Prudential Policy with Distorted Beliefs

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Online Appendix
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B Proofs and Derivations: Section 3

Proof of Lemma 4 [Planner's problem]

The planner's objective is given by the sum of investors' and creditors' expected utility. Formally, ignoring constant terms that depend only on endowments, we have $W = u^{I,P} + u^{C,P}$, where $u^{I,P}$ and $u^{C,P}$ are given by

$$
\begin{aligned}
u^{I,P} &= \left[Q\left(b\right) - 1 + \beta^{I}\int_{b}^{\overline{s}}\left(s-b\right)dF^{I,P}\left(s\right)\right]k - \Upsilon\left(k\right) \\
u^{C,P} &= \left[-Q\left(b\right) + \beta^{C}\left(\int_{b}^{\overline{s}} bdF^{C,P} + \int_{\underline{s}}^{b}\phi sdF^{C,P}\left(s\right)\right)\right]k - \Delta\left(b\right)k,
\end{aligned}
$$

which imply that

$$
W = \left[\underbrace{\beta^{I}\int_{b}^{\overline{s}}\left(s-b\right)dF^{I,P}\left(s\right) + \beta^{C}\left(\int_{b}^{\overline{s}} bdF^{C,P} + \int_{\underline{s}}^{b}\phi sdF^{C,P}\left(s\right)\right)}_{\equiv M^{P}(b)} - 1 - \Delta\left(b\right)\right]k - \Upsilon\left(k\right).
$$

The results in Lemma 1 follow immediately.

Note that a planner that exclusively values the welfare of investors simply maximizes $u^{I,P}$, taking as given $Q\left(b\right)$ as defined in the paper. This is as if the planner decided to set $F^{C,P}\left(s\right) = F^{C}\left(s\right)$. This observation is useful when relating our results to the Hertz scenario in the paper. A planner that assigns different welfare weights to investors and creditors simply maximizes a linear combination of $u^{I,P}$ and $u^{C,P}$.

Proof of Proposition 4 [Marginal welfare effect of varying the leverage cap]

The result follows directly by totally differentiating the characterization of the planner's objective in Lemma 1, applying the envelope theorem, and noting that $\frac{db^{\star}}{d\bar{b}} = 1$ whenever the leverage cap is binding.

Proof of Proposition 5 [Impact of beliefs on optimal regulation: General characterization]

The variational derivative of marginal welfare effects with respect to beliefs F^j for $j \in \{I, C\}$ is

$$\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j = \left[\frac{dM^P\left(\bar{b}\right)}{db} - \frac{d\Delta\left(b\right)}{db}\right]\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]$$
$$+ \left[M^P\left(\bar{b}\right) - \Delta\left(b\right) - M\left(\bar{b}\right)\right]\left[\frac{\delta \frac{dk^\star}{db}}{\delta F^j} \cdot G^j\right] - \left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]\frac{dk^\star\left(\bar{b}\right)}{d\bar{b}}.$$

Notice that we can express optimal investment as $k^\star\left(\bar{b}\right) = \Psi\left(M\left(\bar{b}\right) - 1\right)$, where $\Psi\left(\cdot\right)$ is the inverse function of $\Upsilon'\left(\cdot\right)$. This implies that

$$\frac{\delta k\left(\bar{b}\right)}{\delta F^j} \cdot G^j = \Psi'\left(\cdot\right)\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right] \quad \text{and} \quad \frac{dk\left(\bar{b}\right)}{d\bar{b}} = \Psi'\left(\cdot\right)\frac{dM\left(\bar{b}\right)}{db}.$$

Dividing these two expressions, we get

$$\frac{\frac{\delta k\left(\bar{b}\right)}{\delta F^j} \cdot G^j}{\frac{dk\left(\bar{b}\right)}{d\bar{b}}} = \frac{\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j}{\frac{dM\left(\bar{b}\right)}{db}},$$

or equivalently,

$$\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]\frac{dk^\star\left(\bar{b}\right)}{d\bar{b}} = \frac{dM\left(\bar{b}\right)}{db}\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right].$$

Combining our results, we obtain the required expression.

Whenever the planner's objective is well-behaved in $\bar{b}$, establishing that $\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j > 0$ guarantees that optimal leverage regulation involves a looser leverage cap. Formally, this is the case whenever (i) the planner's objective is quasi-concave in $\bar{b}$ and $\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j > 0$ evaluated at the optimal (second-best) policy or (ii) welfare takes any shape and $\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j > 0$ for all $\bar{b}$. As is standard in normative exercises, the planner's objective need not be quasi-concave without imposing additional restrictions on primitives — even though we find the problem to be well-behaved when simulating the model for standard functional forms and belief distributions.

Proof of Proposition 6 [Impact of beliefs on optimal regulation: Specific scenarios]

First, consider the debt and joint exuberance scenarios. By Proposition 3, debt or joint exuberance (in a hazard rate sense) increases $M(b)$ and also the marginal value of leverage $\frac{dM(b)}{db}$ for all b.

Moreover, we have

$$\frac{dM^P\left(\bar{b}\right)}{db} < \frac{dM\left(\bar{b}\right)}{db} \Rightarrow \left[\frac{dM^P\left(\bar{b}\right)}{db} - \frac{d\Delta(b)}{db} - \frac{dM\left(\bar{b}\right)}{db}\right] < 0$$

and also

$$M^P\left(\bar{b}\right) < M\left(\bar{b}\right) \Rightarrow M^P\left(\bar{b}\right) - \Delta\left(\bar{b}\right) - M\left(\bar{b}\right) < 0.$$

Moreover, for any marginal increase in debt/joint exuberance represented by the distortion G^j, we have

$$\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j > 0, \ \frac{\delta \frac{dM}{d\bar{b}}}{\delta F^j} \cdot G^j > 0.$$

Using Proposition 5,[1] we obtain that

$$\begin{aligned}\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j = \varphi \cdot &\left[\underbrace{\frac{dM^P\left(\bar{b}\right)}{d\bar{b}} - \frac{d\Delta(b)}{d\bar{b}} - \frac{dM\left(\bar{b}\right)}{d\bar{b}}}_{<0}\right]\left[\underbrace{\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j}_{>0}\right] \\ &+ \varphi \cdot \left[\underbrace{M^P\left(\bar{b}\right) - \Delta\left(\bar{b}\right) - M\left(\bar{b}\right)}_{<0}\right]\left[\underbrace{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^j} \cdot G^j}_{>0}\right] < 0,\end{aligned}$$

as required.

Second, consider the equity exuberance scenario. Repeating our steps above, we find

[1]In general, note that $\frac{\delta \frac{dk^\star}{db}}{\delta F^j} \cdot G^j = \Psi'(\cdot) \frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j + \Psi''(\cdot) \frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j$. When adjustment costs are quadratic, $\Psi'(\cdot) = \varphi$ — a scalar defined in the text — and $\Psi''(\cdot) = 0$.

that

$$\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^{j}} \cdot G^{j} = \varphi \cdot \left[\underbrace{\frac{dM^{P}\left(\bar{b}\right)}{d\bar{b}} - \frac{d\Delta\left(b\right)}{d\bar{b}} - \frac{dM\left(\bar{b}\right)}{d\bar{b}}}_{\lesseqgtr 0, \equiv A}\right] \left[\underbrace{\frac{\delta M\left(\bar{b}\right)}{\delta F^{j}} \cdot G^{j}}_{>0, \equiv B}\right]$$
$$+ \varphi \cdot \left[\underbrace{M^{P}\left(\bar{b}\right) - \Delta\left(\bar{b}\right) - M\left(\bar{b}\right)}_{<0, \equiv C}\right] \left[\underbrace{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^{j}} \cdot G^{j}}_{<0, \equiv D}\right].$$

and we can write

$$\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^{j}} \cdot G^{j} > 0 \Leftrightarrow AB + CD > 0 \Leftrightarrow A > -C\frac{D}{B},$$

which is equivalent to

$$\frac{dM^{P}\left(\bar{b}\right)}{d\bar{b}} - \frac{d\Delta\left(b\right)}{d\bar{b}} - \frac{dM\left(\bar{b}\right)}{d\bar{b}} > -\left[M^{P}\left(\bar{b}\right) - \Delta\left(\bar{b}\right) - M\left(\bar{b}\right)\right] \frac{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^{j}} \cdot G^{j}}{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^{j}} \cdot G^{j}}.$$

The results in this proposition follow directly by combining the comparative statics in Propositions 3 with the general characterization in Proposition 5. Our conclusions about optimal policy follow directly, because those results provide signs for $\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^{j}} \cdot G^{j} > 0$ for all $\bar{b}$.

