Premarital Investments and Multidimensional Matching

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Abstract

I expand Gary Becker’s market analysis of marriage behavior to derive a set of new implications about marriage and education. In my model, people can make college and career investments to improve their income and marriage prospects before they enter the marriage market. In the overlapping-generations marriage market, men are characterized by income, and women are characterized by income and reproductive fitness. Women’s reproductive fitness declines as they age, tampering with their career investment incentives. What investments people make, when they enter the marriage market, whom they marry, and how much they get in the marriage market are all endogenously determined. First, the model explains the observed relationships between marriage age and personal income. Second, the model helps explain the college gender gap puzzle that more women than men go to college. Third, the model explains the evolving relationship between wife’s marriage age and husband’s income. Finally, the model shows how different factors, especially uncertainty about future income, delay marriage.

Keywords: premarital investments, multidimensional matching, reproductive fitness, marriage age, the college gender gap

JEL: C7, D1, J1

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1 Introduction

Gary Becker’s idea to analyze marriage behavior in a market framework is revolutionary, but the scope of the analysis is limited. In the static unidimensional matching model of Becker (1973), everyone marries at the same time, and is distinguished only by a fixed, one-dimensional marriage characteristic. The model cannot be used to study marriage age variations, premarital investments that change marriage characteristics (e.g. education), or the relative importance of different characteristics in the marriage market. In this paper, I allow dynamics, premarital investments, and multidimensional matching characteristics to derive a set of new implications about marriage and education.

Let’s start with the relationships between marriage age and personal income. When a person marries and how much he or she earns later in life are systematically related. Figures 1 and 2 illustrate the average personal income of 40-44 year-old American men and women in 1980 and in 2012 by their marriage age. The relationship for men has been persistently inverse-U shaped: those who married between ages 25 and 30 earned significantly more on average than those who married earlier and later, and those who have not married by 40 earned the least on average. The relationship for women was positive in 1980: those who married later earned more on average, and those who have not married by 35 earned the most on average. However, the relationship in 2012 became inverse-U shaped: the women who have not married by 35 earned less on average than those who married earlier.

The two leading theories only partially accounted for the patterns, and are based on the outdated premise of Becker (1973) that marriage allows men to specialize in labor market activities and women in household chores. Keeley (1977, 1979) predicts a negative relationship for men and a positive relationship for women, because higher-income men and lower-income women benefit more from marriage specialization, find partners more easily, and marry earlier. Bergstrom and Bagnoli (1993) predict a positive relationship for men and no relationship for women. They argue that it takes time for men to reveal their wage-earning abilities, so men with higher wage-earning potential delay marriage. All women have their marriage characteristics like childbearing abilities revealed early, and marry at the same time.

I propose the following theory. Consider a dynamic economy in which each agent participates for three periods (think ages 17-24, 25-32, and 33-40). Men and women of heterogeneous abilities are born each period, and can go to college at age 1 to improve their income and marriage prospects. Higher-ability agents improve with higher probabilities. Those who do not go to college enter the marriage market at age 1 and earn a
low income on average. Those who go to college delay entering the marriage market. At age 2, the college graduates learn whether or not their income and marriage prospects improve. Those who learn a good outcome enter the marriage market. Those who do not can make a career investment (e.g. switch occupation, search for a better job opportunity, receive more training, or work harder) and delay entering the marriage market with the hope to improve at age 3. At age 3, they enter the marriage market regardless of the outcome of the career investment. All the investment opportunities are the same for men and women, but making the career investment is more costly for women because their reproductive fitness declines when they reach age 3. In the model’s overlapping-generations marriage market, men and women match and divide their marriage surplus so that no agent wants to leave his or her partner. A couple’s marriage surplus depends

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1Fitness in biology describes the relative reproductive success of a genotype. In this paper, fitness refers to women’s ability to have desired quantity and quality of children. Older women, especially past 30, tend to have lower fertility and more health complications associated with their newborns.
on the husband’s income, the wife’s income, and her fitness level.

The investment decisions, the distributions of marriage characteristics, the matching between men and women of different ages, and the division of marriage surplus between each couple are all endogenously determined. I use Glicksberg’s fixed-point theorem to establish equilibrium existence. Equilibrium strategies are characterized by cutoffs. Men above a certain ability choose to go to college, and they will also make the career investment in case the college investment fails. Women above a certain ability choose to go to college, but only the higher-ability ones among them will make the career investment in case the college investment fails.

The marriage-age/personal-income relationship for men is inverse-U shaped in equilibrium (Section 4.1). Those who do not go to college enter the marriage market at age 1 and earn a low income on average. Those who improve right after college enter the marriage market at age 2 and earn a high income on average. Those who do not improve after college make the career investment and enter the marriage market at age 3; some of them improve, and some of them strike out.

The marriage-age/personal-income relationship for women is positive or inverse-U shaped in equilibrium (Section 4.2). Those who enter the marriage market at age 1 do not go to college, and earn a low income. Those who improve right after college enter the marriage market at age 2. Because it is reproductively costly to make the career investment, some college graduates who do not improve right after college still choose to enter the marriage market at age 2. Those who do choose to make the career investment and enter the marriage market at age 3 are the extremely high-ability women who expect to earn high enough income to compensate for their fitness loss. If the fitness loss at age 3 is significant, more lower-income women will choose to enter the marriage market at age 2, and the relationship between marriage age and personal income is positive. If the fitness loss at age 3 is not significant, women’s strategies are similar to men’s, so the equilibrium relationship is inverse-U shaped.

The model also helps explain the college gender gap puzzle, one of the central puzzles in labor economics (Section 5). A country with more college-educated women than college-educated men was an exception in 1970, but has become the norm among high-income countries (Figure 3). In the model, when women face the same education and labor market opportunities as men, but still face the reproductive constraints, actually more women than men go to college. This result may be surprising at first sight, because women should have less incentives to invest given their reproductive cost. Many female college graduates marry early and earn a low income without making the career investment as a result of the reproductive cost. But remember the division of marriage
surplus is endogenously and competitively determined: as a result of the reproductive cost, high-income reproductively fit women are more scarce and more valuable in the marriage market than high-income men. Women receive endogenously higher marriage gains from going to college, so more of them go to college with the hope to capture the higher marriage gains.

Women’s higher college marriage premium predicted by the model is in line with the theoretical arguments of Chiappori et al. (2009) and the empirical findings of Chiappori et al. (2012a). College-educated women have married at higher rates and have married higher-income husbands, for example. Of course, other factors also contribute to the college gender gap. Goldin et al. (2006) and Becker et al. (2010) attribute the gap to women’s higher average non-cognitive abilities. Gender differences in occupational choices have also received increasing attention (Bronson, 2013; Reijnders, 2014).

The changing relationship between a woman’s marriage age and her husband’s income offers additional support for the improving marriage prospects of college-educated women (Section 6). The women who marry around the median age on average married husbands with the highest income. In 1980, the 40-44 year-old American women who married before 20 had higher-income husbands than those who married after 30. However, the relationship was reversed in 2012: those who married after 30 had higher-income husbands than those who married before 20 (Figure 4). Because of the extra dimension of reproductive fitness, the equilibrium matching is not positive assortative in incomes as in the unidimensional matching model. In the non-assortative equilibrium matching in the current model, when reproductive fitness is less important, men are more inclined to marry high-income, low-fitness women than low-income, high-fitness women.

Finally, I extend the basic model and show that uncertainty about future income deters
Figure 4: Average husband’s income by wife’s marriage age, American women in 1980 and 2012.

One from marrying early (Section 7). People are allowed to marry without realizing their future income in the extended model. However, people do not want to do so. Intuitively, marrying early locks a person to a partner, and because of the competitive nature of the marriage market, the partner can turn out to be a sub-optimal match when income is realized.

None of the previous models can derive a rich set of implications about marriage age, premarital investments, and gender differences in a unified manner. Iyigun and Walsh (2007) and Chiappori et al. (2009) focus on the joint determination of college investments and marriage behavior in a two-period model. Low (2014) focuses on the tradeoff between reproductive capital loss and human capital gain when women take advanced education (e.g. PhD, JD). Siow (1998) and Díaz-Giménez and Giolito (2013) focus on how gender difference in reproductive fitness can generate gender differences in social and economic roles. The paper also extends the result that the socially efficient investments can be supported in equilibrium to the setting with stochastic premarital investments from the deterministic settings (Cole et al., 2001a,b; Peters and Siow, 2002; Iyigun and Walsh, 2007; Dizdar, 2013; Hatfield et al., 2014; Nöldeke and Samuelson, 2014).

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes the equilibrium. Section 4 discusses the relationships between marriage age and personal income. Section 5 discusses the college gender gap. Section 6 discusses the relationship between a wife’s marriage age and the husband’s income. Section 7 extends the model, and shows how income uncertainty delays marriage. Section 8 concludes. Appendices A, B, and C contain proofs, data description, and numerical illustrations.
2 Model

There are an infinite number of discrete periods. At the beginning of each period, unit masses of males and females are born. They are born with heterogeneous abilities $\theta_m \in \Theta_m, \theta_f \in \Theta_f$ distributed according to continuous and strictly increasing distributions $F_m, F_f$ on $[0, 1]$.

2.1 Timing

Each agent learns his or her own ability $\theta$, and participates in the economy for three periods. Each age 1 agent chooses whether to go to college ($I_1$) or not ($N_1$). An agent who does not go to college enters the marriage market, and realizes an income drawn from distribution $P_{mL}, P_{fL}$. An agent who goes to college pays a cost $c_m, c_f$, and delays entering the marriage market.\(^2\)

At the beginning of age 2, each college graduate receives a high signal $H$ with probability $\theta$ or a low signal $L$ with probability $1 - \theta$. A college graduate who receives a high signal enters the marriage market, and realizes an income from distribution $P_{mH}, P_{fH}$ that strictly first order stochastically dominates $P_{mL}, P_{fL}$. A college graduate who receives a low signal can make a career investment.\(^3\) Without making the career investment ($N_2$), an agent would enter the marriage market and realize an income from distribution $P_{mL}, P_{fL}$. An agent making the career investment ($I_2$) would pay cost $c_m, c_f$ and delay entering the marriage market.

At age 3, an ability $\theta$ agent who makes the career investment receives a high signal with probability $\theta$ or a low signal with probability $1 - \theta$. An agent at this point has no choice but to enter the marriage market. An agent with a high signal realizes an income from distribution $P_{mH}, P_{fH}$, and an agent with a low signal realizes an income from distribution $P_{mL}, P_{fL}$.

Figure 5 summarizes the timing of the game. Let $\sigma_{ma}(\theta), \sigma_{fa}(\theta)$ denote an ability $\theta$ agent’s probability of choosing action $I_a$ at age $a$ for $a = 1, 2$. Let $\sigma_m(\cdot) = (\sigma_{m1}(\cdot), \sigma_{m2}(\cdot))$ and $\sigma_f(\cdot) = (\sigma_{f1}(\cdot), \sigma_{f2}(\cdot))$ denote the strategies of males and females, and $\sigma$ the population strategies. Assume that it is strictly dominant for ability $\theta = 1$ males and females to choose $I_a$ at age $a$.\(^4\)

\(^2\)Assume for now that to invest and to marry in the same period is infeasible. Proposition 5 shows that the strategy is strictly dominated.

\(^3\)Making a career investment corresponds to switching a career, receiving additional training, and/or waiting for a better opportunity.

\(^4\)It is sufficient to assume $E[w_{mH} - w_{mL} + S(w_{mH}, w_{fH}, r_1) - S(w_{mH}, w_{fL}, r_1)] > c_m$ and $E[w_{mH} - w_{mL} + \phi S(w_m, w_{fH}, r_+) - \phi S(w_m, w_{fL}, r_+)] > E[S(\bar{w}_m, w_{fH}, r_1) - S(\bar{w}_m, w_{fL}, r_1)] + c_f$. 
age 1
learn ability $\theta$

decides college investment or not
if no college investment, enters MM
if college investment, skips MM

age 2
observes signal H or L
if H, enters MM
if L, decides career investment or not
if no career investment, enters MM
if career investment, skips MM

age 3
observes signal H or L
(observ fitness $r_+$ or $r_-$)
enters MM

Figure 5: The timing of the game. MM stands for the marriage market. The only gender difference is in reproductive fitness.

