

ONLINE APPENDIX FOR “STABLE AND EFFICIENT RESOURCE ALLOCATION WITH  
CONTRACTS”

BOBAK PAKZAD-HURSON

Here, I present additional results and extensions omitted from the body of the paper.

*B1. Top trading cycles*

This section demonstrates an example of the “cook book” provided in Theorem 2 and how it applies to another real-world mechanism, top trading cycles. Unlike  $\varphi^{DA}$  which always yields a stable matching, top trading cycles, denoted  $\varphi^{TTC}$ , is always efficient and group strategyproof (Abdulkadiroğlu and Sönmez, 2003). I will show that these properties carry over to the extended mechanism  $\bar{\varphi}^{TTC}$ . To the best of my knowledge, this is the first top trading cycles mechanism that accommodates contracts. I define  $\bar{\varphi}^{TTC}$  as follows.

**DEFINITION 13:** *The top trading cycles mechanism  $\bar{\varphi}^{TTC}$  proceeds in a sequence of steps  $t = 1, 2, \dots$ :*

*Step 1: Each  $s \in S$  is endowed with a counter,  $C_s^0 = q_s$ . Each  $s \in S$  points to the student named in its highest priority contract. Each student similarly points to the school named in her most preferred contract. The null object points to all students, and has  $q_s = |I|$ . Due to finiteness, there is at least one cycle. Let  $I^1$  and  $S^1$  be the sets of students and schools involved in cycles. Each  $i \in I^1$  is matched to school  $s \in S^1$  to which student  $i$  is pointing via the student’s most preferred contract. For all  $s \in S^1$  set  $C_s^1 = q_s - 1$  and  $C_{s'}^1 = q_{s'}$  for all  $s' \notin S^1$ . Let  $J^1 = I^1$  and let  $W^1 = \{s \in S : C_s^1 = 0\}$ . Define  $X^1 := X \setminus \{\mathcal{X}_{J^1} \cup \mathcal{X}_{W^1}\}$ .*

*Step  $t$ : Each  $s \in S \setminus W^{t-1}$  points to the student named in its highest priority contract in  $X^{t-1}$ . Each student  $i \in I \setminus J^{t-1}$  similarly points to the school named in her most preferred contract in  $X^{t-1}$ . The null object points to all students. There is at least one cycle. Let  $I^t$  and  $S^t$  be the sets of students and schools involved in cycles. Each  $i \in I^t$  is matched to some school  $s \in S^t$  via the student’s most preferred contract, that is, the  $R_i$ -maximal element of  $X_i \cap X_s$ . For all  $s \in S^t$  set  $C_s^t = C_s^{t-1} - 1$  and  $C_{s'}^t = C_{s'}^{t-1}$  for all  $s' \notin S^t$ . Let  $J^t = J^{t-1} \cup I^t$  and let  $W^t = \{s \in S : C_s^t = 0\}$ . Define  $X^t := X \setminus \{\mathcal{X}_{J^t} \cup \mathcal{X}_{W^t}\}$ .*

*The mechanism terminates at some step  $\tau$  when  $W^\tau = S$  or  $J^\tau = I$ , whichever comes first; either no acceptable contracts remain to unmatched students, or all of the seats are filled at every school. The final matching is given by the selections made at, and prior to, step  $\tau$ .*

In the case in which  $|\mathcal{X}_i \cap \mathcal{X}_s| \leq 1$  for all  $s \in S$  and all  $i \in I$ , the algorithm becomes identical to the contract-free top trading cycles mechanism, denoted  $\varphi^{TTC}$  (see Abdulkadiroğlu and Sönmez (2003) for a formal description).

As in Ergin (2002), Kesten (2006) provides an acyclicity condition under which  $\varphi^{TTC}$  is stable (recall that it is always efficient and group strategyproof). Again, I have adapted the definition to be consistent with the inclusion of contracts.