Proof of Proposition 7 [Impact of the planner's beliefs on optimal regulation: Specific scenarios]

Each case in the proposition holds constant the beliefs of creditors and investors. Hence, the terms $M\left(\bar{b}\right) > 0$, $k^{\star}\left(\bar{b}\right) > 0$, and $\frac{dk^{\star}(\bar{b})}{d\bar{b}} \geq 0$ in the marginal welfare effect $\frac{dW}{d\bar{b}}$ (see Proposition 4) are also held fixed. It is then clear that $\frac{dW}{d\bar{b}}$ is increasing in the planner's marginal value $\frac{dM^{P}(\bar{b})}{d\bar{b}}$ of leverage and weakly increasing in the planner's valuation $M^{P}\left(\bar{b}\right)$ of investment.

By a parallel argument to Proposition 3, it follows that (i) $M^{P}\left(\bar{b}\right)$ increases with the planner's equity exuberance, debt exuberance, and joint exuberance, and that (ii) $\frac{dM^{P}(\bar{b})}{d\bar{b}}$ increases with the planner's debt exuberance or joint exuberance but decreases with the planner's equity exuberance. This establishes the claims in the proposition.

C Extensions

C.1 Imperfect knowledge of beliefs

C.1.1 Optimal policy conditional on endogenous investment

For technical consistency, we assume here that Θ is a closed set of pairs of continuous cumulative distribution functions. We can equip this set with the sup norm to make it a metric space, meaning that we define probability measures over the Borel sets in Θ, including the planner's probability measure μ.

Let $I\left(\bar{b},\theta\right) = \left\{\tilde{\theta}\in\Theta : k^{\star}\left(\bar{b},\tilde{\theta}\right) = k^{\star}\left(\bar{b},\theta\right)\right\}$ be the set of beliefs under which the equilibrium response to policy $\bar{b}$ is the same as under θ. For any given θ, a consistent policy $\bar{b}^{\star}$ has to solve the following fixed point problem:

$$\bar{b}^{\star} \in \arg\max_{\bar{b}} \mathbb{E}_{\mu}\left[W\left(\bar{b},k^{\star}\left(\bar{b};\tilde{\theta}\right)\right)\middle| I\left(\bar{b}^{\star},\theta\right)\right]. \tag{1}$$

For any interior solution to the maximization problem in (1), we have the first-order condition

$$\mathbb{E}_{\mu}\left[\left[\frac{dM^{P}\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]k^{\star}\left(\bar{b};\tilde{\theta}\right)+\left[M^{P}\left(\bar{b}\right)-\Delta\left(b\right)-M\left(\bar{b};\tilde{\theta}\right)\right]\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}\middle| I\left(\bar{b}^{\star},\theta\right)\right]=0. \tag{2}$$

A consistent policy $\bar{b}^{\star}$ must also satisfy this condition, and substituting, we obtain

$$\mathbb{E}_{\mu}\left[\left[\frac{dM^{P}\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]k^{\star}\left(\bar{b}^{\star};\tilde{\theta}\right)+\left[M^{P}\left(\bar{b}\right)-\Delta\left(b\right)-M\left(\bar{b}^{\star};\tilde{\theta}\right)\right]\frac{dk^{\star}\left(\bar{b}^{\star};\tilde{\theta}\right)}{d\bar{b}}\middle| I\left(\bar{b}^{\star},\theta\right)\right]=0.$$

The expression used in the paper follows by noting that $k^{\star}\left(\bar{b}^{\star};\tilde{\theta}\right)$ is a constant conditional on the information set $I\left(\bar{b}^{\star},\theta\right)$, and that the mapping between capital and $M\left(\bar{b}^{\star};\tilde{\theta}\right)$ is invertible, so that we can remove all terms except $\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}$ from the expectation operator — note that $\frac{dM^{P}\left(\bar{b}\right)}{db}$ and $M^{P}\left(\bar{b}\right)$ depend on the planner's beliefs, not on investors' or creditors' beliefs.

Alternative policy problem 1/No conditioning on endogenous investment: The marginal welfare effect of varying $\bar{b}$ when the planner does not observe realized investment is given by the equivalent of (2) without conditioning on any information:

$$\mathbb{E}_{\mu}\left[\frac{dW}{d\bar{b}}\right]=\mathbb{E}_{\mu}\left[\left[\frac{dM^{P}\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]k^{\star}\left(\bar{b};\tilde{\theta}\right)+\left[M^{P}\left(\bar{b}\right)-\Delta\left(b\right)-M\left(\bar{b};\tilde{\theta}\right)\right]\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}\right]$$
$$=\left[\frac{dM^{P}\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]\mathbb{E}_{\mu}\left[k^{\star}\left(\bar{b};\tilde{\theta}\right)\right]+\left[M^{P}\left(\bar{b}\right)-\Delta\left(b\right)-\mathbb{E}_{\mu}\left[M\left(\bar{b};\tilde{\theta}\right)\right]\right]\mathbb{E}_{\mu}\left[\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}\right]$$
$$-\mathbb{C}ov_{\mu}\left[M\left(\bar{b};\tilde{\theta}\right),\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}\right].$$

We note that the marginal benefit of varying $\bar{b}$ does not have a certainty equivalent property, because the planner must consider the covariance between agents' valuation of investment $M\left(\bar{b};\tilde{\theta}\right)$ and the policy elasticity $\frac{dk^{\star}\left(\bar{b};\tilde{\theta}\right)}{d\bar{b}}$. Intuitively, it is more valuable to impose a binding leverage cap if beliefs that induce overinvestment also induce a high policy elasticity. For instance, if θ is a scalar parameter so that a higher θ shifts creditors' beliefs in the sense that of hazard rate dominance, then Proposition 3 implies that this covariance is positive, meaning that uncertain debt exuberance further strengthens the case for imposing a binding leverage cap. By contrast, if θ is a parameter that shifts (equity) investors' beliefs in a hazard rate sense, then this covariance is negative, which makes a binding leverage cap (even) less attractive.

Alternative policy problem 2/Commitment: Hauk, Lanteri and Marcet (2021) study a planner who maximizes expected welfare $\mathbb{E}\left[W\left(\tau,s,A\right)\right]$, where τ is a scalar policy variable, $s=h\left(\tau,A\right)$ is a quantity/signal determined in equilibrium, and A is the state of the economy. The planner observes s but not A, and commits to a mapping $\tau=\mathcal{R}\left(s\right)$ based on the observable signal. In our setting, we can map $\tau\to\bar{b}$ to the (binding) leverage requirement, $s\to k^{\star}$ to equilibrium capital investment, and $A\to\theta=\left\{F^{I},F^{C}\right\}$ to any uncertainty about agents' beliefs. Under appropriate regularity conditions, a variational argument in Proposition 2 in Hauk, Lanteri and Marcet (2021) implies that an optimal commitment satisfies the following first-order condition:

$$\mathbb{E}\left[\left.\frac{\frac{\partial W}{\partial b}k^{\star}+\frac{\partial W}{\partial k}\frac{dk^{\star}}{d\bar{b}}}{1-\frac{dk^{\star}}{d\bar{b}}\mathcal{R}'\left(k^{\star}\right)}\right|k^{\star}\right]=0.$$

Evaluating the derivatives, we obtain

$$\mathbb{E}\left[\left.\frac{\left(\frac{dM^{P}\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right)k^{\star}+\left(M^{P}\left(\bar{b}\right)-\Delta\left(b\right)-M\left(\bar{b};\tilde{\theta}\right)\right)\frac{dk^{\star}}{d\bar{b}}}{1-\frac{dk^{\star}}{d\bar{b}}\mathcal{R}'\left(k^{\star}\right)}\right|k^{\star}\right]=0,$$

where all terms inside the expectation are evaluated at $\bar{b} = \mathcal{R}(k^{\star})$. Since the first term in the numerator is known conditional on $k^{\star}$, and defining the random variable $\Omega = \left(1 - \frac{dk^{\star}}{d\bar{b}}\mathcal{R}'(k^{\star})\right)^{-1}$,we can re-write this condition as

$$\begin{aligned}\mathbb{E}\left[\Omega|\, k^{\star}\right]\left(\frac{dM^{P}\left(\bar{b}\right)}{db} - \frac{d\Delta\left(\bar{b}\right)}{db}\right)k^{\star}\\ +\mathbb{E}\left[\Omega|k^{\star}\right]\mathbb{E}\left[\left(M^{P}\left(\bar{b}\right) - \Delta(b) - M\left(\bar{b};\tilde{\theta}\right)\right)\frac{dk^{\star}}{d\bar{b}}\right| k^{\star}\right]\\ +\mathbb{C}ov\left[\Omega, \left(M^{P}\left(\bar{b}\right) - \Delta(b) - M\left(\bar{b};\tilde{\theta}\right)\right)\frac{dk^{\star}}{d\bar{b}}\right] = 0.\end{aligned}$$

