ONLINE APPENDIX

THE RISK-ADJUSTED CARBON PRICE

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Appendix A. Perturbation Theory

A.1. General framework

Perturbation theory is a method for finding an approximate solution to a complicated problem by starting with the exact solution of a related, simpler problem. The problem is thus not solved exactly, but instead so-called 'small' terms are added to adjust the solution of the simpler, exactly solvable problem. Perturbation theory provides a formal framework to control how small these adjustment terms are.

In general, after substitution of the optimality conditions for the forward-looking variables, any HJB equation takes the form

(A1.1) $$\mathcal{F}\left[J(\mathbf{x}),\mathbf{x}\right]=0,$$

where the 'operator' $\mathcal{F}$ is typically nonlinear, includes first- and second-order derivatives of the value function J with respect to the vector of states $\mathbf{x}$, which may include time, and is also a function of $\mathbf{x}$ directly. Provided a (single) small parameter ϵ, defined so that we return to the simpler, exactly solvable problem in the limit $\epsilon \to 0$, can be identified, we can solve this HJB for the value function $J(\mathbf{x})$ using perturbation theory. Following practice in perturbation theory (e.g., Van Dyke, 1975; Kevorkian and Cole, 1996; Bender and Orszag, 1999; Nayfeh, 2004), the solution for the value function then takes the form of a series in ϵ

(A1.2) $$J(\mathbf{x})=J^{(0)}(\mathbf{x})+\epsilon J^{(1)}(\mathbf{x})+\mathcal{O}(\epsilon^2),$$

where the dependence of the value function on the states $\mathbf{x}$ continues to be nonlinear. To be clear, (A1.2) is not a Taylor-series expansion. Instead, (A1.2) expresses the solution as a series of adjustments (depending on the small parameter ϵ) to the so-called zeroth-order solution $J^{(0)}(\mathbf{x})$, which corresponds to the solution of the simpler, exactly solvable problem referred to above. In the limit $\epsilon \to 0$, $J(\mathbf{x}) \to J^{(0)}(\mathbf{x})$. The zeroth-order solution is said to be $\mathcal{O}(1)$ since $\epsilon^0 = 1$ and thus 'not small'. The subsequent terms in (A1.2) adjust the solution if $\epsilon \neq 0$, where the first-order term $\epsilon J^{(1)}(\mathbf{x})$ is the so-called 'leading-order' adjustment to the solution. Formally, in the limit of an infinite number of terms in (A1.2), the solution of the simpler problem plus all its adjustments, provided the series is convergent, become equal to the exact solution of the complicated problem $J(\mathbf{x})$. In practice, only a finite number of terms gives a reasonable approximation, and the series solution is truncated. In (A1.2), the series solution is accurate up to first order in ϵ, and the error is thus $\mathcal{O}(\epsilon^2)$.

Having expanded the value function in (A1.2), we also expand the operator $\mathcal{F}$:

(A1.3) $\mathcal{F} = \mathcal{F}^{(0)} + \epsilon\mathcal{F}^{(1)} + \mathcal{O}(\epsilon^2),$

where $\mathcal{F}^{(0)}$ contains all operations that leave the order unchanged and $\epsilon\mathcal{F}^{(1)}$ contains all operations that increase the order by ϵ. Combining (A1.2) and (A1.3), the general form of the HJB (A1.1) becomes

(A1.4) $\left(\mathcal{F}^{(0)} + \epsilon\mathcal{F}^{(1)} + \mathcal{O}(\epsilon^2)\right)\left[J^{(0)}(\mathbf{x}) + \epsilon J^{(1)}(\mathbf{x}) + \mathcal{O}(\epsilon^2), \mathbf{x}\right] = 0,$ which can be expressed as a series solution itself by expanding out the brackets

(A1.5) $\underbrace{\mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}), \mathbf{x}\right]}_{O(1)} + \underbrace{\epsilon\left(\mathcal{F}^{(0)}\left[J^{(1)}(\mathbf{x}), \mathbf{x}\right] + \mathcal{F}^{(1)}\left[J^{(0)}(\mathbf{x}), \mathbf{x}\right]\right)}_{O(\epsilon)} + \mathcal{O}(\epsilon^2) = 0.$

We note that the term $\epsilon^2\mathcal{F}^{(1)}\left[J^{(1)}(\mathbf{x}), \mathbf{x}\right]$ that arises from (A1.4) is small and of the same order as terms previously ignored in (A1.2) and (A1.3), and can therefore also be ignored in (A1.5); this term is contained in the $\mathcal{O}(\epsilon^2)$ error in (A1.5). Solving the HJB equation using perturbation theory then amounts to solving (A1.5) successively at each order. For the first two orders the resulting two equations are

(A1.6) $\mathcal{O}(1):\ \mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}), \mathbf{x}\right] = 0,$

(A1.7) $\mathcal{O}(\epsilon):\ \mathcal{F}^{(0)}\left[J^{(1)}(\mathbf{x}), \mathbf{x}\right] + \mathcal{F}^{(1)}\left[J^{(0)}(\mathbf{x}), \mathbf{x}\right] = 0.$

We first solve (A1.6) for the zeroth-order solution and then solve (A1.7) for the first-order solution $J^{(1)}(\mathbf{x})$ using the (now known) zeroth-order solution $J^{(0)}(\mathbf{x})$ from (A1.6).

A.2. Perturbation theory applied to our model

To apply the framework introduced in section A.1, we take several steps. First, we identify the small parameter ϵ, which we find by writing the problem in non-dimensional form (section A.2.1). Second, we must choose the structure of our perturbation expansion, depending on how and where the small parameter ϵ appears in the HJB equation (section

A.2.2). Third, we perform the perturbation expansion and then solve the HJB equation at zeroth order (section A.2.3) and first order (section A.2.4), respectively.

A.2.1. Non-dimensional form and identification of the small variable

Following standard practice in the physical sciences, we begin by writing the HJB equation (14) in non-dimensional form[1]. To do so, we normalize the four states K, E, χ and λ by their initial values (at $t=0$):[2]

(A2.1) $$\hat{K} \equiv \frac{K}{K_0},\ \hat{E} \equiv \frac{E}{E_0},\ \hat{\chi} \equiv \frac{\chi}{\chi_0},\ \hat{\lambda} \equiv \frac{\lambda}{\lambda_0},$$

so that all four hatted variables are equal to 1 at $t=0$. We define non-dimensional time $\hat{t} \equiv g_0 t$ with $g_0 \equiv g\left(E=E_0, \chi=\chi_0, \lambda=\lambda_0\right)$ the growth rate of the economy without additional climate change (this growth rate is constant in time). We define the non-dimensional forward-looking variables as

(A2.2) $$\hat{F} \equiv \frac{F}{F_0},\ \hat{C} \equiv \frac{C}{C_0},$$

where $F_0 \equiv A\left(E_0\right)^{\frac{1}{\alpha}}\left((1-\alpha)/b\right)^{\frac{1}{\alpha}} K_0$ and $C_0 \equiv g_0 K_0$ (these are not the initial values of *F* and *C*, as initial values of the forward-looking variables are not known at this stage in the solution procedure). In accordance with the non-dimensional form (A2.1)-(A2.2), we further define $\hat{J} \equiv g_0 J / C_0^{1-\eta}$, $\hat{I} \equiv I/C_0$, $\hat{\Phi} \equiv \Phi/C_0$, $\hat{Y} \equiv Y/C_0$, $\hat{\phi} \equiv \phi/g_0$ and $\hat{i} \equiv i/g_0$, where $\phi \equiv \Phi / K$ and $i \equiv I / K$.

In non-dimensional form, the HJB equation (14) now becomes

(A2.3) $$0 = \max_{\hat{C},\hat{F}}\left[\frac{1}{1-\gamma}\frac{\hat{C}^{1-\gamma}-\hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}_{\hat{t}}+\hat{J}_{\hat{K}}\phi(\hat{i})\hat{K}+\hat{J}_{\hat{E}}(\hat{\mu}\hat{F}e^{-\hat{g}\hat{t}}-\hat{\varphi}\hat{E}) \right.$$
$$\left. +\hat{J}_{\hat{\chi}}\hat{\nu}_{\chi}(\bar{\hat{\chi}}-\hat{\chi})+\hat{J}_{\hat{\lambda}}\hat{\nu}_{\lambda}(\bar{\hat{\lambda}}-\lambda)+\frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2+\frac{1}{2}\hat{J}_{\hat{E}\hat{E}}\hat{\sigma}_E^2+\frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2+\frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2\right]$$

[1] Through normalizing all variables by typical values these variables may take, we remove the physical dimensions (e.g., time) and therefore the measurements units (e.g., years) of these variables in a process known as non-dimensionalization.
[2] We do not distinguish between χ, λ and $\max(\chi,0)$, $\max(\lambda,0)$ here for simplicity, as we show in section IV that the probabilities of χ and λ becoming zero or negative are negligibly small.