2.2 The Marriage Market

A couple’s marriage surplus $S(w_m, w_f, r)$ depends on the husband’s income $w_m$, the wife’s income $w_f$, and her reproductive fitness $r$. A female is reproductively fit ($r_+$) if she enters the marriage market at age 1 or 2, and is fit with probability $\phi$ and unfit ($r_-$) with probability $1 - \phi$ if she enters the marriage market at age 3. Assume that $S$ is non-negative, continuously differentiable, strictly increasing, and strictly supermodular in $w_m$ and $w_f$, and strictly supermodular in $w_m$ and $r$. An unmarried agent gets zero surplus.\(^5\)

In the marriage market, men are characterized by income, and women are characterized by income and reproductive fitness. Let $\mathcal{X}_m = [w_m, \bar{w}_m]$ and $\mathcal{X}_f = [w_f, \bar{w}_f] \times \{r_-, r_+\}$ denote the sets of marriage characteristics. The population of the marriage market is described by measures $\mu_m$ on $\mathcal{X}_m$ and $\mu_f$ on $\mathcal{X}_f$. Because action $I_a$ at age $a$ is assumed to be strictly dominant for ability $\theta = 1$ males and females, the marriage market measures in equilibrium have full supports. $\mu_m$ and $\mu_f$ are equivalently described by the income mass distributions. When the population strategies are $\sigma$, the income mass distributions

\(^5\)The marriage surplus can be thought to result from a household optimization problem. A $w_m$ man and a $(w_f, r)$ woman choose make labor supply and fertility decisions to maximize their total utilities in the form of wage earnings, leisure, and children.
of males, fit females, and unfit females in a period’s marriage market are

\[
G_m(w) = P_{mH}(w) \int \sigma_{m1}(\theta_m)[\theta_m + (1 - \theta_m)\sigma_{m2}(\theta_m)\theta_m]dF_m(\theta_m)
+ P_{mL}(w) \int [1 - \sigma_{m1}(\theta_m) + \sigma_{m1}(\theta_m)(1 - \theta_m)\sigma_{m2}(\theta_m)(1 - \theta_m)]dF_m(\theta_m).
\]

\[
G_{f+}(w) = P_{fH}(w) \int \sigma_{f1}(\theta_f)[\theta_f + (1 - \theta_f)\sigma_{f2}(\theta_f)\theta_f\phi]dF_f(\theta_f)
+ P_{fL}(w) \int [1 - \sigma_{f1}(\theta_f) + \sigma_{f1}(\theta_f)(1 - \theta)\sigma_{f2}(\theta_f)(1 - \theta_f)\phi]dF_f(\theta_f).
\]

\[
G_{f-}(w) = P_{fH}(w) \int \sigma_{f1}(\theta_f)(1 - \theta_f)\sigma_{f2}(\theta_f)(1 - \theta)\phi dF_f(\theta_f)
+ P_{fL}(w) \int \sigma_{f1}(\theta_f)(1 - \theta)\sigma_{f2}(\theta_f)(1 - \theta_f)(1 - \phi)dF_f(\theta_f).
\]

Males and females match and divide their marriage surplus in the marriage market. A (feasible) matching is described by a measure \(\mu\) on \(X_m \times X_f\) such that \(\mu\) has marginals \(\mu_m\) and \(\mu_f\). A marriage payoff function is \(V : X_m \cup X_f \rightarrow \mathbb{R}\). \(V(x)\) denotes the marriage payoff of an agent with marriage characteristic \(x\). Let \(V_m, V_f, V_{f+},\ \text{and} \ V_{f-}\) be \(V\) restricted to \(X_m, X_f, [w_f, \overline{w}_f] \times r_+\), and \([w_f, \overline{w}_f] \times r_-\), respectively. \((\mu, V)\) is a stable outcome of \((\mu_m, \mu_f)\) if \(V(x) \geq 0\) for all \(x\), and for any \(x_m \in \text{supp}(\mu_m)\) and \(x_f \in \text{supp}(\mu_f)\),

\[
V(x_m) + V(x_f) \geq S(x_m, x_f).
\]

That is, no pair of agents can strictly gain by leaving their partners and marrying each other. By Gretsky et al. (1992), a stable outcome always exists, and the total marriage surplus is maximized under stable matching.

Each agent’s lifetime utility is income \(w\) plus marriage payoff \(V\) net any costs paid for investments. All the agents are risk-neutral and do not discount. An income \(w_m\) male who does not go to college gets utility \(w_m + V(w_m)\). An income \(w_m\) male who goes to college but does not make the career investment gets utility \(w_m + V(w_m) - c_m\). An income \(w_m\) male who makes both investments gets utility \(w_m + V(w_m) - 2c_m\). An income \(w_f\) female who does not go to college is fit and has lifetime utility \(w_f + V(w_f, r_+)\). An income \(w_f\) female who goes to college and does not make the career investment has lifetime utility \(w_f + V(w_f, r_+) - c_f\). An income \(w_f\) female who makes both investments gets utility \(w_f + V(w_f, r_-) - 2c_f\) if she is fit, and \(w_f + V(w_f, r_-) - 2c_f\) if she is unfit.
3 Equilibrium

From now on, we focus on the stationary equilibrium of the economy. An equilibrium consists of population strategies $\sigma^*$ and a marriage market outcome $(\mu^*, V^*)$. In equilibrium, agents choose the strategies that maximize their expected utilities given marriage payoffs, and the matching and the marriage payoffs are stable with respect to the distributions of marriage characteristics induced by the optimal strategies. We say the population strategy $\sigma$ is optimal given $V$ if $\sigma_m(\theta) = (\sigma_{m1}(\theta), \sigma_{m2}(\theta))$ and $\sigma_f(\theta) = (\sigma_{f1}(\theta), \sigma_{f2}(\theta))$ maximize the expected utilities of ability $\theta$ males and females when the marriage payoff function is $V$.

**Definition 1.** $(\sigma^*, \mu^*, V^*)$ is an equilibrium if $\sigma^*$ is optimal given $V^*$, and marginals $\mu^*_m$ and $\mu^*_f$ are induced by $\sigma^*$, and $(\mu^*, V^*)$ is a stable outcome of $(\mu^*_m, \mu^*_f)$.

Subsequently in this section, I characterize each component of the equilibrium, and show that a stationary equilibrium always exists. In addition, equilibrium satisfies some efficiency properties.

3.1 Optimal Strategies

The following lemma guarantees higher-ability agents receive higher expected benefits from choosing $I_a$.

**Lemma 1.** Stable marriage payoff functions $V_m$ and $V_f$ are continuous and strictly increasing.

3.1.1 Males’ Optimal Strategies

By the Principle of Optimality, optimal strategies can be solved backwards. An ability $\theta_m$ male who receives a low signal pays cost $c_m$ to make the career investment, and generates expected wage gain $\theta_m(\mathbb{E}w_{mH} - \mathbb{E}w_{mL})$ and expected marriage gain $\theta_m[\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{wL})]$. Hence ability $\theta_m \geq \theta^*_m \equiv [\mathbb{E}w_{mH} - \mathbb{E}w_{mL} + \mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{wL})]/c_m$ men make the career investment after receiving a low signal.

An ability $\theta_m$ male pays cost $c_m$ to go to college, and generates an expected income gain $\theta_m(\mathbb{E}w_{mH} - \mathbb{E}w_{mL})$, an expected marriage gain $\theta_m[\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{wL})]$, and an expected gain from making the career investment. Each male’s net benefit from going to college without the benefit from the career investment is the same as his net benefit of making the career investment. Therefore, every male who goes to college will make the career investment if needed.

Therefore, males’ optimal strategies can be characterized by one cutoff.
Lemma 2 (Males’ Optimal Strategies). Suppose the stable marriage payoff function is $V$. Let
\[
\theta_m^* = \frac{c_m}{\mathbb{E}w_{mH} - \mathbb{E}w_{mL} + \mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{mL})}. \tag{1}
\]
Ability $\theta_m \geq \theta_m^*$ males choose to go to college and would make the career investment. That is,
\[
\sigma_m^*(\theta) = (\sigma_m^1(\theta), \sigma_m^2(\theta)) = \begin{cases} 
(1, 1) & \theta \geq \theta_m^*, \\
(0, 0) & \theta < \theta_m^*.
\end{cases} \tag{2}
\]

3.1.2 Females’ Optimal Strategies

In contrast to males, the tradeoffs females face at age 1 and at age 2 are different, so the sets of females making the college and the career investments will be different. The optimal strategies can still be solved backwards. An ability $\theta_f$ female who does not improve from the college investment and makes the career investment gets an expected wage gain of $\theta_f(\mathbb{E}w_{fH} - \mathbb{E}w_{fL})$. The expected marriage payoff from making the career investment is
\[
\theta_f[\phi \mathbb{E}V_{f+}(w_{fH}) + (1 - \phi)\mathbb{E}V_{f-}(w_{fH})] + (1 - \theta_f)[\phi \mathbb{E}V_{f+}(w_{fL}) + (1 - \phi)\mathbb{E}V_{f-}(w_{fL})].
\]
A female who makes the career investment incurs not only an investment cost but also an opportunity cost from expected fitness loss. Therefore, a female makes the career investment if and only if
\[
\theta_f\mathbb{E}[w_{fH} - w_{fL} + \phi(V_{f+}(w_{fH}) - V_{f+}(w_{fL})) + (1 - \phi)(V_{f-}(w_{fH}) - V_{f-}(w_{fL}))] \\
\geq c_f + (1 - \phi)[\mathbb{E}V_{f+}(w_{fL}) - \mathbb{E}V_{f-}(w_{fL})].
\]
A female’s tradeoff at age 1 is similar to a male’s. An ability $\theta_f$ female who goes to college pays cost $c_f$ and accrues an expected income gain of $\theta_f(\mathbb{E}w_{fH} - \mathbb{E}w_{fL})$, an expected marriage gain of $\theta_f[\mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})]$, and an expected net benefit from making the career investment if needed. The marginal female college investor does not get any net benefit from making the career investment. An ability $\theta_{f1}$ female is indifferent between investing in college and not investing if and only if
\[
\theta_{f1}^{*}[\mathbb{E}w_{fH} - \mathbb{E}w_{fL} + \mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})] = c_f.
\]
Females’ optimal strategies are characterized by two cutoffs.
Lemma 3 (Females’ Optimal Strategies). Suppose the stable marriage payoff function is $V$. Let

$$
\theta_{f1} = \frac{c_f}{\mathbb{E}w_{fH} - \mathbb{E}w_{fL} + \mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})},
$$

and

$$
\theta_{f2}^* = \frac{c_f + (1 - \phi)[\mathbb{E}V_{f+}(w_{fL}) - \mathbb{E}V_{f-}(w_{fL})]}{\mathbb{E}[w_{fH} - w_{fL} + \phi(V_{f+}(w_{fH}) - V_{f+}(w_{fL})) + (1 - \phi)(V_{f-}(w_{fH}) - \mathbb{E}V_{f-}(w_{fL}))]}.
$$

Ability $\theta_f \geq \theta_{f1}^*$ females choose to go to college. Ability $\theta_f \geq \theta_{f2}^* > \theta_{f1}^*$ females would make the career investment. That is,

$$
\sigma_f^*(\theta) = (\sigma_f^1(\theta), \sigma_f^2(\theta)) = \begin{cases} (1, 1) & \theta \geq \theta_{f2}^*, \\ (1, 0) & \theta_{f1}^* \leq \theta < \theta_{f2}^*, \\ (0, 0) & \theta < \theta_{f1}^*. \end{cases}
$$

3.2 Stable Marriage Market Outcome

When the population strategies are characterized by $(\theta^*_m, \theta_{f1}^*, \theta_{f2}^*)$, the induced income distributions of men, fit women, and unfit women are

$$
G_m(w) = P_{mL}(w) \left[ F_m(\theta^*_m) + \int_{\theta^*_m}^1 (1 - \theta)^2 dF_m(\theta) \right] + P_{mH}(w) \int_{\theta^*_m}^1 \theta(2 - \theta) dF_m(\theta)
$$

$$
G_{f+}(w) = P_{fL}(w) \left[ F_f(\theta_{f1}^*) + \int_{\theta_{f1}^*}^{\theta_{f2}^*} (1 - \theta)dF_f(\theta) + \phi \int_{\theta_{f2}^*}^1 (1 - \theta)^2 dF_f(\theta) \right]
$$

$$
+ P_{fH}(w) \left[ \int_{\theta_{f1}^*}^{\theta_{f2}^*} \theta dF_f(\theta) + \phi \int_{\theta_{f2}^*}^1 \theta(1 - \theta)dF_f(\theta) \right]
$$

$$
G_{f-}(w) = (1 - \phi) \left[ P_{fL}(w) \int_{\theta_{f2}^*}^1 (1 - \theta)^2 dF_f(\theta) + P_{fH}(w) \int_{\theta_{f2}^*}^1 \theta(1 - \theta)dF_f(\theta) \right]
$$
In addition, let $G_f(w) = G_{f+}(w) + G_{f-}(w)$ represent females’ income distribution,

$$G_f(w) = P_{fL}(w) \left[ F_f(\theta_f^+) + \int_{\theta_f^+}^{\theta_f^2} (1 - \theta) dF_f(\theta) + \int_{\theta_f^2}^{1} (1 - \theta)^2 dF_f(\theta) \right] + \right.$$ 

$$\left. + P_{fH}(w) \left[ \int_{\theta_f^1}^{1} \theta dF_f(\theta) + \int_{\theta_f^2}^{1} \theta (1 - \theta) dF_f(\theta) \right] \right].$$

(9)

### 3.2.1 Stable Matching

Fix marriage market distributions $G_m$, $G_{f+}$, and $G_{f-}$. Because the surplus is strictly supermodular in a husband’s income and a wife’s reproductive fitness, a fit woman marries a higher-income husband than an unfit woman with the same income. Because the surplus is strictly supermodular in the couple’s incomes, men and fit women, and men and unfit women sort positively. Lemma 4 formally states these sorting patterns.

**Lemma 4 (Dimensional Assortative Matching).** For almost all couples $(w_m, (w_f, r_+))$ and $(w'_m, (w_f, r_-))$ in the stable matching, $w_m \geq w'_m$. For almost all couples $(w_m, (w_f, r))$ and $(w'_m, (w'_f, r))$ in the stable matching, $w_m \geq w'_m$ if and only if $w_f \geq w'_f$.

