for } j \geq 2 \qquad \text{and} \qquad \varphi_\upsilon = \frac{\nu_1}{\nu_0}. \tag{K12}$$

The SDF can be expanded to

$$m_{t,t+1} = \exp(H_1 + H_2x_t + H_3x_{t+1} + H_4\upsilon_t + H_5\upsilon_{t+1}), \tag{K13}$$

where

$$H_1 = H_0 + (\alpha-\rho) D\upsilon + (\alpha-\rho) D b_1 \nu[b_1] \frac{(\varphi_\upsilon - 1)}{\nu_0}\upsilon, \tag{K14}$$

$$H_2 = (\rho-1) - ((\alpha-\rho)\gamma[b_1] + (\rho-1)) \frac{\varphi_g}{\gamma_1}, \tag{K15}$$

$$H_3 = \frac{(\rho-1)}{\gamma_1} + \frac{(\alpha-\rho)\gamma[b_1]}{\gamma_1}, \tag{K16}$$

$$H_4 = (\alpha-\rho) D \left(-b_1 \nu[b_1] \frac{\varphi_\upsilon}{\nu_0} - 1\right), \text{ and} \tag{K17}$$

$$H_5 = (\alpha-\rho) D b_1 \frac{\nu[b_1]}{\nu_0}. \tag{K18}$$

Proceeding, we now solve the eigenfunction problem specified in equations (H1) and (H2). We conjecture that $\log(e_{t+1}) = \tau_0 x_{t+1} + \tau_1 \upsilon_{t+1}$. Hence,

$$\log(m_{t,t+1} e_{t+1}/e_t) = H_1 + (H_2 - \tau_0) x_t + (H_3 + \tau_0) x_{t+1} + (H_4 - \tau_1)\upsilon_t + (H_5 + \tau_1)\upsilon_{t+1} \tag{K19}$$

and

$$\begin{aligned} \log(\zeta) &= H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \frac{1}{2}(H_5+\tau_1)^2\nu_0^2 + (H_2 - \tau_0 + (H_3+\tau_0)\varphi_g) x_t \\ &\quad + \left((H_4-\tau_1) + \frac{1}{2}(H_3+\tau_0)^2\gamma_1^2 + (H_5+\tau_1)\varphi_\upsilon\right)\upsilon_t. \end{aligned} \tag{K20}$$

Using the identification approach, we arrive at the expressions:

$$\log(\zeta) = H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \frac{1}{2}(H_5+\tau_1)^2\nu_0^2 \tag{K21}$$

and

$$\tau_0 = \frac{H_2 + H_3\varphi_g}{1-\varphi_g} \quad \text{and} \quad \tau_1 = \frac{H_4 + \frac{1}{2}(H_3+\tau_0)^2\gamma_1^2 + H_5\varphi_\upsilon}{1-\varphi_\upsilon}. \tag{K22}$$

With these results, we are in a position to state the transitory and permanent components as:

$$\begin{aligned} m^T_{t,t+1} &= \exp\left( H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \tfrac{1}{2}(H_5+\tau_1)^2\nu_0^2 + \tau_0(x_t - x_{t+1}) + \tau_1(\upsilon_t - \upsilon_{t+1}) \right), \\ m^P_{t,t+1} &= \exp\left( \begin{array}{c} -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon) - \tfrac{1}{2}(H_5+\tau_1)^2\nu_0^2 \\ (H_2-\tau_0)x_t + (H_3+\tau_0)x_{t+1} + (H_4-\tau_1)\upsilon_t + (H_5+\tau_1)\upsilon_{t+1} \end{array} \right). \end{aligned} \tag{K23}$$

Setting $H_6 \equiv -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon) - (H_5+\tau_1)^2\nu_0^2/2$, we obtain equation (38) of Proposition 2. ■

**B. Recursive utility model with constant jump intensity:** Consider the consumption growth dynamics with $\nu[B]=0$ (in this case $\upsilon_t = \upsilon$). It can be shown that the SDF reduces to

$$m_{t,t+1} = \exp\left( \begin{array}{c} \chi_0 \\ +(\rho-1)x_t + ((\rho-1)\gamma_0 + (\alpha-\rho)\gamma[b_1])\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ +(\rho-1)(\psi[B]-\psi_0)z_{gt+1} + ((\rho-1)\psi_0 + (\alpha-\rho)\psi[b_1])z_{gt+1} \\ +(\alpha-\rho)Jb_1\eta[b_1]\omega_{ht+1} - (\alpha-\rho)(h_t-h)J \end{array} \right). \tag{K24}$$

