Bargaining over Treatment Choice under Disagreement
ONLINE APPENDIX

Nabil I. Al-Najjar* Robert J. Gary-Bobo†

revised version May 17, 2022

* Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston IL 60208. Al-Najjar's Research Page.

†CREST, ENSAE 5, avenue Henry Le Chatelier, 91764 Palaiseau cedex, France, and University of Paris 1 Panthéon-Sorbonne; CES E-mail:robert.gary-bobo@ensae.fr

Online Appendix. Appendix B (Not for Publication)

1 Extension to Different Utility Functions

We have assumed that experts share the same utility function in order to focus on the role of bargaining in mediating differences in beliefs. This is appropriate in collective decision problems where experts agree on the objectives but have different opinions on how to achieve them.

A bargaining framework can still be used when experts have different utilities as well as different beliefs. The bargaining solution $t^\star$ continues to be well-defined but the analysis becomes less transparent: since utilities now depend on i, the log Nash product is a function of $n \times K$ terms of the form $\delta_i(t)(\theta_k)$, so we cannot characterize $t^\star$ in terms of a planner's belief. While we suspect that our results on commitment, inadmissibility, and under-reaction to information would continue to hold when experts have different utilities, new proofs are needed since the present proofs rely on the characterization of the planner's belief derived in Proposition 1 in a fundamental way.

In summary, the broader point of this paper is the use of Nash bargaining to study collective decisions under disagreement. This requires neither a common utility nor concordant beliefs. The narrower path we pursue in this paper, on the other hand, makes it possible to obtain sharper results about bargaining under disagreement.

2 Further Details for the Hard Choices Example

2.1 Efficiency and Speculative Betting

First we show that $E_{\pi^\star} u(t^\star) > 0$ in Case 2 stated in the body of the paper:

$$\pi^\star \left\{ s : \ell(s) \notin \left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right) \right\} > 0.$$

Define $S^+ = \{s : \ell(s) > \beta/\alpha\}$ and $S^- = \{s : \ell(s) < \frac{\alpha}{\beta}\}$. Appealing to symmetry, each of these events has positive probability.

For any $s \in S^+$, $t^\star$ selects a_1 and $\ell(s) = \frac{\pi^\star(\theta_1 \,|\, s)}{\pi^\star(\theta_2 \,|\, s)} > \frac{\beta}{\alpha}$. Therefore:

$$E_{\pi^\star} \left(u(t^\star) \,\middle|\, s \right) = \pi^\star(\theta_1 \,|\, s)\ \alpha - \pi^\star(\theta_2 \,|\, s)\ \beta > 0.$$

Using a similar argument, we also have $E_{\pi^\star}\left(u(t^\star)\,\middle|\,s\right) > 0$ for any s such that $\ell(s) < \frac{\alpha}{\beta}$. From this, it follows that $E_{\pi^\star}\left(u(t^\star)\,\middle|\,\ell(s) \notin \left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right)\right) > 0$. Since the payoff at the status quo is zero, the conclusion follows.

To show that $t^\star$ dominates the status quo, we first note that the symmetry of the problem implies that the Nash planner puts equal weight on the two states, so

$$E_{\pi^\star}u(t^\star) = \frac{1}{2}\; u(t^\star)(\theta_1) + \frac{1}{2}\; u(t^\star)(\theta_2) > 0.$$

Appealing to symmetry again, we have $u(t^\star)(\theta_1) = u(t^\star)(\theta_2) > 0$, establishing the claim that $t^\star$ dominates a°.

To justify the claim in the body of the paper that the two experts improve their welfare through speculative betting, we again appeal to symmetry to conclude that the Nash planner puts equal weight on the two experts. Thus:

$$E_{\pi^\star}u(t^\star) = \frac{1}{2}\; E_{p_1}u(t^\star) + \frac{1}{2}\; E_{p_2}u(t^\star) > 0,$$

and $E_{p_1}u(t^\star) = E_{p_2}u(t^\star) > 0$. Expert 1's payoff can be expressed as:

$$E_{p_1}u(t^\star) = p_1(S^+)\,\alpha - p_1\,(S^-)\beta,$$

from which it follows that:

$$\frac{p_1\,(S^+)}{p_1\,(S^-)} > \frac{\beta}{\alpha} > 1 \implies p_1\left(S^+\right) > p_1\left(S^-\right).$$

Using a similar calculation, we have

$$\frac{p_2\,(S^+)}{p_2\,(S^-)} < \frac{\alpha}{\beta} < 1 \implies p_2\left(S^+\right) < p_2\left(S^-\right).$$

Thus, the ex ante bargaining solution sets up a bet where Expert 1 bets on S^+ and against S^-, while Expert 2 bets on S^- and against S^+. Disagreement guarantees that they both believe they will achieve higher expected utility relative to the status quo.

