

# GINI AND OPTIMAL INCOME TAXATION BY RANK

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## ONLINE APPENDIX

### *Appendix 1 - Proof of Proposition 2 and Comments*

*Proof:* By (9), Ramsey behavioral term is independent of gross income if and only if we can write:  $v'_z(z; \theta) / v''_{z\theta}(z; \theta) = k(\theta)$ . Using basic integration formulas, we must have  $\log(v'_z(z; \theta)) = \phi(\theta) + c_1(z)$ , where  $\phi(\theta)$  is a primitive of  $1/k(\theta)$  and  $c_1(z)$  is increasing to ensure convexity of  $v(z; \theta)$  with respect to  $z$ . Taking the exponential, we obtain  $v'_z(z; \theta) = \exp(\phi(\theta)) \exp(c_1(z))$ . Integrating the latter with respect to  $z$ , this implies:

$$(28) \quad v(z; \theta) = g(z) \exp(\phi(\theta)) + \gamma(\theta),$$

where  $z \rightarrow g(z)$  is a primitive of  $z \rightarrow \exp(c_1(z))$  and  $\gamma$  is a function independent of  $z$ , which may depend on  $\theta$ . Note that:  $v'_z(z; \theta) = g'(z) \exp(\phi(\theta))$ . Hence, we must have  $g'(z) > 0$ . Moreover,  $v''_{zz}(z; \theta) = g''(z) \exp(\phi(\theta))$  must be positive, requiring  $g''(z) > 0$ . In addition,  $v''_{z\theta}(z; \theta) = g'(z) \phi'(\theta) \exp(\phi(\theta))$  must be negative, implying  $\phi'(\theta) < 0$  given that  $g'(z) > 0$ .  $\square$

A specification of  $v(z; \theta)$  often used in the literature is  $v(z; \theta) = (z/\theta)^{(1+1/\beta)}$  with  $\beta > 0$  (see, e.g., Atkinson and Stiglitz (1980) or Salanié (2005)). It can be rewritten as:

$$v(z; \theta) = z^{1+1/\beta} \theta^{-(1+1/\beta)} = z^{1+1/\beta} \exp \left[ -\left(1 + \frac{1}{\beta}\right) \log(\theta) \right]$$

and therefore verifies all conditions for Proposition 2 to hold. It should be noted that Ramsey behavioral term is constant in this example. However, the fact that Ramsey behavioral term is structural does not preclude it from varying along the type distribution. Indeed, with (28), we check that it is equal to:

$$(29) \quad -\frac{1}{\theta \phi'(\theta)} > 0.$$

For future reference (see Appendix 2), we note that if  $\phi'(\theta)$  is *constant*, then Ramsey behavioral term is *hyperbolic* in  $\theta$ .

### *Appendix 2 - Computability*

When Ramsey behavioral term is structural, Proposition 1 provides optimal marginal tax rate formulas which are completely exogenous and, in many cases, have a closed-form solution. Indeed, combining the expression for Ramsey behavioral term obtained in (29), the definition of the local

Pareto parameter given in (4) and the ABZ Formula (13), we obtain:

$$(30) \quad \frac{MTR(p)}{1 - MTR(p)} = -\phi'(H^{-1}(p)) \underbrace{\frac{1-p}{h(H^{-1}(p))}}_{R(p)} [1 - G(p)].$$

In this expression,  $1/R(p)$  corresponds to the hazard rate for the distribution of productivity. Hence, for any distribution for which  $R(p)$  has a closed form, so does the optimal marginal tax rate formula. This would for example be the case for the exponential, Weibull, Pareto, generalized Gamma and log-normal distributions.

A second insight from (30) is obtained when Ramsey behavioral term is both structural and *hyperbolic*. In that case, (29) implies  $e(p) = 1/(k\theta(p))$ , where  $k$  is a positive scalar. Hence, Formula (30) simplifies as follows:

$$\frac{MTR(p)}{1 - MTR(p)} = kR(p) [1 - G(p)].$$

Optimal tax rates can thus be expressed in a very simple way,  $MTR(p)/(1 - MTR(p))$  being equal to the product of the inverse of the hazard rate and the inequality factor, scaled by  $k$ .

