Monetary Services Aggregation Theory under Choquet Expectation

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Abstract

This paper considers monetary aggregation under uncertainty aversion (perhaps under risk aversion as well). The presence of uncertainty and the agent's attitude towards it are represented by a nonadditive probability measure. The major findings are three-fold: first, the user cost of monetary assets under uncertainty aversion produces useful boundaries. We no longer have covariances, instead, we have inequalities, and our model nests some of the previously derived results. Second, deviating from expected utility does not exclude the existence of a user-cost solution which is analogous to the expected utility representation, but that is only a special case. Third, under Choquet expectation the user costs have an interval within which no trade of monetary assets will occur, such an effect depends solely on uncertainty aversion, not on risk aversion.

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1 Introduction

In this paper we consider monetary aggregation theory under non-expected utility and derive the model implications for monetary asset user costs and optimal portfolio selection, when the agents are uncertainty averse (perhaps risk averse as well). Specifically, by non-expected utility, we mean for one thing that the utility deviates from linear probabilistic additivity, which is pervasively accepted as a representation of rational expectation models. For another, we allow the utility function, in a dynamical context, to exhibit a recursive structure: current period utility depends on expected future utility as well as on current consumption. Under the assumption that this recursive dependence is time-separable, the non-expected utility we use is expected utility under a nonadditive probability measure. The resulting model aims to separate the more subtle "uncertainties" from quantifiable "risk".

The objective of this paper is to unravel the implications of monetary aggregation when consumers' behavior deviates from expected utility. The literature on monetary services aggregation derived from aggregation theory began with Barnett (1978, 1980). Because simple sum monetary aggregates like M1, M2 are inconsistent with economic theory, the idea of separating investment motive from services motive of monetary assets if we want to measure money correctly (Divisia index) has been of central importance to economists and central banks. Development of these disciplines has been burgeoning fruitful results, including among many others, Barnett (1995) who considers monetary aggregation under risk; Barnett, Liu, and Jensen (1997) who connect Divisia to CAPM models, and this connection is further explored by Barnett and Wu (2005) who generate even larger CAPM user cost risk adjustment. Keating and Lee Smith (2018) dexterously test the usefulness of monetary aggregates in Taylor rules in the framework of a rational expectations model. Kelly, Barnett and Keating (2011) show how measurement error in money is associated with liquidity puzzle.

What is unknown in this realm, however, is what will become of the monetary aggregation theory, if people's behavior deviates from expected utility theory. Money assets are durable goods and thus have user cost prices. Deriving those user costs is a fundamental step in producing monetary aggregates from economic aggregation theory. As a result, our research begins by determining the implications of non-expected utility for the user costs of monetary assets.

The expected utility of von Neumann and Morgenstern, which is further rationalized by Savage (1954) using a prior subjective probability that sums up to one, has been the building block of the largest amount of economic models. Yet Allais (1953) and Ellsberg (1961) paradox find that human being's behavior frequently falls outside the prediction of expected utility. One group of models, seeking to generalize expected utility theory, distinguishes between risk and uncertainty, as defined by Knight (1921) and further developed by Bewley (2002). In that literature, risk exists when economic agents know the objective probabilities, which do sum to one. Under uncertainty, the objective probabilities are not known to agents, and the resulting behavior of economic agents need not to be representable by a subjective probability distribution, having the same measure theoretic properties of the unknown probability distribution. Although the subjective joint probability of the union of all possible outcomes is necessarily one, the sum of the probabilities of each those independent, separate outcomes is not necessarily one.[1]

We follow that approach. In particular, the model we use is built on a nonadditive probability measure. This approach has its foundations in Schmeidler's findings: if probability reflects people's willingness to bet, this probability needs not be additive. An axiomatic treatment of nonadditive probability models can be found in Schmeidler (1986, 1989), Gilboa (1987, 2009), and Gilboa and Schmeidler (1989).

We find that nonadditive probability measure yields boundaries to the user cost of monetary assets, depending on whether the marginal utility and rate of return are comonotonic or countermonotonic. This does not mean, however, that user costs under nonadditive expectation are only subject to inequality solutions. If there exists an underlying probability measure to properly define the nonadditive probability from the subjective additive probability, we find that the user cost has a rank-dependent expected utility representation. This solution has an expected utility form, but uses transformed distorted additive probability as weights. The rank dependence is much less restrictive than might appear to be the case, since there is always a permutation to line up the objective function in an ascending/descending order.

We also find that under optimality there is a user cost interval within which the agent will

[1] For a formal definition of nonadditive probability, see the first paragraph of section 2.1 below.

not hold any position in the monetary asset. When the user cost is below the lower limit of this interval, she will want to buy more of the monetary asset. When the user cost is above the upper limit of this interval, she will want to sell the monetary asset (short). The two limits of this interval constitute the reserve prices for transactions, if the agent's belief reflects uncertainty aversion. This result does not hinge on her attitude towards risk. Our model thus is capable of explaining why there are situations under which people are not active in changing their monetary asset portfolios. A reasonable individual may not behave consistently with Savage's model. Maximizing utility under nonadditive prior can provide a useful rationale for observed behavior in the market. When probability becomes additive, the model reduces to von Neumann-Morgenstern expected utility case. The existing publications on monetary aggregation under risk become special cases of our analysis and hence are fomally nested within our theory.

The rest of the paper is organized as follows. In section 2 we introduce the model and the associated nonadditive probability measure, solve for the user cost under uncertainty aversion, and derive the user cost boundaries. In section 3 we find the conditions under which the user cost has a rank-dependent expected utility representation. In section 4 we consider the consumer's problem from an asset pricing perspective and demonstrate our main theorem providing the user cost interval within which no trade will happen. In section 5 we conclude the paper. The appendix contains the mathematical proofs of theorems and useful lemmas.