2.2 Asymptotic Behavior

Here we explore how the solution to the Hard Choices example changes when the number of observations increases. Recall that data takes the form of observations $s = (o^1, \ldots, o^m)$ where m represents sample size. Index the experiment by m so that $S^m = \{o_1, o_2\}^m$ and $q^m(\cdot \,|\, \theta)$ the i.i.d. sampling distribution as explained in the body of the paper. We show that, as the number of observations $m \to \infty$, the ex ante expected utility for both experts approaches the maximum payoff $u(t^{max})$, corresponding to perfect information.

Fix m and let $k(s)$ be the number of times o_1 is drawn in the sample s. Consider the treatment rule $t_m(k(s))$ that depends on s only through $k(s)$. The expected utility of expert i under this treatment is:

$$\begin{aligned} E_{p_i} u(t) = p_i(\theta_1) \sum_{s \in \mathcal{S}^m} & \sigma^{k(s)} (1-\sigma)^{(m-k(s))} u(t(s)) \\ & + p_i(\theta_2) \sum_{s \in \mathcal{S}^m} (1-\sigma)^{k(s)} \sigma^{(m-k(s))} u(t(s)). \end{aligned}$$

Specifically, for $t_m(k)$, we have,

$$\begin{aligned} E_{p_i} u(t_m) = p_i(\theta_1) \sum_{k=0}^{m} & \binom{m}{k} \sigma^k (1-\sigma)^{(m-k)} u(t_m(k)) \\ & + p_i(\theta_2) \sum_{k=0}^{m} \binom{m}{k} (1-\sigma)^k \sigma^{(m-k)} u(t_m(k)). \end{aligned}$$

Let $[m/2]$ be equal to $m/2$ if m is even, and equal to $(m-1)/2$ if m is odd. Consider next the particular treatment rule:

$$t_m(k) = \begin{cases} a_1 & \text{if } k \geq [m/2] \\ a_2 & \text{if } k < [m/2]; \end{cases}$$

Then expected utility can be rewritten as:

$$\begin{aligned} E_{p_i} u(t_m) = & p_i(\theta_1) \left[\alpha \Pr\left(k \geq 1 + \left[\frac{m}{2}\right] \,\middle|\, \theta_1 \right) - \beta \Pr\left(k \leq \left[\frac{m}{2}\right] \,\middle|\, \theta_1 \right) \right] \\ & + p_i(\theta_2) \left[\alpha \Pr\left(k \leq \left[\frac{m}{2}\right] \,\middle|\, \theta_2 \right) - \beta \Pr\left(k \geq 1 + \left[\frac{m}{2}\right] \,\middle|\, \theta_2 \right) \right]. \end{aligned}$$

Now consider a sequence of even values of m, that is, $m = 2r$, r increasing without bound. Since $q > 1/2$, by the Law of Large Numbers, when $m \to +\infty$ we have,

$$\Pr\left[k \leq \frac{m}{2} \,\middle|\, \theta_1 \right] = \Pr\left[\frac{k}{m} \leq \frac{1}{2} \,\middle|\, \theta_1 \right] \to 0,$$

$$\Pr\left[k \geq 1 + \frac{m}{2} \,\middle|\, \theta_1\right] = \Pr\left[\frac{k}{m} \geq \frac{1}{2} + \frac{1}{m} \,\middle|\, \theta_1\right] \to 1.$$

We find similar results for m odd and when the conditioning state is θ_2. We then easily find that for any belief p, the expected payoffs converge, that is, $E_p u(t_m) \to \alpha$. We conclude that the Nash planner cannot achieve a smaller expected utility with the optimal ex ante solution, as m grows without bound. The expert's payoffs must be approaching α too.

3 Pareto Optimality and Admissibility

In assessing the optimality of a treatment rule, it is natural to consider the Pareto criterion:

Definition 2. *Given a profile of beliefs* $\{p_1, \ldots, p_n\}$*, a treatment rule* t Pareto dominates *another treatment rule* t' *if* $E_{p_i} u(t) \geq E_{p_i} u(t')$ *for each expert* i*, with at least one strict inequality.*

Treatment rule t *is* Pareto optimal *relative to a feasible set* T *if it is not Pareto dominated by any other treatment rule* $t' \in T$*.*

A number of authors questioned the appropriateness of the Pareto criterion when agents have different beliefs. See, for example, Mongin (2016), Brunnermeier, Simsek and Xiong (2012), and Gilboa, Samuelson and Schmeidler (2014). This suggests *admissibility* as an attractive alternative:

Definition 3. *A treatment rule* t' dominates *another treatment rule* t *if* $u(t')(\theta) \geq u(t)(\theta)$ *for each state* θ*, with at least one strict inequality.*

Treatment rule t *is* admissible *relative to a feasible set* T *if it is not dominated by any other treatment rule* $t' \in T$*.*

Admissibility is an appealing criterion, commonly used in statistical decision theory and in the treatment choice literature (see, for example, Berger (1985).[1] We first observe the following:

> **Fact:** *A treatment rule* t *that is Pareto optimal relative to a feasible set* T *is admissible.*[2]

[1] The definition of admissibility in our bargaining context coincides with that in the statistics literature under the assumptions of common values and when the experts share the likelihood function $q(s|\theta)$.

[2] It is easy to see that the converse is not true in general.

A key advantage of admissibility is that it is belief-free—in contrast to Pareto optimality which depends on the experts' profile of beliefs. To say that a treatment rule is inadmissible is an unambiguous judgement about its inefficiency since that rule can be improved on in every state and, therefore, in expectation for *any* belief.

We conclude by recalling a well-known result, the Complete Class Theorem (see, *e.g.*, Ferguson (1967)), which we use extensively in this paper, and which characterizes admissible rules as those that are Bayesian:

Definition 4. *A treatment rule t is* Bayesian *if it maximizes expected utility with respect to some prior p on Ω.*

Proposition 5. A Bay*esian treatment rule with respect to a full-support prior p is admissible. Conversely, an admissible treatment rule must be Bayesian.*

4 Proof of Lemma A.1

Proof:

$$\begin{aligned}\rho'(\Delta) &= \frac{\sum_i \frac{-p_i^2}{(p_i\Delta+1)^2}}{\sum_i \frac{1-p_i}{p_i\Delta+1}} + \frac{\sum_i \frac{p_i}{p_i\Delta+1}}{\left(\sum_i \frac{1-p_i}{p_i\Delta+1}\right)^2}\left(\sum_i \frac{(1-p_i)p_i}{(p_i\Delta+1)^2}\right)\\ &= B \times A\end{aligned}$$

where

$$B = \left(\sum_i \frac{1-p_i}{p_i\Delta+1}\right)^{-2} > 0 \tag{1}$$

and

$$A = \left[\left(\sum_i \frac{p_i}{p_i\Delta+1}\right)\left(\sum_i \frac{(1-p_i)p_i}{(p_i\Delta+1)^2}\right) - \left(\sum_i \frac{p_i^2}{(p_i\Delta+1)^2}\right)\left(\sum_i \frac{1-p_i}{p_i\Delta+1}\right)\right]. \tag{2}$$

We show that $A < 0$:

$$
\begin{aligned}
A &= \sum_i \sum_j \frac{p_i p_j (1-p_j)}{(p_i\Delta+1)(p_j\Delta+1)^2} - \sum_i \sum_j \frac{p_i^2(1-p_j)}{(p_i\Delta+1)^2(p_j\Delta+1)} \\
&= \sum_i \sum_j \frac{p_i p_j - p_i p_j^2}{(p_i\Delta+1)(p_j\Delta+1)^2} - \sum_j \sum_i \frac{p_j^2(1-p_i)}{(p_j\Delta+1)^2(p_i\Delta+1)} \\
&= \sum_i \sum_j \frac{p_i p_j - p_j^2}{(p_i\Delta+1)(p_j\Delta+1)^2} \\
&= \sum_i \sum_{\substack{j \\ j\neq i}} \frac{p_j(p_i-p_j)(p_i\Delta+1)}{(p_i\Delta+1)^2(p_j\Delta+1)^2} \\
&= \sum_i \sum_{\substack{j \\ j<i}} \frac{p_j(p_i-p_j)(p_i\Delta+1) + p_i(p_j-p_i)(p_j\Delta+1)}{(p_i\Delta+1)^2(p_j\Delta+1)^2} \qquad (3) \\
&= -\sum_i \sum_{\substack{j \\ j<i}} \frac{(p_i-p_j)^2}{(p_i\Delta+1)^2(p_j\Delta+1)^2},
\end{aligned}
$$

where the last equality follows from cancellation of the numerator in (3) :

$$
\begin{aligned}
p_j(p_i-p_j)(p_i\Delta+1) + p_i(p_j-p_i)(p_j\Delta+1) &= p_j p_i^2\Delta + p_j p_i - p_j^2 p_i\Delta \\
&\quad - p_j^2 + p_i p_j^2\Delta + p_i p_j \\
&\quad - p_i^2 p_j\Delta - p_i^2 \\
&= -(p_i-p_j)^2.
\end{aligned}
$$