Now denote

$$\widetilde{x}_t = (\psi[B]-\psi_0)z_{gt+1}. \tag{K25}$$

The law of motion of $\widetilde{x}_t$ becomes

$$\widetilde{x}_t = \varphi_z\widetilde{x}_{t-1} + \psi_1 z_{gt} \qquad \text{with} \qquad \varphi_z = \frac{\psi_2}{\psi_1} \qquad \text{and} \qquad \psi_{j+2} = \varphi_z\psi_{j+1} \quad \text{for } j \geq 1. \tag{K26}$$

The SDF in equation (K24) reduces to

$$m_{t,t+1} = \exp\left( G_0 + G_1x_t + G_2\widetilde{x}_{t-1} + G_3z_{gt} + G_4h_t + G_5z_{gt+1} + G_6\upsilon^{\frac{1}{2}}\omega_{gt+1} + G_7\omega_{ht+1} \right), \tag{K27}$$

with

$$
\begin{aligned}
G_0 &= \chi_0 + (\alpha - \rho) hJ, & G_1 &= (\rho - 1), \\
G_2 &= (\rho - 1) \varphi_z, & G_3 &= (\rho - 1) \psi_1, \\
G_4 &= -(\alpha - \rho) J, & G_5 &= (\rho - 1) \psi_0 + (\alpha - \rho) \psi [b_1], \\
G_6 &= (\rho - 1) \gamma_0 + (\alpha - \rho) \gamma [b_1], & G_7 &= (\alpha - \rho) J b_1 \eta [b_1].
\end{aligned}
$$

For the eigenfunction problem in equations (H1)-(H2), i.e., $E_t [m_{t,t+1} e_{t+1}] = \zeta e_t$, we conjecture that the eigenfunction is of the form:

$$
e_{t+1} = \exp (\varsigma_0 h_{t+1} + \varsigma_1 z_{gt+1} + \varsigma_2 x_{t+1} + \varsigma_3 \widetilde{x}_t). \tag{K28}
$$

Algebraic manipulation yields the expression:

$$
\begin{aligned}
m_{t,t+1} \frac{e_{t+1}}{e_t} &= \exp \begin{pmatrix} G_0 + \varsigma_0 h - \varsigma_0 \varphi_h h + (G_1 - \varsigma_2 + \varsigma_2 \varphi_g) x_t + G_2 \widetilde{x}_{t-1} + (G_3 - \varsigma_1 + \varsigma_3 \psi_1) z_{gt} \\ + (G_4 - \varsigma_0 + \varsigma_0 \varphi_h) h_t + (\varsigma_3 \varphi_z - \varsigma_3) \widetilde{x}_{t-1} \end{pmatrix} \\
&\times \exp \left( (G_5 + \varsigma_1) z_{gt+1} + (G_6 + \varsigma_2 \gamma_1) \upsilon^{\frac{1}{2}} \omega_{gt+1} + (G_7 + \varsigma_0 \eta_0) \omega_{ht+1} \right).
\end{aligned} \tag{K29}
$$

Upon further manipulation of equation (K29), we get

$$
\zeta = \xi \times \exp \begin{pmatrix} G_0 + \varsigma_0 h - \varsigma_0 \varphi_h h + (G_1 - \varsigma_2 + \varsigma_2 \varphi_g) x_t + G_2 \widetilde{x}_{t-1} + (G_3 - \varsigma_1 + \varsigma_3 \psi_1) z_{gt} \\ + (G_4 - \varsigma_0 + \varsigma_0 \varphi_h) h_t + (\varsigma_3 \varphi_z - \varsigma_3) \widetilde{x}_{t-1} \end{pmatrix}, \tag{K30}
$$

with

$$
\xi = E_t \left( \exp \left( (G_5 + \varsigma_1) z_{gt+1} + (G_6 + \varsigma_2 \gamma_1) \upsilon^{\frac{1}{2}} \omega_{gt+1} + (G_7 + \varsigma_0 \eta_0) \omega_{ht+1} \right) \right). \tag{K31}
$$