The overall matching is more complicated, however. It is non-assortative in incomes.\(^6\) First, the stable matching might not be pure, that is, identical agents may be matched with different mates. In this particular setting, the matching may not be pure because a man may be indifferent between a high-income unfit woman and a low-income fit woman. Mathematically, a matching is pure when the support of $\mu$ can be borne by the graph of a function $\rho : [\bar{w}_m, \bar{w}_m] \rightarrow [(\bar{w}_f, \bar{w}_f) \times \{r_-, r_+\}]$. The surplus function $S$ does not necessarily satisfy the “twisted-buyer” condition in Chiappori et al. (2010) that guarantees pure stable matching.

The following result provides some additional structures about the stable matching. Top-income men match with top-income fit women, and bottom-income men match with bottom-income unfit women. However, for men in the middle of the pack, they are either matched with a fit woman or an unfit woman with higher income. The matching pattern can be categorized into two scenarios depending on whether there exists a middle pack. When the reproductive fitness is relatively important, the middle pack is of measure zero: higher-income men marry fit women and lower-income men marry unfit women. When the fitness loss can be compensated by income gain, the middle pack exists, and some men are indifferent between a high-income unfit wife and a low-income fit woman.

\(^6\)Chiappori et al. (2012b) and Low (2014) have similar bi-dimensional settings in which one dimension (income) takes continuous values and the other (smoker status, reproductive capital) is binary.
Let \( q(w_m) \) denote the probability that a wage \( w_m \) man matches with a fit woman. Let \( w_f^+(w_m) \) and \( w_f^-(w_m) \) denote the fit woman and the unfit woman an income \( w_m \) man marries. Similarly, let \( w_m(w_f, r) \) denote the income of a \((w_f, r)\) woman’s husband. Figure 6 illustrates the two possible stable matching patterns.

**Lemma 5 (Stable Matching).** Suppose the marriage market is described by \( G_m, G_{f+}, G_{f-} \). Let \( w_m^* = G_m^{-1}(G_{f-}([w_f])) \).

1. **(Pure Matching)** If and only if for almost all \( w_m \geq w_m^* \),

\[
S(w_m, G_f^+(G_m(w_m) - G_f^-(w_f)), r_+) + S(w_m^*, w_f, r_-) \\
\geq S(w_m^*, G_f^+(G_m(w_m) - G_f^-(w_f)), r_+) + S(w_m, w_f, r_-),
\]

(a) almost all \( w_m \geq w_m^* \) men marry fit women and almost all \( w_m \leq w_m^* \) men marry unfit women in the stable matching.

2. **(Mixed Assortative Matching)** If (10) does not hold, there exist cutoffs \( w_m^+ > w_m^* \) and \( w_m^- < w_m^* \) where

\[
w_m^+ = \inf \left\{ w \mid S(w_m, G_f^+(G_m(w_m) - G_f^-(w_f)), r_+) + S(w_m^+, w_f, r_-) \geq S(w_m^+, G_f^+(G_m(w_m) - G_f^-(w_f)), r_+) + S(w_m, w_f, r_-) \text{ almost all } w_m > w \right\}
\]
and
\[ w_m^- = \sup \left\{ w \mid S(w_m, G_{f-1}^{-1}(G_m(w_m)), r_-) + S(w_m, w_f, r_-) \geq S(w_m, G_{f-1}^{-1}(G_m(w_m)), r_-) + S(w_m, w_f, r_-) \text{ almost all } w_m > w \right\} \]
such that almost all \( w_m \geq w_m^+ \) men marry fit women and almost all \( w_m \leq w_m^- \) men marry unfit women.

### 3.2.2 Stable Marriage Payoff Function

Stable marriage payoff function \( V \) can be characterized. A \((w_f, r_+)\) woman marries a \(w_m(w_f, r_+)\) man, and gets \( V_{f+}(w_f) = S(w_m(w_f, r_+), w_f, r_+) - V_m(w_m(w_f, r_+)) \). By the stability condition, for any other \( w_m \), \( V_{f+}(w_f) \geq S(w_m, w_f, r_+) - V_m(w_m) \). Therefore, for any fit woman \( w_f \),

\[ V_{f+}(w_f) = \max_{w_m \in \text{supp}(G_m)} [S(w_m, w_f, r_+) - V_m(w_m)] = S(w_m(w_f, r_+), w_f, r_+) - V_m(w_m(w_f, r_+)) \]

By the Envelope Theorem,

\[ V'_{f+}(w_f) = S_2(w_m(w_f, r_+), w_f, r_+) \]

By integration,

\[ V_{f+}(w_f) = V_{f+}(w_f) + \int_{w_f}^{w_f} S_2(w_m(w, r_+), w, r_+)dw. \] (11)

\( V_m \) and \( V_{f-} \) can be similarly characterized,

\[ V_{f-}(w_f) = V_{f-}(w_f) + \int_{w_f}^{w_f} S_2(w_m(w, r_-), w, r_-)dw. \] (12)

\[ V_m(w_m) = V_m(w_m) + \int_{w_m}^{w_m} [q(w)S_1(w, w_{f+}(w), r_+) + (1 - q(w))S_1(w, w_{f-}(w), r_-)]dw. \] (13)

### 3.3 Equilibrium Existence and Efficiency

**Theorem 1.** An equilibrium exists.
By definition, in equilibrium, if the marriage market measures are \((\mu^*_m, \mu^*_f)\), the marriage payoff function \(V^*\) is stable with respect to \((\mu^*_m, \mu^*_f)\), and the population strategies \(\sigma^*\) are optimal with respect to \(V^*\), and the population strategies induce \((\mu^*_m, \mu^*_f)\). That is, \((\mu^*_m, \mu^*_f)\) satisfies \((\mu^*_m, \mu^*_f) \in \Phi_\mu(\Phi_\theta(\partial \Sigma(\mu^*_m, \mu^*_f)))\) where \(\partial \Sigma(\mu^*_m, \mu^*_f)\) denotes the set of stable marriage payoff functions \(V\) of the marriage market \((\mu^*_m, \mu^*_f)\), \(\Phi_\theta(V)\) denotes the optimal cutoffs \((\theta^*_m, \theta^*_f, \theta^*_1, \theta^*_2)\) given the stable marriage payoff function \(V\), and \(\Phi_\mu(\sigma)\) denotes the marriage market measures \((\mu^*_m, \mu^*_f)\) induced by strategies \(\sigma\). An equilibrium exists if the composite map has a fixed point. Glicksberg’s fixed-point theorem, an extension of Kakutani’s fixed-point theorem to locally convex infinite-dimensional space, is used to show that the composite map has a fixed point. The theorem states that if a map from infinite-dimensional locally convex topological space to itself is non-empty, convex-valued, compact-valued, and upper-hemicontinuous, then there is a fixed point. The more difficult step is to show the upper-hemicontinuity of the composite map. The space of continuous functions is not compact, but the space of stable payoff functions is proven to be compact by invoking the Arzelà-Ascoli Theorem.

Furthermore, the equilibrium strategies are constrained efficient; that is, even a benevolent social planner who can dictate the investments of one side of the population would not be able to improve the total welfare. Moreover, there always exists an equilibrium in which people choose the welfare-maximizing strategies.

**Theorem 2.** Equilibrium strategies are constrained efficient. There exists an efficient equilibrium.

Efficiency of private investments has been shown in many similar settings (Cole et al., 2001a; Peters and Siow, 2002; Iyigun and Walsh, 2007; Chiappori et al., 2009; Dizdar, 2013; Nöldeke and Samuelson, 2014; Hatfield et al., 2014). However, the current result does not directly follow from the previous results in the settings in which investments yield deterministic returns. When the investments yield deterministic returns, a man and a woman who would match after investing can contract on their pre-matching investments. In this setting, such contracts cannot be signed pairwise, because the investment outcome not only changes one’s own marriage type but also possibly changes one’s partner. Nonetheless, the underlying economic principles are comparable. In the deterministic settings, the stable marriage payoffs internalize the social gains. In this setting, the expected stable marriage payoffs internalize the expected social gains.
Recall the observed relationships between marriage age and personal income. The relationship for men is inverse-U shaped: those who married between ages 25 and 30 earned more than those who married earlier and later (Figure 1). The relationship for women was positive in 1980: women who married later earned more on average. These patterns are also similar in other countries (Canada in Figure 15, and Brazil in Figure 16). However, the relationship among women became inverse-U shaped in 2012: those who have married in their late thirties, the group who previously earned the most, earned less than those who have married in their twenties (Figure 2).\footnote{\textsuperscript{7}Figure 14 shows that the relationships do not change qualitatively if we restrict to look at the women who earn positive incomes.}

The model can explain these patterns. The relationship for males is always inverse-U shaped in equilibrium. The relationship for females is positive when the fitness decline at age 3 is sharp ($\phi$ small). It tends towards inverse-U shaped when the fitness decline is not as significant ($\phi$ large).

\section*{4 Males’ Personal Income by Marriage Age}

\textbf{Proposition 1 (Males’ Average Personal Income by Marriage Age).} The equilibrium relationship between men’s marriage age and their personal income is inverse-U shaped: $E_{\omega_{m1}} < E_{\omega_{m3}} < E_{\omega_{m2}}$ where $E_{\omega_{ma}}$ denotes the average personal income of those who marry at age $a$ in equilibrium.

Figure 7 illustrates men’s equilibrium strategies, marriage age, and income.

Those who marry at age 1 are the ability $\theta_{m} < \theta_{m}^{*}$ men, and they do not go to college. They all draw their income from the low income distribution $P_{mL}$, so $E_{\omega_{m1}} = E_{\omega_{mL}}$.\footnote{\textsuperscript{7}Figure 14 shows that the relationships do not change qualitatively if we restrict to look at the women who earn positive incomes.}
Figure 8: Men’s average personal income by marriage age, $\phi = 0.6$ and $\phi = 0.8$.

Those who marry at age 2 are the ability $\theta_m \geq \theta^*_m$ men who improve after a college investment. They all draw from the high income distribution $P_{mH}$, so $E w^*_m = E w_{mH}$.

Those who marry at age 3 do not improve from the college investment and make the career investment. They draw either from the high income distribution or from the low income distribution at age 3. The proportion of those who draw from the high income distribution is

$$p^*_m = \frac{\int_{\theta_m}^1 (1 - \theta) \theta dF_m(\theta)}{\int_{\theta_m}^1 (1 - \theta) dF_m(\theta)}.$$

Therefore, the average income of those who marry at age 3, $E w^*_m = p^*_m E w_{mH} + (1 - p^*_m) E w_{mL}$, is strictly between $E w_{mL}$ and $E w_{mH}$. Therefore, $E w^*_m < E w^*_m < E w^*_m$. Figure 8 numerically illustrates the persistence of the relationship when the parameters of the model change.

The upward-sloping portion of the relationship from age 1 to age 2 comes from the fact that college investment is time-intensive and delays marriage. Those men who marry early are the persons who do not have a bright labor market outlook and choose to marry as early as possible. Those men who can have bright labor market outlook make effort and take time to improve their chance of success and delay marriage.

The downward-sloping portion of the relationship from age 2 to age 3 formalizes the effect in the concluding remarks of Bergstrom and Schoeni (1996), the first paper documenting males’ inverse-U shaped relationship between marriage age and personal income: “Some of these men who marry very late in life ... may be persons whose successes in life have not met the expectations that led them to postpone marriage and who continue to postpone marriage until their true worth is recognized.”

If there is an excess supply of marriageable men, then the lowest-income men will stay
unmarried. This prediction is also consistent with empirical observations. This resonates with another effect in the concluding remarks of Bergstrom and Schoeni (1996): “There may also be a considerable number of males who are such poor marriage material, that any female whom they would wish to marry would prefer being single to marrying one of these males.”

### 4.2 Females’ Personal Income by Marriage Age

**Proposition 2** (Females’ Average Personal Income by Marriage Age). When $\phi$ is sufficiently small, the equilibrium relationship between women’s marriage age and their personal income is positive: $E_{w\star f1} < E_{w\star f2} < E_{w\star f3}$. When $\phi$ is sufficiently large, the relationship is inverse-U shaped: $E_{w\star f1} < E_{w\star f3} < E_{w\star f2}$.

Figure 9 illustrates women’s equilibrium strategies, marriage age, and labor market outcome.

Ability $\theta_f < \theta_{f1}$ women marry at age 1 and earn a low average income, $E_{w\star f1} = E_{w fL}$.

Those who marry at age 2 consist of all the females with ability $\theta_f$ between $\theta_{f1}$ and $\theta_{f2}^*$, and the ability $\theta_f \geq \theta_{f2}^*$ females who improve after the college investment. The proportion of women who draw from the high wage distribution is

$$p_{f2H} = \frac{\int_{\theta_{f1}}^{\theta_{f2}} \theta dF_f(\theta)}{\int_{\theta_{f1}}^{\theta_{f2}} (1-\theta) dF_f(\theta) + \int_{\theta_{f1}}^{\theta_{f2}} \theta dF_f(\theta)}.$$

The average income is $E_{w\star f2} = p_{f2H} E_{w fH} + (1 - p_{f2H}) E_{w fL}$.