2 The Model

2.1 Utility Function and Uncertainty Averse

When we say the probability is nonadditive, we mean that if A and B are two disjoint events in the sample space Ω, such that $A \cup B = \Omega$, their probabilities being $v(A)$ and $v(B)$ respectively, $v(A) + v(B) \neq 1$, although $v(A \cup B) = 1$. As explained below, uncertainty aversion will imply $v(A) + v(B) < 1$. Under a nonadditive probability measure, the proper way to define an integral is no longer Riemann but Chquet. Under these conditions, Riemann integration suffers from discontinuity, nonmonotonicity, and ambiguity (dependence upon the way we write utility functions). Suppose there is a function $f \geq 0$, then the Choquet (1954) integral integrates over rectangles horizontally:

$$\int f dv = \int_0^{+\infty} v\left(s\,|f\left(s\right) \geq t\,\right) dt,$$

where the right hand side is a standard Riemann integral. Choquet integral has many attractive properties, such as reflecting linear translations multiplied by a positive coefficient. But generally it is not additive, unless the functions under evaluation are comonotonic, a property that will be relevant to some of our subsequent results.

Uncer uncertainty, the utility function under our consideration is in the form of nonexpected utility[2] as follows:

$$V_t = U\left(c_t, \mathbf{m}_t, E_t^C V_{t+1}\right) = u\left(c_t, \mathbf{m}_t\right) + \beta \int V_{t+1} dv, \tag{1}$$

where c_t is the date t consumption of goods, $\mathbf{m}_t$ is the vector of monetary assets, and $E_t^C V_{t+1}$ is the expected future utility, conditional on all information at time t. We use a superscript C on the expectation operator to denote Choquet expectation. In this uncertainty context, $U\left(\cdot\right)$ is the aggregator function through which current consumption, all monetary assets, and expected future utility are aggregated. We follow canonical Macroeconomic models to allow time separability, where β is the discount factor and V_{t+1} is tomorrow's utility in each of tomorrow's states. Without the separability assumption, the discount factor would be the derivative of $U\left(\cdot\right)$ with respect to its third argument.

We further assume there exists a linearly homogenous aggregator function $M_t = M\left(\mathbf{m}_t\right)$, such that:

$$u\left(c_t, \mathbf{m}_t\right) = F\left[c_t, M\left(\mathbf{m}_t\right)\right]. \tag{2}$$

In this paper, additive probability is denoted by P, while capacity (nonadditive probability or "charge") is denoted by v, so that $\int\left(\cdot\right) dv$ is Choquet integral.

More formally, suppose that S is a finite set of states of nature, and in every period there are a finite number of n different states. Let $\mathcal{F}$ be the σ-algebra generated by the events on

[2] Distinguishing attitudes towards risk from behavior towards intertemporal substitution is beyond the scope of this paper. Once we include monetary assets in the utility function, the effects of Epstein and Zin (1989) or Weil's (1990) generalized isoelastic utility are much harder to find. But it could be a topic worth pursuing.

S. Then capacity v on a measurable space $(S,\mathcal{F})$ is a real-valued set function $v:\mathcal{F}\rightarrow[0,1]$ such that $v(\phi)=0$, $v(S)=1$, and $v(A)\leq v(B)$ for all $A\subseteq B\in\mathcal{F}$. An example of capacity could be $v=P^{\alpha}$. In this case $\alpha\in(0,1)\cup\{1\}\cup(1,+\infty)$ measures the agent's attitude towards uncertainty. If $\alpha=1$, then capacity reduces to an additive prior, and the probability measure is both concave and convex. We emphasize that the fact that probability is nonadditive itself represents both the presence of uncertainty and the agent's attitude towards it.

Using the example from the beginning of this section, $v(A)+v(B)<1=v(A\cup B)$ is equivalent to concluding that the agent's decisions reflect uncertainty aversion.[3] Schmeidler (1986, 1989) defines uncertainty aversion in terms of probability capacity by:

$$v(A)+v(B)\leq v(A\cup B)+v(A\cap B), \tag{3}$$

although that definition is not universally accepted. That condition is also known as supermodularity, convexity, or 2-monotonicity of v.

The states of nature are a natural partition of the sample space S. If today's nature is denoted by s, we denote the nature tomorrow by s'. With a somewhat informal notation for V, it can be useful to rewrite the utility function (1) in terms of states for any given sequence $\mathbf{X}=\{c_t,\mathbf{m}_t\}_{t=0}^{T}$, as

$$V(\mathbf{X})=\lim_{T\rightarrow\infty}\left\{u(c_0,\mathbf{m}_0)+\beta\int\left[u(c_1,\mathbf{m}_1)+\cdots+\beta\int u(c_T,\mathbf{m}_T)\,v_{s_{T-1}}(ds_T)\cdots\right]v_s(ds_1)\right\}. \tag{4}$$

This facilitates the calculation of Choquet integral using Riemann integrals.

2.2 Equilibrium

The agent holds two types of assets, monetary assets and nonmonetary assets. Nonmonetary assets provide only investment return, while monetary assets provide both investment return

[3] We avoid use of the word "ambiguity", which is usually defined to mean that the agent vaguely perceives the probability of a particular state in a range. This possibility is out of the scope of this paper.

and monetary service flows, which we seek to measure. The budget constraints are:

$$W_t = p_t c_t + \sum_{i=1}^{L} p_t m_{it} + \sum_{j=1}^{K} p_t k_{jt} \tag{5}$$

$$W_{t+1} = \sum_{i=1}^{L} R_{i,t+1} p_t m_{it} + \sum_{j=1}^{K} \tilde{R}_{j,t+1} p_t k_{jt} + y_{t+1} \tag{6}$$

where W_t is the agent's wealth in period t, p_t is the true cost of living index, c_t is consumption of goods, and y_{t+1} is income from all other sources, received at the beginning of $t+1$. The variables, m_{it} and k_{jt}, denote the quantities of monetary asset i and nonmonetary asset j respectively. The interest rate $R_{i,t+1}$ is the gross rate of return of holding the monetary asset m_{it} between periods t and $t+1$, while the interest rate $\tilde{R}_{j,t+1}$ is the gross return of nonmonetary asset k_{jt} from t to $t+1$. Suppose L and K are the number of two types of assets in the agent's portfolio, since nonmonetary assets do not provide service flows, other than their investment rates of return, it follows that $\tilde{R}$ is higher than R. Combining equation (5) and (6) yields the following flow of funds equation:

$$p_t c_t = \sum_{i=1}^{L} \left[R_{it} p_{t-1} m_{i,t-1} - p_t m_{it}\right] + \sum_{j=1}^{K} \left[\tilde{R}_{jt} p_{t-1} k_{j,t-1} - p_t k_{jt}\right] + y_t. \tag{7}$$