One may observe that

$$
\begin{aligned}
\xi &= (E_t \exp ((G_5 + \varsigma_1) z_{gt+1})) \left( E_t \exp \left( (G_6 + \varsigma_2 \gamma_1) \upsilon^{\frac{1}{2}} \omega_{gt+1} \right) \right) (E_t ((G_7 + \varsigma_0 \eta_0) \omega_{ht+1})) \qquad \text{(K32)} \\
&= E_t \left( \exp \left( \left( (G_5 + \varsigma_1) \theta + \frac{1}{2} (G_5 + \varsigma_1)^2 \delta^2 \right) j \right) \right) \exp \left( \frac{1}{2} (G_6 + \varsigma_2 \gamma_1)^2 \upsilon + \frac{1}{2} (G_7 + \varsigma_0 \eta_0)^2 \right)
\end{aligned}
$$

and

$$E_t\left(\exp\left(\left(\left(G_5+\varsigma_1\right)\theta+\frac{1}{2}\left(G_5+\varsigma_1\right)^2\delta^2\right)j\right)\right)=\exp\left(G_8h_t\right), \tag{K33}$$

with

$$G_8=e^{\left(\left(G_5+\varsigma_1\right)\theta+\frac{1}{2}\left(G_5+\varsigma_1\right)^2\delta^2\right)}-1. \tag{K34}$$

As a consequence, equation (K32) simplifies to

$$\xi \;=\; \exp\left(G_8h_t+\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2\right). \tag{K35}$$

We substitute equation (K35) in equation (K30) and rearrange to obtain:

$$\begin{aligned}\log(\varsigma) \;=\;& G_0+\varsigma_0h-\varsigma_0\varphi_hh+\left(G_1-\varsigma_2+\varsigma_2\varphi_g\right)x_t\\ &+\left(G_3-\varsigma_1+\varsigma_3\psi_1\right)z_{gt}+\left(G_4-\varsigma_0+\varsigma_0\varphi_h+G_8\right)h_t\\ &+\left(\varsigma_3\varphi_z-\varsigma_3+G_2\right)\widetilde{x}_{t-1}+\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2.\end{aligned} \tag{K36}$$

Using the identification approach, we then have

$$\log(\varsigma)=G_0+\varsigma_0h\left(1-\varphi_h\right)+\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2 \tag{K37}$$

and

$$\begin{aligned}G_1-\varsigma_2+\varsigma_2\varphi_g &= 0, \qquad G_4-\varsigma_0+\varsigma_0\varphi_h+G_8 = 0,\\ G_3-\varsigma_1+\varsigma_3\psi_1 &= 0, \qquad \varsigma_3\varphi_z-\varsigma_3+G_2 = 0.\end{aligned} \tag{K38}$$

Finally, we get

$$\varsigma_0 = \frac{G_8+G_4}{1-\varphi_h}, \qquad \varsigma_1 = G_3+\varsigma_3\psi_1, \qquad \varsigma_2 = \frac{G_1}{1-\varphi_g}, \qquad \varsigma_3 = \frac{G_2}{1-\varphi_z}. \tag{K39}$$

The transitory component is, therefore, $m_{t,t+1}^{T} = \zeta \exp(e_t - e_{t+1})$, and we obtain:

$$m_{t,t+1}^{T} = \zeta \exp\left(\varsigma_0 (h_t - h_{t+1}) + \varsigma_1 (z_{gt} - z_{gt+1}) + \varsigma_2 (x_t - x_{t+1}) + \varsigma_3 (\widetilde{x}_{t-1} - \widetilde{x}_t)\right). \tag{K40}$$

We can establish the relation in equation (39) of Proposition 3 by setting $G_9 \equiv -\frac{1}{2}(G_6 + \varsigma_2\gamma_1)^2 \upsilon - \frac{1}{2}(G_7 + \varsigma_0\eta_0)^2$. ■

Table 1
**Sharpness of our entropy bounds on $m_{t,t+1}$ and $m^P_{t,t+1}$ when SDFs correctly price each of the $N+2$ assets**