The women who marry at age 3 have not improve from either the college investment and or the career investment at age 2. The proportion of those who improve in the labor market is
market at age 3 after making the career investment is

\[ p_{f3H}^* = \frac{\int_{\theta_{f2}}^{1} (1 - \theta) \theta dF_f(\theta)}{\int_{\theta_{f2}}^{1} (1 - \theta) dF_f(\theta)}. \]

The average income is \( Ew_{f3}^* = p_{f3H}^* Ew_{fH} + (1 - p_{f3H}^*) Ew_{fL} \).

Whether \( Ew_{f2}^* \) or \( Ew_{f3}^* \) is bigger, equivalently whether \( p_{f2H}^* \) or \( p_{f3H}^* \) is bigger, depends on the parameters of the model, especially \( \phi \). When \( \phi \) is sufficiently small so that women’s fitness declines sharply, the relationship is positive. Many of those who marry at age 2 are low wage earners. Only extremely high ability unlucky women reinvest and delay marriage into the third period, resulting in an average wage close to \( Ew_{fH} \). \( Ew_{f2}^* < Ew_{f3}^* \). On the other hand, when \( \phi \) is sufficiently large so that women’s fitness declines less sharply, \( \theta_{f1}^* \) is close to \( \theta_{f2}^* \). In the extreme case when \( \phi = 1, \theta_{f1}^* = \theta_{f2}^* \) and \( p_{f2H}^* = 1 \). As a result, when \( \phi \) is large, among the women who marry at age 2, not many will earn low wage, so \( Ew_{f2}^* > Ew_{f3}^* \).

Figure 10 illustrates the relationship shifting from being positive to inverse-U shaped as \( \phi \) increases from 0.6 and 0.8, while the other parameters are kept the same.

Ample evidence suggests that the relative importance of reproductive fitness has declined in the marriage market in the United States. The declining importance results from interplay of social and economic changes that push up the supply of and push down the demand for women’s reproductive capital. Evidently, the total fertility rate sharply declined from early 1960s to early 1980s. Although it has been steady around 2 since then, the change in the ages of the mothers has been dramatic. Fertility rates among late age groups of 30-34, 35-39, and 40-44 have grown exponentially, while the fertility
rates among the younger groups all dropped. On one hand, the supply of reproductive capital by women is much higher. Advances in medical technology and better health services result in higher probability of staying fit, so women of older ages can have children with much ease and less adverse effects. On the other hand, the demand for reproductive capital by men is lower. For instance, the desired family size has decreased. In the United States, the desired number of children has declined from 3.6 to 2.6 from 1960 to 2010. Demand for women’s reproductive fitness decreases because increases in parents’ income and higher opportunity cost of raising kids prevent bigger family size. More families shift from demand for quantity of children to demand for quality (Becker and Lewis, 1973).

5 The College Gender Gap

Women have caught up with, and more surprisingly, overtaken men in college enrollment. In most developed countries and many developing countries, significantly more women than men go to college today. In 1970, only five out of 120 countries in the world had more 30-34-year-old college women than college men. That number has leaped to 50 in 2000 and 67 in 2010 (Figure 3). Out of the seventeen OECD countries with consistent higher education data, four had more female college graduates in 1985, but fifteen did by 2002 (Goldin et al., 2006). Every inhabited continent has countries with more women than men enrolled in college. The United States experienced the college gender gap reversal in the early 1990s and the gap has been growing steadily (Figure 11). In fall 2013, 58% of college freshmen in the United States were females.

Women’s increase in college enrollment rate is not surprising: more women broke free from their traditional household roles and participated in the labor force. What is particularly puzzling is that more women than men go to college. I offer an explanation. Consider the following result.

**Proposition 3 (The College Gender Gap).** When the ability distributions, the labor market opportunities, and the investment costs are the same for the two genders, and the marriage surplus is symmetric in wages, yet women still face fitness loss, strictly more women than men go to college. That is, if \( F_m = F_f \), \( P_{mL} = P_{fL} \), \( P_{mH} = P_{fH} \), \( c_m = c_f \), \( S(w_m,w_f,r) = S(w_f,w_m,r) \), and \( \phi_3 < 1 \), then \( 1 - F_f(\theta_f^*) > 1 - F_m(\theta_m^*) \).

The result could be surprising at first sight, as women should have relatively less incentive to invest because of the fitness loss. The endogenously determined marriage...
market values play a key role in the result. The marriage matching market is a competitive market where supply and demand determine the agents’ values and payoffs. Exactly because women face a disadvantage if they marry late, fewer women than men can fulfill their labor market potential: all the men who go to college make the career investment, but only some of the women who go to college make the career investment. This makes high-wage fit women relatively more scarce than high-wage men. The scarcity raises women’s relative marriage value. More women than men go to college to capture the higher marriage gain.

The following lemma captures the idea that a more scarce characteristics is more valuable in the marriage market.

**Lemma 6.** Suppose \( S(w_m, w_f, r) = S(w_f, w_m, r) \), and \( G_m \) strictly first order stochastically dominates \( G_f \). For almost all \( w' \) and \( w < w' \),

\[
V_m(w') - V_m(w) < V_f+(w') - V_f+(w).
\]

Let me demonstrate the lemma with the case in which there are only high wage \( w_H \) and low wage \( w_L \) in the marriage market. First order stochastic dominance in this case simply means that there are more high wage men than high wage women. I show that \( V_m(w_H) - V_m(w_L) < V_f+(w_H) - V_f+(w_L) \). Figure 12 illustrates the two possible cases of equilibrium matching in Lemma 5.
Figure 12: Stable matching when $\mu_m(w_L) > \mu_f(w_H \times \{r_-, r_+\})$.

First, there are always matches between $w_H$ males and $(w_H, r_+)$ females, so

$$V_m(w_H) + V_{f_+}(w_H) = S(w_H, w_H, r_+).$$

Second, because there are more $w_H$ wage men than $w_H$ women, there must be matches between $w_H$ males and $(w_L, r_+)$ females, so

$$V_m(w_H) + V_{f_+}(w_L) = S(w_H, w_L, r_+).$$

Third, also because there are more $w_H$ wage men than $w_H$ women, there are no matches between $w_L$ males and $(w_H, r_+)$ females,

$$V_m(w_L) + V_{f_+}(w_H) < S(w_L, w_H, r_+).$$

The first and third conditions together imply

$$V_m(w_H) - V_m(w_L) < S(w_H, w_H, r_+) - S(w_L, w_H, r_+),$$

and the first and the second conditions together imply

$$V_{f_+}(w_H) - V_{f_+}(w_L) = S(w_H, w_H, r_+) - S(w_H, w_L, r_+) = S(w_H, w_H, r_+) - S(w_L, w_H, r_+),$$

where the second equality follows from the assumption that the marriage surplus is symmetric in wages.

The following corollary follows from Lemma 6. In words, when high-income women are more scarce than high-income men in the sense of first-order stochastic dominance, an investment that returns the same gain in the labor market brings higher gain in the marriage market to women.
Corollary 1. Suppose \( S(w_m, w_f, r) = S(w_f, w_m, r) \), and \( G_m \) strictly first order stochastically dominates \( G_f \). When \( P_{mH} = P_{fH} \) and \( P_{mL} = P_{fL} \),

\[
\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{mL}) < \mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL}).
\]

The corollary is key to prove Proposition 3. The proof of Proposition 3 is by contradiction. Suppose that more men than women go to college. First, it implies that men get higher marriage gain, i.e. \( \mathbb{E}V_m^*(w_{mH}) - \mathbb{E}V_m^*(w_{mL}) \geq \mathbb{E}V_{f+}^*(w_{fH}) - \mathbb{E}V_{f+}^*(w_{fL}) \). Second, it implies that more men than women make the career investment. However, more men than women go to college and try, so more men end up drawing income from high-income distribution. By Corollary 1, \( \mathbb{E}V_m^*(w_{mH}) - \mathbb{E}V_m^*(w_{mL}) < \mathbb{E}V_{f+}^*(w_{fH}) - \mathbb{E}V_{f+}^*(w_{fL}) \), contradicting the previous implication.

Chiappori et al. (2012a) empirically show that American women’s average college marriage premium has been consistently higher than men’s. Although bigger average marriage premium is correlated with higher college enrollment rate, it is neither sufficient nor necessary for more women than men to go to college in the current model. The college marriage premium is \( \theta \mathbb{E}V_{f+}^*(w_H) - \theta \mathbb{E}V_{f+}^*(w_L) - c \) for \( \theta_{f1} \leq \theta < \theta_{f2} \) women, and is \( (2 - \theta)[\theta \mathbb{E}V_{f+}^*(w_H) - \theta \mathbb{E}V_{f+}^*(w_L) - c] - (1 - \phi)[\mathbb{E}V_{f+}^*(w_L) - \mathbb{E}V_{f-}^*(w_L)] \) for \( \theta \geq \theta_{f2} \) women. It is \( (2 - \theta)[\theta \mathbb{E}V_m^*(w_H) - \theta \mathbb{E}V_m^*(w_L) - c] \) for all \( \theta \geq \theta_{m} \) men. Each male college graduate can expect to reap additional reinvestment benefits, but only some females can reap the reinvestment benefits. Therefore, women’s college marital premium on average can be lower than men’s, but still more women go to college than men.

It should also be noted that surplus supermodularity is a fairly crucial assumption for the result. If the surplus function is strictly submodular, there cannot be more women than men in college (shown in Appendix A.12). The transition of a couple’s labor market activities as substitutes to as complements in the household production provides another reason for the college gender gap reversal. In the recent decades, as technological advances free women from daily chores, the role of the household may have transitioned from a unit of labor specialization to one of cooperation. The marriage surplus, resulting from an underlying intra-household optimization problem, may have changed from being submodular to supermodular in the couple’s wage earnings. The college gender gap reversal consequently accompanies the change of husband’s and wife’s wages from substitutes to complements in household production.
6 Husband’s Income by Wife’s Marriage Age

The model predicts the changing relationship between wife’s marriage age and husband’s income. Figure 4 illustrates the relationships between husband’s income and wife’s marriage age for 40-44 year old Americans in 1980 and 2012. Women who married around the median age always marry husbands with higher income on average. In 1980, the women who married before 20 had significantly higher-income husbands than those who married late. The relationship has been reversed in 2012: those who married later than 30 have higher-income husbands than those who married really early but still married lower-income husbands than those who married after 30.

**Proposition 4 (Average Husband’s Income by Wife’s Marriage Age).** *The equilibrium relationship between a wife’s marriage age and a husband’s income is inverse-U shaped*: $E_{w^*m}(x_{f1}^*) < E_{w^*m}(x_{f2}^*)$ and $E_{w^*m}(x_{f3}^*) < E_{w^*m}(x_{f2}^*)$. When (10) holds and/or $\phi$ is sufficiently small, the income of husbands of the women who marry at age 1 is higher than the income of husbands of the women who marry at age 3: $E_{w^*m}(x_{f1}^*) > E_{w^*m}(x_{f3}^*)$. When (10) does not hold and/or $\phi$ is sufficiently large, the income of husbands of the women who marry at age 1 is lower than the income of husbands of the women who marry at age 3: $E_{w^*m}(x_{f1}^*) < E_{w^*m}(x_{f3}^*)$.

The women who marry at age 1 have low incomes but are reproductively fit, so their average husband’s income is

$$E_{w^*m}(x_{f1}^*) = E_{w^*m}(w_{fL}, r_+).$$

Those who marry at age 2 are fit, but draw from either the high-income or the low-income distribution. Their husbands’ income on average is

$$E_{w^*m}(x_{f2}^*) = p_{f2H}^*E_{w^*m}(w_{fH}, r_+) + (1 - p_{f2H}^*)E_{w^*m}(w_{fL}, r_+).$$

The average income of the husbands of the women who marry at age 3 is

$$E_{w^*m}(x_{f3}^*) = p_{f3H}^*\phi E_{w^*m}(w_{fH}, r_+) + (1 - \phi)E_{w^*m}(w_{fH}, r_-) + (1 - p_{f3H}^*)\phi E_{w^*m}(w_{fL}, r_+) + (1 - \phi)E_{w^*m}(w_{fL}, r_-).$$

The women who marry at age 2 earn higher incomes on average than those who marry at age 1 and are more reproductively fit than those who marry at age 3. Therefore, they marry the husbands with the highest income on average.

The relative importance of reproductive fitness in the marriage market determines
whether those who marry at age 1 or at age 3 marry higher-income husbands on average. When the relative importance of reproductive fitness is high so that the equilibrium matching is the pure matching, reproductively fit women marry higher husbands than almost all unfit women,

\[
\mathbb{E}w^*_m(w_{fL}, r_-) < \mathbb{E}w^*_m(w_{fH}, r_-) < \mathbb{E}w^*_m(w_{fL}, r_+) < \mathbb{E}w^*_m(w_{fH}, r_+) .
\]

In addition, if \( \phi \) is small, majority of those who marry at age 3 have low reproductive fitness. As a result, they marry lower-income husbands than those who marry at age 1.