Hence, the individual's consumption of goods is funded each period from the proceeds from rolling over the monetary assets and nonmonetary assets, and from all other income. Note that equation (7) is the one used in Barnett (1980) and Barnett, Liu and Jensen (1997) to facilitate comparison of our results with the existing literature.

The agent maximizes the lifetime discounted utility (4), subjects to the flow of funds constraint (7). The resulting Bellman equation is:

$$\begin{aligned} V_s(W_t) &= \sup_{\{c_t, \mathbf{m}_t, \mathbf{k}_t\}} \left\{ u(c_t, \mathbf{m}_t) + \beta \int V_{s'}(W_{t+1})\, v_s(ds') \right\} \qquad (8) \\ s.t.\;\; p_t c_t &= \sum_{i=1}^{L} \left[R_{it} p_{t-1} m_{i,t-1} - p_t m_{it}\right] + \sum_{j=1}^{K} \left[\tilde{R}_{jt} p_{t-1} k_{j,t-1} - p_t k_{jt}\right] + y_t \end{aligned}$$

Here $V_s(W_t)$ denotes Bellman value function. The agent is also subject to the following transversality condition:

$$\lim_{t\to\infty} \beta^t \frac{\partial V^*}{\partial W_t^*} W_t^* = 0, \tag{9}$$

with $*$ denoting the solution value from the optimization.

After substituting from the Benveniste-Scheinkman equation, the first order conditions (Euler equations) with respect to consumption become:

$$\frac{\partial u}{\partial c_t} = \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}} \right], \tag{10}$$

while the first order conditions with respect to monetary assets become:

$$\frac{\partial u}{\partial m_{it}} = \frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} R_{i,t+1} \frac{p_t}{p_{t+1}} \right]. \tag{11}$$

The contemporaneous real user-cost price of the services of monetary asset i is the marginal rate of substitution between monetary asset and consumption,

$$\pi_{it} = \frac{\frac{\partial u}{\partial m_{it}}}{\frac{\partial u}{\partial c_t}} = \frac{\frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} R_{i,t+1} \frac{p_t}{p_{t+1}} \right]}{\beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}} \right]} = \frac{\frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1} \right]}{\beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{r}_{j,t+1} \right]}. \tag{12}$$

For notational convenience, we convert the nominal gross returns, $R_{i,t+1}$ and $\tilde{R}_{j,t+1}$, to the corresponding real gross rates of return, $r_{i,t+1} = R_{i,t+1} \frac{p_t}{p_{t+1}}$ and $\tilde{r}_{j,t+1} = \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}}$. Since the expectation $E_t^C(\cdot)$ is not additive, it is the Choquet integral.

Also note that under the weak separability condition (2), we have:

$$\frac{\partial u}{\partial m_{it}} = \frac{\partial F}{\partial M_t} \frac{\partial M_t}{\partial m_{it}}.$$

Substituting the definition of the user cost, we acquire:

$$\frac{\partial M_t}{\partial m_{it}} = \pi_{it} \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}}.$$

Taking the total differential of the monetary aggregator function, $M_t = M(\mathbf{m}_t)$, yields:

$$dM_t = \sum_{i=1}^{L} \frac{\partial M_t}{\partial m_{it}} dm_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} dm_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} m_{it} d \log m_{it}. \tag{13}$$

Since $M(\mathbf{m}_t)$ is linearly homogenous of degree one, Euler theorem applies to M_t,

$$M_t = \sum_{i=1}^{L} \frac{\partial M_t}{\partial m_{it}} m_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} m_{it}. \tag{14}$$

Dividing equation (13) by (14) yields the Divisia index

$$d \log M_t = \sum_{i=1}^{L} s_{it} d \log m_{it}, \tag{15}$$

where $s_{it} = \frac{\pi_{it} m_{it}}{\sum_{l=1}^{L} \pi_{lt} m_{lt}}$ is the user cost valued expenditure share. We conclude that the resulting Divisia quantity index is in exactly the same form as in Barnett (1980), with the only difference being that the user costs now are computed under a nonadditive probability measure.

2.3 User-Cost Boundaries

We now return to the user-cost, π_{it}, in equation (12). Because the expectation is non-additive, we no longer have $E_t\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] = Cov\left(\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1}\right) + E_t\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t\left(r_{i,t+1}\right)$. Instead, we have the following theorem:

Theorem 1 *If* $\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1} \geq 0$ *are comonotonic, then*

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] \geq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right). \tag{16}$$

If v *is submodular, while* $\frac{\partial u}{\partial c_{t+1}}$ *and* $r_{i,t+1}$ *are countermonotonic, then:*

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] \leq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right). \tag{17}$$

The proof of this theorem is in the appendix. Comonotonicity is defined as follows. For every pair of states $s_1', s_2' \in S$,

$$\left[\frac{\partial u}{\partial c_{t+1}}\left(s_1'\right) - \frac{\partial u}{\partial c_{t+1}}\left(s_2'\right)\right]\left[r_{i,t+1}\left(s_1'\right) - r_{i,t+1}\left(s_2'\right)\right] \geq 0. \tag{18}$$

That is, the marginal utility and the rate of return increase or decrease at the same time. Countermonotonicity just reverses the direction of the above inequality. Hence, under nonadditive probabilities, we do not have covariances, but we have inequalities. Equation (17) corresponds to uncertainty loving which is unusual, but it is not as unlikely as it seems to be. If we think about gain-loss asymmetry, when people particularly hate to lose what they have already had, such an extreme loss aversion might lead people to behave in an uncertainty loving way in the domain of losses.