**DEFINITION 14:** *A priority structure  $(\succeq, q)$  is Kesten acyclic if there do not exist distinct  $r, s \in S$ , distinct  $i, j, k \in I$ , and distinct  $(i, r, t)$ ,  $(j, r, u)$ ,  $(k, r, v)$ ,  $(k, s, w)$ ,  $(i, s, q)$ ,  $(j, s, z) \in \mathcal{X}$  such that the following are satisfied:*

*C) Cycle condition:  $(i, r, t) \succ_r (j, r, u) \succ_r (k, r, v)$ ,  $(k, s, w) \succ_s (i, s, q)$  and  $(k, s, w) \succ_s (j, s, z)$ ,*

*S) Scarcity condition: There exists a set of contracts  $Y_r \subset \mathcal{X}_r \setminus \{\mathcal{X}_i \cup \mathcal{X}_j \cup \mathcal{X}_k\}$  such that*

*S.1)  $|Y_r| = q_r - 1$ ,*

*S.2) For every  $x \in Y_r$  either  $x \succ_r (i, r, t)$ , OR  $x \succ_r (j, r, u)$  and  $(k, s, w) \succ_s x'$  for some  $x' \in \mathcal{X}_s \cap \mathcal{X}_{x_1}$ , and*

*S.3)  $|Y_r \cap \mathcal{X}_\ell| \leq 1$  for all  $\ell \in I$ .*

This scarcity condition is slightly different than Ergin's (see Definition 1). *S.1)* says that there exists a set of contracts that is smaller than school  $r$ 's capacity by one that has *S.2)* higher priority at school  $r$  than that held by student  $i$ , OR a higher priority than student  $j$ 's contract and there is a contract naming the student in contract  $x$  with a lower priority than that held by  $k$  at school  $s$  and *S.3)* that no student is named in more than one of these contracts at each school. Note that Kesten acyclicity is a stronger restriction than Ergin acyclicity.

The following proposition states that  $\varphi^{TTC}$  requires this stronger acyclicity condition to guarantee the desired properties. It is stated here without proof, but is the main result of Kesten (2006).

**PROPOSITION 7:** *For any  $X \in \mathcal{Y}$ ,  $\varphi^{TTC}$  is stable with respect to  $(R, \succeq, q, X)$  if and only if  $(\succeq, q)$  is Kesten acyclic.*

It is possible to show, following the proof of Abdulkadiroğlu and Sönmez (2003) that  $\bar{\varphi}^{TTC}$  is efficient and group strategyproof (Abdulkadiroğlu and Sönmez, 2003). However, proving these properties directly is not necessary to apply the main result; I need merely show that  $\bar{\varphi}^{TTC}$  is an extension of  $\varphi^{TTC}$  (it is straightforward that  $\varphi^{TTC}$  is contract neutral). That  $\bar{\varphi}^{TTC}$  is an extension of  $\varphi^{TTC}$  can be seen as  $\bar{\varphi}^{TTC}$  is constructed in the same way as the general extension constructed in the proof of Proposition 3. The result is stated formally in the following proposition.

**PROPOSITION 8:**  *$\bar{\varphi}^{TTC}$  is an extension of  $\varphi^{TTC}$ .*

Finally, I provide the following result, which is a corollary of Proposition 8 and Theorem 2, and generalizes the main result of Kesten (2006) to allow for contracts. Since Kesten acyclicity is stronger than Ergin acyclicity, this domain is smaller than the domain over which  $\bar{\varphi}^{DA}$  is efficient and group strategyproof.

**COROLLARY 1:**  $\bar{\varphi}^{TTC}$  is stable, efficient, and group strategyproof if and only if  $(\succeq, q)$  satisfies Kesten acyclicity and is student-lexicographic.

This section demonstrates how to use the results of this paper to apply previous maximal domain results to the setting with contracts. First, I construct a mechanism which is an extension of  $\varphi^{TTC}$ . Kesten (2006) provides the necessary and sufficient priority restriction of which  $\varphi^{TTC}$  is always stable and efficient. Because  $\varphi^{TTC}$  satisfies contract neutrality, I am able to apply Theorem 2 to  $\bar{\varphi}^{TTC}$  to generalize the main result of Kesten (2006).