On the other hand, if the relative importance of reproductive fitness is not as high so that there is mixed assortative matching in equilibrium, it is possible that higher income can compensate lower fitness, so that higher-income fit women may marry higher-income husbands than low-income unfit women,

\[
\mathbb{E}w^*_m(w_{fL}, r_-) < \mathbb{E}w^*_m(w_{fL}, r_+) < \mathbb{E}w^*_m(w_{fH}, r_-) < \mathbb{E}w^*_m(w_{fH}, r_+) .
\]

In addition, when \( \phi \) is large, the average income of the husbands of the women who marry at age 3 is close to

\[
 p_{f3H}^* \mathbb{E}w^*_m(w_{fH}, r_+) + (1 - p_{f3H}^*) \mathbb{E}w^*_m(w_{fL}, r_+) ,
\]

and is larger than \( \mathbb{E}w^*_m(x_{f1}^* ) = \mathbb{E}w^*_m(w_{fL}, r_+) .\)

Figure 13 illustrates the relationships. The observation is consistent with the earlier claim that reproductive fitness has declining relative importance in the marriage market. Higher-income men in general married low-income fit women rather than high-income

Figure 13: Average husband’s income by wife’s marriage age, \( \phi = 0.6 \) and \( \phi = 0.8 \).
unfit women in the previous generations, but now higher-income men married high-income fit women rather than low-income fit women. As more and more women enter the labor force, their wage-earning ability is more valued than their contribution to household chores. More husbands want wiser wives rather than healthier wives.

7 Income Uncertainty Delays Marriage

So far in the model, making the college investment and the career investment by assumption preclude people from entering the marriage market in that period. In this section, I extend the model and allow men and women to marry even when income uncertainty is unresolved. It turns out that these strategies, even when feasible, are strictly dominated.

Suppose that men and women can enter the marriage market when men go to college or make the career investment, and when women go to college (making the career investment for women remains to delay their marriage). Their investment strategies are publicly known and they can commit to the investments. If they choose to enter the marriage market early, their income is not realized. A male’s marriage type is represented by a probability measure $q_m \in \Delta(\mathcal{X}_m)$ instead of $x_m \in \mathcal{X}_m$, and a female’s marriage type is $q_f \in \Delta(\mathcal{X}_f)$ instead of $x_f \in \mathcal{X}_f$. The set of marriage types is $\Delta \equiv \Delta(\mathcal{X}_m) \cup \Delta(\mathcal{X}_f)$ instead of $\mathcal{X}$. The expected marital surplus of a type $q_m$ husband and a type $q_f$ wife is $\tilde{S}(q_m, q_f) \equiv \int_{\mathcal{X}_m} \int_{\mathcal{X}_f} S(x_m, x_f) dq_mdq_f$. I will abuse notation, and let $x_m$ represent degenerate $q_m$, i.e. $q_m(x_m) = 1$, and $\tilde{S}(x_m, q_f) \equiv \int S(x_m, x_f) dq_f$. Let $\tilde{\mu}_m$ and $\tilde{\mu}_f$ represent the measure on $\Delta(\mathcal{X}_m)$ and $\Delta(\mathcal{X}_f)$, respectively. Let $\tilde{\mu}$ be the matching measure on $\Delta(\tilde{\mathcal{X}}_m) \times \Delta(\tilde{\mathcal{X}}_f)$ such that $\tilde{\mu}(X_m \times \Delta(\mathcal{X}_f)) = \tilde{\mu}_m(X_m)$ for all Borel sets $X_m \in \Delta(\mathcal{X}_m)$ and $\tilde{\mu}(\Delta(\mathcal{X}_m) \times X_f) = \tilde{\mu}_f(X_f)$ for all Borel sets $X_f \in \Delta(\mathcal{X}_f)$. Let $\tilde{V} : \Delta \to \mathbb{R}_+$ be the extended marriage payoff function. A marriage market outcome $(\tilde{\mu}, \tilde{V})$ is stable if

$$\tilde{V}(p_m) + \tilde{V}(p_f) \geq \tilde{S}(p_m, p_f) \quad \forall (p_m, p_f) \in \text{supp}(\tilde{\mu}_m) \times \text{supp}(\tilde{\mu}_f).$$

Consider an ability $\theta$ man who decides between (A1) to invest then to marry and (A2) to invest and to marry in the same period. The two actions give him the same expected wage and incur the same cost. The only difference is in the marriage payoffs. A man’s expected marriage payoff from A1 is

$$\theta \mathbb{E}\tilde{V}(w_{mH}) + (1 - \theta) \mathbb{E}\tilde{V}(w_{mL}) = \int \tilde{V}(w) d(\theta p_{mH} + (1 - \theta) p_{mL})$$

---

9See Borch (1962); Wilson (1968); Chiappori and Reny (2006) for settings with income uncertainty.
where $w_mH$ and $w_mL$ are random variables with probability measures $p_{mH}$ and $p_{mL}$. A man’s expected marriage payoff from A2 is

$$\tilde{V}(\theta p_{mH} + (1 - \theta) p_{mL}).$$

**Proposition 5** (Income Uncertainty Delays Marriage). To invest then to marry strictly dominates to invest and to marry in the same period. That is,

$$\int \tilde{V}(w_m)d(\theta p_{mH} + (1 - \theta) p_{mL}) > \tilde{V}(\theta p_{mH} + (1 - \theta) p_{mL}). \quad (14)$$

Intuitively, marrying early locks a man to a partner. If he waits to marry, he can choose to marry a wife who can produce marriage surplus more efficiently with him and who can provide him higher marriage payoff. A type $\theta p_{mH} + (1 - \theta) p_{mL}$ man would marry a type $q_f$ woman, and gets

$$\tilde{V}(\theta p_{mH} + (1 - \theta) p_{mL}) = \tilde{S}(\theta p_{mH} + (1 - \theta) p_{mL}, q_f) - \tilde{V}(q_f).$$

By the stability condition, the marriage payoff of each $w_m$ male weakly exceeds what he would get if he marries $q_f$,

$$\tilde{V}(w_m) \geq \tilde{S}(w_m, q_f) - \tilde{V}(q_f).$$

Integrating all the possible $w_m$,

$$\int \tilde{V}(w_m)d(\theta p_{mH} + (1 - \theta) p_{mL}) \geq \int [\tilde{S}(w_m, q_f) - \tilde{V}(q_f)]d(\theta p_{mH} + (1 - \theta) p_{mL}),$$

where the right hand side equals how much the man would get if he marries early, $\tilde{V}(\theta p_{mH} + (1 - \theta) p_{mL})$.

I have shown that a man prefers to marry after knowing whether he improves or not after college rather than to marry while in college. The essential effect is that an agent would like to resolve his income uncertainty before he marries. Therefore, even if he makes the career investment, he would want to wait to marry until all uncertainty is resolved. The same applies to a woman who makes a college investment and decides whether to marry earlier or later.

Although both effects delay marriage, the income-uncertainty effect here is conceptually different from the Bergstrom-Bagnoli signaling effect. In Bergstrom and Bagnoli (1993), a man delays marriage from the first period to let his privately known wage-
earning ability become public information in the second period. In this model, although all the information is public, income uncertainty coupled with the competitive structure of the marriage market still deters people from marrying early.

Evidence is in line with the theoretical predictions. Many people choose to marry only after they finish their schooling and start working. Among people aged 11-21 in 1979, 79% of women and 84% of men finished their schooling before marriage (Browning et al., 2014). On the flip side, 21% of women and the 16% of men do marry before schooling is over. A possible reason for early marriage is that they enjoy flow consumption of marriage while their uncertainty about future income is quite low. In the model, entering the marriage market and investing at the same time can be a preferable strategy if there is discounting.

8 Conclusion

This paper enriches the static unidimensional matching model to incorporate dynamics, premarital investments, and multidimensional matching. Previously exogenous investment decisions and distributions of marriage characteristics are endogenized. Using the extended model, I am able to shed light on an array of patterns related to marriage age and gender differences in education and income that have not been satisfactorily addressed. First, the model is consistent with males’ inverse-U shaped relationship between marriage age and personal income, and the shift in females’ relationship from positive to inverse-U. Second, the model attributes the college gender gap to women’s endogenously higher college marriage premium. Third, the stable matching is non-assortative, but explains how higher-income husbands become increasingly inclined to marry older high-income women than younger low-income women. Finally, I extend the basic model and show that uncertainty about future can deter people from marrying early.

The current framework can serve as stepping stone for future work in at least three directions. First, cohabitation and divorce should be incorporated. These marriage institutions have become more common, but their causes and effects are not as thoroughly understood as we would like. Second, schooling and career decisions in real life can be more intricate than modeled; for example men and women differ vastly in occupational choices. Incorporating more dynamics and uncertainty to the current model is worthwhile. Deeper theoretical investigations of stochastic premarital investments are needed. They may yield interesting results. For example, in another paper, I investigate the choices among investments that give the same expected return but differ in their uncertainties, and find that quite surprisingly, people would prefer to choose investments
that have bigger uncertainties before they enter the competitive matching market (Zhang, 2014). Third, to the best of my knowledge, there has been no attempt to estimate the class of investment-and-matching models. Techniques used to estimate static matching market, developed by Choo and Siow (2006) for example, can be adopted.
References


Keeley, Michael C., “The Economics of Family Formation: An Investigation of the Age at First Marriage,” _Economic Inquiry_, April 1977, 15, 238–250.


Appendix

A  Proofs

A.1  Proof of Lemma 1

Gretsky et al. (1992, Theorem 6) shows continuity of stable marriage payoff functions. Only strict monotonicity needs to be shown. $V_m(w_m) + V_f(x_f) = S(w_m, x_f)$ if $(w_m, x_f) \in \text{supp}(G)$, and by stability condition, $V_m(w_m) + V_f(x_f) \geq S(w_m, x_f)$ for all $x_f$,

$$V_m(w_m) = \sup_{x_f \in \text{supp}(\mu_f)} [S(w_m, x_f) - V_f(x_f)].$$

Since $S(\cdot, \cdot, \cdot)$ is strictly increasing in each argument, $S(w_m, w_f, r) - V_f(w_f, r)$ is strictly increasing in $w_m$ for fixed $(w_f, r)$. By the Theorem of Maximum, $V_m$ is strictly increasing. Strict monotonicity of $V_f$ can be similarly shown. QED

A.2  Proof of Lemma 2

An ability $\theta$ male’s Bellman equation is

$$\max_{N_1, I_1} \left\{ \mathbb{E}[V_m(w_{mL}) + w_{mL}], -c_m + \theta \mathbb{E}[V_m(w_{mH}) + w_{mH}] + (1 - \theta) \times \right.$$  

$$\max_{N_2, I_2} \left\{ \mathbb{E}[V_m(w_{mL}) + w_{mL}], -c_m + \theta \mathbb{E}[V_m(w_{mH}) + w_{mH}] + (1 - \theta) \mathbb{E}[V_m(w_{mL}) + w_{mL}] \right\} \right\}$$

which can be rearranged as

$$\mathbb{E}[V_m(w_{mL}) + w_{mL}] +$$

$$\max_{N_1, I_1} \left\{ 0, \theta \mathbb{E}[V_m(w_{mH}) + w_{mH}] - \theta \mathbb{E}[V_m(w_{mL}) + w_{mL}] - c_m + (1 - \theta) \mathbb{E}[V_m(w_{mL}) + w_{mL}] - c_m \right\}.$$  

By Bellman’s Principle of Optimality, optimal strategies can be solved backwards.