Barnett, Liu, and Jensen (1997) proved, in their Theorem 1, that the user cost of services of monetary assets under risk aversion has an additional adjustment term not appearing in the risk free user cost. That adjustment term is about covariances, as in all CCAPM risk adjustments. The Barnett, Liu, and Jensen's risk adjusted user cost is a special case of our result. If the probability measure is additive, so that uncertainty is removed, risk aversion is all that is left. Then the Choquet expectation in equation (12) becomes the linearly additive expectation, and covariances appear.

When the agent is not only risk averse but also uncertainty averse, then equation (12) cannot be further simplified by collecting covariances. We end up with inequalities giving rise to boundaries on user costs. In the next section, we will see that equality solutions do exist for $E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right]$, but those again hold as special cases of Choquet expectation. Our case nests Barnett, Liu and Jensen's (1997) result. If we further assume away both uncertainty aversion and risk aversion, we will have the perfect certainty case. Then equation (12) reduces to the user cost derived in Barnett (1980).

It is convenient to work on rates of returns, $r_{i,t+1}$, which are usually assumed to be stationary, so that taking averages is meaningful. But marginal utility, $\frac{\partial u}{\partial c_{t+1}}$, is not observed and difficult to estimate. Therefore we reinterpret equation (12) in terms of a stochastic discount factor, which, although still not observable, is much easier to estimate. We assume the agent has not passed the blissful point, so that $\frac{\partial u}{\partial c_t} > 0$. Given date t information, uncertainty at time t has been resolved, and $\frac{\partial u}{\partial c_t}$ can be treated as a constant. By the positive homogeneity of Choquet integral, equation (12) can be written as

$$\pi_{it} = \frac{1 - E_t^C \left[\beta \frac{\partial u / \partial c_{t+1}}{\partial u / \partial c_t} r_{i,t+1} \right]}{E_t^C \left[\beta \frac{\partial u / \partial c_{t+1}}{\partial u / \partial c_t} \tilde{r}_{j,t+1} \right]} = \frac{1 - E_t^C \left[Q_{t+1} r_{i,t+1} \right]}{E_t^C \left[Q_{t+1} \tilde{r}_{j,t+1} \right]}, \tag{19}$$

where we denote by $Q_{t+1} = \beta \frac{\partial u / \partial c_{t+1}}{\partial u / \partial c_t}$ the pricing kernel. Note that from equation (10) and (11) we have respectively

$$1 = E_t^C \left[Q_{t+1} \tilde{r}_{j,t+1} \right], \tag{20}$$

$$\pi_{it} = 1 - E_t^C \left[Q_{t+1} r_{i,t+1} \right]. \tag{21}$$

Based on equation (19), a reinterpretation of Theorem 1 is that, if $Q_{t+1}, r_{i,t+1} \geq 0$ and $Q_{t+1}, \tilde{r}_{j,t+1} \geq 0$ are both comonotonic, then:

$$\pi_{it} \leq \frac{1 - E_t^C \left(Q_{t+1} \right) E_t^C \left(r_{i,t+1} \right)}{E_t^C \left(Q_{t+1} \right) E_t^C \left(\tilde{r}_{j,t+1} \right)}. \tag{22}$$

If v is submodular, Q_{t+1} and the rate of return on both the monetary and non-monetary assets are countermonotonic, we have:

$$\pi_{it} \geq \frac{1 - E_t^C \left(Q_{t+1} \right) E_t^C \left(r_{i,t+1} \right)}{E_t^C \left(Q_{t+1} \right) E_t^C \left(\tilde{r}_{j,t+1} \right)}. \tag{23}$$

Since returns tend to move together, the dual satisfaction of comonotonic (or countermonotonic) with Q_{t+1} is not restrictive.

Therefore when the probability measure is nonadditive, Choquet expectation produces boundaries to the user cost of monetary assets. More specifically, assume the expected consumption is before the blissful point, the real rates of return on both types of assets are positive, and the substitution effect on intertemporal consumption dominates (so that the comonotonicity between Q_{t+1} and $r_{i,t+1}$ ($\tilde{r}_{j,t+1}$) is satisfied). Then the calculated user cost should be lower than $\frac{1 - E_t^C (Q_{t+1}) E_t^C (r_{i,t+1})}{E_t^C (Q_{t+1}) E_t^C (\tilde{r}_{j,t+1})}$. On the other hand, if the agent were uncertainty loving, meanwhile the income effect wins out in the intertemporal allocation of consumption (so that the countermonotonicity between Q_{t+1} and $r_{i,t+1}$ $(\tilde{r}_{j,t+1})$ is satisfied). Then any calculated user cost would be incorrect, if it were lower than $\frac{1-E_t^C(Q_{t+1})E_t^C(r_{i,t+1})}{E_t^C(Q_{t+1})E_t^C(\tilde{r}_{j,t+1})}$.