### B2. Multi-unit Demand

This section demonstrates the applicability of the theory developed in this paper to the context of multi-unit demand (many-to-many matching). Following Kojima (2013), I consider students who can be matched to a number of “courses” (for similarity of notation, I use the letter  $s$  to denote a generic course and  $S$  to denote the set of courses). Each student  $i \in I$  has a quota  $p_i \geq 1$  of courses she can attend, with the restriction that each student and each course can only be matched together under a single contract. For example, a contract could specify whether a student audits a particular course, takes the course on a pass/fail basis, or takes the course for a letter grade. Student preferences are responsive. A *course allocation problem* is  $(I, R, p, S, \succeq, q, \mathcal{X})$ . The rest of the notation generally carries through from above.<sup>11</sup> In the non-contractual setting, stable, efficient and, strategyproof mechanisms are very limited under multi-unit demand; multi-unit deferred acceptance  $\varphi^{MDA}$  is equivalent to a serial dictatorship under the necessary domain restriction to ensure that the mechanism is stable, efficient, and strategyproof (Kojima, 2013).<sup>12</sup> The necessary restriction is that, excluding top students who are guaranteed seats in any course, the priority structure must rank all students in the same position across all courses. The definition, adapted from Kojima (2013), is presented below.

**DEFINITION 15:** A priority structure  $(\succeq, q)$  is essentially homogeneous if there exist no distinct  $r, s \in S$ , distinct  $i, j \in I$ , and distinct  $(i, r, u)$ ,  $(j, r, v)$ ,  $(j, s, w)$ ,  $(i, s, t) \in \mathcal{X}$  such that:

C) *Cycle condition:*  $(i, r, u) \succ_r (j, r, v)$  and  $(j, s, w) \succ_s (i, s, t)$  and,

<sup>11</sup>The one exception is that the “no blocking” condition of stability now requires the contract  $x$  to be preferred by student  $i$  to some  $y \in \mu(i)$ .

<sup>12</sup>Deferred acceptance extends in the natural way to the multi-unit demand case. At each step, every student  $i$  has  $p_i$  offers out to courses.

S) *Scarcity condition: There exist (possibly empty) disjoint sets of contracts  $Y_r, Y_s \subset \mathcal{X}$  such that*

$$S.1) |Y_r| = q_r - 1 \text{ and } |Y_s| = q_s - 1,$$

$$S.2) x \succ_r (j, r, v) \text{ for every } x \in Y_r \text{ and } y \succ_s (i, s, t) \text{ for every } y \in Y_s, \text{ and}$$

$$S.3) |Y_r \cap \mathcal{X}_\ell| + |Y_s \cap \mathcal{X}_\ell| \leq 1 \text{ for all } \ell \in I.$$

This is the second strongest acyclicity condition presented in this paper (following Definition 6). Kojima (2013) proves that essential homogeneity of the priority structure, efficiency of  $\varphi^{MDA}$ , and strategyproofness of  $\varphi^{MDA}$  are all equivalent.

The proof of Theorem 2 in this paper goes through with minimal modification to include the general case of many-to-many matching. This allows for the following result, which generalizes the main result of Kojima (2013) to include contracts.

**COROLLARY 2:** *The following are all equivalent:*

1)  $\bar{\varphi}^{MDA}$  is efficient,

2)  $\bar{\varphi}^{MDA}$  is strategyproof,

3)  $(\succeq, q)$  is essentially homogeneous and student lexicographic.

**Proof:**

The “necessity” argument follows from Theorem 3 (since single-unit demand is a special case of multi-unit demand), and Kojima (2013). The “sufficiency” argument follows similar logic as in Theorem 3 by noting that  $\bar{\varphi}^{MDA}$  meets the necessary regularity conditions.

□

### B3. Substitutable priorities

The analysis thus far has assumed that school priorities are responsive, meaning that a school’s ranking over sets of contracts is uniquely defined by its ranking over individual contracts. For example, if school  $s$  gives highest priority to contract  $x$  and second highest priority to contract  $y$ , then of all the sets of contracts of size two, it gives highest priority to the set  $\{x, y\}$ . It is well known that allowing for general complementarities in school priorities may lead to non-existence of stable matchings (Kelso and Crawford (1982), Hatfield and Milgrom (2005)). Nevertheless, stability is guaranteed if school priorities are *substitutable*, meaning that if a contract is accepted by a school from a set of available contracts, it must also be accepted when only a subset of those available contracts are presented to the school. Indeed, substitutable priorities have a natural interpretation in school choice settings as they allow school systems to achieve diversity goals by favoring sets of applicants which include students from underrepresented groups (see Abdulkadiroğlu and Sönmez, (2003), Echenique and Yenmez (2015), and

Aygün and Turhan (2020)). In the contract setting, substitutable priorities also allow for schools, as an example, to favor applicant sets listing a wide range of majors to a set of applicants who all wish to study physics, even if each of the physics students individually have higher priority than all others. This is a powerful tool for schools, as there may be internal constraints on the number of students that can study any particular major.