At age 2, an ability $\theta$ male who does not improve from the college investment makes
the career investment if

$$\theta \mathbb{E}[V_m(w_{mH}) + w_{mH}] - \theta \mathbb{E}[V_m(w_{mL}) + w_{mL}] - c_m \geq 0.$$ 

Now solve for the male’s optimal college investment decision at age 1. Every male’s expected benefit from a college investment equates the expected benefit from the career investment. Therefore, any male who does not benefit from a college investment will not benefit from making the career investment. An ability $$\theta$$ male makes the college investment if

$$\theta \mathbb{E}[V_m(w_{mH}) + w_{mH}] - \theta \mathbb{E}[V_m(w_{mL}) + w_{mL}] - c_m \geq 0.$$ 

QED

A.3 Proof of Lemma 3

An ability $$\theta$$ female’s maximal utility is

$$\max_{N_1,N_1} \left\{ \mathbb{E}[V_{f+}(w_{fL}) + w_{fL}], -c_f + \theta \mathbb{E}[V_{f+}(w_{fH}) + w_{fH}] + (1 - \theta) \times \right.$$ 

$$\max_{N_2,N_2} \left\{ \mathbb{E}[V_{f+}(w_{fL}) + w_{fL}], \theta \mathbb{E}[V_{f+}(w_{fH}) + w_{fH}] + \theta (1 - \phi) \mathbb{E}[V_{f-}(w_{fH}) + w_{fH}] \right.$$ 

$$+ (1 - \theta) \phi \mathbb{E}[V_{f+}(w_{fL}) + w_{fL}] + (1 - \theta)(1 - \phi) \mathbb{E}[V_{f-}(w_{fL}) + w_{fL}] - c_f \right\}$$

which can be rearranged as

$$\mathbb{E}[V_{f+}(w_{fL}) + w_{fL}] +$$

$$\max_{N_1,N_1} \left\{ 0, -c_f + \theta (\mathbb{E}w_{fH} - \mathbb{E}w_{fL}) + \theta [\mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})] + (1 - \theta) \times \right.$$ 

$$\max_{N_2,N_2} \left\{ 0, -c_f + \theta (\mathbb{E}w_{fH} - \mathbb{E}w_{fL}) + \theta \phi \mathbb{E}V_{f+}(w_{fH}) + \theta (1 - \phi) \mathbb{E}V_{f-}(w_{fH}) \right.$$ 

$$+ (1 - \theta) \phi \mathbb{E}V_{f+}(w_{fL}) + (1 - \theta)(1 - \phi) \mathbb{E}V_{f-}(w_{fL}) - \mathbb{E}V_{f+}(w_{fL}) \right\}$$

By the Principle of Optimality, the optimal decisions can be solved backwards. An ability $$\theta$$ female makes the career investment at age 2 if

$$\theta \phi \mathbb{E}[V_{f+}(w_{fH}) + w_{fH}] + \theta (1 - \phi) \mathbb{E}[V_{f-}(w_{fH}) + w_{fH}]$$

$$+ (1 - \theta) \phi \mathbb{E}[V_{f+}(w_{fL}) + w_{fL}] + (1 - \theta)(1 - \phi) \mathbb{E}[V_{f-}(w_{fL}) + w_{fL}]$$

$$\geq c_f + \mathbb{E}[V_{f+}(w_{fL}) + w_{fL}]$$

A-2
which can be rearranged as
\[
\theta E\{\phi[V_f+(w_{fH}) - EV_f+(w_{fL})] + (1 - \phi)[V_f-(w_{fH}) - V_f-(w_{fL})] + (w_{fH} - w_{fL})\} \\
\geq \ c_f + (1 - \phi)[EV_f+(w_{fL}) - EV_f-(w_{fL})].
\]

Because of the declined reproductive capital, every female gets weakly lower utility gain from making the career investment than from the college investment. An ability \( \theta \) female makes a college investment if
\[
\theta[E V_f+(w_{fH}) + (w_{fH}) - E V_f+(w_{fL}) + (w_{fL})] + \ (1 - \phi)[E V_f+(w_{fL}) - E V_f-(w_{fL})].
\]

QED

A.4 Proof of Lemma 4

The proof is by contradiction. Suppose that there is a positive measure of couples of \((w_m, (w'_{f}, r))\) and \((w'_m, (w_f, r))\) such that \(w_m > w'_m\) and \(w_f > w'_f\). The surplus generated by the two matched couples is \(S(w_m, w'_f, r) + S(w'_m, w_f, r)\). The surplus generated by swapping the partners is \(S(w_m, w_f, r) + S(w'_m, w'_f, r)\). The surplus generated by the swap is strictly higher by wage supermodularity. If there is a positive measure of such couples, the total surplus cannot be maximized under \(\mu\), contradicting the surplus-maximizing property of the stable matching \(\mu\). The second part of the lemma can be proved with similar arguments. QED

A.5 Proof of Lemma 5

First, I show that (10) is a necessary condition for the pure matching to be stable. Suppose (10) does not hold. Then there exists \(\epsilon > 0\) such that for almost all \(w' \in (w_m - \epsilon, w_m + \epsilon)\), \(w \in (w^*_m - \epsilon, w^*_m)\), and \(w_f \in (\bar{w}_f - \epsilon, \bar{w}_f)\),
\[
S(w', G_f^{-1}(G_m(w) - G_f-(w_f)), r_+) + S(w, w_f, r_-) \\
< \ S(w, G_f^{-1}(G_m(w) - G_f-(w_f)), r_+) + S(w', w_f, r_-).
\]

Thus it is more efficient to have couples \((w', (G_f^{-1}(G_m(w') - G_f-(w_f)), r_+))\) and \((w, w_f, r_-)\). The pure matching described is not efficient and thus not stable.

Next, I show that (10) is sufficient for the pure matching to be stable. By (10), it is never more efficient for \((\bar{w}_f, r_+)\) to marry any \(w_m > w^*_m\) than to marry \(w^*_m\). The highest
income husband a \((\overline{w}_f, r_-)\) can marry is \(w^*_m\). By Lemma 4, for almost all \(w_f, (w_f, r_+)\) marries a higher income husband than \((w_f, r_-)\) does, and \((\overline{w}_f, r_-)\) marries a higher income husband than \((w_f, r_-)\) does. Therefore, the lowest income husband a \((\overline{w}_f, r_-)\) can marry is also \(w^*_m\).

Finally, I show that thresholds \(w^*_m\) and \(w^-\) exist (regardless of whether (10) is satisfied). By complementarity of \(w_f\) and \(r\), \(S_1(\overline{w}_m, \overline{w}_f, r_+) > S_1(\overline{w}_m, \overline{w}_f, r_-)\). By the smoothness of the surplus function, for some \(\epsilon > 0\), for almost all \(w > \overline{w}_m - \epsilon\) and \(w_f > \overline{w}_f - \epsilon\),

\[
S_1(w, w_f, r_+) > S_1(w, \overline{w}_f, r_-).
\]

That is, for \(w_m < w'_m < \overline{w}_m - \epsilon\),

\[
\int_{w_m}^{w'_m} S_1(w, w_f, r_+)dw > \int_{w_m}^{w'_m} S_1(w, \overline{w}_f, r_-)dw,
\]

or equivalently,

\[
S(w'_m, w_f, r_+) - S(w_m, w_f, r_+) > S(w'_m, \overline{w}_f, r_-) - S(w_m, \overline{w}_f, r_-).
\]

Therefore, it is more efficient for couples \((w'_m, w_f, r_+)\) and \((w_m, w_f, r_+)\) than for couples \((w_m, \overline{w}_f, r_-)\) and \((w'_m, \overline{w}_f, r_-)\). The existence of \(w^-\) can be similarly shown from \(S_1(w_m, w_f, r_+) > S_1(\overline{w}_m, \overline{w}_f, r_-)\). QED

### A.6 Proof of Theorem 1

For notational convenience, in and only in this proof, \(\mu\) denote the marriage market measure on \(\mathcal{X}_m \cup \mathcal{X}_f\) such that \(\mu\) restricted to \(\mathcal{X}_m\) is \(\mu_m\) and restricted to \(\mathcal{X}_f\) is \(\mu_f\).

Let \(C(X)\) denote the Banach space of continuous functions on set \(X\) equipped with the supremum norm, i.e \(\|f\| = \sup_{x \in X} |f(x)|\). Let \(V\) denote the set of non-negative and strictly increasing \(V \in C(\mathcal{X}_m \times \mathcal{X}_f)\). A stable marriage payoff function \(V \in V\) by Lemma 1. Let \(\mathcal{M}(X)\) denote the Banach space of measures on set \(X\) equipped with the weak topology, that is, the weakest topology for which \(\mu \mapsto \int f d\mu\) are continuous for all \(f \in C(X)\). Let \(\mathcal{M}\) denote the set of measure \(\mu \in \mathcal{M}(\mathcal{X}_m \cup \mathcal{X}_f)\) such that \(\mu_m(\mathcal{X}_m) = 1\) and \(\mu_f(\mathcal{X}_f) = 1\). A stationary marriage market measure \(\mu \in \mathcal{M}\).

Define the value function \(\Sigma(\mu)\) as

\[
\min \left\{ \int V d\mu \mid V(x_m) + V(x_f) \geq S(x_m, x_f) \forall x_m \in \text{supp}(\mu_m), x_f \in \text{supp}(\mu_f) \right\}
\]
and the superdifferential correspondence \( \partial \Sigma(\mu_m, \mu_f) \) as

\[
\left\{ V \in C(\mathcal{X}_m \cup \mathcal{X}_f) \left| \int Vd\mu - \int Vd\mu' \leq \Sigma(\mu) - \Sigma(\mu') \forall \mu' \right. \right\}.
\]

By Gretsky et al. (1992), \( \partial \Sigma(G) \) can be redefined as

\[
\left\{ V \in C(\mathcal{X}_m \cup \mathcal{X}_f) \left| \int Vd\mu = \Sigma(\mu), \int Vd(\mu') \geq \Sigma(\mu') \forall \mu' \right. \right\}. \tag{15}
\]

and corresponds to the set of stable marriage payoff functions of \( \mu \).

Define the composite mapping \( \Phi = \Phi_\mu \circ \Phi_\theta \circ \partial \Sigma : \mathcal{M} \to \mathcal{M} \) where \( \Phi_\theta : \mathcal{V} \to \Theta_m \times \Theta_f \times \Theta_f \) is the optimal cutoffs function defined by (1), (3) and (4), and \( \Phi_\mu : \Theta_m \times \Theta_f \times \Theta_f \to \mathcal{M} \) is the marriage market distributions function described by (6), (7) and (8).

To prove equilibrium existence, it suffices to show \( \Phi \) has a fixed point \( \mu^* \), i.e. \( \mu^* \in \Phi(\mu^*) \). By Glicksberg (1952) fixed point theorem, it suffices to show \( \mathcal{M} \) is non-empty, convex, and compact, \( \Phi(\mu) \) is non-empty, convex, and compact for every \( \mu \in \mathcal{M} \), and \( \Phi : \mathcal{M} \to \mathcal{M} \) is upper hemicontinuous.

If \( \mu_n \to \mu \), then \( V_n \in \partial \Sigma(\mu_n) \) has a convergent subsequence. To show that \( \{V_n\} \in \mathcal{V} \) has a convergent subsequence, by the Arzelà-Ascoli Theorem, it suffices to show that \( \{V_n\} \) is a uniformly bounded, equicontinuous sequence of real-valued functions on \( \mathcal{X}_m \cup \mathcal{X}_f \). It is uniformly bounded because \( V_n(x) \leq S(w_m, w_f, r_+) \) for any \( x \) and \( n \). Equicontinuity follows from the following argument. For any \( x_m > x_m' \),

\[
V_n(x_m) - V_n(x_m') = \sup_{x_f \in \text{supp}(\mu_n) \cap \mathcal{X}_f} [S(x_m, x_f) - V_n(x_f)] - \sup_{x_f \in \text{supp}(\mu_n) \cap \mathcal{X}_f} [S(x_m', x_f) - V_n(x_f)].
\]

For each \( n \), there exists \( x_{fn} \in \text{supp}(\mu_n) \cap \mathcal{X}_f \) such that

\[
V_n(x_m) = S(x_m, x_{fn}) - V_n(x_{fn}).
\]

Therefore,

\[
V_n(x_m) - V_n(x_m') = S(x_m, x_{fn}) - V_n(x_{fn}) - \sup_{x_f \in \text{supp}(\mu_n) \cap \mathcal{X}_f} [S(x_m', x_f) - V_n(x_f)] \\
\leq [S(x_m, x_{fn}) - V_n(x_{fn})] - [S(x_m', x_{fn}) - V_n(x_{fn})] \\
= S(x_m, x_{fn}) - S(x_m', x_{fn}) = \int S_1(\tilde{x}_m, x_{fn})d\tilde{x}_m \\
\leq \sup_{\tilde{x}_m \in \mathcal{X}_m, \tilde{x}_f \in \mathcal{X}_f} S_1(\tilde{x}_m, \tilde{x}_f) \Vert(x_m - x_m')\Vert.
\]
\[ K \equiv \sup_{\tilde{\ell}_m, \tilde{\ell}_f} S_1(\tilde{x}_m, \tilde{x}_f) \] is bounded because \( S \) is continuously differentiable on compact sets \( \mathcal{X}_m \) and \( \mathcal{X}_f \). For any \( \epsilon \), there exists \( \delta = \epsilon/K \) such that \(|x_m - x'_m| < \delta \) implies \(|V_n(x_m) - V_n(x'_m)| < \epsilon \). Similarly, \( V_n(w_f, r_+) - V_n(w'_f, r_+) \leq K(w_f - w'_f) \).

By definition, \( \Phi : M \rightarrow M \) is upper hemicontinuous if \( \forall \mu_n, \mu'_n \in M, V_n \in \mathcal{V}, \theta_n \in \Theta_n \times \Theta_f \times \Theta_f : \mu_n \rightarrow \mu, V_n \rightarrow V, \theta_n^* \rightarrow \theta^*, \mu'_n \rightarrow \mu', \) and \( V_n \in \partial \Sigma(\mu_n) \), and \( \mu'_n = \Phi_\mu(\Phi_\theta(V_n)) \) imply \( \mu' \in \Phi(\mu) \). It suffices to show that \( \partial \Sigma(\cdot) \) is upper hemicontinuous, and \( \Phi_\theta \) and \( \Phi_\mu \) are continuous.