3 Rank-Dependent Representation

The existence of derived boundaries is not the only result under nonadditive probabilities. In this section we show that under some circumstances, there exists a linear solution for equation (21). Suppose $\mathbf{P} = (P_1, P_2, \cdots, P_n)^T$ is an additive probability vector satisfying $\sum_{s=1}^{n} P_s = 1$, and suppose there is a probability measure μ such that for some nondecreasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, the capacity $v = f(\mu)$ is well-defined. Then a new, additive, probability vector $\mathbf{P}^{\uparrow}$ is permissible to order events as follows:

$$\begin{aligned}\mathbf{P}^{\uparrow} &= \left(P_1^{\uparrow}, P_2^{\uparrow}, \cdots, P_{n-1}^{\uparrow}, P_n^{\uparrow}\right)^T \qquad (24)\\ &= \left[1 - f\left(\sum_{s\geq 2} P_s\right), f\left(\sum_{s\geq 2} P_s\right) - f\left(\sum_{s\geq 3} P_s\right), \cdots, f\left(\sum_{s\geq n-1} P_s\right) - f(P_n), f(P_n)\right]^T.\end{aligned}$$

If the agent is uncertainty loving, $f(\cdot)$ should be convex, in this case higher states are weighted less. Such a transformed probability is tailored for accumulative lottery outcomes, where $x_1 \leq \cdots \leq x_n$ are in a lottery $(x_1, P_1; \cdots; x_n, P_n)$. This observation is a reason we choose the notation $\uparrow$ on the left side of equation (24). Take the distorted probability as an example, in which $f(\mu) = \mu^{\alpha}$, where $\alpha \in (0, +\infty)$, and μ is the probability measure relative to which the additive probabilities P_s are given. Then $P_t^{\uparrow} = \left(\sum_{s=t}^{n} P_s\right)^{\alpha} - \left(\sum_{s=t+1}^{n} P_s\right)^{\alpha}$, so that the above probability vector $\mathbf{P}^{\uparrow}$ becomes

$$\mathbf{P}^{\uparrow} = \left[1 - \left(\sum_{s\geq 2} P_s\right)^{\alpha}, \left(\sum_{s\geq 2} P_s\right)^{\alpha} - \left(\sum_{s\geq 3} P_s\right)^{\alpha}, \cdots, \left(\sum_{s\geq n-1} P_s\right)^{\alpha} - (P_n)^{\alpha}, (P_n)^{\alpha}\right]^T.$$

The higher states are weighted less in this example when, $\alpha > 1$.

Similarly, we define another probability vector, $\mathbf{P}^{\downarrow}$, for decumulative outcomes $x_1 \geq \cdots \geq x_n$

as follows:

$$\begin{aligned}\mathbf{P}^{\downarrow} &= \left(P_1^{\downarrow}, P_2^{\downarrow}, \cdots, P_{n-1}^{\downarrow}, P_n^{\downarrow}\right)^T \qquad (25)\\ &= \left[f\left(P_1\right), f\left(\sum_{s\leq 2} P_s\right) - f\left(P_1\right), \cdots, f\left(\sum_{s\leq n-1} P_s\right) - f\left(\sum_{s\leq n-2} P_s\right), 1 - f\left(\sum_{s\leq n-1} P_s\right)\right]^T\end{aligned}$$

If the agent is uncertainty loving, higher states are weighted more. This approach is also the method proposed by Yaari (1987) to deal with the violation of continuity and monotonicity in Kahneman and Tversky's (1979) prospect theory. We therefore have the following lemma showing that Choquet expectation has an expected utility solution, but with a transformed probability measure on ordered utilities.

Lemma 2 *Suppose P is an additive probability measure, for any capacity $v = f(\mu)$ that is well supported by the probability measure μ, and for any nonnegative function $u \in \mathbb{R}_+^n$, the Choquet integral has a rank-dependent expected utility representation:*

$$\int u_s v(ds) = \mathbf{u}^T \mathbf{P}^{\uparrow} = \sum_{s=1}^{n} u_s P_s^{\uparrow} \quad \textit{if } u \textit{ is weakly increasing in } s, \qquad (26)$$

$$\int u_s v(ds) = \mathbf{u}^T \mathbf{P}^{\downarrow} = \sum_{s=1}^{n} u_s P_s^{\downarrow} \quad \textit{if } u \textit{ is weakly decreasing in } s, \qquad (27)$$

where $\mathbf{P}^{\uparrow}$ and $\mathbf{P}^{\downarrow}$ are state-reweighted probability vectors defined above.

The proof of the lemma is in the appendix. With this result, if $Q_{t+1}r_{i,t+1}$ is weakly increasing in s', as can always be done by permutation, we have

$$\pi_{it} = 1 - E_t^C\left[Q_{t+1}r_{i,t+1}\right] = 1 - \sum_{s'=1}^{n} Q_{t+1}r_{i,t+1}P_{s'}^{\uparrow}. \qquad (28)$$

If $Q_{t+1}r_{i,t+1}$ is weakly decreasing in s', then

$$\pi_{it} = 1 - E_t^C\left[Q_{t+1}r_{i,t+1}\right] = 1 - \sum_{s'=1}^{n} Q_{t+1}r_{i,t+1}P_{s'}^{\downarrow}. \qquad (29)$$

Therefore, in addition to deriving inequality bounds, we also have an alternative solution.

Choquet expectation relative to v coincides with an expected utility model defined by $f(\cdot)$. This expected utility requires rank dependence, so that the product $Q_{t+1}(s')\, r_{i,t+1}(s')$ must be either weakly increasing or weakly decreasing in s'. It's important to emphasize that the correspondence between Choquet expectation and the rank-dependent representation does not always exist. Rather, the rank-dependent expected utilities are a special case of Choquet expected utility, a case in which the underlying probability measure μ exists and contains sufficient information to define v. The Ellsberg paradox is a violation of this condition and therefore has no rank-dependence representation. In all those cases, there does not exist an underlying measure which provides us all we need to know about events. Potentially, Choquet expectation is more general, in that it allows us to work on scenarios during which our capabilities of defining probabilities are limited.