School priorities are also often *acceptant* in that schools favor filling as many seats as possible. A likely objective of providing as much education to the best students, subject to capacity constraints, demands that schools have acceptant priorities. Indeed, acceptant priorities are necessary if a market designer wishes to ensure efficiency, for otherwise an empty seat at a school may be denied to a student who wishes to attend.

A natural question is whether it is possible to guarantee stability, efficiency and group strategyproofness in the contracts setting when the priority structure is acceptant and substitutable. This section will show that the answer is yes. More specifically, I show that (with a slight generalization of notation) Theorem 2 extends to this more general framework, so acyclic and student-lexicographic priorities are again necessary and sufficient to guarantee stability, efficiency and group strategyproofness when contracts are added to the model. This allows me to generalize the result of Kumano (2009), and show that  $\bar{\varphi}^{DA}$  is stable, efficient, and group strategyproof under acceptant substitutable priorities if and only if the priority structure is Ergin acyclic and student lexicographic.

#### SUBSTITUTABLE PRIORITY STRUCTURES

Here, I introduce acceptant substitutable priorities and modify the setting introduced in Section 2 to apply to this new context.

$(\succeq, q) = (\succeq_s, q_s)_{s \in S}$  is a *priority structure* over a feasible set of contracts  $\mathcal{X}$  where  $\succeq_s$  is an exogenous tuple of linear orders in which  $\succeq_s$  represents school  $s$ 's complete, transitive, and antisymmetric ranking of *sets* of contracts  $A \subset \mathcal{X}_s$ . Again,  $\succ_s$  is the asymmetric subset of  $\succeq_s$ . Note that the priority ranking does not depend on the feasibility of a set of contracts. Nevertheless, it is important to denote which sets of contracts are actually available to which a school can be matched. This is done by adding a *choice function*  $C_s(\cdot)$ . The choice function satisfies  $\forall A \subset \mathcal{X}_s, C_s(A) \subset A$ , and feasibility, meaning that a)  $|C_s(A)| \leq q_s$  and b)  $\forall i \in I$  and if  $\exists x, y \in C_s(A) \cap X_i$  then  $x = y$ . Feasibility requires that a) the chosen set does not exceed the size of the school, and b) that no student is chosen for multiple contracts. The relation of the choice function to the priority order is that  $C_s(A) = Y$  if and only if  $Y \succeq_s Z$  for all feasible  $Z \subset A$ . Throughout, I use the notation  $I_A$  to denote the set of students who are named in a contract in  $A$ , that is,  $I_A = \{i \in I : \exists x \in A \cap X_i\}$ .

Although the choice function defines the highest priority set of contracts from an available set, it does not specify how a school chooses this highest priority

set of contracts. More specifically, the choice function does not yet restrict the priority structure to exclude complementarities. The following definition formally describes the acceptant substitute property.

DEFINITION 16: *A priority structure  $(\succeq, q)$  is acceptant substitutable if:*

$$A) \forall s \in S, \forall A \subset \mathcal{X}_s, |C_s(A)| = \min\{|I_A|, q_s\}, \text{ and}$$

$$S) \forall s \in S, \forall A, A' \subset \mathcal{X}_s \text{ with } A' \subset A, C_s(A) \cap A' \subset C_s(A').$$

This definition states that a school's priorities must *A)* accept as many students as possible, and must *S)* accept any contract from a set  $A'$  when the same contract was accepted from a superset of  $A'$ . The acceptant property is necessary for efficiency; without it, efficiency can not be guaranteed since some seats may be left unfilled, and it would be a Pareto improvement to allow amenable students to fill these extra seats.