Noting that $M\left(\bar{b};\tilde{\theta}\right)$ is also known conditional on $k^{\star}$, we can further simplify to obtain

$$\begin{aligned}\left(\frac{dM^{P}\left(\bar{b}\right)}{db} - \frac{d\Delta\left(\bar{b}\right)}{db}\right)k^{\star}\\ +\left(M^{P}\left(\bar{b}\right) - \Delta(b) - M\left(\bar{b};\tilde{\theta}\right)\right)\left\{\mathbb{E}\left[\frac{dk^{\star}}{d\bar{b}}\right| k^{\star}\right] + \mathbb{C}ov\left[\hat{\Omega}, \frac{dk^{\star}}{d\bar{b}}\right]\right\} = 0,\end{aligned}$$

where $\hat{\Omega} = \frac{\Omega}{E[\Omega|k^{\star}]}$. This condition is the same as in the case discussed in the paper, except for the final term, which depends on $Cov\left[\Omega, \frac{dk^{\star}}{d\bar{b}}\right]$. As discussed in more detail in Hauk, Lanteri and Marcet (2021), Ω corrects the first-order condition for the change in the probability distribution of $k^{\star}$ when the planner alters her ex-ante commitment. If this change of measure is correlated with policy elasticities, it introduces an additional marginal welfare effect. By contrast, a planner in a "consistent" equilibrium, in the sense that we discuss in the text, does not take this change into account since she is able to adjust her policy *ex-post*, after $k^{\star}$ is realized.

C.1.2 Robust optimal policy

Applying the envelope theorem to the planner's problem, and noting that the constraint set Θ does not depend on $\bar{b}$, we can write the marginal welfare effect of varying $\bar{b}$ as

$$\begin{aligned}\frac{d}{d\bar{b}}\min_{\theta\in\Theta} W\left(\bar{b}, k^{\star}\left(\bar{b};\theta\right)\right) = \left[\frac{dM^{P}\left(\bar{b}\right)}{db} - \frac{d\Delta\left(\bar{b}\right)}{db}\right]k^{\star}\left(\bar{b};\hat{\theta}\left(\bar{b}\right)\right)\\ + \left[M^{P}\left(\bar{b}\right) - \Delta(b) - M\left(\bar{b}\right)\right]\frac{\partial k^{\star}\left(\bar{b};\hat{\theta}\left(\bar{b}\right)\right)}{\partial\bar{b}},\end{aligned}$$

where $\hat{\theta}\left(\bar{b}\right)$ denotes the solution to the minimization problem. Thus, the planner considers the same inframarginal and incentive effects as in the baseline model, but now evaluates them at the worst-case beliefs.

To gain further insights, we characterize the worst-case beliefs in two steps. First, we note that by Lemma 4, welfare for any given $\bar{b}$ depends on beliefs only through their impact on $k^\star$. In the case with quadratic adjustment costs, welfare is a quadratic function of $k^\star\left(\bar{b};\theta\right)$. Thus, we can equivalently define the worst-case belief as the value of θ that induces the largest deviation from the planner's preferred value of $k^\star$. Moreover, noting that $k^\star$ is a linear function of the market valuation $M\left(b;\theta\right)$, we can write the worst case beliefs in terms of the largest mis-valuation of capital investments:

$$\hat{\theta}\left(\bar{b}\right) = \arg\max_{\theta\in\Theta}\left|M\left(\bar{b};\theta\right) - M^P\left(\bar{b}\right)\right|, \tag{3}$$

We formally define the set of plausible beliefs Θ in the planner's problem as follows:

$$\Theta = \left\{\left(m^I, m^C\right) : \sum_{j\in\{I,C\}} \mathbb{E}^{j,P}\left[m^j \log m^j\right] \leq \mathcal{D},\, \mathbb{E}^{j,P}\left[m^j\right] = 1\right\}.$$

This set contains all beliefs such that the sum of relative entropies across investors and creditors, relative to the planner's beliefs, is at most $\mathcal{D}$. The worst-case beliefs in problem (3) must also solve the dual problem

$$\begin{aligned}\min_{\theta} \sum_{j\in\{I,C\}} \mathbb{E}^{j,P}\left[m^j \log m^j\right] \text{ subject to}& \\ k^\star\left(\bar{b};\theta\right) &= \mathcal{K}, \\ \mathbb{E}^{j,P}\left[m^j\right] &= 1.\end{aligned}$$

where $\mathcal{K}$ is the value of $k^\star\left(\bar{b};\theta\right)$ that is achieved by the maximum in problem (3). Equivalently, noting that there is a one-to-one relationship between $k^\star\left(\bar{b};\theta\right)$ and the market valuation $M\left(\bar{b};\theta\right)$, and expanding the expected values, we can write this problem as

$$\min_{\theta} \sum_{j\in\{I,C\}} \int_{\underline{s}}^{\bar{s}} m^j(s) \log m^j(s)\, dF^{j,P}(s) \text{ subject to}$$

$$\beta^I \int_b^{\bar{s}} (s-b)\, m^I(s)\, dF^{I,P}(s)$$

$$+\beta^C \left(\int_b^{\bar{s}} b m^C(s)\, dF^{C,P}(s) + \phi \int_{\underline{s}}^{b} s m^C(s)\, dF^C(s) \right) = \mathcal{M}, \ \left(\upsilon^0\right)$$

$$\int_{\underline{s}}^{\bar{s}} m^I(s)\, dF^{I,P}(s) = 1, \ \left(\upsilon^I\right)$$

$$\int_{\underline{s}}^{\bar{s}} m^C(s)\, dF^{C,P}(s) = 1, \ \left(\upsilon^C\right),$$

where the variables in brackets denote the relevant Lagrange multipliers. Taking the first-order conditions yields:

$$1 + \log m^I(s) = \upsilon^I + \upsilon^0 \beta^I \max\left\{ s - \bar{b}, 0 \right\} \tag{4}$$

$$1 + \log m^C(s) = \begin{cases} \upsilon^C + \upsilon^0 \beta^C \bar{b}, & s \geq \bar{b} \\ \upsilon^C + \upsilon^0 \beta^C \phi s, & s < \bar{b}, \end{cases} \tag{5}$$

where $m^j(s)$ denotes the proportional distortion (i.e., a Radon-Nikodym derivative) of agent j's beliefs relative to the planner's, and where υ^0, υ^I, and υ^C are Lagrange multipliers defined in the Online Appendix.

Equation (4) characterizes the worst-case distortion $m^I(s)$ to investors' beliefs. For a given $\bar{b}$, investors' beliefs about default states with $s < b$ do not affect market valuations or capital investment. Hence, it is optimal in those states to implement a fixed distortion $m^I(s) = e^{\upsilon^I - 1}$.[2] The optimal distortion in solvent states, by contrast, scales with the value of investors' equity claim $s - \bar{b}$. Equation (5) characterizes the worst case for creditors' beliefs. Once again, in order to maximize capital distortions, these conditions imply that belief distortions are scaled with the value of creditors' claims in each (default or solvent) state of the world. In conclusion, we find that a planner gears robust optimal policy towards belief distortions that most heavily affect market valuations. The worst-case belief distortions correlate with the market valuations of debt and equity claims.

[2] If the worst-case capital distortion in problem (3) features overinvestment with $M\left(\bar{b};\theta\right) > M^P\left(\bar{b}\right)$, then this fixed distortion satisfies $m^I(s) < 1$, and vice versa.