Note that equation (20) also features a similar rank-dependent solution:

$$\sum_{s'=1}^{n} Q_{t+1}\tilde{r}_{j,t+1}P_{s'}^{\uparrow} = 1 \text{ if } Q_{t+1}\tilde{r}_{j,t+1} \text{ is weakly increasing in } s', \tag{30}$$

$$\sum_{s'=1}^{n} Q_{t+1}\tilde{r}_{j,t+1}P_{s'}^{\downarrow} = 1 \text{ if } Q_{t+1}\tilde{r}_{j,t+1} \text{ is weakly decreasing in } s'. \tag{31}$$

These two equations provide a useful guidance for estimating the stochastic discount factor when uncertainty aversion is involved. We can compare equation (20) with the classical asset pricing theory under additive priors. In that case, returns should follow

$$1 = E_t\left[Q_{t+1}\tilde{r}_{t+1}\right]. \tag{32}$$

That is, one dollar paid today is weighted against how many dollars or units of consumption the agent will get in return tomorrow. Nevertheless, if the decision also involves attitude towards uncertainty, we now see that equation (32) becomes $1 = E_t^C\left[Q_{t+1}\tilde{r}_{t+1}\right]$. With the implication of Lemma 2, it becomes clear that even if people manage to evaluate this true equation, as in (30) and (31), the result would still be a special case of our more general theory.

4 Monetary Asset Choice under Uncertainty Aversion: A No Trade Interval

In this section we reverse the perspective by looking at the monetary asset portfolio choice problem. Given v is a probability measure, the value of expected discounted real rate of return exhibits linearity and translation invariance; that is, $E_t^C\left[\alpha Q_{t+1} r_{i,t+1} + \beta\right] = \alpha E_t^C\left[Q_{t+1} r_{i,t+1}\right] + \beta$ if $\alpha \geq 0$, $\beta \in \mathbb{R}$. But property does not hold when α is negative. Therefore, we consider $-E_t^C\left[-Q_{t+1} r_{i,t+1}\right]$ instead, giving rise to the following lemma:

Lemma 3 *If the agent is uncertainty averse, the Choquet expected value satisfies:*

$$-E_t^C\left[-Q_{t+1} r_{i,t+1}\right] > E_t^C\left[Q_{t+1} r_{i,t+1}\right]. \tag{33}$$

The proof of this lemma is relegated to the appendix. Intuitively, adding a constant to a random variable or multiplying a random variable by a positive number will linearly shift the Choquet expectation. This relationship does not hold for negative multipliers. The nonadditivity of the probability causes an asymmetric effect, it turns out $-E_t^C\left[-Q_{t+1} r_{i,t+1}\right] > E_t^C\left[Q_{t+1} r_{i,t+1}\right]$. It is this interval that leads to non-transaction of the monetary asset i. There will be a range of discounted returns from $E_t^C\left[Q_{t+1} r_{i,t+1}\right]$ to $-E_t^C\left[-Q_{t+1} r_{i,t+1}\right]$, within which the agent neither want to buy nor to sell the monetary asset. If the discounted return $E_t^C\left[Q_{t+1} r_{i,t+1}\right]$ is larger than 1, she will want to buy the monetary asset. If the discounted return $-E_t^C\left[-Q_{t+1} r_{i,t+1}\right]$ is lower than 1, she will want to sell this monetary asset (short).

To prove this result, we assume the utility function, $u \geq 0$, is twice continuously differentiable with $u' > 0$, $u'' < 0$. We use Jensen's inequality to prove this result. But first we need to verify whether Jensen's inequalities hold under a nonadditive probability measure.

Lemma 4 *Let $(S, \mathcal{F}, \mathcal{V})$ be a nonadditive probability space, $s' \in S$, and let $v \in \mathcal{V}$ be capacity. Suppose $x_{t+1}(s')$ is choquet integrable. If u is a concave function on $[0, +\infty)$, then Jensen's inequality follows:*

$$u\left\{E_t^C\left[x_{t+1}(s')\right]\right\} \geq E_t^C\left\{u\left[x_{t+1}(s')\right]\right\}. \tag{34}$$

Proof: We produce a second order Taylor expansion of $u(x_{t+1}(s'))$ around $E_t^C(x_{t+1}(s'))$:

$$u(x) = u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2,$$

taking Choquet expectation on both sides, we have:

$$\begin{aligned}
E^C[u(x)] &= E^C\left\{u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2\right\} \\
&= u\left[E^C(x)\right] + E^C\left\{u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2\right\} \\
&\leq u\left[E^C(x)\right] + E^C\left\{u'\left[E^C(x)\right]\left[x - E^C(x)\right]\right\} \\
&= u\left[E^C(x)\right] + u'\left[E^C(x)\right]E^C\left[x - E^C(x)\right] \\
&= u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[E^C(x) - E^C(x)\|v\|\right] \\
&= u\left[E^C(x)\right],
\end{aligned}$$

where $\xi = \lambda x_{t+1}(s') + (1-\lambda)E_t^C[x_{t+1}(s')]$ with $\lambda \in [0,1]$. The convex case can be proved similarly. *Q.E.D.*

The second line holds because of comonotonic additivity, the constant is comonotonic with any variable. The third line holds because the Choquet integral is monotone and $u''(\xi) < 0$, $\left[x - E^C(x)\right]^2 \geq 0$. The fourth line is true because of positive homogeneity, $u'\left[E^C(x)\right]$ is a constant. The rest holds because of translation invariance.

So Jensen's equalities are still satisfied under nonadditive probabilities, and they are satisfied in multivariate case. We now have the main result, as follows:

Theorem 5 *Consider a risk neutral or risk averse agent with wealth W_t, who is considering investing m_{it} on a monetary asset, yielding a real rate of return $r_{i,t+1}$. Suppose the two conditions in Lemma 4 are satisfied. Denoting the Choquet expected discounted rate of return by $E_t^C[Q_{t+1}r_{i,t+1}]$, she will buy this monetary asset if $1 < E_t^C[Q_{t+1}r_{i,t+1}]$, or equivalently $\pi_{it} < 0$. She will sell the asset (short) if $1 > -E_t^C[-Q_{t+1}r_{i,t+1}]$, or equivalently $\pi_{it} > -E_t^C[-Q_{t+1}r_{i,t+1}] - E_t^C[Q_{t+1}r_{i,t+1}]$.*