Having now defined acceptant substitutes, I address whether it is possible to guarantee stability,<sup>13</sup> efficiency, and group strategyproofness with contracts. One important point to make is that the set of responsive priorities is a subset of the set of acceptant substitutable priorities (Kumano, 2009). Therefore, some form of acyclic and student-lexicographic priorities will again be necessary for the existence of a stable and efficient matching. These concepts, however, need to be redefined to account for acceptant substitutable priorities. Indeed, the former definition of student-lexicographic priorities is not well-defined with these more general priority structures. Before giving the new definition of student-lexicographic priorities, I provide the following example, which illustrates the intuition of a cycle within the priority order of a single school that conflicts stability and efficiency.

EXAMPLE 5: *There is one school with two seats, and there are three students,  $i, j, k$ . Consider the priority order over feasible sets of contracts within  $\mathcal{X}$  and preferences (where  $x^\ell, y^\ell \in \mathcal{X}_\ell$  for  $\ell \in \{i, j, k\}$ ):*

*School  $s$ :  $\{x^i, y^j\} \succ \{x^i, x^k\} \succ \{x^i, x^j\} \succ \{y^j, x^k\} \succ \{x^j, x^k\} \succ \{x^i\} \succ \{y^j\} \succ \{x^k\} \succ \{x^j\} \succ \emptyset$*   
*Student  $i$ :  $x^i P_i \emptyset$*   
*Student  $j$ :  $x^j P_j y^j P \emptyset$*   
*Student  $k$ :  $x^k P_k \emptyset$*

<sup>13</sup>The “no blocking pair” definition has to be slightly restated to account for the substitutable priority structure. An allocation is stable if it is individually rational and there are no blocking pairs. Formally, a matching  $\mu \in \mathcal{M}_X$  is *stable with respect to  $(R, \succeq, q, X)$*  if:

*IR)*  $\forall i \in I \mu(i) R_i \emptyset$ , and

*NB)*  $\nexists (i, s) \in I \times S$  with  $x \in X_i \cap X_s$  such that  $x P_i \mu(i)$  and  $x \in C_s(\mu(s) \cup \{x\})$ .

$\mu$  is *stable* if it is stable with respect to  $(R, \succeq, q, \mathcal{X})$ .

This priority structure is acceptant substitutable.<sup>14</sup> Since all students prefer to be matched and student  $j$  most prefers contract  $x^j$ , the efficient matchings are  $\{x^i, x^k\}$ ,  $\{x^i, x^j\}$  and  $\{x^j, x^k\}$ . However, the unique stable matching is  $\{x^i, y^j\}$ . ■

Intuitively, the reason that there are no stable and efficient matchings in Example 5 is that the sets of contracts involving students  $i$  and  $j$  are interrupted in the priority order by  $\{x_i, x_k\}$ . Similarly to Example 2, the problem is that the sets of contracts naming the same students are not back-to-back in the priority order, and so the school can distinguish between the contracts that a student selects.

The generalized setting of acceptant substitutable priorities is similar conceptually to responsive preferences. Nevertheless, several definitions in Sections II and III need to be translated into the more general setting. Again, the root of the following definitions is the notion of indistinguishable (sets of) contracts.

DEFINITION 17:  $Y, Z \subset \mathcal{X}_s$  with  $Y \succ_s Z$  and  $I_Y = I_Z$  are indistinguishable sets of contracts to school  $s$  if  $Y$  and  $Z$  are feasible and either:

- a)  $|I_Y| \leq q_s - 1$ , or
- b) There does not exist  $W \subset \mathcal{X}_s$  with  $I_W \neq I_Y$  such that  $Y \succ_s W \succ_s Z$ .

A priority structure  $(\succeq, q)$  is set-lexicographic if for every school  $s$ , any two sets of feasible contracts  $Y, Z \subset \mathcal{X}_s$  with  $I_Y = I_Z$  are indistinguishable to school  $s$ .

a) takes advantage of the acceptant property. If a school can accommodate all interested students, then it is not important how sets of contracts of size smaller than the capacity of the school are ranked. This corresponds to the scarcity condition in Definition 2. Similarly, b) is analogous to the “back-to-back” condition in Definition 2.

As in the case of responsive priorities, when all sets of contracts involving the same students are indistinguishable to the school named in those contracts, the priority structure is set-lexicographic.