C.2 Monetary policy

In the model with monetary policy described in Section 4.2, investors' problem is equivalent to the proof of Lemma 1, except that the price of debt is given by

$$Q\left(b,r\right)=\beta\left(r\right)\left(\int_{b}^{\overline{s}}bdF^{C}\left(s\right)+\phi\int_{\underline{s}}^{b}sdF^{C}\left(s\right)\right),$$

with β^{C} now given by $\beta\left(r\right)=\frac{1}{1+r}$. All our results in Section 2 of the paper therefore apply once we replace the market value of investment $M\left(b\right)$ in investors' objective function with

$$M\left(b,r\right)=\beta^{I}\int_{b}^{\overline{s}}\left(s-b\right)dF^{I}\left(s\right)+\beta\left(r\right)\left(\int_{b}^{\overline{s}}bdF^{C}\left(s\right)+\phi\int_{\underline{s}}^{b}sdF^{C}\left(s\right)\right).$$

A parallel argument to Lemma 4, taking into account the deadweight loss $\mathcal{L}\left(r\right)$ of interest rate distortions, establishes that the planner's problem is

$$\max_{\overline{b},r}W\left(b^{\star}\left(\overline{b},r\right),k^{\star}\left(\overline{b},r\right),r\right),$$

where the welfare function is

$$W\left(b,k,r\right)=\left[M^{P}\left(b,r\right)-\Delta\left(b\right)-1\right]k-\Upsilon\left(k\right)-\mathcal{L}\left(r\right).$$

Taking the total derivative with respect to r yields the expression used in the paper. Finally, totally differentiating investors' first-order conditions yields

$$\frac{dk^{\star}\left(\overline{b},r\right)}{dr}=\frac{1}{\Upsilon''\left(k^{\star}\right)}\frac{dM\left(\overline{b},r\right)}{dr}=\frac{1}{\Upsilon''\left(k^{\star}\right)}\frac{\beta'\left(r\right)}{\beta\left(r\right)}Q\left(b,r\right),$$

which for a given value of r is increasing in the bond price $Q\left(b,r\right)$. We conclude that, as claimed in the text, $\frac{dk^{\star}\left(\overline{b},r\right)}{dr}$ is not diminished by equity exuberance (which leaves $Q\left(b,r\right)$ unchanged), and is increased by debt exuberance (which increases $Q\left(b,r\right)$).

C.3 Alternative micro-foundations for externalities

C.3.1 Ex-post government bailouts

With government bailouts, investors' problem is equivalent to the proof of Lemma 1, except for investors' date 1 budget constraints and the equilibrium price of debt, which are now given by

$$
\begin{aligned}
c_1^I(s) &= n_1^I(s) + \max\{s + t(b,s) - b, 0\}\, k, \forall s \\
Q(b) &= \beta^C \left(\int_{s^\star(b)}^{\bar{s}} b\, dF^C(s) + \int_{\underline{s}}^{s^\star(b)} (\phi s + t(b,s))\, dF^C(s) \right),
\end{aligned}
$$

where $s = s^\star(b)$ solves the equation $s + t(b,s) - b = 0$, and is uniquely defined as long as $t(b,s)$ is decreasing in s and increasing in b. All our results in Section 2 therefore apply once we replace the market value of investment $M(b)$ in investors' objective function with

$$
M(b) = \beta^I \int_{s^\star(b)}^{\bar{s}} (s + t(b,s) - b)\, dF^I(s) + \beta^C \left(\int_{s^\star(b)}^{\bar{s}} b\, dF^C + \int_{\underline{s}}^{s^\star(b)} (\phi s + t(b,s))\, dF^C(s) \right).
$$

A parallel argument to Lemma 4 establishes that the planner's problem is

$$
\max_{\bar{b}} W\left(b^\star\left(\bar{b}\right), k^\star\left(\bar{b}\right)\right),
$$

where the welfare function is

$$
W(b,k) = \left[M^P(b) - \Delta(b) - 1\right] k - \Upsilon(k),
$$

with the externality $\Delta(b)$ defined in the paper, and with the planner's valuation of investment defined as

$$
\begin{aligned}
M^P(b) = \beta^I & \int_{s^\star(b)}^{\bar{s}} (s + t(b,s) - b)\, dF^{I.P}(s) \\
&+ \beta^C \left(\int_{s^\star(b)}^{\bar{s}} b\, dF^{C,P} + \int_{\underline{s}}^{s^\star(b)} (\phi s + t(b,s))\, dF^{C,P}(s) \right).
\end{aligned}
$$

With these alternative definitions, we can repeat the steps leading to Propositions 4 and 5 to establish the marginal welfare effects that we used in the paper.

Finally, taking variational derivatives and integrating by parts, we obtain the effect of belief distortions on market valuations as

$$
\begin{aligned}
\frac{\delta M}{\delta F^I} \cdot G^I &= \beta^I \int_{s^\star(b)}^{\bar{s}} (s + t(b,s) - b)\, dG^I(s) \\
&= -\beta^I \int_{s^\star(b)}^{\bar{s}} \left(1 + \frac{\partial t(b,s)}{\partial s}\right) G^I(s)\, ds,
\end{aligned}
$$

and

$$\frac{\delta M}{\delta F^C}\cdot G^C = \beta^C\left(\int_{s^\star(b)}^{\bar{s}} b dG^C(s) + \int_{\underline{s}}^{s^\star(b)}(\phi s + t(b,s))\,dG^C(s)\right)$$
$$= -\beta^C\left((1-\phi)\,s^\star(b)\,G^C(s^\star(b)) + \int_{\underline{s}}^{s^\star(b)}\left(\phi + \frac{\partial t(b,s)}{\partial s}\right)G^C(s)\,ds\right).$$

These expressions show that, if bailouts satisfy $\frac{\partial t(b,s)}{\partial s}\leq 0$, then the presence of bailouts attenuates the sensitivity of the market valuation $M(b)$ to the changes in beliefs $G^I(s)$ and $G^C(s)$. Moreover, if bailouts are convex in s, so that $\frac{\partial t(b,s)}{\partial s}$ is larger in absolute value for low s, then the attenuation effect is skewed towards belief distortions in bad states. Intuitively, bailouts imply that agents' beliefs about downside risk become less important for market valuation.

We further obtain the effect of belief distortions on the marginal value of leverage as

$$\frac{\delta\frac{dM}{db}}{\delta F^I}\cdot G^I = \beta^I\left(-\int_{s^\star(b)}^{\bar{s}} dG^I(s) + \int_{s^\star(b)}^{\bar{s}}\frac{\partial t(b,s)}{\partial b}dG^I(s)\right)$$
$$= \beta^I\left(\left(1-\frac{\partial t(b,s^\star(b))}{\partial b}\right)G^I(s^\star(b)) - \int_{s^\star(b)}^{\bar{s}}\frac{\partial^2 t(b,s)}{\partial b\partial s}G^I(s)\,ds\right),$$

and

$$\frac{\delta\frac{dM}{db}}{\delta F^C}\cdot G^C = -\beta^C G^C(s^\star(b))\frac{\partial s^\star(b)}{\partial b}\left(1+\frac{\partial t(b,s^\star(b))}{\partial s} + (1-\phi)s^\star(b)\frac{g^C(s^\star(b))}{G^C(s^\star(b))}\right)$$
$$-\beta^C\int_0^{s^\star(b)}\frac{\partial^2 t(b,s)}{\partial b\partial s}G^C(s)\,ds.$$

If bailouts satisfy $\frac{\partial t(b,s)}{\partial b}\geq 0$, then the effect of changes in beliefs over the marginal default state $s^\star(b)$ on the marginal valuation $\frac{dM}{db}$ is attenuated towards zero by the presence of bailouts. In addition, both variational derivatives of $\frac{dM}{db}$ contain a term with the sign of $-\frac{\partial^2 t(b,s)}{\partial b\partial s}G^j(s)$ for $j\in\{I,C\}$. These terms arise because changes in beliefs affect investors' strategic incentive to take on leverage in order to increase bailouts. If the strategic incentive $\frac{\partial t(b,s)}{\partial b}$ is decreasing in s, then optimism increases $\frac{dM}{db}$. Bailouts are often modeled as a convex function of the shortfall $b-s$ of asset values from debt obligations. This directly implies $\frac{\partial^2 t(b,s)}{\partial b\partial s}\leq 0$.