To show that \( \partial \Sigma : M \rightarrow \mathcal{V} \) is weak-to-norm upper hemicontinuous is to show that \( \forall \mu_n \in M, V_n \in \mathcal{V}, \) if \( \mu_n \rightarrow \mu, V_n \rightarrow V, V_n \in \partial \Sigma(\mu_n) \) then \( V \in \partial \Sigma(\mu) \). By (15), to show \( V \in \partial \Sigma(\mu) \) is equivalent to show that \( \int V d\mu = \Sigma(\mu) \) and \( \int V d\mu' \geq \Sigma(\mu') \) for all \( \mu' \in M(\mathcal{X}) \). First, I show that \( \int V d\mu = \Sigma(\mu) \). Since \( V_n \in \partial \Sigma(\mu_n) \), \( \int V_n d\mu_n = \Sigma(\mu_n) \) for all \( n \), and

\[
|\int V d\mu - \Sigma(\mu)| = |\int V d\mu - \int V_n d\mu_n + \Sigma(\mu_n) - \Sigma(\mu)|
\]

\[
\leq |\int V d\mu - \int V_n d\mu_n| + |\Sigma(\mu_n) - \Sigma(\mu)|
\]

\[
\leq |\int V d\mu - \int V d\mu_n| + |\int V d\mu_n - \int V_n d\mu_n| + |\Sigma(\mu_n) - \Sigma(\mu)|
\]

\[
\leq |\int V d\mu - \int V d\mu_n| + 2\|V - V_n\| + |\Sigma(\mu_n) - \Sigma(\mu)|.
\]

Fix any \( \epsilon \). Since \( \mu_n \rightarrow \mu \), there is \( N_1(\epsilon) \) such that for all \( n > N_1(\epsilon), |\int V d\mu - \int V d\mu_n| \leq \epsilon/3 \). Since \( V_n \rightarrow V \), there is \( N_2(\epsilon) \) such that for all \( n > N_2(\epsilon), \|V - V_n\| \leq \epsilon/6 \). Since \( \Sigma(\mu) \) is weakly continuous, there is \( N_3(\epsilon) \) such that for all \( n > N_3(\epsilon), |\Sigma(\mu_n) - \Sigma(\mu)| \leq \epsilon/3 \). Therefore, given any \( \epsilon \), for all \( n > N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon), N_3(\epsilon)\}, |\int V d\mu - \int V d\mu_n| + 2\|V - V_n\| + |\Sigma(\mu_n) - \Sigma(\mu)| \leq \epsilon \). Therefore, \( |\int V d\mu - \Sigma(\mu)| = 0 \). Second, I show that \( \int V d\mu' \geq \Sigma(\mu') \) for all \( \mu' \in M \). Since \( V_n \rightarrow V \), it has to be that \( V(x_m) + V(x_f) \geq S(x_m, x_f) \) for all \( x_m \in \mathcal{X}_m \) and \( x_f \in \mathcal{X}_f \). Otherwise if \( V(x_m) + V(x_f) < S(x_m, x_f) \), then \( \lim_{n \rightarrow \infty} \|V - V_n\| > 0 \). \( \int V d\mu' \geq \Sigma(\mu') \) by definition of \( \Sigma \).

Next, \( \Phi_\theta : \mathcal{V} \rightarrow \Theta_m \times \Theta_f \times \Theta_f \) is continuous. Take \( V \) and \( V' \). \( \theta^*_m = c_m / \mathbb{E}[w_{m} H - \ldots] \)
\[ w_{mL} + V_m(w_{mH}) - V_m(w_{mL}) \] and \[ \theta_m' = c_m / \mathbb{E}[w_{mH} - w_{mL} + V_m'(w_{mH}) - V_m'(w_{mL})]. \]

\[
\begin{align*}
|\theta_m^* - \theta_m'| &= \frac{c_m|\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{mL})| - |\mathbb{E}V_m'(w_{mH}) - \mathbb{E}V_m'(w_{mL})|}{\mathbb{E}[w_{mH} - w_{mL} + V_m(w_{mH}) - V_m(w_{mL})]} \\
&\leq \frac{2c_m}{|\mathbb{E}w_{mH} - \mathbb{E}w_{mL}|} \|V - V'\|.
\end{align*}
\]

For any \( \varepsilon > 0 \) there exists \( \delta = \varepsilon |\mathbb{E}w_{mH} - \mathbb{E}w_{mL}|^2 / (2c_m) \) such that for all \( V \) and \( V' \), \( \|V - V'\| \leq \delta \) implies \( |\theta_m^* - \theta_m'| \leq \varepsilon. \) \( \theta_f^* \) and \( \theta_f' \) can be similarly shown to be continuous.

Finally, \( \Phi_\mu : \Theta_m \times \Theta_f \to \mathcal{M} \) is weakly continuous. It suffices to show that \( G_m(w), G_f^+(w), G_f^-(w) \) in (6), (7), (8) are continuous in \( (\theta_m^*, \theta_f^*, \theta_f') \) (Billingsley, 1999). Since \( F_m(\theta) \) and \( F_f(\theta) \) are assumed to be continuous and strictly increasing, for each \( w, G_m(w), G_f^+(w), \) and \( G_f^-(w) \) are continuous in \( (\theta_m^*, \theta_f^*, \theta_f^*) \).

To complete the proof, check that \( \Phi(\mu) \) is non-empty, convex, and compact for every \( \mu \in \mathcal{M} \). \( \Phi(\mu) \) is non-empty since each of the mappings is non-empty. It is convex because \( \partial \Sigma(\mu) \) is convex: \( V, V' \in \partial \Sigma(\mu) \) implies that for any \( \lambda \in (0, 1), \int [\lambda V + (1 - \lambda)V']d\mu = \lambda \Sigma(\mu) + (1 - \lambda)\Sigma(\mu) = \Sigma(\mu) \) and \( \int [\lambda V + (1 - \lambda)V']d\mu' = \lambda \int Vd\mu' + (1 - \lambda) \int V'd\mu' \geq \lambda \Sigma(\mu') + (1 - \lambda)\Sigma(\mu') = \Sigma(\mu') \) for all \( \mu' \). For every \( \mu \), if \( S \) is continuous, then \( \partial \Sigma(\mu) \) is a norm-compact set (Gretsky et al., 1992, Theorem 7). Since \( \Phi_\theta \) and \( \Phi_\mu \) are continuous, \( \Phi(\mu) \) is compact. QED

### A.7 Proof of Theorem 2

The expected social welfare \( SW(\theta_m, \theta_f, \theta_f^*) \) when the population strategies are characterized by \( (\theta_m, \theta_f^*, \theta_f^*) \) is

\[
\begin{align*}
\Sigma(\Phi_\mu(\theta_m, \theta_f^*, \theta_f^*)) + \mathbb{E}w_{mL} + (\mathbb{E}w_{mH} - \mathbb{E}w_{mL}) \int_{\theta_m}^{1} \theta(2 - \theta)dF_m(\theta) \\
+ \mathbb{E}w_{fL} + (\mathbb{E}w_{fH} - \mathbb{E}w_{fL}) \left[ \int_{\theta_f^1}^{\theta_f^2} \theta dF_f(\theta) + \int_{\theta_f^2}^{1} \theta(2 - \theta)dF_f(\theta) \right] \\
- c_m \left[ \int_{\theta_m}^{1} \theta dF_m(\theta) + 2 \int_{\theta_m}^{1} (1 - \theta)dF_m(\theta) \right] - c_f \left[ \int_{\theta_f^1}^{1} \theta dF_f(\theta) + 2 \int_{\theta_f^2}^{1} (1 - \theta)dF_f(\theta) \right].
\end{align*}
\]
\( \theta^*_m, \theta^*_{f1} \) and \( \theta^*_{f2} \) satisfy the optimality conditions
\[
\begin{align*}
\theta^*_m & \left[ Ew_{mH} - Ew_{mL} + EV_m(w_{mH}) - EV_m(w_{mL}) \right] = c_m \\
\theta^*_{f1} & \left[ Ew_{fH} - Ew_{fL} + EV_{f+}(w_{fH}) - EV_{f+}(w_{fL}) \right] = c_f \\
\theta^*_{f2} & \left[ w_{fH} - w_{fL} + \phi(V_{f+}(w_{fH}) - V_{f+}(w_{fL})) + (1 - \phi)(V_{f-}(w_{fH}) - V_{f-}(w_{fL})) \right] - (1 - \phi)(EV_{f+}(w_{fL}) - EV_{f-}(w_{fL})) = c_f 
\end{align*}
\]
where \( V \in \partial \Sigma(\Phi_\mu(\theta^*_m, \theta^*_{f1}, \theta^*_{f2})) \). They coincide with (1), (3) and (4), the equations that determine the equilibrium cutoffs. Therefore, the equilibrium strategies are constrained efficient. That is, a social planner cannot improve the social welfare by changing only one side’s investment patterns.

The socially efficient investment strategies can be supported in equilibrium. \( SW \) is continuous on a compact set, so a maximum exists. Since ability \( \theta = 1 \) agents always benefit from taking \( I_a \) by assumption, maximum exists in an interior point \( (\theta^*_m, \theta^*_{f1}, \theta^*_{f2}) \). The optimality conditions, equivalently the equilibrium conditions, are satisfied. \( \text{QED} \)

A.8 Proof of Proposition 2

Ability \( \theta_f < \theta^*_{f1} \) women marry at age 1 and earn an average wage of \( Ew^*_{f1} = Ew_{fL} \).

The women who marry at age 2 are either ability \( \theta_f \geq \theta^*_{f1} \) women who improve or ability \( \theta^*_{f1} \leq \theta_f \leq \theta^*_{f2} \) women who do not improve from the college investment and will not make the career investment. The mass of the first type of women is \( \int_{\theta^*_{f1}}^{1} \theta dF_f(\theta) \). The mass of the second type is \( \int_{\theta^*_{f1}}^{\theta^*_{f2}} (1 - \theta) dF_f(\theta) \). The proportion of women who marry at age 2 and draw from high wage distribution is denoted by
\[
p^*_{f2H} = \frac{\int_{\theta^*_{f1}}^{1} \theta dF_f(\theta)}{\int_{\theta^*_{f1}}^{\theta^*_{f2}} (1 - \theta) dF_f(\theta) + \int_{\theta^*_{f1}}^{\theta^*_{f2}} (1 - \theta) dF_f(\theta)}.
\]
The average wage of the women who marry at age 2 is
\[
Ew^*_{f2} = p^*_{f2H} Ew_{fH} + (1 - p^*_{f2H}) Ew_{fL}.
\]
The women who marry at age 3 are all ability \( \theta_f \geq \theta^*_{f2} \) women who make the career investment. Among them, mass \( \int_{\theta^*_{f2}}^{1} (1 - \theta) dF_f(\theta) \) draw from high wage, and mass
\[ f_{\theta_{f2}}^1 (1 - \theta)(1 - \theta) dF_f(\theta) \] draw from low wage. Therefore, the proportion of women drawing from the high wage distribution is

\[
p_{f3H}^* = \frac{\int_{\theta_{f2}}^1 (1 - \theta) \theta dF_f(\theta)}{\int_{\theta_{f2}}^1 (1 - \theta) dF_f(\theta)}.
\]

The average wage of the women who marry at age 3 is

\[
\mathbb{E}w_{f3}^* = p_{f3H}^* \mathbb{E}w_{fH} + (1 - p_{f3H}^*) \mathbb{E}w_{fL}.
\]

When \( \phi \) is sufficiently small, \( \theta_{f2}^* \) is close to 1, so \( p_{f3H}^* \) is close to 1 and \( p_{f3H}^* > p_{f2H}^* \). Therefore, \( \mathbb{E}w_{f3}^* > \mathbb{E}w_{f2}^* \). When \( \phi \) is sufficiently large, \( \theta_{f2}^* \) is small and is close to \( \theta_{f1}^* \). \( p_{f2H}^* \) is close to 1, so \( p_{f2H}^* > p_{f3H}^* \). QED

### A.9 Proof of Lemma 6

By (11),

\[
V_{f+}(w') - V_{f+}(w) = \int_{w}^{w'} S_2(w_w(z, r_+), z, r_+) dz.
\]

By the wage symmetry assumption,

\[
V_{f+}(w') - V_{f+}(w) = \int_{w}^{w'} S_1(z, w_m(z, r_+), r_+) dz.
\]

Because \( G_m \) first order stochastically dominates \( G_f \), for almost all \( z \), \( w_m(z, r_+) > z \). By complementarity, that implies \( S_1(z, w_m(z, r_+), r_+) < S_1(z, z, r_+) \). Therefore,

\[
V_{f+}(w') - V_{f+}(w) > \int_{w}^{w'} S_1(z, z, r_+) dz.
\]

By (13),

\[
V_m(w') - V_m(w) = \int_{w}^{w'} [q(z) S_1(z, w_{f+}(z), r_+) + (1 - q(z)) S_1(z, w_{f-}(z), r_-)] dz.
\]

Because \( G_m(\cdot) \) first stochastically dominates \( G_f(\cdot) \), if \( q(z) > 0 \), \( w_{f+}(z) < z \). By wage complementarity, \( S_1(z, w_{f+}(z), r_+) \leq S_1(z, z, r_+) \). In addition, \( S_1(z, w_{f-}(z), r_-) < S_1(z, z, r_+) \) for almost all \( z \) such that \( q(z) = 0 \). By wage complementarity, \( S_1(z, w, r_+) < S_1(z, z, r_+) \).
Combining these facts,

\[
\int_{w'}^{w} [q(z)S_1(z, w_f(z), r_+) + (1 - q(z))S_1(z, w_f(z), r_-)] \, dz \\
< \int_{w}^{w'} [q(z)S_1(z, z, r_+) + (1 - q(z))S_1(z, z, r_+)] \, dz = \int_{w}^{w'} S_1(z, z, r_+) \, dz.
\]

Therefore, \(V_{f+}(w') - V_{f+}(w) > \int_{w}^{w'} S_1(z, z, r_+) \, dz > V_m(w') - V_m(w)\). QED