Reported are the lower entropy bounds with the one-sided $p$-values in $\langle . \rangle$. Our lower entropy bounds on $m_{t,t+1}$ and $m^P_{t,t+1}$ are based on equations (21) of Theorem 3 and (24) of Theorem 4, respectively, and rely on the ability of the SDF to correctly price *each of the* $N+2$ assets (the risk-free bond, the long-term discount bond, and $N$ risky assets). The Backus, Chernov, and Zin (2014, equation (5)) lower bound on the entropy of $m_{t,t+1}$ (denoted by BCZ) is based on the expression: $E[\log(R^m_{t,t+1})]$, while the Alvarez and Jermann (2005, equation (4)) lower bound on the entropy of $m^P_{t,t+1}$ (denoted by AJ) is based on the expression: $E[\log(R^m_{t,t+1})] - E[\log(R_{t,t+1,\infty})]$, where $R^m_{t,t+1}$ is the return on a single risky asset or a benchmark portfolio (i.e., which we proxy, for instance, by the value-weighted equity market return or equally weighted portfolio of 25 Fama-French size and book-to-market portfolios). Moreover, $R_{t,t+1,\infty}$ is the return on an infinite-maturity bond, which we proxy by the return of a 30-year Treasury bond. $R^f_t$ is the gross return of the three-month Treasury bond. We employ different assets and $N$ in the construction of the bounds. For example, in Panel I, the $N$ risky assets are based on two data sets: SET A contains the value-weighted market returns, together with the 25 Fama-French size and book-to-market portfolios, while SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011 (966 observations). To compute these $p$-values, we first use the block bootstrap with a block size of 20 to generate 50,000 samples from the original data. Then we compute the lower bounds in each sample and tabulate the proportion of bootstrap samples for which the lower bound is less than zero.

| | Lower bound on $m_{t,t+1}$ | | Lower bound on $m^P_{t,t+1}$ | |
|---|---|---|---|---|
| | Bound | $p$-value | Bound | $p$-value |
| *Panel I. SDF correctly prices each of the $N+2$ assets, and we set $N=26$* | | | | |
| (a) Market, 25 size & B/M | **0.023** | ⟨0.000⟩ | **0.021** | ⟨0.000⟩ |
| (b) Market, 25 size & momentum | **0.037** | ⟨0.003⟩ | **0.035** | ⟨0.003⟩ |
| *Panel II. SDF correctly prices each of the $N+2$ assets, and we set $N=25$* | | | | |
| (c) 25 size & B/M | **0.022** | ⟨0.000⟩ | **0.020** | ⟨0.000⟩ |
| (d) 25 size & momentum | **0.029** | ⟨0.000⟩ | **0.027** | ⟨0.000⟩ |
| *Panel III. SDF correctly prices each of the $N+2$ assets, and we set $N=11$* | | | | |
| (e) Market, 10 momentum | **0.020** | ⟨0.000⟩ | **0.018** | ⟨0.001⟩ |
| *Panel IV. SDF correctly prices each of the $N+2$ assets, and we set $N=2$* | | | | |
| (f) Market, Low Momentum | **0.010** | ⟨0.000⟩ | **0.008** | ⟨0.000⟩ |
| (g) Market, high Momentum | **0.014** | ⟨0.010⟩ | **0.012** | 0.011 |
| *Panel V. SDF correctly prices each of the $N+2$ assets, and we set $N=1$* | | | | |
| | (BCZ, Eq. 5) | | (AJ, Eq. 4) | |
| (h) Market portfolio only | **0.005** | ⟨0.005⟩ | **0.003** | ⟨0.066⟩ |
| (i) EWI portfolio of 25 size & B/M | **0.007** | ⟨0.001⟩ | **0.005** | ⟨0.018⟩ |
| (j) EWI portfolio of 25 size & momentum | **0.007** | ⟨0.001⟩ | **0.005** | ⟨0.021⟩ |

Table 2
**Relevance of our entropy bounds on $m_{t,t+1}^2$ and $(m_{t,t+1}^P)^2$**
The logic of this test is that when the SDF (its permanent component) is lognormally distributed with no time-variation in the conditional volatility of the SDF (its log permanent component), then $L[m_{t,t+1}^2] = 4L[m_{t,t+1}]$ (or $L[(m_{t,t+1}^P)^2] = 4L[m_{t,t+1}^P]$). Guided by Theorem 2, the ratio of the lower bound on $L[(m_{t,t+1}^2]$ to four times the lower bound on $L[m_{t,t+1}]$ is equal to 1 (similarly for the permanent component of the SDFs). Accordingly, we define

$$\Pi_m \equiv \frac{2(E[\log(\mathbf{a}'\mathbf{R}_{t+1})] - \log\left((E[q_t])^{-1}\right)) + \log(1 + \mathbf{y}'\Sigma^{-1}\mathbf{y}/(E[q_t])^2)}{4(E[\log(\mathbf{a}'\mathbf{R}_{t+1})] - \log((E[q_t])^{-1}))} - 1 \quad \text{and}$$
$$\Pi_{m^P} \equiv \frac{2(E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})]) + \log(1 + \mathbf{y}_P'\Sigma_P\mathbf{y}_P)}{4(E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})])} - 1,$$