C.3.2 Fire sales/Pecuniary externalities

Detailed description of the environment: We consider an extension of our baseline model with three time periods $t=0,1,2$. There are three type of agents: Investors, creditors and households. Investors' and creditors' preferences are given by

$$
\begin{aligned}
U^I &= c_0^I + \beta^I \mathbb{E}^I \left[c_1^I + c_2^I \right] \\
U^C &= c_0^C + \beta^C \mathbb{E}^C \left[c_1^C + c_2^C \right].
\end{aligned}
$$

Investors' budget constraints at each date are

$$
\begin{aligned}
c_0^I &= n_0^I - k_0 - \Upsilon\left(k_0\right) + Q\left(b, \lambda\right) k_0 \\
c_1^I &= q\left(k_0 - k_1\right) - \xi k_0 \\
c_2^I &= \begin{cases} sk_1 - b\lambda k_0, & s \in \mathcal{N} \\ 0, & s \in \mathcal{D}. \end{cases}
\end{aligned}
$$

Investors are endowed with n_0^I at date 0. They raise $Q\left(b, \lambda\right) k_0$ at date 0 from creditors, where $\lambda = 1 - \frac{\xi}{q}$ represents the market price of capital at date 1, which investors take as given. This finances their expenditure on capital and consumption. At date 1, investors have no endowment and sell $k_0 - k_1$ units of capital and pay the reinvestment requirement ξ per unit of k_0. At date 2, in non-default states (denoted $s \in \mathcal{N}$), investors consume the difference between the value of their remaining capital and the face value of their bonds. As described in the text, we write b for the normalized face per unit of λk.

Creditors' budget constraints are

$$
\begin{aligned}
c_0^C &= n_0^C - hQ\left(b, \lambda\right) k_0 \\
c_1^C &= 0 \\
c_2^C &= h \begin{cases} b\lambda k_0, & s \in \mathcal{N} \\ \phi s k_1, & s \in \mathcal{D}. \end{cases}
\end{aligned}
$$

Creditors buy a fraction h of investors' debt at date 0. They have no endowment or consumption at date 1. At date 2, they are repaid the face value of their debt if investors do not default, and extract a fraction ϕ of the value of remaining capital otherwise.

Households are active at dates 1 and 2. Their preferences are given by

$$
U^H = c_1^H + \mathbb{E}^H \left[c_2^H \right].
$$

Their budget constraints are

$$
\begin{aligned}
c_1^H &= n_1^H - qk_1^H \\
c_2^H &= F\left(k_1^H\right).
\end{aligned}
$$

Households have an endowment at date 1, which they can spend on purchasing capital k_1^H. Capital purchases yield consumption $F\left(k_1^H\right)$ at date 2. We assume that $F\left(k_1^H\right)$ is concave and satisfies $F'(0) < (1-\phi)\mathbb{E}^I[s]$. The latter inequality is a sufficient condition ensuring that, in equilibrium, investors will never sell more capital to households than is necessary to cover their reinvestment need.

Equilibrium characterization: At date 1, the non-negativity constraint for investors' consumption always binds. Setting $c_1^I = 0$, we obtain

$$
k_1 = \left(1 - \frac{\xi}{q}\right) k_0 \equiv \lambda k_0.
$$

This implies that investors' asset values at date 2 are

$$
sk_1 = \lambda s k_0.
$$

Optimally, the following condition therefore determines investors' default choice:

$$
\begin{aligned}
s \in \mathcal{D} &\Leftrightarrow sk_1 < b\lambda k_0 \\
&\Leftrightarrow s < b.
\end{aligned}
$$

By creditors' optimality conditions, the price of debt at date 0, per unit of k_0, satisfies

$$
\begin{aligned}
Q(b,\lambda) &= \beta^C \left[\int_b^{\bar{s}} \lambda b dF^C(s) + \phi \int_{\underline{s}}^{b} \lambda s dF^C(s)\right] \\
&= \lambda \beta^C \left[\int_b^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{b} s dF^C(s)\right].
\end{aligned}
$$

Following the steps leading to Lemma 1 in the baseline model, we now find that investors' problem can be re-written as

$$
\max_{b,k_0} \left[\lambda M(b) - 1\right] k_0 - \Upsilon(k_0) \text{ subject to } b \leq \bar{b},
$$

where $M(b)$ is defined as in the paper. Investors' first-order conditions determining the optimal choice of b and k_0 are therefore

$$\lambda \frac{dM(b)}{db} = \mu \tag{6}$$

$$\lambda M(b) - 1 = \Upsilon'(k_0). \tag{7}$$

In addition, households' first-order condition for capital purchases, which is given by

$$q = F'\left(k_1^H\right),$$

and substituting the market clearing requirement that $k_1^H = k_0 - k_1$, we obtain the equilibrium relationship

$$q = F'(k_0 - k_1) = F'\left(\frac{\xi}{q} k_0\right) = F'((1-\lambda) k_0).$$

Using the definition $1 - \frac{\xi}{q} = \lambda$, we can write $q = \frac{\xi}{1-\lambda}$, which implies that

$$\xi = (1-\lambda) F'((1-\lambda) k_0). \tag{8}$$

For any given $\bar{b}$, equilibrium is obtained by solving equations (6), (7) and (8) for the unknown variables $k_0 = k_0^\star\left(\bar{b}\right)$, $b = b^\star\left(\bar{b}\right)$ and $\lambda = \lambda^\star\left(\bar{b}\right)$. We can further write (7) as

$$k_0 = \Psi(\lambda M(b) - 1),$$

where $\Psi(.)$ is the inverse of $\Upsilon'(.)$, and substitute into (8) to obtain

$$\xi = (1-\lambda) F'((1-\lambda) \Psi(\lambda M(b) - 1)),$$

This equation defines an implicit mapping from market valuations $M(b)$ to equilibrium prices λ. Let $\lambda = \Lambda(M(b))$ denote this mapping. Whenever the leverage cap is binding, we can now write equilibrium prices as

$$\lambda^\star\left(\bar{b}\right) = \Lambda\left(M\left(\bar{b}\right)\right), \tag{9}$$

which we will employ in our welfare analysis below.

Welfare analysis: The planner maximizes the sum of agents' utilities in equilibrium. Ignoring exogenous endowments and repeating the steps leading to Lemma 4, we can write and simplify welfare as follows:

$$\begin{aligned} W = U^I + U^C + U^H &\propto \left[\lambda M^P(b) - 1\right] k_0 - \Upsilon\left(k_0\right) + F\left(k_1^H\right) - q k_1^H \\ &= \left[\lambda M^P(b) - 1\right] k_0 - \Upsilon\left(k_0\right) + F\left(k_1^H\right) - \frac{\xi}{1-\lambda} k_1^H \\ &= \left[\lambda M^P(b) - 1\right] k_0 - \Upsilon\left(k_0\right) + F\left(\left(1-\lambda\right) k_0\right) - \xi k_0 \\ &\equiv W\left(b, k_0, \lambda\right). \end{aligned}$$

where the third line follows by substituting the market clearing condition $k_1^H = k_0 - k_1$. Notice that we can write

$$\begin{aligned} \frac{\partial W}{\partial \lambda} &= M^P\left(b\right) k_0 + \underbrace{\left\{F'\left(k_1^H\right) - q\right\}}_{=0} \frac{dk_1^H}{d\lambda} \\ &= M^P\left(b\right) k_0, \end{aligned}$$

where the second line follows by substituting households' first-order condition for capital purchases. The planner sets $\bar{b}$ to maximize

$$W\left(b^\star\left(\bar{b}\right), k_0^\star\left(\bar{b}\right), \lambda^\star\left(\bar{b}\right)\right).$$

Whenever the leverage cap is binding, totally differentiating this expression yields

$$\begin{aligned} \frac{dW}{d\bar{b}} &= \left[\lambda \frac{dM^P\left(\bar{b}\right)}{d\bar{b}} + M^P\left(\bar{b}\right) \frac{d\lambda}{d\bar{b}}\right] k_0^\star\left(\bar{b}\right) + \left[\lambda M^P\left(\bar{b}\right) - 1 - \Upsilon'\left(k_0^\star\left(\bar{b}\right)\right)\right] \frac{dk_0^\star\left(\bar{b}\right)}{d\bar{b}} \\ &= \left[\lambda \frac{dM^P\left(\bar{b}\right)}{d\bar{b}} + M^P\left(\bar{b}\right) \frac{d\lambda}{d\bar{b}}\right] k_0^\star\left(\bar{b}\right) + \lambda \left[M^P\left(\bar{b}\right) - M\left(\bar{b}\right)\right] \frac{dk_0^\star\left(\bar{b}\right)}{d\bar{b}}, \end{aligned}$$

where the second line substitutes investors' first-order condition. Dividing both sides by $\lambda > 0$ yields the marginal welfare effects used in the paper.