**A.10 Proof of Corollary 1**

It suffices to show that \([\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{mL})] - [\mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})] < 0.\)

\[
\begin{align*}
\left[\mathbb{E}V_m(w_{mH}) - \mathbb{E}V_m(w_{mL})\right] - \left[\mathbb{E}V_{f+}(w_{fH}) - \mathbb{E}V_{f+}(w_{fL})\right] \\
= \left[\int V_m(w) dP_H(w) - \int V_m(w) dP_L(w)\right] - \left[\int V_{f+}(w) dP_H(w) - \int V_{f+}(w) dP_L(w)\right] \\
= \left[\int V_m(P_H^{-1}(z)) dz - \int V_m(P_L^{-1}(z)) dz\right] - \left[\int V_{f+}(P_H^{-1}(z)) dz - \int V_{f+}(P_L^{-1}(z)) dz\right] \\
= \int \left\{\left[V_m(P_H^{-1}(z)) - V_m(P_L^{-1}(z))\right] - \left[V_{f+}(P_H^{-1}(z)) - V_{f+}(P_L^{-1}(z))\right]\right\} \, dz.
\end{align*}
\]

By Lemma 6, for each \(z \in [0, 1],\)

\[
\left[V_m(P_H^{-1}(z)) - V_m(P_L^{-1}(z))\right] - \left[V_{f+}(P_H^{-1}(z)) - V_{f+}(P_L^{-1}(z))\right] < 0.
\]

QED

**A.11 Proof of Proposition 3**

When \(P_{mH} = P_{fH} \equiv P_H, P_{mL} = P_{fL} \equiv P_L, F_m = F_f \equiv F,\) and \(c_m = c_f \equiv c,\) the three equilibrium cutoffs are

\[
\begin{align*}
\theta_m^* &= \frac{c}{[\mathbb{E}w_H - \mathbb{E}w_L] + [\mathbb{E}V_m^*(w_H) - \mathbb{E}V_m^*(w_L)]}, \\
\theta_{f1}^* &= \frac{c}{[\mathbb{E}w_H - \mathbb{E}w_L] + [\mathbb{E}V_{f+}^*(w_H) - \mathbb{E}V_{f+}^*(w_L)]}, \\
\theta_{f2}^* &= \frac{c + (1 - \phi)[\mathbb{E}V_{f+}^*(w_L) - \mathbb{E}V_{f+}^*(w_L)]}{[\mathbb{E}w_H - \mathbb{E}w_L] + \phi[\mathbb{E}V_{f+}^*(w_H) - \mathbb{E}V_{f+}^*(w_L)] + (1 - \phi)[\mathbb{E}V_{f-}^*(w_H) - \mathbb{E}V_{f-}^*(w_L)]}.
\end{align*}
\]
The proof is by contradiction. Suppose that weakly more men go to college than women
in equilibrium: \(1 - F(\theta_m^*) \geq 1 - F(\theta_{f1}^*)\). Hence, \(\theta_m^* \leq \theta_{f1}^*\). First, \(\theta_m^* \leq \theta_{f1}^*\) implies

\[
E V^*_m(w_H) - E V^*_m(w_L) \geq E V^*_{f1}(w_H) - E V^*_{f1}(w_L)
\]  \hspace{1cm} (16)

Second, since \(\theta_{f2}^* > \theta_{f1}^*, \theta_m^* \leq \theta_{f1}^* < \theta_{f2}^*\). Hence, more men draw from high wage distribution than more women do in equilibrium, i.e.

\[
\int_{\theta_m^*}^{1} \theta dF(\theta) + \int_{\theta_m^*}^{1} (1-\theta) dF(\theta) > \int_{\theta_{f1}^*}^{1} \theta dF(\theta) + \int_{\theta_{f2}^*}^{1} (1-\theta) dF(\theta).
\]

Therefore, \(G_m^*\) strictly first order stochastically dominates \(G_f^*\). By Corollary 1,

\[
E V^*_m(w_H) - E V^*_m(w_L) < E V^*_{f1}(w_H) - E V^*_{f1}(w_L),
\]

contradicting (16). QED

A.12 No College Gender Gap Under Wage Submodularity

Suppose that the high wage distribution is degenerate \(w_H\), and that the low wage distribution is degenerate \(w_L\). Suppose \(S\) is submodular in wages, \(S(w_H, w_H, r) + S(w_L, w_L, r) \leq S(w_H, w_L, r) + S(w_L, w_H, r)\), and all the assumptions in Proposition 3 hold. There cannot be strictly more women making college investments than men. There is a positive mass of matches between high-wage men and low-wage, fit women. Therefore, they divide up their marriage surplus, \(V_m(w_H) + V_{f1}(w_L) = S(w_H, w_L, r_+)\). There is a positive mass of matches between low-wage men and high-wage, fit women. They divide up their marriage surplus, \(V_m^*(w_L) + V_{f1}^*(w_H) = S(w_L, w_H, r_+)\). Subtracting the two equations,

\[
[V_m^*(w_H) - V_m^*(w_L)] - [V_{f1}^*(w_H) - V_{f1}^*(w_L)] = S(w_H, w_L, r_+) - S(w_L, w_H, r_+) = 0.
\]

Therefore, \(\theta_m^* = \theta_{f1}^*\).

A.13 Proof of Proposition 4

The average husband’s income of the women who marry at age 1 is

\[
E w_m(x_{f1}^*) = E[w_m^*(w_{fL}, r_+)].
\]
Those who marry at age 2 have an average husband’s income of
\[
\mathbb{E} w^*_m(x^*_f2) = p^*_{f2H}\mathbb{E} w^*_m(w_{fH}, r_+) + (1 - p^*_{f2H})\mathbb{E} w^*_m(w_{fL}, r_+).
\]

By Lemma 4, \( w^*_m(w^*_f, r_+) > w^*_m(w_f, r_+) \) for almost all \( w^*_f > w_f \), so \( \mathbb{E}[w^*_m(w_{fH}, r_+)] > \mathbb{E}[w^*_m(w_{fL}, r_+)] \). Therefore, \( \mathbb{E} w^*_m(x^*_f2) > \mathbb{E} w^*_m(x^*_f1) \). Those who marry at age 3 have an average spousal income of
\[
\mathbb{E} w_m(x^*_f3) = p^*_{f3H}[\phi\mathbb{E} w^*_m(w_{fH}, r_+) + (1 - \phi)\mathbb{E} w^*_m(w_{fH}, r_-)]
+ (1 - p^*_{f3H})[\phi\mathbb{E} w^*_m(w_{fL}, r_+) + (1 - \phi)\mathbb{E} w^*_m(w_{fL}, r_-)].
\]

Regardless of the stable matching, by Lemma 4, \( \mathbb{E} w^*_m(w_{fL}, r_-) < \mathbb{E} w^*_m(w_{fH}, r_-) \) and \( \mathbb{E} w^*_m(w_{fL}, r_+ < \mathbb{E} w^*_m(w_{fH}, r_+) \).

When (10) holds, \( w^*_m(w^*_f, r_+) > w^*_m(w_f, r_-) \) for almost all \( w^*_f, w_f \). Hence,
\[
\mathbb{E} w^*_m(w_{fL}, r_+) < \mathbb{E} w^*_m(w_{fH}, r_) < \mathbb{E} w^*_m(w_{fL}, r_+) < \mathbb{E} w^*_m(w_{fH}, r_+).
\]

When \( \phi = 0 \),
\[
\mathbb{E} w^*_m(x^*_f3) = p^*_{f3H}\mathbb{E}[w^*_m(w_{fH}, r_+)] + (1 - p^*_{f3H})\mathbb{E}[w^*_m(w_{fL}, r_-)].
\]

\( \mathbb{E} w^*_m(x^*_f3) < \mathbb{E} w^*_m(w_{fL}, r_+) \). When \( \phi \) is sufficiently small, \( \mathbb{E} w^*_m(x^*_f3) < \mathbb{E} w^*_m(x^*_f1) < \mathbb{E} w^*_m(x^*_f2) \).

The difference between the average spousal incomes of the women who marry at age 1 and the women who marry at age 3 is
\[
\mathbb{E} w^*_m(x^*_f3) - \mathbb{E} w^*_m(x^*_f1)
= p^*_{f3H}[\phi\mathbb{E} w^*_m(w_{fH}, r_+) + (1 - \phi)\mathbb{E} w^*_m(w_{fH}, r_-)]
+ (1 - p^*_{f3H})[\phi\mathbb{E} w^*_m(w_{fL}, r_+) + (1 - \phi)\mathbb{E} w^*_m(w_{fL}, r_-)] - \mathbb{E} w^*_m(w_{fH}, r_+)
= p^*_{f3H}\phi[\mathbb{E}[w^*_m(w_{fH}, r_+) - w^*_m(w_{fL}, r_+)] - p^*_{f3H}(1 - \phi)\mathbb{E}[w^*_m(w_{fL}, r_+) - w^*_m(w_{fL}, r_-)]
+ (1 - p^*_{f3H})(1 - \phi)\mathbb{E}[w^*_m(w_{fH}, r_-) - \mathbb{E} w^*_m(w_{fL}, r_+)].
\]

When (10) does not hold, it is possible that
\[
\mathbb{E} w^*_m(w_{fL}, r_-) < \mathbb{E} w^*_m(w_{fL}, r_+) < \mathbb{E} w^*_m(w_{fH}, r_-) < \mathbb{E} w^*_m(w_{fH}, r_+).
\]

When \( \phi \) is sufficiently large, \( \mathbb{E} w^*_m(x^*_f3) > \mathbb{E} w^*_m(x^*_f1). \) QED
A.14 Proof of Proposition 5

A type $\theta p_{mH} + (1 - \theta) p_{mL}$ man would marry a type $q_f$ woman in the stable matching, and yields marriage payoff

$$\tilde{V}(\theta p_{mH} + (1 - \theta) p_{mL}) = \tilde{S}(\theta p_{mH} + (1 - \theta) p_{mL}, q_f) - \tilde{V}(q_f).$$

By the stability condition, the marriage payoff of each type $w_m$ man weakly exceeds what he would get if he marries $q_f$, $\tilde{V}(w_m) \geq \tilde{S}(w_m, q_f) - \tilde{V}(q_f)$. Because the distribution of female types is not degenerate, for almost all $w_m$,

$$\tilde{V}(w_m) > \tilde{S}(w_m, q_f) - \tilde{V}(q_f).$$

A man who enters the marriage market after income uncertainty is resolved gets

$$\int \tilde{V}(w_m) d(\theta p_{mH} + (1 - \theta) p_{mL}) > \int [\tilde{S}(w_m, q_f) - \tilde{V}(q_f)] d(\theta p_{mH} + (1 - \theta) p_{mL})$$

$$= \int \tilde{S}(w_m, q_f) d(\theta p_{mH} + (1 - \theta) p_{mL}) - \tilde{V}(q_f)$$

$$= \int \tilde{S}(\theta p_{mH} + (1 - \theta) p_{mL}, q_f) dq_f$$

where the last equality follows from the definition of $\tilde{S}$. QED
Figure 14: Average personal income by marriage age, working American women in 1980 and 2012.

B Data Descriptions

The Americans born in 1936-1940 are the 40-44 year-olds in the 1% sample of the 1980 US Census via IPUMS-USA (Ruggles et al., 2010). Age at marriage (AGEMARR) and total personal income (INCTOT) are directly asked on the form. INCTOT is bottom-coded and top-coded. Spouse’s income is calculated by FTOTINC - INCTOT. To be comparable with 2010, I dropped all the people who have married more than once, which constituted 18% of the sample (including them does not change the results).

The Americans born in 1968-1972 are the 40-44 year-olds in the 1% sample of the 2012 American Community Survey via IPUMS-USA. Age at marriage is calculated for those who married once in the current marital status (YRMARR). Those who married twice or thrice (no one in the sample has married more than thrice) constitute about 16% of the sample and are dropped because age at first marriage cannot be computed for this subsample. Spouse’s income is calculated by FTOTINC - INCTOT.

The relationships between marriage age and personal income among the women who earn a positive income are illustrated in Figure 14. They do not differ qualitatively from the relationships of the entire sample.

The relationships between marriage age and personal income are similar in other countries and can be explained by the model. Figures 15 and 16 illustrate the relationships among the Canadians born in 1937-1941, and the Brazilians born in 1947-1951. The relationship was inverse-U shaped for men, and positive for women.

The Canadian dataset is from 1981 Census of Canada via IPUMS-International. Age at first marriage was directly asked and recorded as AGEMARR. Personal income is INCTOT. Questions related to family income or spouse’s income are not asked.
Figure 15: Average personal income by marriage age, Canadians in 1981.

Figure 16: Average personal income by marriage age, Brazilians in 1991.

The Brazilian dataset is from 1991 Censo Demográfico via IPUMS-International. Age at first marriage was directly asked and recorded as AGEMARR. Personal income is INC-TOT. Questions related to family income or spouse’s income are not asked.
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</table>

**Table 1**: Parameters in the numerical illustrations.

### C  Numerical Illustrations

Table 1 presents the parameters used in the numerical illustrations of Figures 8, 10 and 13. The only difference between the old and new regimes is $\phi$. 