■

5 Proof of Lemma A.4

Proof: From Lemma A.1, the function ρ is strictly decreasing in Δ:

$$\rho'(\Delta) = BA < 0,$$

where the terms A, B are defined in (2) and (1). Next, using (11) in the main text, we write ρ as:

$$\rho(\Delta) = BC,$$

where

$$C = \left(\sum_i \frac{p_i}{p_i \Delta + 1} \right) \left(\sum_i \frac{1 - p_i}{p_i \Delta + 1} \right).$$

Thus, the derivative of ζ with respect to Δ at a fixed (δ_1, δ_2) may be written as:

$$\begin{aligned} \zeta' &= \frac{\delta_2}{\delta_1} \left[\rho'(\Delta)(1 + \Delta) + \rho(\Delta) \right] \\ &= \frac{\delta_2}{\delta_1} B[A(1 + \Delta) + C]. \end{aligned}$$

Since both B and δ_2/δ_1 are positive, the sign of ζ' is the same as that of $A(1 + \Delta) + C$.

Recall from Lemma A.1 that

$$A = -\sum_i \sum_{\substack{j \\ j<i}} \frac{(p_i - p_j)^2}{(p_i \Delta + 1)^2 (p_j \Delta + 1)^2}.$$

Furthermore,

$$\begin{aligned} C &= \sum_i \sum_j \frac{p_i(1 - p_j)}{(p_i \Delta + 1)(p_j \Delta + 1)} \\ &= \sum_i \sum_{\substack{j \\ j \neq i}} \frac{(p_i - p_i p_j)(p_i \Delta + 1)(p_j \Delta + 1)}{(p_i \Delta + 1)^2 (p_j \Delta + 1)^2} + \sum_k \frac{p_k(1 - p_k)}{(p_k \Delta + 1)^2} \\ &= \sum_i \sum_{\substack{j \\ j<i}} \frac{(p_i - 2p_i p_j + p_j)(p_i \Delta + 1)(p_j \Delta + 1)}{(p_i \Delta + 1)^2 (p_j \Delta + 1)^2} + \sum_k \frac{p_k(1 - p_k)}{(p_k \Delta + 1)^2}. \end{aligned}$$

Since the second term in the last equality is obviously positive, the desired conclusion obtains if the following function of Δ, denoted g_{ij}, is positive for every i and $j < i$, that is,

$$g_{ij}(\Delta) = \tilde{A}_{ij}(\Delta + 1) + \tilde{C}_{ij}(\Delta) > 0,$$

where, by definition,

$$\tilde{A}_{ij} = -(p_i - p_j)^2, \quad \text{and} \quad \tilde{C}_{ij}(\Delta) = (p_i - 2p_i p_j + p_j)(p_i \Delta + 1)(p_j \Delta + 1).$$

We find that:

$$\begin{aligned} g_{ij}(\Delta) &= 4p_ip_j\Delta + p_i(1-p_j) + p_j(1-p_i) \\ &\quad + p_ip_j\Delta[p_i\Delta - 2p_ip_j\Delta + p_j\Delta - 2p_i - 2p_j] \\ &= 4p_ip_j\Delta + p_i(1-p_j) + p_j(1-p_i) \\ &\quad + p_ip_j\Delta^2[p_i - 2p_ip_j + p_j] - 2p_ip_j\Delta[p_i + p_j] \\ &= p_i(1-p_i) + p_j(1-p_j) \\ &\quad + p_ip_j\Delta^2\big[p_i(1-p_j) + p_j(1-p_i)\big] + 4p_ip_j\Delta\left[1 - \frac{p_i+p_j}{2}\right]. \end{aligned}$$

It is easy to check that $g_{ij}(\Delta)$ is quadratic and convex with respect to Δ and reaches its global minimum at point Δ_{ij}^{min}, where,

$$\Delta_{ij}^{min} = \frac{-(2-p_i-p_j)}{p_i(1-p_j)+p_j(1-p_i)}.$$

It is also easy to check that $\Delta_{ij}^{min} \leq -1$ since this inequality is equivalent to $1-p_i \geq p_j(1-p_i)$, or $p_j \leq 1$, a true statement. We now evaluate g_{ij} at point $\Delta = -1$ and we find,

$$g_{ij}(-1) = \tilde{C}_{ij}(-1) = [p_i(1-p_j) + p_j(1-p_i)]\,(1-p_i)(1-p_j) > 0.$$

This last result ends the proof of the Lemma. ■