Further welfare analysis The response of equilibrium prices to $\bar{b}$, using the mapping derived in Equation (9), can be expressed as

$$\frac{d\lambda}{d\bar{b}} = \Lambda'\left(M\left(\bar{b}\right)\right) \frac{dM\left(\bar{b}\right)}{db}.$$

To obtain a tractable characterization of this effect to belief distortions, we use a log-linear approximation

$$\log \Lambda(M) \simeq \lambda_0 - \lambda_1 M.$$

We now obtain

$$\frac{d\log\lambda}{d\bar{b}} = \underbrace{\frac{\Lambda'\left(M\left(\bar{b}\right)\right)}{\Lambda\left(M\left(\bar{b}\right)\right)}}_{=-\lambda_1} \frac{dM\left(\bar{b}\right)}{d\hat{b}}.$$

Hence, we find that

$$\frac{\delta\left(\frac{d\log\lambda}{d\bar{b}}\right)}{\delta F^j} \cdot G^j = -\lambda_1 \left[\frac{\delta \frac{dM\left(\bar{b}\right)}{db}}{\delta F^j} \cdot G^j\right].$$

We note that, as discussed in the paper, this effect scales with the variational derivative of the marginal value of leverage, which we have characterized in Lemma 2.

C.4 Endogenous belief distortions

When beliefs are endogenous to aggregate capital investments K, which individual investors take as given, we write the market value of capital (i.e., the equivalent to $M(b)$ in Lemma 1) as

$$M(b;K) = \beta^I \int_b^{\bar{s}} (s-b)\, dF^I(s;K) + \beta^C \left[\int_b^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{b} s dF^C(s;K)\right].$$

The first-order conditions of an individual investor are, as in the baseline model,

$$\frac{\partial M(b;K)}{\partial b} = \mu$$
$$M(b;K) = 1 + \Upsilon'(k).$$

Substituting the consistency requirement that $k = K$ and totally differentiating, we obtain

$$\frac{\partial M\left(\bar{b};k\right)}{\partial b} + \frac{\partial M\left(\bar{b};k\right)}{\partial K}\frac{dk}{d\bar{b}} = \Upsilon''(k)\frac{dk}{d\bar{b}}$$
$$\Rightarrow \frac{dk^\star}{d\bar{b}} = \frac{\partial M\left(\bar{b};k^\star\right)}{\partial b}\left(\Upsilon''(k) - \frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-1}. \tag{10}$$

Similarly, the total variational derivative with respect to beliefs F^j satisfies

$$\frac{\delta M\left(\bar{b};k\right)}{\delta F^j}\cdot G^j+\frac{\partial M\left(\bar{b};k\right)}{\partial K}\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right]=\Upsilon''(k)\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right]$$
$$\Rightarrow\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j=\left[\frac{\delta M\left(\bar{b};k^\star\right)}{\delta F^j}\cdot G^j\right]\left(\Upsilon''(k)-\frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-1}. \tag{11}$$

We can further compute, with quadratic adjustment costs,

$$\begin{aligned}\frac{\delta\frac{dk^\star}{d\bar{b}}}{\delta F^j}\cdot G^j&=\frac{\delta\left(\frac{\partial M\left(\bar{b};k\right)}{\partial b}\left(\Upsilon''(k)-\frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-1}\right)}{\delta F^j}\cdot G^j\\&=\left[\frac{\delta\frac{\partial M}{\partial b}}{\delta F^j}\cdot G^j\right]\left(\Upsilon''(k)-\frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-1}\\&\quad+\frac{\partial M\left(\bar{b};k\right)}{\partial b}\left(\Upsilon''(k)-\frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-2}\frac{\delta\frac{\partial M}{\partial K}}{\delta F^j}\cdot G^j.\end{aligned} \tag{12}$$

Repeating the steps leading to Proposition 4, we find that the marginal welfare effect of varying $\bar{b}$ is given by

$$\frac{dW}{d\bar{b}}=\left[\frac{dM^P\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]k^\star\left(\bar{b}\right)+\left[M^P\left(\bar{b}\right)-\Delta\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\frac{dk^\star}{d\bar{b}}.$$

Taking variational derivatives, we then obtain

$$\begin{aligned}\frac{\delta\frac{dW}{d\bar{b}}}{\delta F^j}\cdot G^j&=\left[\frac{dM^P\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}\right]\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right]\\&\quad+\left[M^P\left(\bar{b}\right)-\Delta\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\left[\frac{\delta\frac{dk^\star}{d\bar{b}}}{\delta F^j}\cdot G^j\right]\\&\quad-\frac{dk^\star}{d\bar{b}}\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right].\end{aligned} \tag{13}$$

Now we divide (10) by (11) to obtain

$$\frac{dk^\star}{d\bar{b}}\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right]=\frac{\partial M\left(\bar{b};k^\star\right)}{\partial b}\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right].$$

Substituting into (13) establishes that

$$\begin{aligned}\frac{\delta\frac{dW}{d\bar{b}}}{\delta F^j}\cdot G^j=&\left[\frac{dM^P\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}-\frac{\partial M\left(\bar{b};k^\star\right)}{\partial b}\right]\left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j}\cdot G^j\right]\\&+\left[M^P\left(\bar{b}\right)-\Delta\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\left[\frac{\delta\frac{dk^\star}{d\bar{b}}}{\delta F^j}\cdot G^j\right].\end{aligned}\tag{14}$$

Finally, substituting (11) and (12) into (14), we get

$$\begin{aligned}\frac{\delta\frac{dW}{d\bar{b}}}{\delta F^j}\cdot G^j=&\underbrace{\left(\Upsilon''\left(k\right)-\frac{\partial M\left(\bar{b};k\right)}{\partial K}\right)^{-1}}_{\text{Multiplier }A}\left\{\left[\frac{dM^P\left(\bar{b}\right)}{db}-\frac{d\Delta\left(\bar{b}\right)}{db}-\frac{\partial M\left(\bar{b};k^\star\right)}{\partial b}\right]\left[\frac{\delta M\left(\bar{b};k^\star\right)}{\delta F^j}\cdot G^j\right]\right.\\&\left.+\left[M^P\left(\bar{b}\right)-\Delta\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\left[\frac{\delta\frac{\partial M}{\partial b}}{\delta F^j}\cdot G^j\right]\right\}\\&+\left[M^P\left(\bar{b}\right)-\Delta\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\frac{\partial M\left(\bar{b};k^\star\right)}{\partial b}\left(\Upsilon''\left(k\right)-\frac{\partial M\left(\bar{b};k^\star\right)}{\partial K}\right)^{-2}\frac{\delta\frac{\partial M}{\partial K}}{\delta F^j}\cdot G^j.\end{aligned}\tag{15}$$

The first two lines are the same as in the baseline model, but are multiplied by the factor A defined in the text. The final term, which arises only if $M(b;K)$ is not linear in K, is also amplified, and increases the magnitude of the incentive effect if a distortion increases the responsiveness of M to K.

D Additional Proofs and Derivations

D.1 Regularity conditions

Note that investors always find it optimal to choose non-negative leverage in equilibrium, since

$$\left.\frac{dM}{db}\right|_{b=0}=\beta^C-\beta^I>0.$$

Therefore, for a given leverage constraint $\bar{b}$, our problem always features a solution for leverage in $\left[0,\bar{b}\right]$ and a finite solution for investment, since $\frac{d^2V}{dk^2}=-\Upsilon''\left(k\right)<0$. A sufficient condition that guarantees a finite solution without leverage regulation is that creditors perceive the net present value of investment to be negative if there is always default, that is, $\beta^C \phi \mathbb{E}^C [s] < 1$, since

$$\lim_{b \to \infty} M(b) = \beta^C \phi \mathbb{E}^C [s].$$

This sufficient condition extends directly to the environment with bailouts in Section 4, after imposing that bailouts are bounded above, $t(b,s) \leq \bar{t}$, and that investment has negative net present value if always in distress, even under the maximum bailout, $\beta^C \left(\phi \mathbb{E}^C [s] + \bar{t}\right) < 1$.