Observe that when the hazard rate  $R(p)$  is constant (e.g., when the productivity distribution is exponential), the inequality term  $1 - G(p)$  explains all variations in the optimal marginal tax rate function. Even when the latter is not constant, the marginal tax formula is extremely simple, with:

$$MTR(p) = \frac{1 - G(p)}{1 - G(p) + kR(p)}.$$

This opens the way to a short path to recover the whole optimal tax schedule, as well as the gross-income/net-income allocation. The whole gross income path  $p \rightarrow z(p)$  is obtained from the individual first-order condition (6). Integrating marginal tax rates from rank 0 to  $p$ , we get:

$$(31) \quad T(z(p)) = T(0) + \int_0^p MTR(\pi) d\pi,$$

that must verify:

$$\int_0^1 T(z(p)) dp = E,$$

where  $E$  is an exogenous amount of expenditures to finance. This allows us to recover the value of the intercept of the tax function,  $T(z(0))$ . Substituting the latter into (31), net incomes are then computed as  $x(p) = z(p) - T(z(p))$ .

### **Appendix 3 - Proof of Proposition 6**

By the taxation principle, defining a non-linear income tax schedule  $T$  is equivalent to specifying a  $(c, z)$ -combination for any  $\theta$ , with  $T = z - c$ , subject to incentive-compatibility constraints. In addition, for any given  $z(\theta)$  and  $V(\theta)$ , there is a unique corresponding consumption level  $c(\theta)$

solution to (5), which satisfies  $\partial c/\partial V = 1/u'(c)$  and  $\partial c/\partial z = v'_z(z;\theta)/u'(c)$ .

Consequently, the optimal income tax schedule will be fully characterized if we find  $\theta \rightarrow (\bar{z}(\theta), V(\theta))$  maximizing  $\mathcal{W}$ , as written in (12), subject to (i)  $\int_{\underline{\theta}}^{\bar{\theta}} (\bar{z}(\theta) - \bar{c}(\theta)) \geq E$ , (ii)  $dV(\theta)/d\theta = -v'_\theta(\bar{z}(\theta); \theta)$  and (iii)  $\bar{z}$  non decreasing. As discussed above, we will not take constraint (iii) explicitly into account when solving for the optimal solution. We rely on optimal control theory to solve this problem. We let  $\bar{z}(\theta)$  be the control variable and  $V(\theta)$  be the state variable. We call  $\mu(\theta)$  the co-state variable and  $\eta$  the Lagrange multiplier of the government's budget constraint. The corresponding Hamiltonian is:

$$\mathcal{H} = \{\lambda(H(\theta)V(\theta) + \eta \cdot [\bar{z}(\theta) - \bar{c}(\theta)])\} h(\theta) - \mu(\theta)v'_\theta(\bar{z}(\theta); \theta).$$

The necessary conditions for a maximum are:

$$(32) \quad \frac{\partial \mathcal{H}}{\partial \bar{z}} = 0 \Leftrightarrow \lambda(H(\theta)) \frac{\partial V(\theta)}{\partial \bar{z}} h(\theta) + \eta \left[ 1 - \frac{\partial \bar{c}(\theta)}{\partial z} \right] h(\theta) - \mu(\theta)v''_{\theta z}(\bar{z}(\theta); \theta) = 0,$$

$$(33) \quad \frac{\partial \mathcal{H}}{\partial V} = -\mu'(\theta) \Leftrightarrow \mu'(\theta) = -\lambda(H(\theta))h(\theta) + \frac{\eta}{u'(\bar{c}(\theta))}h(\theta),$$

$$(34) \quad \mu(\underline{\theta}) = \lim_{\theta \rightarrow \underline{\theta}} \mu(\theta) = 0.$$