We only sketch the proof. Suppose the agent spends m_{it} on this monetary asset. Then by Jensen's inequality

$$E_t^C \{u [W_t - m_{it} + m_{it} \cdot Q_{t+1} r_{i,t+1}]\} \leq u \left\{E_t^C [W_t - m_{it} + m_{it} \cdot Q_{t+1} r_{i,t+1}]\right\} \leq u (W_t) . \quad (35)$$

The last inequality holds, if $E_t^C [Q_{t+1} r_{i,t+1}] \leq 1$. Therefore the individual is at least as well off not buying anything as holding a positive position in monetary asset i. Analogous arguments give rise to selling the asset, if $1 > -E_t^C [-Q_{t+1} r_{i,t+1}]$. In this circumstance,

$$\begin{aligned} & 1 - \left[-E_t^C [-Q_{t+1} r_{i,t+1}]\right] \quad (36) \\ = \quad & 1 - E_t^C [Q_{t+1} r_{i,t+1}] - \left[-E_t^C [-Q_{t+1} r_{i,t+1}]\right] + E_t^C [Q_{t+1} r_{i,t+1}] > 0, \end{aligned}$$

since $\pi_{it} = 1 - E_t^C [Q_{t+1} r_{i,t+1}]$, this condition is equal to $\pi_{it} > -E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]$, and by Lemma 3 this difference is positive. *Q.E.D.*

Hence $\left[0, -E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]\right]$ is a range of user costs with no trade under uncertainty aversion. If the user cost π_{it} is lower than zero, we conclude $1 < E_t^C [Q_{t+1} r_{i,t+1}]$, so the return tomorrow is larger than the one dollar spent on it today, and she will buy it. If π_{it} is larger than $-E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]$, then $1 > -E_t^C [-Q_{t+1} r_{i,t+1}]$ and the uncertainty premium is not enough to compensate for the cost of holding the monetary asset. She will want to sell it. This range of user cost depends only on the beliefs and attitude towards uncertainty, not on the attitude towards risk.

5 Concluding Remarks

In this paper we consider the monetary services aggregation theory under uncertainty, as distinguished by Knight (1921) from risk. The presence of, and the agent's attitude towards uncertainty is represented by the nonadditivity of the probability measure. We acquire three primary conclusions. First, different from CCAPM risk adjusted user costs incorporating covariances and subject to the "risk premium puzzle" critique, we find that the uncertainty adjusted user cost, in its most general form, produces boundaries. The previously derived perfect certainty user cost and the risk adjusted user cost are special cases of ours, if the probability measure becomes additive. Second, we are able to derive an expected utility analogous solution using transformed additive probabilities, the result will be rank-dependent and will require the existence of an underlying probability measure that contains sufficient information to properly define the capacity. This is a special case of Choquet expectation. Third, user costs under uncertainty produce an interval within which no transactions of monetary assets will occur. This effect is brought about solely by uncertainty aversion, not by risk aversion captured by the utility function.

Appendix

Proof of Theorem 1:

Let $\frac{\partial u}{\partial c_{t+1}}$ and $r_{i,t+1}$ be comonotonic in the sense that for each pair of states, $s_1', s_2' \in S$,

$$\left[\frac{\partial u}{\partial c_{t+1}}\left(s_1'\right) - \frac{\partial u}{\partial c_{t+1}}\left(s_2'\right)\right]\left[r_{i,t+1}\left(s_1'\right) - r_{i,t+1}\left(s_2'\right)\right] \geq 0.$$

Suppose $E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] < \infty$ and both $\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1} \geq 0$. For any given s_0', we have by comonotonicity:

$$\left[\frac{\partial u}{\partial c_{t+1}} - \frac{\partial u}{\partial c_{t+1}}\left(s_0'\right)\right]\left[r_{i,t+1} - r_{i,t+1}\left(s_0'\right)\right] \geq 0.$$

That is:

$$\frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}}\left(s_0'\right) \cdot r_{i,t+1}\left(s_0'\right) \geq \frac{\partial u}{\partial c_{t+1}}\left(s_0'\right) \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}\left(s_0'\right).$$

Since Choquet Expectation is monotone, we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}}\left(s_0'\right) \cdot r_{i,t+1}\left(s_0'\right)\right] \geq E_t^C\left[\frac{\partial u}{\partial c_{t+1}}\left(s_0'\right) \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}\left(s_0'\right)\right].$$

Given s_0', $\frac{\partial u}{\partial c_{t+1}}(s_0')$ and $r_{i,t+1}(s_0')$ are constants, by translatability[4], we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}}\cdot r_{i,t+1}\right]+\frac{\partial u}{\partial c_{t+1}}(s_0')\cdot r_{i,t+1}(s_0')\|v\|\geq E_t^C\left[\frac{\partial u}{\partial c_{t+1}}(s_0')\cdot r_{i,t+1}+\frac{\partial u}{\partial c_{t+1}}\cdot r_{i,t+1}(s_0')\right].$$

By positive homogeneity, since $\frac{\partial u}{\partial c_{t+1}}(s_0')$ and $r_{i,t+1}(s_0')\geq 0$, we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\right]+\frac{\partial u}{\partial c_{t+1}}(s_0')\cdot r_{i,t+1}(s_0')\|v\|\geq\frac{\partial u}{\partial c_{t+1}}(s_0')E_t^C(r_{i,t+1})+r_{i,t+1}(s_0')E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right),$$

and this holds for any $s_0'\in S$. That is,

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\right]+\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\|v\|\geq\frac{\partial u}{\partial c_{t+1}}E_t^C(r_{i,t+1})+r_{i,t+1}E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right).$$

When both $E_t^C(r_{i,t+1})$ and $E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right)$ are finite, apply translatability again to acquire

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\right]\|v\|+E_t^C\left[\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\right]\|v\|\geq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right)E_t^C(r_{i,t+1})+E_t^C(r_{i,t+1})E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right).$$

Dividing the norm on both sides, we find

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}}r_{i,t+1}\right]\geq\frac{1}{\|v\|}E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right)E_t^C(r_{i,t+1}).$$

Since v is a probability measure, $\|v\|=1$. This proves part 1 of the theorem.