$$+\hat{J}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\hat{J}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi+\hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda+\hat{J}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi$$
$$+\hat{J}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda+\hat{J}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda,$$

where $\hat{I}=\hat{Y}-\hat{b}\hat{F}-\hat{C}=\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})\hat{K}^{\alpha}\hat{F}^{1-\alpha}-\hat{b}\hat{F}-\hat{C}$ and $\hat{\phi}=\hat{i}-(1/2)\hat{\omega}\hat{i}^2-\hat{\delta}$ with $\hat{A}\equiv AF_0^{1-\alpha}/g_0K_0^{1-\alpha}$. The resulting non-dimensional parameters are

$$\text{(A2.4)}\quad \begin{gathered}\hat{\rho}\equiv\frac{\rho}{g_0},\ \hat{b}\equiv\frac{bF_0}{g_0K_0},\ \hat{\omega}\equiv g_0\omega,\ \hat{\delta}\equiv\frac{\delta}{g_0},\ \hat{g}\equiv\frac{g}{g_0},\ \hat{\mu}\equiv\frac{\mu F_0}{g_0E_0},\ \hat{\varphi}\equiv\frac{\varphi}{g_0},\ \hat{\nu}_\chi\equiv\frac{\nu_\chi}{g_0},\\ \hat{\bar{\chi}}\equiv\frac{\bar{\chi}}{\chi_0},\ \hat{\nu}_\lambda\equiv\frac{\nu_\lambda}{g_0},\ \hat{\bar{\lambda}}=\frac{\bar{\lambda}}{\lambda_0},\ \hat{\sigma}_K\equiv\frac{\sigma_K}{\sqrt{g_0}},\ \hat{\sigma}_E\equiv\frac{\sigma_E}{\sqrt{g_0}E_0},\ \hat{\sigma}_\chi\equiv\frac{\sigma_\chi}{\sqrt{g_0}\chi_0},\ \ \hat{\sigma}_\lambda\equiv\frac{\sigma_\lambda}{\sqrt{g_0}\lambda_0}.\end{gathered}$$

Except for $\hat{b}$, $\hat{\mu}$, $\bar{\chi}$ and $\hat{\bar{\lambda}}$, which respectively measure the relative cost of fossil fuel use, the relative contribution of new emissions to the total atmospheric carbon stock, and the ratios of the steady-state and initial values of the climate sensitivity and the climate damage parameters, the non-dimensional parameters in (A2.4) measure the different rates in the economy relative to the growth rate g_0. We assume all these non-dimensional parameters are $O(1)$. That is, they are not small parameters, and their effects must be fully accounted for in our solutions and cannot be approximated using perturbation theory.

Having assumed that all non-dimensional parameters in (A2.4) are $\mathcal{O}(1)$, it is not immediately obvious how we can use perturbation theory to simplify the solutions to the HJB equation (A2.3). However, one additional non-dimensional parameter arises when we define non-dimensional damages and total factor productivity[3]

$$\text{(A2.5)}\quad \hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\equiv\hat{\lambda}^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}\quad\text{and}\quad\hat{A}\equiv\hat{A}^*(1-\epsilon\hat{D})=\hat{A}^*(1-\epsilon\hat{\lambda}^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}),$$

where $\hat{D}\equiv D/D_0=D/\epsilon$, $\hat{A}\equiv AF_0^{1-\alpha}/\left(g_0K_0^{1-\alpha}\right)$ and $\hat{A}^*\equiv A^*F_0^{1-\alpha}/\left(g_0K_0^{1-\alpha}\right)$.

Assumption A: The final additional non-dimensional parameter, which we will assume to be the small parameter of our problem, is defined as

$$\text{(A2.6)}\quad \epsilon\equiv D_0\equiv\lambda_0^{1+\theta_\lambda}\chi_0^{1+\theta_{\chi T}}\left(\frac{E_0}{S_{PI}}\right)^{1+\theta_{ET}}.$$

[3] The term 'normalization' or 'scaling' is perhaps more appropriate than 'writing in non-dimensional form' for the damage ratio *D*, which is already non-dimensional. We avoid this ambiguity by using the three terms interchangeably.

The small parameter ϵ equals the initial damage ratio D_0, which is known a priori and empirically also small (see section IV). The limit $\epsilon \to 0$ thus corresponds to the case in which climate damages are zero. In this limit, the value function does not depend on the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$, and all terms in the HJB equation (A2.3) involving derivatives with respect to $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ disappear:

$$\text{(A2.7)}\quad \max_{\hat{C},\hat{F}}\left[\frac{1}{1-\gamma}\frac{\hat{C}^{1-\gamma}-\hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}_{\hat{t}}+\hat{J}_{\hat{K}}\phi(\hat{i})\hat{K}+\frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2\right]=0,$$

which can be solved for the value function in closed form (e.g., Pindyck & Wang, 2013) to give the solution to the simpler, exactly solvable problem we perturb here. Note, however, that a small but non-zero value of the small parameter ϵ reduces total factor productivity via the damage ratio according to (A2.5) and introduces the solution's dependence on $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$. In the original HJB equation (A2.3), ϵ will change the investment level $\hat{i}$, which directly affects the term $\hat{J}_{\hat{K}}\phi(\hat{i})\hat{K}$ and indirectly all others. Perturbation theory now allows us to re-introduce ϵ into (A2.7) in a controlled fashion. Before we do so, we emphasize that except for the small variable ϵ, all (hatted) variables in the HJB (A2.4) are $\mathcal{O}(1)$.

A.2.2. Perturbation expansion

We now seek a perturbation series solution for the value function of the following form:

$$\text{(A2.8)}\quad \hat{J}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})=\hat{J}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right)+\epsilon\hat{J}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})+\mathcal{O}(\epsilon^2),$$

where the structure of the solution is based on the underlying HJB (A2.3), as explained below. Because the zeroth-order solution will only be affected by the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ through the total factor productivity of capital (A2.5), the zeroth-order solution only depends on these three states through the damage ratio $\hat{D}=\hat{\lambda}^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}$. Moreover, in the functional dependence of the zeroth-order solution $\hat{J}^{(0)}$, the $\mathcal{O}(1)$ damage ratio $\hat{D}$ is always multiplied by the small parameter ϵ (i.e. the functional dependence is on $D=\epsilon\hat{D}$). Importantly, as a result, changes in the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ have a smaller effect on the zeroth value function than changes in the capital stock $\hat{K}$. To illustrate this, consider the expected rate of change of the zeroth-order value function through its total derivative:

$$\text{(A2.9)}\qquad \frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{J}^{(0)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})\right]=\underset{\mathcal{O}(1)}{\hat{J}^{(0)}_{\hat{t}}}+\underset{\mathcal{O}(1)}{\hat{J}^{(0)}_{\hat{K}}}\underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{t}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(1)}$$
$$+\underset{\mathcal{O}(\epsilon)}{\epsilon}\underset{\mathcal{O}(1)}{\hat{J}^{(0)}_{D}}\left(\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{E}}}\underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{E}\right]}_{\mathcal{O}(1)}+\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{\chi}}}\underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{\chi}\right]}_{\mathcal{O}(1)}+\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{\lambda}}}\underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{\lambda}\right]}_{\mathcal{O}(1)}\right)+\ldots,$$

where we have left out the stochastic terms for ease of exposition. The contributions to the rate of change of the zeroth-order value function (left-hand side of (A2.9)) from changes in time $\hat{t}\equiv g_0 t$ (the first term on the right-hand side) and in the capital stock $\hat{K}$ (the second term) are $\mathcal{O}(1)$, whereas the contributions from changes in the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ (the remaining terms on the right-hand side) are $\mathcal{O}(\epsilon)$ and thus smaller than the first two terms by a factor ϵ. The functional dependence of $\hat{J}^{(0)}$ on the climatic states is said to be 'slow'. When solving the HJB equation (A2.3) using perturbation methods, this means that some of the derivatives of $\hat{J}^{(0)}$ with respect to the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be ignored because their order in ϵ is too high. For the first-order term $\hat{J}^{(1)}$ in (A2.8), we do not assume a slow dependence on any of the states a priori. Precisely which terms in (A2.3) will be included at which order is considered in detail in sections A.2.3 and A.2.4 below, where we set out to find the zeroth- and first-order solutions.

To cast the HJB equation (A2.3) into the form $\mathcal{F}\left[J(\mathbf{x}),\mathbf{x}\right]=0$ in (A1.1), we must first find solutions for the forward-looking variables. The optimality conditions of (A2.3) with respect to $\hat{C}$ and $\hat{F}$ are, respectively,

$$\text{(A2.10)}\quad \frac{\hat{C}^{-\gamma}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}}-\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}=0\;\Rightarrow\;\hat{C}=\left(\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}\right)^{-\frac{1}{\gamma}}\left((1-\eta)\hat{J}\right)^{-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}},$$

$$\text{(A2.11)}\quad \hat{J}_{\hat{K}}\left((1-\alpha)\hat{A}\hat{K}^{\alpha}\hat{F}^{-\alpha}-\hat{b}\right)\phi'\left(\hat{i}\right)+\hat{J}_{\hat{E}}e^{-\hat{g}\hat{t}}\hat{\mu}=0\;\Rightarrow\;\hat{F}=\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K},$$

where we define the optimal SCC in non-dimensional form as $\hat{P}\equiv F_0P/(g_0K_0)$, which is given by

(A2.12) $\hat{P} = -\hat{\mu}\dfrac{\hat{J}_{\hat{E}}}{\phi'(\hat{i})\hat{J}_{\hat{K}}}.$

Upon substituting equations (A2.10)-(A2.11) into the HJB equation (A2.3) and recognizing that $\hat{J}, \hat{C}, \hat{F}, \hat{A}, \hat{i}$ and $\hat{P}$ are functions of the state variables collected in $\mathbf{x}$, we obtain an equation of the form $\mathcal{F}\left[J(\mathbf{x}),\mathbf{x}\right]=0$ as in (A1.1).