The notion of extension follows in a straightforward way from the base model:

DEFINITION 18: A set of feasible contracts  $Z \subset \mathcal{X}_s$  is irrelevant if there exists  $Y \subset \mathcal{X}_s$  with  $I_Y = I_Z$  such that:

- a)  $Y \succ_s Z$ , and
- b) for each  $i \in I_Y$ ,  $y \in Y_i \cap Y_s$ , and  $z \in Z_i \cap Z_s$ ,  $yR_iz$ .

<sup>14</sup>It clearly satisfies the acceptant property. To see that it is also substitutable, note that for any set of contracts  $A' \neq \{x^j, y^j\}$  with  $|A'| \leq 2$ ,  $C_s(A') = A'$  and so the definition of substitutability has no bite. Similarly, the case in which  $|A'| = 4$  has no bite since the only superset of  $A'$  is  $A'$  itself. Therefore, it suffices to consider  $A = \{x^i, x^j, y^j, x^k\}$  with  $A'$  being any of the  $\binom{4}{3} = 4$  subsets of size 3, and  $A' = \{x^j, y^j\}$  with  $A$  being any of the three supersets which contain  $A'$ . None of these cases violate Definition 16.

For a subset of students  $I'$  and some  $s \in S$ , let  $Y_{I',s}$  be a maximal collection of sets of contracts that are indistinguishable to  $s$  with respect to  $(\succeq, q)$ , that is, all  $Y, Z \in Y_{I',s}$  are indistinguishable to  $s$  and there is no  $Y_{I',s} \subset Z_{I',s}$  such that all  $Y, Z \in Z_{I',s}$  are indistinguishable to  $s$ . Let  $X^{Y_{I',s}}$  be the  $\succeq_s$ -maximal set in  $Y_{I',s}$  and let  $Y^{Y_{I',s}} \in Y_{I',s}$  denote the set containing the  $R_i$ -maximal contract in  $Y_{I',s}$  for all  $i \in I'$  if such a set exists. Let  $\tilde{\succeq}_s^{Y_{i,s}}$  permute the rankings of  $X^{Y_{I',s}}$  and  $Y^{Y_{I',s}}$ , that is, for any  $W, U \subset \mathcal{X}_s \setminus \{X^{Y_{I',s}}, Y^{Y_{I',s}}\}$ :

- a) If  $Y^{Y_{I',s}}$  exists, then  $X^{Y_{I',s}} \tilde{\succeq}_s^{Y_{I',s}} W$  if and only if  $Y^{Y_{I',s}} \succeq_s W$ ,
- b) If  $Y^{Y_{I',s}}$  exists, then  $Y^{Y_{I',s}} \tilde{\succeq}_s^{Y_{I',s}} W$  if and only if  $X^{Y_{I',s}} \succeq_s W$ ,
- c) If  $Y^{Y_{I',s}}$  does not exist, then  $X^{Y_{I',s}} \tilde{\succeq}_s^{Y_{I',s}} W$  if and only if  $X^{Y_{I',s}} \succeq_s W$ , and
- d)  $U \tilde{\succeq}_s^{Y_{I',s}} W$  if and only if  $U \succeq_s W$ .

I write  $\tilde{\succeq}$  to denote the permuted priority ranking as defined above for all  $I', s$  and all non-empty maximal sets of indistinguishable contracts  $Y_{I',s}$ .

**DEFINITION 19:** *Let  $\varphi$  be a contract-free mechanism. Take any  $(R, \succeq, q)$  such that  $Y^{Y_{I',s}}$  exists for all non-empty maximal sets of indistinguishable contracts  $Y_{I',s}$ ,  $s \in S$ . Let  $Q^{\mathcal{X}}$  be the collection of all irrelevant sets of contracts associated with  $(R, \tilde{\succeq})$ , and suppose  $\varphi(R, \tilde{\succeq}, q, \mathcal{X} \setminus Q^{\mathcal{X}})$  is well-defined. Then  $\bar{\varphi}$  is an extension of  $\varphi$  if  $\bar{\varphi}(R, \succeq, q, \mathcal{X}) = \varphi(R, \tilde{\succeq}, q, \mathcal{X} \setminus Q^{\mathcal{X}})$ .*

I now state Theorem 4, which is a translation of Theorem 2. It states that set-lexicographic priorities and an (mechanism specific) acyclicity condition are necessary and sufficient to ensure stability, efficiency and group strategyproofness with acceptant substitutable priorities when contracts are added. The reasoning behind this theorem follows from the logic of Theorem 2.