If v is submodular and $\frac{\partial u}{\partial c_{t+1}}$ and $r_{i,t+1}$ are countermonotonic, the second part of the theorem follows from the same logic. This completes the proof.

Proof of Lemma 2:

Suppose u is weakly increasing in $s\in S$. Given a monotone measure space $(S,\mathcal{F},\mathcal{V})$, we denote $\{u_s\geq t\}=\{s\,|u(s)\geq t\}$ for any $t>0$. The Choquet integral of u over S with respect

[4] Choquet integrals are translatable for any real number β, such that $E_t^C(X+\beta)=E_t^C(X)+\beta\|v\|$, if v is a monotone measure on the measurable space $(S,\mathcal{F})$.

to a real monotone measure v is:

$$\begin{aligned}
\int u_s v\left(ds\right) &= \int_0^{+\infty} v\left(\left\{i\left|u_i \geq t\right.\right\}\right) dt \\
&= \int_0^{u_1} v\left(\left\{i\left|u_i \geq t\right.\right\}\right) dt + \sum_{s=2}^{n} \int_{u_{s-1}}^{u_s} v\left(\left\{i\left|u_i \geq t\right.\right\}\right) dt + \int_{u_n}^{+\infty} v\left(\left\{i\left|u_i \geq t\right.\right\}\right) dt \\
&= \int_0^{u_1} v\left(\left\{1,2,\cdots,n\right\}\right) dt + \sum_{s=2}^{n} \int_{u_{s-1}}^{u_s} v\left(\left\{s,s+1,\cdots,n\right\}\right) dt + \int_{u_n}^{+\infty} v\left(\phi\right) dt \\
&= u_1 f\left(\sum_i P_i\right) + \sum_{s=2}^{n} \left(u_s - u_{s-1}\right) f\left(\sum_{i\geq s} P_i\right) \\
&= \sum_{s=1}^{n} u_s f\left(\sum_{i\geq s} P_i\right) - \sum_{s=2}^{n} u_{s-1} f\left(\sum_{i\geq s} P_i\right) \\
&= u_n f\left(P_n\right) + \sum_{s<n} u_s f\left(\sum_{i\geq s} P_i\right) - \sum_{s<n} u_s f\left(\sum_{i>s} P_i\right) \\
&= \sum_{s=1}^{n} u_s P_s^{\uparrow}.
\end{aligned}$$

The weakly decreasing case of u can be proven likewise. *Q.E.D.*

Proof of Lemma 3: Now we prove the fact that $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right] > E_t^C\left[Q_{t+1}r_{i,t+1}\right].$ Denoting by $A\left(t\right) = \left\{s' \in S\left|Q_{t+1}\left(s'\right) r_{i,t+1}\left(s'\right) \geq t\right.\right\}$, by definition

$$E_t^C\left[Q_{t+1}\left(s'\right) r_{i,t+1}\left(s'\right)\right] = \int_{-\infty}^{0} \left[v\left(A\left(t\right)\right) - 1\right] dt + \int_0^{+\infty} v\left(A\left(t\right)\right) dt.$$

Base on the definition $A\left(t\right)$, consider the event $-Q_{t+1}\left(s'\right) r_{i,t+1}\left(s'\right) > t$:

$$\begin{aligned}
&\left\{s' \in S\left|-Q_{t+1}\left(s'\right) r_{i,t+1}\left(s'\right) > t\right.\right\} \\
=\ &\left\{s' \in S\left|Q_{t+1}\left(s'\right) r_{i,t+1}\left(s'\right) < -t\right.\right\} \\
=\ &\Omega \backslash A\left(-t\right) \\
=\ &A\left(-t\right)^c.
\end{aligned}$$

Here this superscript lower case c means complement of $A\left(-t\right)$, it should not be confused with the upper case C superscript notation for Choquet. We have therefore

$$\begin{aligned}E_t^C\left[-Q_{t+1}\left(s'\right)r_{i,t+1}\left(s'\right)\right] &= \int_{-\infty}^{0}\left[v\left(A\left(-t\right)^c\right)-1\right]dt+\int_{0}^{+\infty}v\left(A\left(-t\right)^c\right)dt \\ &= -\int_{\infty}^{0}\left[v\left(A\left(z\right)^c\right)-1\right]dz-\int_{0}^{-\infty}v\left(A\left(z\right)^c\right)dz \\ &= \int_{-\infty}^{0}v\left(A\left(t\right)^c\right)dt+\int_{0}^{+\infty}\left[v\left(A\left(t\right)^c\right)-1\right]dt.\end{aligned}$$

Furthermore, $E_t^C\left[Q_{t+1}\left(s'\right)r_{i,t+1}\left(s'\right)\right]+E_t^C\left[-Q_{t+1}\left(s'\right)r_{i,t+1}\left(s'\right)\right]$ yields,

$$\begin{aligned}& E_t^C\left[Q_{t+1}\left(s'\right)r_{i,t+1}\left(s'\right)\right]+E_t^C\left[-Q_{t+1}\left(s'\right)r_{i,t+1}\left(s'\right)\right] \\ = & \int_{-\infty}^{+\infty}\left[v\left(A\left(t\right)\right)+v\left(A\left(t\right)^c\right)-1\right]dt,\end{aligned}$$

by the fact that the probability is nonadditive, particularly the agent is uncertainty averse, $v\left(A\right)+v\left(A^c\right)<1$, thus $\int_{-\infty}^{+\infty}\left[v\left(A\left(t\right)\right)+v\left(A\left(t\right)^c\right)-1\right]dt<0$. This proves $-E_t^C\left[-Qr\right]>E_t^C\left[Qr\right]$. *Q.E.D.*

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