By substituting our series solution for the value function (A2.8) into (A2.12), the leading-order estimate of the optimal SCC is given by:

(A2.13) $\hat{P} = -\hat{\mu}\dfrac{\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}}{\phi'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}} + \mathcal{O}(\epsilon^2),$

which is accurate up to $\mathcal{O}(\epsilon^2)$. For completeness, we note that both $\hat{J}_{\hat{E}}^{(0)}$ and $\hat{J}_{\hat{E}}^{(1)}$ are $\mathcal{O}(\epsilon)$. We therefore need to obtain both the zeroth- and the first-order solution for the value function to obtain a consistent leading-order estimate of the SCC.

A.2.3. Zeroth-order solution (see also appendix B)

Substituting the series solution for the value function (A2.8) into the HJB equation (A2.3), in which we have substituted for the forward-looking variables from (A2.10)-(A2.11), and collecting zeroth-order terms in ϵ, we obtain a nonlinear second-order ordinary differential equation (as the dependence on time has disappeared) given by (B1) in Appendix B, which we can write generally as $\mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}),\mathbf{x}\right]=0$ (cf. (A1.6)). We can solve $\mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}),\mathbf{x}\right]=0$ to give a solution of the form (see Appendix B):

(A2.14) $\hat{J}^{(0)} = \psi_0(\epsilon\hat{D})K^{1-\eta},$

where the function $\psi_0(\epsilon\hat{D})$ captures the slow dependence on the climatic states through $\hat{D} = \lambda^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}$ (see explanation in beginning of section A.2.2) and is given by (B3) in Appendix B.

A.2.4. First-order solution (see also Appendix C)

We proceed to collect terms in the HJB that are first order in ϵ (cf. (A1.7)). First, we ignore those derivatives of the zeroth-order value function with respect to the climatic states

$\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ that result in terms of $\mathcal{O}(\epsilon^2)$ and higher. Second, we perform Taylor-series expansions in ϵ (about $\epsilon = 0$) of any nonlinear function of the value function, again ignoring those terms of order $\mathcal{O}(\epsilon^2)$ and higher. To illustrate this second step, consider the optimality condition (A2.10), which becomes at $\mathcal{O}(\epsilon)$

(A2.15) $$\epsilon\hat{C}^{(1)} = \hat{C}^{(1)}\left(-\frac{1}{\gamma}\frac{\phi''\left(\hat{i}^{(0)}\right)}{\phi'\left(\hat{i}^{(0)}\right)}\epsilon\hat{i}^{(1)} - \frac{1}{\gamma}\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right)$$

where we have used the product and chain rules of differentiation repeatedly ((A2.10) is the product of three functions), noting that we have used the expansions $\hat{C} = \hat{C}^{(0)} + \epsilon\hat{C}^{(1)} + \mathcal{O}(\epsilon^2)$, $\hat{i} = \hat{i}^{(0)} + \epsilon\hat{i}^{(1)} + \mathcal{O}(\epsilon^2)$, $\hat{J}_{\hat{K}} = \hat{J}_{\hat{K}}^{(0)} + \epsilon\hat{J}_{\hat{K}}^{(1)} + \mathcal{O}(\epsilon^2)$ and, of course, $\hat{J} = \hat{J}^{(1)} + \epsilon\hat{J}^{(1)} + \mathcal{O}(\epsilon^2)$. A single nonlinear term in the HJB equation can thus give rise to multiple terms upon expansion. Performing the first and second step explicitly and consistently is straightforward yet cumbersome, and details are given in Appendix C. Because we have chosen $\hat{D} = \lambda^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}$ to be a product of power functions, we can solve the resulting partial differential equation in closed form.

A.3. Result A

Combining the zeroth- and first-order solutions for the value function according to (A2.13), we obtain the following (dimensional) leading-order estimate of the optimal SCC (corresponding its non-dimensional equivalent (C3.19) in Appendix C). We present results in dimensional form here, so that they can be referred to directly by the reader of the main paper.

Result A: The optimal risk-adjusted SCC is:

(A3.1) $$P = \frac{\mu\,\Theta(E,\chi,\lambda)Y\big|_{P=0}}{r^*}\left(1 - \frac{\Omega}{E^{\theta_{ET}}\chi^{1+\theta_{ET}}\lambda^{1+\theta_\lambda}K^{1-\eta}}\right) + \mathcal{O}(\epsilon^2),$$

where $\Theta \equiv D_E/(1-D)$ and $r^* \equiv r^{(0)} - g^{(0)} = \rho + (\gamma-1)(g^{(0)} - \eta\sigma_K^2/2)$. Further,

(A3.2) $$\Omega = \mathrm{E}_t\left[\int_t^\infty \Gamma e^{-r_\Omega(s-t)}\mathrm{d}s\right] \text{ with } r_\Omega \equiv r^* - (\eta-1)\left(\phi(i^{(0)}) - \eta\sigma_K^2/2\right) + \varphi,$$

where $\phi \equiv \Phi/K = i - \omega i^2/2 - \delta$, $i = I/K$. The term Γ is given dimensionally by

$$\Gamma \equiv \Big((1+\theta_{ET})\varphi \mathrm{X}\Lambda - \nu_\chi(\bar{\chi}-\chi)\mathrm{X}_\chi\Lambda - \nu_\lambda(\bar{\lambda}-\lambda)\mathrm{X}\Lambda_\lambda$$

$$-\frac{1}{2}\sigma_\chi^2 \mathrm{X}_{\chi\chi}\Lambda - \frac{1}{2}\mathrm{X}\Lambda_{\lambda\lambda}\sigma_\lambda^2 - (1-\eta)\mathrm{X}_\chi\Lambda\rho_{K\chi}\sigma_K\sigma_\chi$$

(A3.3)

$$-(1-\eta)\mathrm{X}\Lambda_\lambda\rho_{K\lambda}\sigma_K\sigma_\lambda - \mathrm{X}_\chi\Lambda_\lambda\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \Big) K^{1-\eta}E^{\theta_{ET}}$$

$$-\theta_{ET}\mu A^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} \mathrm{X}\Lambda K^{2-\eta}E^{\theta_{ET}-1}e^{-g^{(0)}t} - \frac{1}{2}\theta_{ET}(\theta_{ET}-1)\sigma_E^2 \mathrm{X}\Lambda K^{1-\eta}E^{\theta_{ET}-2}$$

$$-\Big((1-\eta)\theta_{ET}\mathrm{X}\Lambda\rho_{KE}\sigma_K\sigma_E + \mathrm{X}_\chi\Lambda\rho_{E\chi}\sigma_E\sigma_\chi + \mathrm{X}\Lambda_\lambda\rho_{E\lambda}\sigma_E\sigma_\lambda\Big)K^{1-\eta}E^{\theta_{ET}-1},$$

where $\mathrm{X} \equiv \chi^{1+\theta_{ET}}$ and $\Lambda \equiv \lambda^{1+\theta_\lambda}$. □

The term in (A3.1) in front of the brackets is the net present value of marginal damages if only economic growth or asset return uncertainty is considered, and the atmospheric carbon stock does not decay; the second term in the large brackets is the mark-up for carbon stock, climate sensitivity and damage ratio uncertainties and carbon stock decay. The integral to evaluate Ω is discounted with a rate r_Ω that differs from r^* in that it corrects for net growth in the capital stock (including a term depending on risk aversion and the volatility of the capital stock) and the rate of decay of atmospheric carbon.

The optimal SCC given in (A3.1) is proportional to world GDP, which is given to leading order by its value when there is no climate policy ($P=0$) and depends on the stock of atmospheric carbon and the climate sensitivity and damage ratio parameters through the function $\Theta(E,\chi,\lambda)$. It depends on preferences (ρ, γ and η), geophysical parameters (μ, φ and ν_χ), and the properties of the stochastic processes driving GDP, the carbon stock, climate sensitivity and damages. The optimal SCC depends on the growth-corrected return on capital r^*, which is given to leading order by its value when there is no climate policy ($P=0$). The expected return on investment $r^{(0)}$ is the risk-free rate, $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} -(1+\gamma)\eta\sigma_K^2/2,$ plus the risk premium $\eta\sigma_K^2$.[4]

Result A indicates that the absolute error in our expression for the optimal SCC is $\mathcal{O}(\epsilon^2)$ and that the error as fraction of the SCC (which is $\mathcal{O}(\epsilon)$ itself) is thus $\mathcal{O}(\epsilon)$. Consistently, we can ignore the slow dependence of the discount rate on the atmospheric carbon stock (via the marginal productivity of capital) when evaluating the discounting integral in Result

[4] The investment and growth rates of GDP are given to leading order by their values without climate policy (cf. (C7)). Implicitly, we get from the Euler and capital accumulation equations $i^{(0)} = Y_K\big|_{P=0} - q^{(0)}\left(\rho + (\gamma-1)\left(\phi(i^{(0)}) - \eta\sigma_K^2/2\right)\right)$ with $Y_K\big|_{P=0} = \alpha A(E,\chi,\lambda)^{1/\alpha}\left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$ and $g^{(0)} = i^{(0)} - \omega(i^{(0)})^2/2 - \delta \equiv \phi(i^{(0)})$. Tobin's q is $q(i) = 1/\phi'(i)$.