where $\mathbf{R}_{t,t+1}$ is a vector of risky asset returns, $R_{t,t+1,\infty}$ is the return on an infinite-maturity discount bond, $\mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}$, $\mathbf{y} \equiv \Sigma^{-1}(\mathbf{1} - E[q_t]E[\mathbf{R}_{t,t+1}])$, and $\mathbf{y}_P \equiv \Sigma_P^{-1}(\mathbf{1} - E[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}])$. In our implementation, we proxy $R_{t,t+1,\infty}$ by the monthly return of a 30-year Treasury bond. $\Sigma_P$ is the variance co-variance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$, whereas $\Sigma$ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$.

The entropy calculations are based on the SDF correctly pricing each of the $N+2$ assets, and our computation of $\Pi_m$ and $\Pi_{m^P}$ relies on three data sets for $\mathbf{R}_{t,t+1}$: SET A contains the value-weighted market returns together with the 25 Fama-French size and book-to-market portfolios (i.e, $N = 26$), SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios (i.e, $N = 26$), while SET C contains only the value-weighted equity market returns (i.e, $N = 1$). The sample period is from July 1931 to December 2011 (966 observations). To compute the $p$-values reported in parentheses, we employ a block bootstrap with a block size of 20 to generate $\widehat{b}$=50,000 samples from the original data. We then compute $\Pi_b = \Pi$ for $b = 1, \ldots, \widehat{b}$, the cross-sectional average $\overline{\Pi}$, and the standard error $\text{se}(\Pi) = \text{std}(\Pi)/\sqrt{\widehat{b}}$ of $\Pi$. Accordingly, we compute the $t$ statistic as $\left(\overline{\Pi} - 0\right)/se(\Pi)$. The absolute value of the $t$-statistic is then used to compute the two-sided $p$-value.

| | Testing $H_0$: $\Pi_m = 0$ versus $H_a$: $\Pi_m \neq 0$ | | |
|---|---|---|---|
| | Market, 25 size & B/M (SET A, $N = 26$) | Market, 25 size & momentum (SET B, $N = 26$) | Market only (SET C, $N = 1$) |
| $\Pi_m$ | 56.17% | 33.26% | 21.89% |
| | (0.000) | (0.000) | (0.000) |
| $\Pi_{m^P}$ | 51.85% | 33.05% | 13.79% |
| | (0.000) | (0.000) | (0.021) |

Table 3

**Model comparisons based on the lower entropy bounds**

Reported are the entropies of $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$ for the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The one-sided $p$-values shown in square brackets represent the proportion of replications for which the model-based entropy exceeds, in 50,000 replications, the lower bound on the entropy computed from observed asset prices. Our lower entropy bounds on $m^P_{t,t+1}$ and $m_{t,t+1}$ are based on equations (20) and (19) of Theorem 2 and rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and $N$ risky assets). The $N$ risky assets are based on SET B, which contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011. The lower entropy bounds on $(m_{t,t+1})^2$ and $m_{t,t+1}$ are analogously obtained based on Theorem 2. We focus on SET B, as it corresponds to the maximum lower bound on entropy measures (as in our Table 1). Panels C and D present the variance, skewness, and kurtosis of $m^P_{t,t+1}$ and $m_{t,t+1}$ that are consistent with model parameterizations in Table Appendix-I. The one-sided $p$-values $\langle . \rangle$ reported below the lower entropy bounds, represent the proportion of bootstrap samples for which the lower bound is less than zero.

| | Habit model | Recursive utility models | | |
|---|---|---|---|---|
| | DH | RU-SV | RU-CJI | Lower entropy bound (Set B) |
| *Panel A: Entropies of $m^2_{t,t+1}$ and $m_{t,t+1}$* | | | | |
| $L[m^2_{t,t+1}]$ | 0.0785 | 0.0869 | 1.4331 | 0.1956 |
| | [0.000] | [0.000] | [1.000] | $\langle 0.003 \rangle$ |
| $L[m_{t,t+1}]$ | 0.0196 | 0.0217 | 0.0190 | 0.0367 |
| | [0.000] | [0.000] | [0.000] | $\langle 0.003 \rangle$ |
| *Panel B: Entropies of $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$* | | | | |
| $L[(m^P_{t,t+1})^2]$ | 0.0811 | 0.095 | 1.4858 | 0.1851 |
| | [0.000] | [0.000] | [1.000] | $\langle 0.003 \rangle$ |
| $L[m^P_{t,t+1}]$ | 0.0203 | 0.0237 | 0.0197 | 0.0348 |
| | [0.000] | [0.000] | [0.000] | $\langle 0.003 \rangle$ |
| *Panel C: Moments of the $m_{t,t+1}$ distribution* | | | | |
| Variance | 0.0403 | 0.0444 | 3.3438 | |
| Skewness | 0.6041 | 0.6476 | $+\infty$ | |
| Kurtosis | 3.6447 | 3.8061 | $+\infty$ | |
| *Panel D: Moments of the $m^P_{t,t+1}$ distribution* | | | | |
| Variance | 0.0415 | 0.0487 | 3.2480 | |
| Skewness | 0.6142 | 0.6778 | $+\infty$ | |
| Kurtosis | 3.6654 | 3.8786 | $+\infty$ | |