In order to explore the quasi-concavity of the investors' objective, it is useful to normalize $\frac{dM}{db}$, characterized in the paper, as follows:

$$J(b) = \frac{\frac{dM}{db}}{\beta^C \left(1 - F^C(b)\right)} \equiv 1 - \frac{\beta^I}{\beta^C} \frac{1 - F^I(b)}{1 - F^C(b)} - (1-\phi)\, b \frac{f^C(b)}{1 - F^C(b)},$$

where the normalization is valid for any non-zero level of b. Therefore, it follows that the quasi-concavity of the investors' objective can be established by characterizing the conditions under which $J'(b)$ is negative. Note that

$$J'(b) = -\frac{\beta^I}{\beta^C} \frac{\partial}{\partial b} \left(\frac{1 - F^I(b)}{1 - F^C(b)} \right) - (1-\phi) \left[\frac{f^C(b)}{1 - F^C(b)} + b \frac{\partial}{\partial b} \left(\frac{f^C(b)}{1 - F^C(b)} \right) \right].$$

There are two sufficient conditions that, when jointly satisfied, guarantee that $J'(b) < 0$. First, when the hazard rate of creditors' beliefs is monotone increasing, then

$$\frac{\partial}{\partial b} \left(\frac{f^C(b)}{1 - F^C(b)} \right) > 0.$$

Second, if investors are more optimistic than creditors in the hazard-rate sense, then

$$\frac{\partial}{\partial b} \left(\frac{1 - F^I(b)}{1 - F^C(b)} \right) > 0.$$

Therefore, when both conditions are satisfied, we have $J'(b) < 0$, which yields the result. We formally state this result as Lemma 1.

Lemma 1. ***(Single-peaked objective function without bailouts)*** *Suppose that there is no bailout, and:*

1. *Equity investors are weakly more optimistic than creditors in the hazard-rate order;*
2. *Creditors' hazard rate $\frac{f^C(s)}{1 - F^C(s)}$ is increasing in s.*

Then $M(b)$ is single peaked.

Notice that the solution for optimal leverage can be expressed in general as follows:

$$b = \frac{1}{(1-\phi)\frac{f^C(b)}{1-F^C(b)}}\left(1-\frac{\beta^I}{\beta^C}\frac{1-F^I(b)}{1-F^C(b)}\right).$$

Note also that whenever $\beta^I = \beta^C$, $\left.\frac{dM}{db}\right|_{b=0} = 0$, but the rest of the results remain valid.

D.2 Variations and cumulative distribution functions

For simplicity, we drop the superscript j and work with $F(s)$ and $G(s)$ in this appendix. Recall that a function $F(s)$ is a cumulative distribution function if and only if it is non-decreasing, right-continuous, and satisfies $F(\underline{s}) = 0$ and $F(\overline{s}) = 1$. We say a variation $G(s)$ of beliefs is valid if, for small enough ε, the perturbed belief $F(s) + \varepsilon G(s)$ remains a cumulative distribution function.

Definition 1. A right-continuous function $G(s)$ is a *valid variation* of a cumulative distribution function $F(s)$ if $G(\overline{s}) = G(\underline{s}) = 0$, and there exists an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in [0, \bar{\varepsilon}]$, the following conditions are satisfied:

1. $F(s) + \varepsilon G(s)$ is non-decreasing in s;

2. $0 \leq F(s) + \varepsilon G(s) \leq 1$, $\forall s$.

The following lemma shows that our regularity conditions in the baseline model are sufficient to guarantee that all variations are valid.

Lemma 2. *(Regularity conditions on belief variations) If (i) $F(s)$ and $G(s)$ are continuously differentiable, (ii) $f(s) = F'(s) > 0$, and (iii) $G(\overline{s}) = G(\underline{s}) = 0$, then $G(s)$ is a valid variation of $F(s)$.*

Proof. By assumption, $f(s) = F'(s)$ and $g(s) = G'(s)$ are continuous and therefore bounded on the interval $[\underline{s}, \overline{s}]$, so that we can define $\underline{f} = \inf\{g(s)|\, s \in [\underline{s}, \overline{s}]\} > 0$ and $\underline{g} = \inf\{f(s)|\, s \in [\underline{s}, \overline{s}]\}$. For all s, we have

$$F'(s) + \varepsilon G'(s) = f(s) + \varepsilon g(s) \geq \underline{f} + \varepsilon \underline{g},$$

Hence, $F(s) + \varepsilon G(s)$ is non-decreasing for all $\varepsilon \leq \underline{f}/|\underline{g}| \equiv \bar{\varepsilon}$.

Moreover, note that, for all $\varepsilon \leq \bar{\varepsilon}$, and for all s, we have

$$F(s) + \varepsilon G(s) = F(\underline{s}) + \varepsilon \underbrace{G(\underline{s})}_{=0} + \int_{\underline{s}}^{s} \underbrace{(f(s) + \varepsilon g(s))}_{\geq 0,\, \forall \varepsilon \leq \bar{\varepsilon}} ds \geq F(\underline{s}) = 0,$$

and similarly,

$$F(s) + \varepsilon G(s) = F(\bar{s}) + \varepsilon \underbrace{G(\bar{s})}_{=0} - \int_s^{\bar{s}} \underbrace{(f(s) + \varepsilon g(s))}_{\geq 0,\, \forall \varepsilon \leq \bar{\varepsilon}} ds \leq F(\bar{s}) = 1.$$

Hence, $0 \leq F(s) + \varepsilon G(s) \leq 1$ for all $\varepsilon \leq \bar{\varepsilon}$, as required. □

D.3 First-best corrective policy

The first-best problem when the planner can control both b and k is

$$\max_{b,k} W(b,k) = \left[M^P(b) - \Delta(b) - 1\right] k - \Upsilon(k),$$

with first-order conditions

$$\begin{aligned} \frac{dM^P(b^1)}{db} - \frac{d\Delta(b^1)}{db} &= 0 \\ M^P\left(b^1\right) - \Delta\left(b^1\right) - 1 &= \Upsilon'\left(k^1\right), \end{aligned}$$

where we denote by b^1 and k^1 the first-best leverage and investment. Formally, we consider an equilibrium with Pigouvian taxes $\tau = (\tau_k, \tau_b)$, where investors pay $\tau_k k + \tau_b b$ at date 0 to the government, which is then rebated as a lump sum to either investors or creditors. Investors solve

$$V(\tau) = \max_{b,k} [M(b) - 1] k - \Upsilon(k) - \tau_k k - \tau_b b,$$

with first-order conditions

$$\begin{aligned} \frac{dM(b)}{db} k &= \tau_b \\ M(b) - 1 &= \Upsilon'(k) + \tau_k. \end{aligned}$$

It follows that the corrective policy that achieves the first-best solution is

$$\begin{aligned} \tau_b &= \left[\frac{dM(b^1)}{db} - \left(\frac{dM^P(b^1)}{db} - \frac{d\Delta(b^1)}{db}\right)\right] k^1 \\ \tau_k &= M\left(b^1\right) - \left(M^P\left(b^1\right) - \Delta\left(b^1\right)\right). \end{aligned}$$

D.4 Properties of hazard-rate dominant perturbations

We often rely on the following two properties of hazard-rate dominant variations/perturbations.

Property 1 The hazard rate after an arbitrary perturbation of the form described in Section 2 of the paper is given by $h(s) = \frac{f(s)+\varepsilon g(s)}{1-(F(s)+\varepsilon G(s))}$. Its derivative with respect to ε takes the form

$$\frac{dh(s)}{d\varepsilon} = \frac{g(s)}{1-(F(s)+\varepsilon G(s))} + \frac{(f(s)+\varepsilon g(s))\,G(s)}{(1-(F(s)+\varepsilon G(s)))^2}.$$

In the limit in which $\varepsilon \to 0$, for hazard-rate dominance to hold, it must be the case that $\lim_{\varepsilon\to 0} \frac{dh(s)}{d\varepsilon} < 0$, therefore

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{dh(s)}{d\varepsilon} &= \frac{g(s)}{1-F(s)} + \frac{f(s)}{1-F(s)}\frac{G(s)}{1-F(s)} < 0 \\
&\iff g(s) + \frac{f(s)}{1-F(s)}G(s) < 0 \\
&\iff \frac{g(s)}{G(s)} + \frac{f(s)}{1-F(s)} > 0 \\
&\iff \frac{f(s)}{1-F(s)} > -\frac{g(s)}{G(s)},
\end{aligned}$$

where in the second-to-last line the sign of the inequality flips because $G(s)$ is negative, since hazard-rate dominance implies first-order stochastic dominance.