A. As $\epsilon \to 0,$ the SCC in Result A becomes exact. Generally, a closed-form solution to the 5-dimensional integral (over time and to evaluate the 4-dimensional expectations operator over the stochastic states in Ω) is unavailable, so Result A must be evaluated numerically.[5]

A.4. Results 1 and 2 (see also Appendix D)

To simplify Result A, we make three additional assumptions.

Assumption I: The future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through its dependence on the stochastic capital stock.

Assumption II: We include only the leading-order effects of uncertainty by performing an additional perturbation expansion.[6]

Assumption III: We set the initial and steady-state values of the damage ratio parameter λ_0 and $\bar{\lambda}$ to be equal, so deterministic damages are not subject to a delay. (We do not make the same assumption for the climate sensitivity parameter χ).

Owing to *Assumptions I-III*, we can derive closed-form solutions for the optimal risk-adjusted SCC by evaluating the 4-dimensional integral in Result A explicitly with all details in Appendix D. In doing so, we derive Results 1 and 2. We show in Appendix F that Results 1 and 2 only have minimal quantitative errors compared to Result A and that *Assumptions I, II and III* are therefore justified ex post.

A.4.1. Result 1

Result 1 gives the simpler case under two additional assumptions.

Assumption IV: Proportional reduced-form damages ($\theta_{ET} = 0$).

Assumption V: An initial climate sensitivity parameter that is equal to its steady-state value ($\chi_0 = \bar{\chi}$).

So-called reduced-form damages (or simply 'damages' below) are obtained when the temperature-carbon stock relationship $T(E)$ is substituted into the damage-temperature relationship $D(T)$, and damages become a direct function of the carbon stock: $D(E)$. Under *Assumption IV*, damages are proportional to the atmospheric carbon stock (i.e.

[5] This requires five-dimensional numerical integration over the probability space corresponding to the four states and with respect to time. If the processes are independent, the integrals over the probability space of states can be evaluated independently.

[6] For completeness, we note that in the limit of small uncertainty in which Results 1 and 2 are valid, the atoms of probability associated with all non-negativity constraints disappear.

$D \propto E$), and marginal damages are constant (i.e. $D_E \neq f(E)$) and thus unaffected by future emissions. Under *Assumption V*, the deterministic climate sensitivity parameter does not vary with time (i.e. $\mu_\chi = \chi_0 = \bar{\chi}$). We emphasize that expected climate sensitivity ($\mathrm{E}_t\left[T_2\right] = \mathrm{E}_t\left[\chi(s)^{1+\theta_\chi}\right]$) does increase the further we are looking into the future $(s > t)$ due to increased uncertainty on longer horizons and the convex dependence of climate sensitivity T_2 on the climate sensitivity parameter χ. Under *Assumption V* only deterministic delays in the climate sensitivity are thus ignored motivated by simplicity alone. Result 1 and its 'risk adjustments' are given in dimensional form in (17) in the paper.

A.4.2. Result 2

Result 2 allows for convex damages ($\theta_{ET} \neq 0$) and an initial climate sensitivity parameter that differs from its steady-state value ($\chi_0 \neq \bar{\chi}$) and thus relaxes *Assumptions IV and V*. In addition to 'risk adjustments', the SCC in Result 2 given by (20) in the paper includes additional so-called 'correction factors', which can be evaluated as simple, one-dimensional integrals. We distinguish two types of so-called 'correction factors', denoted by the symbol Υ with subscripts again denoting the state variable(s) from which the risk originates: for $\theta_{ET} \neq 0$ and for $\chi_0 \neq \bar{\chi}$.

Result 1 and 2 are different in three ways. First, the correction factors in Result 2, $\Upsilon_{\theta_{ET} \neq 0}$ and $\Upsilon_{\chi_0 \neq \bar{\chi}}$, provide deterministic corrections for $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$, respectively. Second, in Result 2, the adjustments for uncertainty in the carbon stock, climate sensitivity, the damage ratio and the interaction between the two are now multiplied by their respective correction factors (Υ). Third, the effective discount rate r^* in Result 1 is replaced by $r^\star \equiv r^* + (1+\theta_{ET})\varphi$ (and $r^{\star\star} \equiv r^\star + (\eta - 1)\sigma_K^2 - \varphi$) in Result 2. The risk adjustments in Result 2 are given by:

(A4.1) $$\Delta_{EE} = -\frac{1}{2}\theta_{ET}(1-\theta_{ET})\left(\frac{\sigma_E}{E}\right)^2 \frac{1}{r^\star - 2\varphi}\Upsilon_{EE},$$

(A4.2) $$\Delta_{\chi\chi} = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{\left(\sigma_\chi/\chi_0\right)^2}{r^\star + 2\nu_\chi}\Upsilon_{\chi\chi}, \quad \Delta_{\lambda\lambda} = \frac{1}{2}\theta_\lambda(1+\theta_\lambda)\frac{\left(\sigma_\lambda/\lambda_0\right)^2}{r^\star + 2\nu_\lambda}\Upsilon_{\lambda\lambda},$$

(A4.3) $$\Delta_{\chi\times\lambda} = \frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\sigma_\lambda^2\sigma_\chi^2}{2\nu_\chi 2\nu_\lambda}\Upsilon_{\chi\times\lambda}.$$

The adjustments for correlated climate and economic risk are

$$\text{(A4.4)}\ \Delta_{\mathrm{CK}} = -(\eta-1)\sigma_K \left(\theta_{ET} \frac{\rho_{KE} \frac{\sigma_E}{E}}{(r^\star - \varphi)} \Upsilon_{KE} + (1+\theta_{\chi T}) \frac{\rho_{K\chi} \frac{\sigma_\chi}{\chi_0}}{r^\star + \nu_\chi} \Upsilon_{K\chi} + (1+\theta_\lambda) \frac{\rho_{K\lambda} \frac{\sigma_\lambda}{\lambda_0}}{r^\star + \nu_\lambda} \Upsilon_{K\lambda} \right).$$

The adjustment for correlated climate sensitivity and damage ratio risk is

$$\text{(A4.5)}\quad \begin{aligned} \Delta_{\mathrm{CC}} &= \theta_{ET}(1+\theta_{\chi T}) \frac{\rho_{E\chi}\sigma_E \sigma_\chi / \chi_0}{(r^\star + \nu_\chi)E} \frac{r^\star}{r^\star - \varphi} \Upsilon_{E\chi} \\ &+ \left(1+\theta_\lambda\right) \left(\theta_{ET} \frac{\rho_{E\lambda}\sigma_E \sigma_\lambda / \lambda_0}{(r^\star + \nu_\lambda)E} \frac{r^\star}{r^\star - \varphi} \Upsilon_{E\lambda} + (1+\theta_{\chi T}) \frac{\rho_{\chi\lambda}\sigma_\chi \sigma_\lambda / \left(\chi_0 \lambda_0\right)}{r^\star + \nu_\chi + \nu_\lambda} \Upsilon_{\chi\lambda} \right). \end{aligned}$$

The correction factors (Υ) in (A.4.1)-(A4.5) multiply a risk adjustment (Δ) and must be linearly combined with unity, so that, for example, $\Upsilon_{\chi\chi} \equiv 1 + \Upsilon_{\chi\chi,\theta_{ET}\neq 0} + \Upsilon_{\chi\chi,\chi_0 \neq \bar{\chi}}$. These combined correction factors are equal to unity when $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$ (e.g., $\Upsilon_{\chi\chi} \equiv 1$). We give the correction factors in terms of dimensional quantities given in (D3.4)-(D3.5) in Appendix D.