Table 4

**Entropy-based measures of the transitory component of the SDF**

Reported are the entropies of $(m^T_{t,t+1})^2$ and $m^P_{t,t+1}$ for three asset pricing models: the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The data-based $L[(m^T_{t,t+1})^2]$ and $L[m^T_{t,t+1}]$ rely on the expressions in equations (25) and (26), whereby we proxy $R_{t,t+1,\infty}$ by the return of a 30-year Treasury bond. The two-sided bootstrap $p$-values, shown in curly brackets, allow to test whether the average value of the model-implied entropy across the 50,000 replications is equal to the entropy-based measures computed from bond returns. Panel B presents the mean and standard deviation of the returns of the risk-free bond and the long-term implied by each model. Our replications are consistent with model parameterizations in Table Appendix-I.

| | *Model-implied entropies* | | | |
|---|---|---|---|---|
| | DH | RU-SV | RU-CJI | Data implied |
| *Panel A: Transitory component of the SDF* | | | | |
| $L[(m^T_{t,t+1})^2]$ | $2.8\times10^{-3}$ | $0.2\times10^{-3}$ | $0.046\times10^{-3}$ | $4.8\times10^{-3}$ |
| | {0.000} | {0.000} | {0.000} | ⟨0.000⟩ |
| $L[m^T_{t,t+1}]$ | $0.7\times10^{-3}$ | $0.1\times10^{-3}$ | $0.012\times10^{-3}$ | $0.4\times10^{-3}$ |
| | {0.000} | {0.000} | {0.000} | ⟨0.000⟩ |
| *Panel B: Returns of the risk-free and the long-term discount bonds* | | | | |
| Mean of risk-free return | -0.0304 | 0.0112 | -0.0160 | 0.0355 |
| Std. Dev. of risk-free return | 0.0342 | 0.0030 | 0.0006 | 0.0311 |
| Mean of long-term bond return | -0.0225 | -0.0124 | -0.0153 | 0.0584 |
| Std. Dev. of long-term bond return | 0.4446 | 0.1323 | 0.0006 | 0.0355 |

Table 5

**Entropy-based measures of codependence**

Reported are the entropy-based codependence measures for three asset pricing models: the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The data-based $L[m^P_{t,t+1} m^T_{t,t+1}] - L[m^P_{t,t+1}] - L[m^T_{t,t+1}]$ is inferred from the Treasury yield curve, as described in equation (27). The $p$-values shown in curly brackets allow to test whether the average entropy-based codependence across the 50,000 values is equal to its data counterparts. The data-based $L[(m^P_{t,t+1} m^T_{t,t+1})^2] - L[(m^P_{t,t+1})^2] - L[(m^T_{t,t+1})^2]$ employs the expression on the right-hand side of equation (28) of Theorem 5. The construction of the upper bound relies on the risk-free bond, the long-term discount bond, and Set A. Specifically, Set A contains the value-weighted market returns together with the 25 Fama-French size and book-to-market portfolios. The reported one-sided $p$-values, shown as $\lfloor . \rfloor$, represent the proportion of replications for which the model entropy-based codependence do not exceed, in 50,000 replications, the upper bound on the codependence computed from asset returns. Our replications are consistent with model parameterizations in Table Appendix-I.