The envelope theorem applied to the individual utility maximization program implies  $\partial V/\partial \bar{z} = 0$ . In addition,  $\partial c(\theta)/\partial \bar{z} = 1 - T'(\bar{z}(\theta))$ . Therefore, (32) may be rewritten as:

$$(35) \quad \eta T'(\bar{z}(\theta))h(\theta) - \mu(\theta)v''_{\theta z}(\bar{z}(\theta); \theta) = 0 \Leftrightarrow T'(\bar{z}(\theta)) = \frac{\mu(\theta)v''_{\theta z}(\bar{z}(\theta); \theta)}{\eta h(\theta)}.$$

Dividing by  $1 - T'(\bar{z}(\theta))$  obtained from (6), the latter is equivalent to:<sup>1</sup>

$$(36) \quad \frac{T'(\bar{z}(\theta))}{1 - T'(\bar{z}(\theta))} = \frac{u'(\bar{c}(\theta))v''_{\theta z}(\bar{z}(\theta); \theta) \mu(\theta)}{v'_z(\bar{z}(\theta); \theta)\eta h(\theta)}.$$

We now manipulate the different optimality conditions to find expressions for  $\mu(\theta)$  and  $\eta$ . First, note that by definition,  $\int_{\underline{\theta}}^{\bar{\theta}} \mu'(t)dt = \mu(\bar{\theta}) - \mu(\underline{\theta})$ . Because  $\mu(\bar{\theta}) = 0$  by (34),  $\mu(\theta) = -\int_{\underline{\theta}}^{\bar{\theta}} \mu'(t)dt$ . Substituting the Euler equation (33) into the latter yields:

$$(37) \quad \mu(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \left[ \lambda(H(t)) - \frac{\eta}{u'(\bar{c}(t))} \right] dH(t).$$

Re-expressing the latter in terms of  $p$ , and using the fact that  $\Lambda(1) = 1$ ,

$$(38) \quad \mu(\theta(p)) = 1 - \Lambda(p) - \eta \int_p^1 \frac{1}{u'(c(p))} dp.$$

<sup>1</sup>Incentive-compatibility requires  $T'(z(\theta)) < 1$ , implying  $1 - T'(z(\theta)) \neq 0$ .

Using (24), this may be rewritten as:

$$(39) \quad \mu(\theta(p)) = 1 - \Lambda(p) - \eta \frac{1-p}{m(p)}.$$

Evaluating the latter at  $p = 0$ , where  $\mu(\theta(p)) = \mu(\underline{\theta}) = 0$  by (34) and  $\lambda(p) = 0$ , and rearranging, we obtain:

$$(40) \quad \eta = m(0).$$

Plugging (40) into (39) yields:

$$(41) \quad \mu(\theta(p)) = 1 - \Lambda(p) - (1-p) \frac{m(0)}{m(p)}.$$

Combining (36), (40) and (41) with the definitions of  $MTR(p)$ ,  $c(p)$ ,  $z(p)$  and  $e(p)$  (see Equation (9)), we obtain:

$$(42) \quad \frac{T'(\bar{z}(\theta))}{1 - T'(\bar{z}(\theta))} = \frac{u'(\bar{c}(\theta)) \theta v'_z(\bar{z}(\theta); \theta)}{m(0) v''_{\theta z}(\bar{z}(\theta); \theta)} \frac{1-p}{\theta h(\theta)} \frac{\mu(\theta)}{1-p} = \frac{1}{\mathcal{B}_i(p)} \frac{1}{e(p)} \frac{1}{\alpha(p)} [\mathcal{B}_c(p) + 1 - G(p)],$$

which completes the proof.

## REFERENCES

- Atkinson, A. B. and J. E. Stiglitz (1980). *Lectures on Public Economics*. McGraw-Hill Inc.
- Salanié, B. (2005). *The Economics of Contracts*. Cambridge, MA: MIT Press.