□

Convexity of damages ($\theta_{ET} > 0$) (*Assumption IV*) causes Result 2 to be different from Result 1 in four ways. First, it changes the normalized marginal damage ratio $\Theta\left(E\right)$. From (6), we obtain $\Theta_E(E) = \left(1/S_{PI}\right)^2 (1+\theta_{ET})\theta_{ET}(E/S_{PI})^{\theta_{ET}-1} \chi^{1+\theta_{\chi T}} \lambda^{1+\theta_\lambda}$ to leading order in our small parameter. With convex damages ($\theta_{ET} > 0$), the normalized marginal damage ratio thus rises with the stock of atmospheric carbon. The time path for the carbon price is then steeper than that of world GDP. Its effect on the deterministic SCC is captured through the correction factor $\Upsilon_{\theta_{ET}\neq 0} > 0$, reflecting the more harmful effect of future emissions (when the stock is higher). Second, convex damages boost the effective discount rate $r^\star = \rho + (\gamma - 1)g^{(0)} + (1+\theta_{ET})\varphi$, because the marginal damage of a unit of CO_2 decays more quickly than the unit itself, depressing the SCC. Combining the first and second effects, the net effect on the SCC is positive for small decay rates of atmospheric carbon. Third, a new adjustment (A4.1) needs to be made for carbon stock uncertainty. For damages that are not too convex ($0 < \theta_{ET} < 1$), this adjustment is negative, reflecting concave marginal damages $D_E \propto (1+\theta_{ET})E^{\theta_{ET}}$ with $D_{EEE} \propto \theta_{ET}(\theta_{ET}-1)(1+\theta_{ET})E^{\theta_{ET}-2} < 0,$ which holds for $0 < \theta_{ET} < 1$ (see section IV). We emphasize that with proportional damages $(\theta_{ET} = 0),$ the adjustment to the SCC for carbon stock uncertainty is zero in Result 1. Fourth, the adjustments for the other two climatic uncertainties in (A4.2) are now multiplied by correction factors that are greater than unity, reflecting rising marginal damages due to future emissions, as in the deterministic case. The same applies to the terms adjusting for correlations in (A4.4)-(A4.5), with new correlation terms with the carbon stock arising there. Finally, Result 2 allows for a higher-order term (A4.3), which is may be non-negligibly small if $\theta_{\chi T}$ is large enough (see also Appendix D).

The effect of the initial climate sensitivity parameter differing from its steady-state value $(\chi_0 \neq \overline{\chi})$ (*Assumption V*) is captured as follows. The normalized marginal ratio Θ is evaluated at the initial (low) temperature. The term multiplying $\Upsilon_{\chi_0 \neq \overline{\chi}} > 0$ is positive and captures this delayed deterministic temperature rise. Similarly, all the adjustments are corrected by their respective correction factors to take this delayed deterministic temperature increase into account.

A.5. Comparison with other types of perturbation theory in economics

The type of perturbation theory we apply is different (but not fundamentally so) from the types of perturbation theory that are commonly applied in (macro-)economics and finance (e.g., Judd, 1996, 1998). Typically, in this literature, the value function is expanded in powers of the states themselves (e.g., $J(K) = \sum_{n=1}^{N} c_n (K-K_0)^n$), sometimes preceded by a transformation of variables using a logarithm or power function). Examples are Judd and Guu (1997), Schmitt-Grohé and Uribe (2004) and van Binsbergen et al. (2012). Our approach is different, as we retain the nonlinear dependence on the states without approximation at every order in ϵ, which is made possible because of our use of power functions. Sometimes in the literature, the relative standard deviation of the stochastic process is the small parameter of the perturbation expansion (e.g., Judd and Guu, 2001).[7] Both methods (expansion in the states and expansion in the small relative standard deviation) can be combined (e.g., Boragan Aruoba, Fernandez-Villaverde and Rubio-Ramirez, 2006). Although we choose a different small parameter, our approach is similar to Judd and Guu (2001) with one fundamental difference: we also make use of the concept of slow functional dependence and slow derivatives (from slow-fast dynamics in the

[7] We also use this type of perturbation expansion to take only leading-order climatic uncertainty into account (i.e. *Assumption II*) when we obtain Results 1 and 2 from *Result A*.

differential equations literature), which we have not seen applied in economics although it has been used in financial mathematics (e.g., Fouque, Papanicolaou and Sircar (2000)).

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Appendix B: Derivation of Zeroth-Order Solution (For Online Publication)

After substituting the series solution for the value function (A2.8), the Hamilton-Jacobi-Bellman equation (A2.3) can be written at $\mathcal{O}(1)$ as

$$\text{(B1)}\quad \frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}\frac{\eta-\gamma}{1-\eta}}-\hat{\rho}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}}}{(1-\gamma)\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}^{(0)}_{\hat{t}}+\hat{J}^{(0)}_{\hat{K}}\hat{\phi}(\hat{i}^{(0)})\hat{K}$$
$$+\frac{1}{2}\hat{J}^{(0)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2=0,$$

where we have substituted for the forward-looking variables $\hat{C}$ and $\hat{F}$ at $\mathcal{O}(1)$ from (A2.10) and (A2.11) and we have used

$$\text{(B2)}\quad \underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(1)}=\hat{\phi}(\hat{i}^{(0)})\hat{K}\,.$$

In (B1)-(B2), $\hat{i}^{(0)}$ is the (constant) optimally chosen investment rate. We note that there is no variation with time $\hat{t}$ in equation (B1), so $\hat{J}^{(0)}_{\hat{t}}=0$, and (B1) is a second-order ordinary differential equation in $\hat{K}$. Equation (B1) has a power-law solution of the form $J^{(0)}=\psi_0\hat{K}^{1-\eta}$, and following some algebraic manipulation we obtain

$$\text{(B3)}\quad \hat{J}^{(0)}=\psi_0\hat{K}^{1-\eta}\ \text{ with }\ \psi_0=\frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-(1-\eta)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right)^{-\gamma\frac{1-\eta}{1-\gamma}},$$

where $\psi_0=\psi_0(\epsilon\hat{D})$ is a slow function of $\hat{D}$ through $\hat{i}^{(0)}=\hat{i}^{(0)}(\epsilon\hat{D})$. From the first-order optimality condition for $\hat{C}$, i.e. (A2.10), at $\mathcal{O}(1)$, we obtain

$$\text{(B4)}\quad \hat{C}^{(0)}=\hat{c}^{(0)}\hat{K}\ \text{ with }\ \hat{c}^{(0)}=\frac{1}{\hat{\phi}'(\hat{i}^{(0)})}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$$

where $q\left(\hat{i}\right)=1/\phi'\left(\hat{i}\right)$ denotes Tobin's q, the price of capital in consumption terms.[8]
We can thus write the value function (B3) as

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[8] The value of the capital stock is $\hat{q}\hat{K}$, or dimensionally qK, where $\hat{q}=1/\hat{\phi}'(\hat{i})=1/\phi'(i)$ is already a fraction and is left unchanged by the normalization (cf. $\hat{q}=q$ or $\omega i=\hat{\omega}\hat{i}$).

$$\text{(B5)} \qquad J^{(0)} = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}} \hat{K}^{1-\eta}.$$

Substituting in for $\hat{F}$ from the first-order optimality condition (A2.11), we obtain from $\hat{I} = \hat{Y} - \hat{C} - \hat{b}\hat{F}$:

$$\text{(B6)} \qquad \hat{i}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{c}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{q}^{(0)}\left(\hat{\rho} - (1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$$

where $\hat{r}^{(0)}_{\text{mpk}}(\epsilon\hat{D}) \equiv \hat{Y}^{(0)}_{\hat{K}} - \hat{\delta} = \alpha\hat{A}\left(\epsilon\hat{D}\right)^{\frac{1}{\alpha}}\left((1-\alpha)/\hat{b}\right)^{\frac{1-\alpha}{\alpha}} - \hat{\delta}$ denotes the marginal productivity of capital net of depreciation[9] at zeroth order, which is a slow function of $\hat{D}$ through its dependence on the total factor productivity. Equation (B6) implicitly defines the optimally chosen investment rate $\hat{i}^{(0)}$. From (B4), the leading-order endogenous growth rate of capital and hence of consumption is

$$\text{(B7)} \qquad \hat{g}^{(0)} = \underbrace{\frac{1}{\hat{K}}\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(1)} = \hat{\phi}(\hat{i}^{(0)}) \text{ and hence } \hat{g}^{(0)} = \hat{\phi}\left(\hat{i}^{(0)}\right) = 1.$$

In equilibrium, the marginal propensity to consume $\hat{c}^{(0)}/\hat{q}^{(0)}$ equals the expected return on investment $\hat{r}^{(0)}$ minus the growth rate of the economy $\hat{g}^{(0)}$. The expected return on investment $\hat{r}^{(0)}$, in turn, equals the sum of the risk-free rate $\hat{r}^{(0)}_{\text{rf}}$ and the risk premium $\Delta\hat{r}^{(0)}$. Hence, $\hat{c}^{(0)}/\hat{q}^{(0)} = \hat{r}^{(0)} - \hat{g}^{(0)} = \hat{r}^{(0)}_{\text{rf}} + \Delta\hat{r}^{(0)} - \hat{g}^{(0)}$, and with a risk premium of $\Delta\hat{r}^{(0)} = \eta\hat{\sigma}_K^2$ in the absence of any climate risk at zeroth-order, the risk-free rate is:

$$\text{(B8)} \qquad \hat{r}^{(0)}_{\text{rf}} = \hat{\rho} + \gamma\hat{g}^{(0)} - (1+\gamma)\eta\hat{\sigma}_K^2/2 \,.$$

Although $\hat{J}^{(0)}_{\hat{E}}$ can be computed from (B5), a consistent leading-order estimate of the optimal SCC also requires $\hat{J}^{(1)}_{\hat{E}}$ and thus the next order in the perturbation expansion.