| | *Model-implied entropy codependence* | | | | |
|---|---|---|---|---|---|
| | DH | RU-SV | RU-CJI | Data implied | Upper bound (Set A) |
| $L[m^P_{t,t+1} m^T_{t,t+1}] - L[m^P_{t,t+1}] - L[m^T_{t,t+1}]$ | -0.0014 | -0.0021 | -0.0007 | 0.0015 | |
| | $\{0.000\}$ | $\{0.000\}$ | $\{0.000\}$ | $\langle 0.031 \rangle$ | |
| $L[(m^P_{t,t+1} m^T_{t,t+1})^2] - L[(m^P_{t,t+1})^2] - L[(m^T_{t,t+1})^2]$ | -0.0054 | -0.0083 | -0.0028 | | 0.1222 |
| | $\lfloor 0.002 \rfloor$ | $\lfloor 0.000 \rfloor$ | $\lfloor 0.000 \rfloor$ | | $\langle 0.000 \rangle$ |

Table Appendix-I

**Parameters employed in model implementation**

Displayed in this table are the parameters that govern preferences and the dynamics of consumption growth. These parameters are adopted from Tables 2, 3, and 4 of Backus, Chernov, and Zin (2014), and likewise $\log(g)$ and $\eta_0$ are taken from their page 16. Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). We use US annual real personal consumption expenditures as a proxy for aggregate consumption over the sample period of 1931:07 to 2011:12 (966 observations). To compare model implications with the data, we simulate a finite sample of consumption growth, $c_{t+1}/c_t$, over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12}\log(c_{t+j}/c_{t+j-1}))$. The reported model mean, standard deviation, and autocorrelation are based on the annualized consumption growth.

| Parameter | DH | RU-SV | RU-CJI | Data implied 1931:07 to 2011:12 |
|---|---|---|---|---|
| *Panel A: Preferences* | | | | |
| $\rho$ | -9.0000 | 0.3333 | 0.3333 | |
| $\alpha$ | | -9.0000 | -9.0000 | |
| $\beta$ | 0.9980 | 0.9980 | 0.9980 | |
| $\varphi_h$ | 0.9000 | | | |
| $s$ | 0.5000 | | | |
| *Panel B: Consumption growth dynamics* | | | | |
| $\gamma_0$ | 1.0000 | 1.0000 | 1.0000 | |
| $\log(g)$ | 0.0015 | 0.0015 | 0.0015 | |
| $\eta_0$ | 0.1000 | | | |
| $\gamma_1$ | 0.0271 | 0.0271 | 0.0281 | |
| $\varphi_g$ | 0.9790 | 0.9790 | 0.9690 | |
| $\upsilon^{1/2}$ | 0.0099 | 0.0099 | 0.0079 | |
| $\nu_0$ | | $0.23\times10^{-5}$ | | |
| $\varphi_\upsilon$ | | 0.9870 | | |
| $h$ | | | 0.0008 | |
| $\theta$ | | | -0.1500 | |
| $\delta$ | | | 0.1500 | |
| $\psi_0$ | | | 1.0000 | |
| $b_1$ | | 0.9977 | 0.9979 | |
| *Panel C: Consumption growth* | | | | |
| Mean (annualized) | 1.0192 | 1.0190 | 1.0189 | 1.0339 |
| Std. Dev. (annualized) | 0.0416 | 0.0415 | 0.0369 | 0.0287 |
| Autocorrelation | 0.2424 | 0.2433 | 0.1771 | 0.2386 |

Table Appendix-II

**Impact of alternative jump parameterizations in the RU-CJI model**

Here we vary $\theta$, $\delta$, and $h$ that govern the distribution of jumps (see equation (36)) in the consumption growth dynamics for the RU-CJI model. We keep other parameters of the RU-CJI model to those specified in Table Appendix-I. For each set of parameters, the reported values are averages across 50,000 replications. For each replication, we simulate the path of consumption growth $c_{t+1}/c_t$ over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12} \log(c_{t+j}/c_{t+j-1}))$. The reported model mean and standard deviation are based on the annualized consumption growth. For each parameter set, we also report the average values of entropy $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$, as well as the central moments of the permanent component of the SDF. The bolded parameter set corresponds to Backus, Chernov, and Zin (2014, Model (4), Table 4).