Property 2 Hazard-rate dominance implies that a perturbation increases $\frac{1-F(s)}{1-F(b)}$, where $s > b$. This implies that

$$\lim_{\varepsilon\to 0^+} \frac{\partial}{\partial\varepsilon}\left(\frac{1-F(s)-\varepsilon G(s)}{1-F(b)-\varepsilon G(b)}\right) = \frac{(-G(s))(1-F(b)) - (1-F(s))(-G(b))}{(1-F(b))^2} \geq 0,$$

or equivalently

$$(-G(s))(1-F(b)) \geq (1-F(s))(-G(b)). \tag{16}$$

D.5 Binding equity constraint

Whenever the investors' date 0 non-negativity constraint is binding, the total amount of equity is effectively fixed to n_0^I, and Lemma 1 ceases to hold. When the date 0 non-negativity constraint binds, the problem that investors face can be expressed as

$$\max_{b,k} \beta^I \int_{s^\star(b)}^{\overline{s}} (s-b)\,dF^I(s)\,k,$$

where $k = \frac{n_0^I}{1-Q(b)}$ and $Q(b) = \beta^C \left(\int_{s^\star(b)}^{\overline{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^\star(b)} s dF^C(s)\right)$. Intuitively, investors maximize the leverage return on their initial wealth n_0^I. Under natural regularity conditions, the solution to this problem is given by the first-order condition on b

$$\frac{1 - Q(b^{\star})}{Q_b(b^{\star})} = \frac{\int_{s^{\star}(b^{\star})}^{\overline{s}} (s - b^{\star})\, dF^I(s)}{\int_{s^{\star}(b^{\star})}^{\overline{s}} dF^I(s)}, \tag{17}$$

where $Q_b(b) = \beta^C \left(\int_{s^{\star}(b)}^{\overline{s}} dF^C(s) - (1-\phi)\, s^{\star}(b)\, f^C(s^{\star}(b)) \right)$. Equation (17) is the counterpart of Equation (11) in Simsek (2013), after accounting for the cost of distress associated with bankruptcy. In this appendix, to highlight the differences with Simsek (2013), we focus on the case of equity exuberance, although our approach can be used to study other scenarios. Formally, we consider the case in which $F^C(s) = F^{C,P}(s) = F^{I,P}(s)$.

In order to understand whether equilibrium leverage increases or decreases in response to a perturbation in investors' leverage, it follows from Equation (17) that it is sufficient to characterize the behavior of $T(b) \equiv \frac{\int_{s^{\star}(b)}^{\overline{s}} s dF^I(s)}{\int_{s^{\star}(b)}^{\overline{s}} dF^I(s)} = \int_b^{\overline{s}} (s-b) \frac{f^I(s)}{1-F^I(b)} ds$. The change in $T(b)$ induced by a change in investors' beliefs in the direction G^I is given by

$$\frac{\delta T}{\delta F^I} \cdot G^I = \frac{\int_b^{\overline{s}} (s-b) \left[g^I(s) \left(1 - F^I(b)\right) - f^I(s) \left(-G^I(b)\right) \right] ds}{\left(1 - F^I(b)\right)^2}.$$

If $\frac{\delta T}{\delta F^I} \cdot G^I$ is positive (negative), leverage will increase (decrease). This characterization allows to consider any perturbation of beliefs. However, if we are interested in hazard-rate dominant perturbations, it can be shown that when investors become more optimistic in a hazard-rate sense and they are constrained on the amount of equity issued, leverage increases in equilibrium. Formally, $\frac{\partial T}{\partial F^I} \cdot G^I \geq 0$ if

$$\left(\int_b^{\overline{s}} (s-b) g^I(s)\, ds \right) \left(1 - F^I(b)\right) - \left(\int_b^{\overline{s}} (s-b)\, f^I(s)\, ds \right) \left(-G^I(b)\right) \geq 0,$$

which is equivalent to

$$\left(\int_b^{\overline{s}} \left(-G^I(s)\right) ds \right) \left(1 - F^I(b)\right) - \left(\int_b^{\overline{s}} \left(1 - F^I(s)\right) ds \right) \left(-G^I(b)\right) \geq 0,$$

which follows by integrating (16) over $s \in [b, \overline{s}]$. This argument is an alternative way to formalize some of the main results in Simsek (2013), in particular Theorems 4 and 5.

Finally, we can consider the normative implications of this case. In this scenario, the planner's objective can be written as $\beta^I \int_{s^{\star}(b)}^{\overline{s}} (s-b)\, dF^{I,P}(s)\, k$. With a single degree of freedom, since b and k are connected via the date 0 budget constraint of investors, it is straightforward to show that an increase in optimism by investors in the hazard-rate sense calls for tightening leverage regulations.

D.6 Alternative modeling assumptions

D.6.1 Outside equity issuance

We consider an extension of our baseline model in which, in addition to investors and creditors, there are shareholders (denoted S) who are able to invest in outside equity claims against investors' cash flows. The lifetime utility of a representative shareholder is $c_0^S + \beta^S \mathbb{E}^S \left[c_1^S(s) \right]$, where $\mathbb{E}^S$ is the expectation under shareholders' beliefs $F^S(s)$. For simplicity, we continue to assume segmented markets: creditors do not invest in equity, and shareholders do not invest in bonds.

In addition to leverage b, investors choose a share $\sigma \in [0,1]$ of equity to retain, and sell a share $1-\sigma$ of equity claims to shareholders. The market value of outside equity in equilibrium is then given by

$$P^S(b,\sigma) = (1-\sigma)\,\beta^S \int_b^{\overline{s}} (s-b)\, dF^S(s).$$

By contrast, the market value of debt $Q(b)$ remains unchanged from the baseline model, since the payoff to debtholders is unaffected by inside or outside ownership of equity shares. Repeating the steps leading to Lemma 1 in the text, we find that the following reformulation of the investors' problem characterizes the equilibrium:

Lemma 3. *[Investors' problem with outside equity issuance] Investors solve the following problem to decide their optimal investment, outside equity issuance, and leverage choices at date* 0*:*

$$\begin{aligned} V\left(\bar{b}\right) = & \max_{b,k;\sigma\in[0,1]} \left[M(b,\sigma)-1\right]k - \Upsilon(k) \\ & s.t. \quad b \le \bar{b} \qquad\qquad\qquad (\mu), \end{aligned}$$

where μ *denotes the Lagrange multiplier on the leverage constraint imposed by the* *government (reformulated as $bk \leq \bar{b}k$), and $M(b,\sigma)$ is given by*

$$M(b,\sigma) = \underbrace{\sigma\,\beta^I \int_{s^\star(b)}^{\bar{s}} (s-b)\,dF^I(s)}_{\textit{inside equity}} + \underbrace{(1-\sigma)\,\beta^S \int_{s^\star(b)}^{\bar{s}} (s-b)\,dF^S(s)}_{\textit{outside equity}} \tag{18}$$
$$\underbrace{+\,\beta^C\left(\int_{s^\star(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^\star(b)} s dF^C(s)\right)}_{\textit{debt}}.$$

Lemma 3 shows that investors continue to maximize the same objective as in the baseline model, but must first solve an auxiliary maximization problem in Equation (18), which determines the optimal value σ of the share of equity retained by insiders. The auxiliary problem is clearly linear in σ. Hence, for any given choice of b, it is either optimal to retain all shares ($\sigma = 1$) or sell all shares to outsiders ($\sigma = 0$), depending on the differences between insiders' and outsiders' discount factors and beliefs.

This result clarifies how our main results are affected by outside equity issuance. On the one hand, if inside and outside shareholders have the same preferences and beliefs, then investors are indifferent between all values of σ, and their problem reduces to the exact same problem as in the baseline model. In this case, all of our positive and normative results on the marginal effects of changes in beliefs carry over without modification.

On the other hand, if there are differences in preferences or belief disagreements between insiders and outsiders, then investors' choices are affected only by marginal changes in the beliefs of (outside) shareholders if it is optimal to sell all shares with $\sigma = 1$, and only by marginal changes in their own beliefs if $\sigma = 0$. However, all of our results on the effects of equity exuberance continue to remain true after a modification to the definition of exuberance, namely, that *both* investors' beliefs $F^I(s)$ and outsider shareholders' beliefs $F^S(s)$ become more optimistic in the sense of hazard-rate dominance.

D.6.2 Collateralized credit

In the body of the paper, we consider an environment in which creditors can seize from investors the full gross return on investment in case of default. If we assume that capital trades at a price $q(s)$ at date 1, and that credit is collateralized exclusively by the market value of the investment at date 1, we can reformulate the relevant equations to accommodate collateralized borrowing as follows:

$$\begin{aligned} c_1^I(s) &= n_1^I(s) + sk + \max\{q(s) - b, 0\}k, \ \forall s \\ Q(b) &= \beta^C \left(\int_{s^\star(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^\star(b)} q(s) dF^C(s) \right), \end{aligned}$$

where $s^\star(b)$ now solves $q(s^\star) = b$. It is straightforward to extend our results to this case.

References

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