[9] Dimensionally, we have $r^{(0)}_{\text{mpk}} = \hat{r}^{(0)}_{\text{mpk}} g_0$.

Appendix C: Derivation of First-Order Solution (For Online Publication)

To derive the first-order solution, we follow the following steps. We first find the evolution equations for $\hat{K}$ and $\hat{E}$ (section C1). We then solve the multi-variate Ornstein-Uhlenbeck process that describes all our states (section C2). In section C3, we will substitute all these results into the HJB equation and retain only terms at $\mathcal{O}(\epsilon)$. It will become clear there that we only need to derive the evolution equation for $\hat{K}$ up to $\mathcal{O}(\epsilon)$ and for $\hat{E}$ at $\mathcal{O}(1)$ in section C1. The terms associated with uncertainty of the climatic variables and their covariances only need to be derived at $\mathcal{O}(1)$ in section C2.

C1. Expected evolution equations for $\hat{K}$ and $\hat{E}$

We consider the expected evolution equations of the states $\hat{K}$ and $\hat{E}$ at $\mathcal{O}(\epsilon)$ and $\mathcal{O}(1)$, respectively. At this order, we have for the expected evolution of $\hat{K}$:

$$\text{(C1.1)} \quad \underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(\epsilon)} = \hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{I}^{(1)} = -\hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{C}^{(1)} = \frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma - \dfrac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\hat{K}\left(\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$$

where the first identity makes use of $\hat{\Phi} = \epsilon\hat{I}^{(1)} - \hat{\omega}\epsilon\hat{I}^{(1)}\hat{I}^{(0)}/\hat{K} = \epsilon\hat{I}^{(1)}\hat{\phi}'(\hat{i}^{(0)})$ at $\mathcal{O}(\epsilon)$. We further note from $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C}$ that $\hat{I}^{(1)} = -\hat{C}^{(1)}$, since production net of fossil fuel costs is unaffected by the SCC in our formulation:

$$\text{(C1.2)} \quad \frac{\partial}{\partial\hat{P}}\left[\hat{Y} - \hat{b}\hat{F}\right]\Big|_{\hat{P}=0} = \frac{\partial}{\partial\hat{P}}\left[\hat{A}^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K} - \hat{b}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}\right]\Bigg|_{\hat{P}=0} = 0.$$

The identity in (C1.2) relies on the Cobb-Douglas nature of the production function. The third identity in (C1.1) follows from a Taylor-series expansion of $\hat{C}$, given by (A2.10), with respect to the small parameter ϵ (about $\epsilon = 0$):

$$(\text{C1.3}) \qquad \epsilon\hat{c}^{(1)} = \hat{c}^{(0)}\left(-\frac{1}{\gamma}\frac{\phi''}{\hat{\phi}'\left(\hat{i}^{(0)}\right)}\epsilon\hat{i}^{(1)} - \frac{1}{\gamma}\frac{\epsilon\hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}} - \frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right).$$

Noting that $\hat{i}^{(1)} = -\hat{c}^{(1)},$ we can rearrange this linear equation to give

$$(\text{C1.4}) \qquad \hat{c}^{(1)} = \frac{\hat{c}^{(0)}}{1-\frac{1}{\gamma}\frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\left(-\frac{1}{\gamma}\frac{\hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}} - \frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}\frac{\hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$$

which is used in the third identity in (C2.1).

For $\hat{E},$ we have at $\mathcal{O}(1)$:

$$(\text{C1.5}) \qquad \underbrace{\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{E}\right]}_{\mathcal{O}(1)} = \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}.$$

C2. Solution to multi-variate Ornstein-Uhlenbeck process at $\mathcal{O}(1)$

We define $\hat{k} \equiv k \equiv \log\left(K/K_0\right),$ so the vector of states $\mathrm{d}\mathbf{x} = \left\{\mathrm{d}\hat{k}, \mathrm{d}\hat{E}, \mathrm{d}\hat{\chi}, \mathrm{d}\hat{\lambda}\right\}^T$ is described by a multi-variate Ornstein-Uhlenbeck process (9), which in non-dimensional form is

$$(\text{C2.1}) \qquad \mathrm{d}\mathbf{x} = \boldsymbol{\alpha}\mathrm{d}t - \mathbf{v}\circ\left(\mathbf{x}-\bar{\boldsymbol{\mu}}\right)\mathrm{d}\hat{t} + \mathbf{S}\mathrm{d}\hat{\mathbf{W}}_{\hat{t}},$$

where we note that we have not included time $\hat{t}$ in the vector $\hat{\mathbf{x}}$ (unlike in Appendix A). The growth rate vector (10), relevant to the capital and atmospheric carbon stock processes only, is given in non-dimensional form by

$$(\text{C2.2}) \qquad \begin{aligned} \boldsymbol{\alpha} &= \left(\frac{1}{\hat{K}}\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right] - \frac{1}{2}\hat{\sigma}_K^2, \frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{E}\right], 0, 0\right)^T, \\ &= \left(\hat{\phi}(\hat{i}) - \frac{1}{2}\hat{\sigma}_K^2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}\hat{t}}, 0, 0\right)^T, \end{aligned}$$

the mean reversion rate vector by $\mathbf{v} = \left(0, \hat{\varphi}, \nu_\chi, \nu_\lambda\right)^T$, the vector of means by $\hat{\bar{\mu}}^T = (0, 0, \hat{\bar{\chi}}, \hat{\bar{\lambda}})^T$, and the covariance matrix $\mathbf{SS}^T$ has the form

$$\text{(C2.3)} \qquad \frac{1}{\mathrm{d}\hat{t}} \mathrm{E}_{\hat{t}}\left[\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \hat{\sigma}_K^2 & \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\ \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \hat{\sigma}_E^2 & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda \\ \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \hat{\sigma}_\chi^2 & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \\ \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda & \hat{\sigma}_\lambda^2 \end{pmatrix}.$$

We begin by integrating the multi-variate Ornstein-Uhlenbeck process (C2.1), including only terms at zeroth order, so that the coefficients are constant, and a closed-form solution is available. It will become apparent in section C3 that $\mathcal{O}(1)$ solutions to (C2.1) are sufficient to obtain the HJB equation at $\mathcal{O}(\epsilon)$. Specifically, we have at $\mathcal{O}(1)$ that $\boldsymbol{\alpha}^{(0)} = \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{1/\alpha} \hat{A}^{1/\alpha}, 0, 0\right)^T$, where we have relied on the solution for $\hat{K}$ from the zeroth-order problem (cf. (B7)). The slow dependence of productivity $\hat{A}$ on the states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be neglected when integrating with respect to time at $\mathcal{O}(1)$. For constant coefficients, (C2.1) can be integrated to give:

$$\text{(C2.4)} \qquad \mathbf{x}(t) = \bar{\boldsymbol{\mu}} + \boldsymbol{\alpha}t + e^{\mathbf{v}\hat{t}} \circ \left(\mathbf{x}_0 - \bar{\boldsymbol{\mu}}\right) + \int_0^{\hat{t}} e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\mathrm{d}\mathbf{W}_{\hat{u}}.$$

The quantity $\mathbf{x}(\hat{t})$ is therefore normally distributed with covariance matrix $\boldsymbol{\Sigma}(t)$:

$$\text{(C2.5)} \qquad \boldsymbol{\Sigma}(t) = \int_0^{\hat{t}} \left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)\left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)^T \mathrm{d}\hat{u} =$$

$$
\begin{pmatrix}
\hat{\sigma}_K^2 \hat{t} & \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\lambda\hat{t}}\right) \\
\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\hat{\sigma}_E^2}{2\hat{\varphi}}\left(1-e^{-2\hat{\varphi}\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{\nu}_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\hat{t}}\right) \\
\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\nu_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\hat{\sigma}_\chi^2}{2\nu_\chi}\left(1-e^{-2\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) \\
\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\lambda\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) & \frac{\hat{\sigma}_\lambda^2}{2\hat{\nu}_\lambda}\left(1-e^{-2\hat{\nu}_\lambda\hat{t}}\right)
\end{pmatrix}.
$$

C3. The Hamilton-Jacobi-Bellman equation

Substituting for the forward-looking variables $\hat{C}$ from (A2.10) and $\hat{F}$ from (A2.11), the Hamilton-Jacobi-Bellman equation (A2.3) becomes at $\mathcal{O}(\epsilon)$:

(C3.1)
$$
\begin{aligned}
&\underset{\mathcal{O}(\epsilon)}{\hat{f}^*(\hat{J})} + \epsilon\hat{J}_{\hat{t}}^{(1)} + \epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{\imath}^{(0)}) + \frac{\hat{\phi}'\left(\hat{\imath}^{(0)}\right)\hat{c}^{(0)}}{\gamma-\hat{c}^{(0)}\phi''/\hat{\phi}'(\hat{\imath}^{(0)})}\left(\epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)}+(\eta-\gamma)\epsilon\hat{J}^{(1)}\right) \\
&+\left(\hat{J}_{\hat{E}}^{(0)}+\epsilon\hat{J}_{\hat{E}}^{(1)}\right)\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{1/\alpha}\hat{A}^{1/\alpha}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\hat{\varphi}\hat{E}\right)+\left(\hat{J}_{\hat{\chi}}^{(0)}+\epsilon\hat{J}_{\hat{\chi}}^{(1)}\right)\hat{\nu}_\chi(\hat{\bar{\chi}}-\hat{\chi}) \\
&+\left(\hat{J}_{\hat{\lambda}}^{(0)}+\epsilon\hat{J}_{\hat{\lambda}}^{(1)}\right)\hat{\nu}_\lambda(\hat{\bar{\lambda}}-\hat{\lambda})+\frac{1}{2}\epsilon\hat{J}_{\hat{K}\hat{K}}^{(1)}\hat{K}^2\hat{\sigma}_K^2+\frac{1}{2}\left(\hat{J}_{\hat{E}\hat{E}}^{(0)}+\epsilon\hat{J}_{\hat{E}\hat{E}}^{(1)}\right)\hat{E}^2\hat{\sigma}_E^2 \\
&+\frac{1}{2}\left(\hat{J}_{\hat{\chi}\hat{\chi}}^{(0)}+\epsilon\hat{J}_{\hat{\chi}\hat{\chi}}^{(1)}\right)\hat{\sigma}_\chi^2+\frac{1}{2}\left(\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(0)}+\epsilon\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(1)}\right)\hat{\sigma}_\lambda^2+\left(\hat{J}_{\hat{K}\hat{E}}^{(0)}+\epsilon\hat{J}_{\hat{K}\hat{E}}^{(1)}\right)\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E \\
&+\left(\hat{J}_{\hat{K}\hat{\chi}}^{(0)}+\epsilon\hat{J}_{\hat{K}\hat{\chi}}^{(1)}\right)\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi+\left(\hat{J}_{\hat{K}\hat{\lambda}}^{(0)}+\epsilon\hat{J}_{\hat{K}\hat{\lambda}}^{(1)}\right)\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\
&+\left(\hat{J}_{\hat{E}\hat{\chi}}^{(0)}+\epsilon\hat{J}_{\hat{E}\hat{\chi}}^{(1)}\right)\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi+\left(\hat{J}_{\hat{E}\hat{\lambda}}^{(0)}+\epsilon\hat{J}_{\hat{E}\hat{\lambda}}^{(1)}\right)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda+\left(\hat{J}_{\hat{\chi}\hat{\lambda}}^{(0)}+\epsilon\hat{J}_{\hat{\chi}\hat{\lambda}}^{(1)}\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda=0,
\end{aligned}
$$

where we have used the identity $\partial/\partial\hat{k}=\hat{K}\,\partial/\partial\hat{K}$ (chain rule), substituted the evolution equations for $\hat{K}$ at subsequent orders ((B2) and (C1.1)) and $\hat{E}$ at zeroth-order (C1.5), and defined $\hat{f}^*(J)\equiv\hat{f}\left(\hat{C}^*,\hat{J}\right)$ with $\hat{C}$ optimally chosen. From (1) and (A2.10), $\hat{f}^*(J)$ is

(C3.2)
$$
\hat{f}^* = \frac{1}{1-\gamma}\left(\hat{\phi}'(\hat{\imath})\hat{J}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}-\frac{1-\eta}{1-\gamma}\hat{\rho}\hat{J}.
$$

A Taylor-series expansion of $\hat{f}^*(J)$ in ϵ (about $\epsilon = 0$) gives

(C3.3)

$$\underset{\mathcal{O}(\epsilon)}{\hat{f}^*} = \frac{\left(\hat{\phi}'(\hat{\imath}^{(0)})\hat{J}_{\hat{K}}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'(\hat{\imath}^{(0)})}\epsilon\hat{\imath}^{(1)} - \frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\gamma-\eta}{(1-\gamma)(1-\eta)}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right) - \frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)}$$

$$\frac{\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'\left(\hat{\imath}^{(0)}\right)}\frac{\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{\imath}^{(0)})}}\left(\epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)} + (\eta-\gamma)\epsilon\hat{J}^{(1)}\right) - \epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)} + \frac{\gamma-\eta}{(1-\gamma)}\epsilon\hat{J}^{(1)}\right) - \frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)},$$

where we have substituted for $\hat{\imath}^{(1)} = -\hat{c}^{(1)}$ from (C2.4) and used the identity:

(C3.4)
$$\frac{\left(\hat{\phi}'(\hat{\imath}^{(0)})\hat{J}_{\hat{K}}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\hat{K}\hat{J}_{\hat{K}}^{(0)}} = \hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}.$$

Substituting from (C3.2), two of the terms in (C3.1) simplify to

(C3.5)
$$\underset{\mathcal{O}(\epsilon)}{\hat{f}^*} + \underbrace{\hat{J}_{\hat{K}}^{(0)}\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(\epsilon)} = -\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}(\eta-\gamma) + (1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)}.$$

Using (C3.5), (C3.1) can be written as an equation with (derivatives of) the unknown first-order value function on the left-hand side and (derivatives of) the known zeroth-order value function on the right-hand side (cf. (A1.7)):

(C3.6)
$$\begin{aligned}
&-\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}(\eta-\gamma) + (1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)} + \epsilon\hat{J}_{\hat{t}}^{(1)} + \epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{\imath}^{(0)}) \\
&\epsilon\hat{J}_{\hat{E}}^{(1)}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \varphi\hat{E}\right) + \epsilon\hat{J}_{\hat{\chi}}^{(1)}\hat{v}_{\chi}\left(\hat{\bar{\chi}} - \chi\right) + \epsilon\hat{J}_{\hat{\lambda}}^{(1)}\hat{v}_{\lambda}\left(\hat{\bar{\lambda}} - \hat{\lambda}\right) \\
&+\frac{1}{2}\epsilon\hat{J}_{\hat{K}\hat{K}}^{(1)}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{E}\hat{E}}^{(1)}\hat{E}^2\hat{\sigma}_E^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{\chi}\hat{\chi}}^{(1)}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(1)}\hat{\sigma}_{\lambda}^2 \\
&\epsilon\hat{J}_{\hat{K}\hat{E}}^{(1)}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \epsilon\hat{J}_{\hat{K}\hat{\chi}}^{(1)}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \epsilon\hat{J}_{\hat{K}\hat{\lambda}}^{(1)}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} \\
&+\epsilon\hat{J}_{\hat{E}\hat{\chi}}^{(1)}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \epsilon\hat{J}_{\hat{E}\hat{\lambda}}^{(1)}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \epsilon\hat{J}_{\hat{\chi}\hat{\lambda}}^{(1)}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda} = -\hat{G}\left(\hat{t}, \hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}\right),
\end{aligned}$$

where we will refer to the right-hand side as (minus) the ‘forcing’. The forcing is defined as

$$\hat{G}(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}) \equiv \hat{J}^{(0)}_{\hat{E}}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\varphi\hat{E}\right)+\hat{J}^{(0)}_{\hat{\chi}}\hat{\nu}_{\chi}(\hat{\bar{\chi}}-\hat{\chi})+$$

$$\text{(C3.7)}\quad \hat{J}^{(0)}_{\hat{\lambda}}\hat{\nu}_{\lambda}(\hat{\bar{\lambda}}-\hat{\lambda})+\frac{1}{2}\hat{J}^{(0)}_{\hat{E}\hat{E}}\hat{\sigma}_E^2+\frac{1}{2}\hat{J}^{(0)}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2+\frac{1}{2}\hat{J}^{(0)}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2+\hat{J}^{(0)}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\hat{J}^{(0)}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}$$

$$+\hat{J}^{(0)}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}+\hat{J}^{(0)}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}+\hat{J}^{(0)}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}+\hat{J}^{(0)}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}.$$

To obtain derivatives of the zeroth-order value function with respect to $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$, we first differentiate with respect to the marginal productivity of capital $\hat{r}^{(0)}_{\text{mpk}}$, which slowly depends on these three variables via $\hat{D}$ (i.e. the chain rule of differentiation). From (B5), we obtain:

$$\text{(C3.8)}\qquad \frac{\partial\hat{J}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\hat{J}^{(0)}\left(-(1-\eta)\frac{\hat{\phi}''(\hat{i}^{(0)})}{\hat{\phi}'(\hat{i}^{(0)})}+\gamma\frac{1-\eta}{\hat{c}^{(0)}}\right)\frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}.$$

Since the investment rate is implicitly defined, we obtain from (B6) by implicit differentiation:

$$\text{(C3.9)}\qquad \frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\frac{1}{\gamma-\hat{c}^{(0)}\hat{\phi}''(\hat{i}^{(0)})/\hat{\phi}'(\hat{i}^{(0)})}.$$

Combining equations (C3.8) and (C3.9), we obtain

$$\text{(C3.10)}\qquad \frac{\partial\hat{J}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\hat{J}^{(0)}\frac{1-\eta}{\hat{c}^{(0)}}=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}.$$