| $\theta$ | $\delta$ | $h$ | Entropies | | Moments of $m^P_{t,t+1}$ | | | $\frac{c_{t+1}}{c_t}$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | | $L[m^P]$ | $L[(m^P)^2]$ | Variance | Skewness | Kurtosis | Mean | Std. Dev. |
| -0.15 | 0.02 | 0.0002 | 0.011 | 0.046 | 0.024 | 3.62E+00 | 1.10E+02 | 1.0190 | 0.033 |
| -0.15 | 0.02 | 0.0004 | 0.011 | 0.050 | 0.027 | 5.92E+00 | 1.93E+02 | 1.0187 | 0.034 |
| -0.15 | 0.02 | 0.0008 | 0.012 | 0.057 | 0.033 | 8.95E+00 | 3.20E+02 | 1.0188 | 0.035 |
| -0.15 | 0.07 | 0.0002 | 0.011 | 0.053 | 0.030 | 3.03E+01 | 6.96E+04 | 1.0187 | 0.033 |
| -0.15 | 0.07 | 0.0004 | 0.012 | 0.062 | 0.039 | 4.51E+01 | 2.49E+06 | 1.0187 | 0.034 |
| -0.15 | 0.07 | 0.0008 | 0.013 | 0.082 | 0.057 | 6.20E+01 | 3.95E+09 | 1.0188 | 0.036 |
| -0.15 | 0.15 | 0.0002 | 0.013 | 0.403 | 0.459 | 4.82E+195 | +∞ | 1.0187 | 0.034 |
| -0.15 | 0.15 | 0.0004 | 0.015 | 0.764 | 1.083 | +∞ | +∞ | 1.0187 | 0.035 |
| **-0.15** | **0.15** | **0.0008** | **0.020** | **1.486** | **3.248** | **+∞** | **+∞** | **1.0189** | **0.037** |
| -0.07 | 0.02 | 0.0002 | 0.011 | 0.043 | 0.022 | 5.63E-01 | 4.86E+00 | 1.0187 | 0.033 |
| -0.07 | 0.02 | 0.0004 | 0.011 | 0.043 | 0.022 | 6.78E-01 | 6.30E+00 | 1.0187 | 0.033 |
| -0.07 | 0.02 | 0.0008 | 0.011 | 0.044 | 0.023 | 8.98E-01 | 9.05E+00 | 1.0187 | 0.033 |
| -0.07 | 0.07 | 0.0002 | 0.011 | 0.044 | 0.023 | 3.46E+00 | 2.94E+02 | 1.0186 | 0.033 |
| -0.07 | 0.07 | 0.0004 | 0.011 | 0.046 | 0.024 | 6.03E+00 | 5.81E+02 | 1.0187 | 0.033 |
| -0.07 | 0.07 | 0.0008 | 0.011 | 0.049 | 0.027 | 1.01E+01 | 1.17E+03 | 1.0187 | 0.034 |
| -0.07 | 0.15 | 0.0002 | 0.012 | 0.115 | 0.096 | 1.79E+19 | +∞ | 1.0187 | 0.033 |
| -0.07 | 0.15 | 0.0004 | 0.012 | 0.188 | 0.177 | 3.64E+36 | +∞ | 1.0187 | 0.034 |
| -0.07 | 0.15 | 0.0008 | 0.014 | 0.333 | 0.356 | 3.24E+71 | +∞ | 1.0188 | 0.036 |
| -0.02 | 0.02 | 0.0002 | 0.011 | 0.043 | 0.022 | 4.48E-01 | 3.39E+00 | 1.0187 | 0.033 |
| -0.02 | 0.02 | 0.0004 | 0.011 | 0.043 | 0.022 | 4.54E-01 | 3.42E+00 | 1.0187 | 0.033 |
| -0.02 | 0.02 | 0.0008 | 0.011 | 0.043 | 0.022 | 4.64E-01 | 3.49E+00 | 1.0187 | 0.033 |
| -0.02 | 0.07 | 0.0002 | 0.011 | 0.043 | 0.022 | 9.93E-01 | 3.55E+01 | 1.0187 | 0.033 |
| -0.02 | 0.07 | 0.0004 | 0.011 | 0.044 | 0.022 | 1.52E+00 | 6.65E+01 | 1.0187 | 0.033 |
| -0.02 | 0.07 | 0.0008 | 0.011 | 0.045 | 0.023 | 2.49E+00 | 1.25E+02 | 1.0187 | 0.033 |
| -0.02 | 0.15 | 0.0002 | 0.011 | 0.069 | 0.048 | 9.03E+05 | +∞ | 1.0187 | 0.033 |
| -0.02 | 0.15 | 0.0004 | 0.012 | 0.096 | 0.075 | 4.10E+09 | +∞ | 1.0187 | 0.034 |
| -0.02 | 0.15 | 0.0008 | 0.013 | 0.149 | 0.131 | 1.40E+17 | +∞ | 1.0188 | 0.036 |