**THEOREM 4:** *Let  $\varphi$  be a contract-free mechanism. Let  $\varphi$ -acyclicity be the weakest acyclicity restriction on an acceptant substitutable priority structure  $(\succeq, q)$  such that contract-free mechanism  $\varphi$  is stable, efficient, and group strategyproof with respect to  $(R, \succeq, q, X)$  for all  $X \in \mathcal{Y}$ . Let  $\bar{\varphi}$  be an extension of  $\varphi$ . Then*

- 1)  $\bar{\varphi}$  is stable and efficient if and only if  $(\succeq, q)$  is set-lexicographic and satisfies  $\varphi$ -acyclicity, and
- 2) Suppose  $\varphi$  is contract neutral.  $\bar{\varphi}$  is stable and group strategyproof if and only if  $(\succeq, q)$  is set-lexicographic and satisfies  $\varphi$ -acyclicity.

**Proof:** Follows the same logic as the proof of Theorem 2.

□

## APPLICATION TO DEFERRED ACCEPTANCE

I apply the previous theorem to exactly characterize the set of acceptant substitutable priority structures over which  $\bar{\varphi}^{DA}$  is stable, efficient, and group strategyproof. The punch line of this section is that Ergin acyclicity (which also needs to be translated to accommodate substitutable priorities) and set-lexicographic priorities are necessary and sufficient to guarantee the desired properties.

First, I give a definition of set-Ergin acyclicity. The intuition is identical to Ergin acyclicity presented at the beginning of this paper, but accommodates the more general priority structure.

**DEFINITION 20:** *An acceptant substitutable priority structure  $(\succeq, q)$  is set-Ergin acyclic if there exist no distinct  $r, s \in S$ , distinct  $i, j, k \in I$ , distinct  $(i, r, t), (j, r, u), (k, r, v), (k, s, w), (i, s, q) \in \mathcal{X}$ , and (possibly empty) disjoint sets of contracts  $Y_r, Y_s \subset \mathcal{X} \setminus \{\mathcal{X}_i \cup \mathcal{X}_j \cup \mathcal{X}_k\}$  such that:*

*C) Cycle condition:*

- C.1)  $(j, r, u) \notin C_r(Y_r \cup \{(j, r, u), (i, r, t)\})$ ,*
- C.2)  $(k, r, v) \notin C_r(Y_r \cup \{(j, r, u), (k, r, v)\})$ , and*
- C.3)  $(i, s, q) \notin C_s(Y_s \cup \{(i, s, q), (k, s, w)\})$ .*

*S) Scarcity condition:*

- S.1)  $|Y_r| = q_r - 1$  and  $|Y_s| = q_s - 1$ , and*
- S.2)  $|Y_r \cap \mathcal{X}_\ell| + |Y_s \cap \mathcal{X}_\ell| \leq 1$  for all  $\ell \in I$ .*

The following proposition states that without contracts, set-Ergin acyclicity, efficiency and group strategyproofness are all equivalent for  $\varphi^{DA}$ . It is presented without proof, but is the main result of Kumano (2009).

**PROPOSITION 9:** *Let  $(\succeq, q)$  be acceptant substitutable. For any  $X \in \mathcal{Y}$ ,  $\varphi^{DA}$  is stable, efficient, and group strategyproof with respect to  $(R, \succeq, q, X)$  if and only if  $(\succeq, q)$  is set-Ergin acyclic.*

By combining Theorem 4 and Proposition 9, I generalize the result of Kumano (2009) to include contracts.

**COROLLARY 3:** *Let  $(\succeq, q)$  be an acceptant substitutable priority structure. The following are all equivalent:*

- 1)  $\bar{\varphi}^{DA}$  is efficient,*
- 2)  $\bar{\varphi}^{DA}$  is group strategyproof,*
- 3)  $(\succeq, q)$  is set-Ergin acyclic and set-lexicographic.*

Therefore, when contracts are added to an acceptant substitutable priority structure, it is again necessary and sufficient for the priorities to be lexicographic and Ergin acyclic if a market designer wishes to guarantee efficiency and group strategyproofness of student-proposing deferred acceptance.