Using the chain rule of differentiation, we find the individual terms that contribute to the forcing (C3.7) at $\mathcal{O}(\epsilon)$:

$$\text{(C3.11)}\qquad \begin{aligned}\underset{\mathcal{O}(\epsilon)}{\hat{J}^{(0)}_{\hat{E}}}&=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\frac{\partial\hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}}\ \text{ and}\\ \underset{\mathcal{O}(\epsilon)}{\hat{J}^{(0)}_{\hat{E}\hat{E}}}&=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\frac{\partial^2\hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}^2},\end{aligned}$$

and similarly for derivatives with respect to $\hat{\chi}$ and $\hat{\lambda}$, as well as cross-derivatives. From the zeroth-order solution $\hat{r}^{(0)}_{\text{mpk}} = \alpha \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{1/\alpha}\left((1-\alpha)/\hat{b}\right)^{(1-\alpha)/\alpha} - \hat{\delta}$, we obtain

(C3.12a)
$$\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*(1+\theta_{ET})\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}^2} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\theta_{ET}(1+\theta_{ET})\hat{E}^{\theta_{ET}-1}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

(C3.12b)
$$\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}} = -\epsilon \hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}^2} = -\epsilon \hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

(C3.12c)
$$\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}} = -\epsilon \hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda}),$$
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}^2} = -\epsilon \hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda}),$$

(C3.12d)
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}\partial \hat{\chi}} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*(1+\theta_{ET})\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}\partial \hat{\lambda}} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*(1+\theta_{ET})\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}\partial \hat{\lambda}} = -\epsilon \hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$

where have used the following short hands $\hat{\mathrm{X}} \equiv \hat{\chi}^{1+\theta_{\chi T}}$ and $\hat{\Lambda} \equiv \hat{\lambda}^{1+\theta_{\lambda}}$, so $\hat{D} = \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})$. Equations (C3.11) and (C3.12) can be substituted into (C3.7):

(C3.13)
$$
\begin{aligned}
\hat{G}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = {} & -\epsilon\hat{A}(\hat{E},\hat{\chi})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^{*}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \\
& \Big[\Big(-(1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\Lambda}+\hat{\nu}_{\chi}(\bar{\hat{\chi}}-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}+\hat{\nu}_{\lambda}(\bar{\hat{\lambda}}-\hat{\lambda})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}+\frac{1}{2}\hat{\sigma}_{\chi}^{2}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}+\frac{1}{2}\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^{2} \\
& +(1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}+(1-\eta)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}+\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}} \\
& +(1+\theta_{ET})\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}}e^{-\hat{g}^{(0)}\hat{t}}+\frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\hat{\sigma}_{E}^{2}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1} \\
& +\Big((1-\eta)(1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}+(1+\theta_{ET})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \\
& +(1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Big].
\end{aligned}
$$

Because we are ultimately interested in $\hat{J}_{\hat{E}}^{(1)}$ for the computation of the social cost of carbon, we first differentiate (C3.6) with respect to $\hat{E}$ and seek a solution for $\hat{J}_{\hat{E}}^{(1)}$ of the form $\hat{J}_{\hat{E}}^{(1)}=\psi_1\left(1+\theta_{ET}\right)\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})$, which gives (from (C3.6)):[10]

(C3.14) $$\hat{J}_{\hat{E}}^{(1)}=\psi_1\left(1+\theta_{ET}\right)\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) \Rightarrow -\hat{r}_{\Omega}\hat{\Omega}+\frac{1}{\mathrm{d}\hat{t}}\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{\Omega}\right]=-\hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}),$$

where we have introduced the effective discount rate

(C3.15) $$\hat{r}_{\Omega}\equiv\hat{r}^{(0)}-\hat{g}^{(0)}+(1-\eta)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_{K}^{2}\right)+\hat{\varphi},$$

and the coefficient

(C3.16) $$\psi_1\equiv\hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\lambda)^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}.$$

The normalized forcing is defined by[11]

[10] Dimensionally, we have $\Omega=E_0^{\theta_{ET}}\chi_0^{1+\theta_{\chi T}}\lambda_0^{1+\theta_{\lambda}}K_0^{1-\eta}\hat{\Omega}$.

[11] Dimensionally, we have $\Gamma=E_0^{1+\theta_{ET}}\chi_0^{1+\theta_{\chi T}}\lambda_0^{1+\theta_{\lambda}}K_0^{1-\eta}g_0\hat{\Gamma}$.

$$\text{(C3.17)}\quad \hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) \equiv \Big((1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\Lambda} - \hat{\nu}_{\chi}(\hat{\bar{\chi}}-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda} - \hat{\nu}_{\lambda}(\hat{\bar{\lambda}}-\hat{\lambda})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}} - \frac{1}{2}\hat{\sigma}_{\chi}^2\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}$$

$$-\frac{1}{2}\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 - (1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} - (1-\eta)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} - \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}$$

$$-\theta_{ET}\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}-1}e^{-\hat{g}^{(0)}\hat{t}} - \frac{1}{2}(\theta_{ET}-1)\theta_{ET}\hat{\sigma}_E^2\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-2}$$

$$-\theta_{ET}\Big((1-\eta)\hat{\mathrm{X}}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}.$$

Equation (C3.14) has the closed-form solution:

$$\text{(C3.18)}\qquad \hat{\Omega} = \mathrm{E}_{\hat{t}}\left[\int_{\hat{t}}^{\infty}\hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{s})e^{-\hat{r}_{\Omega}(\hat{s}-\hat{t})}\mathrm{d}\hat{s}\right].$$

We can now compute the SCC according to $\hat{P} = -\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)\Big/\phi'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}$:

$$\text{(C3.19)}\quad \hat{P} = \frac{\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\Big|_{\hat{P}=0}}{\hat{r}^*}\left(1 - \frac{\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})}{\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})\hat{K}^{1-\eta}}\right) \text{ with } \hat{\Theta} \equiv \frac{\epsilon\hat{D}_{\hat{E}}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}{1-\hat{D}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)},$$

where we have introduced $\hat{r}^* \equiv \hat{r}^{(0)} - \hat{g}^{(0)}$. Dimensionally, equations (C3.17), (C3.18) and (C3.19) correspond to Result A.

Appendix D: Leading-Order Effects of Uncertainty (For Online Publication)

To evaluate the 4-dimensional integral in Result A in closed form and thus derive Results 1 and 2, we take three steps. First, we evaluate the expected carbon stock dynamics as a function of time in section D1. Second, in section D2, we evaluate the forcing (C3.17)-(C3.18) of the first-order problem in Appendix C. In this section, we invoke three assumptions: we ignore the uncertainty in the carbon stock arising from the uncertainty of future emissions (*Assumption I*), we take account of climatic uncertainty only to leading order (*Assumption II*), and we set $\lambda_0 = \bar{\lambda}$ (*Assumption III*). Finally, we combine the zeroth- and first-order value functions and evaluate our leading-order estimate of the SCC in section D3. This is known as Result 2. Result 2 further simplifies to Result 1 for proportional damages ($\theta_{ET} = 0$) (*Assumption IV*) and with the initial climate sensitivity parameter equal to its steady-state value ($\chi_0 = \bar{\chi}$) (*Assumption V*).

D1. Expected carbon stock dynamics

The expected value of the carbon stock is governed by the differential equation (C1.5) with solution

(D1.1) $$\mathrm{E}_{\hat{t}}\left[\hat{E}(\hat{s})\right] = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s}) + \hat{\mu}^*\hat{K}(\hat{t})\left[1-\exp(-\hat{\varphi}\Delta\hat{s})\right]/\hat{\varphi} = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s})\hat{e}(\Delta\hat{s}),$$

with new short hands $\hat{\mu}^* \equiv \hat{\mu}\left((1-\alpha)/\hat{b}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}$, $\Delta\hat{s} \equiv \hat{s} - \hat{t}$ and $\hat{e}(\Delta\hat{s}) = 1 + (\hat{\mu}^*\hat{K}(\hat{t})/\hat{E}(\hat{t}))(\exp(\hat{\varphi}\Delta\hat{s}) - 1)/\hat{\varphi}$. Dimensionally, we define μ^* so that $\mu F^{(0)} = \mu^* K$, where μ does not have units and μ^* has units TtC^{-1}year^{-1}. We can then obtain $\mu^* = \mu\left(A(1-\alpha)/b\right)^{1/\alpha}$ or $\hat{\mu}^* = \left(K_0/g_0E_0\right)\mu^*$.

D2. Forcing of the first-order problem with only leading-order uncertainty

To identify only leading-order contributions of uncertainty, we expand in $\Delta\hat{\chi} \equiv \hat{\chi} - \mathrm{E}_{\hat{t}}\left[\hat{\chi}\right]$, $\Delta\hat{\lambda} \equiv \hat{\lambda} - \mathrm{E}_{\hat{t}}\left[\hat{\lambda}\right]$ and $\Delta\hat{E} \equiv \hat{E} - \mathrm{E}_{\hat{t}}\left[\hat{E}\right]$ with the corresponding covariance matrix given by (C2.5) (*Assumption II*). As in footnote 15 of the paper, we will use short-hand notation for the expected values of $\hat{\chi}$ and $\hat{\lambda}$, namely $\hat{\mu}_{\chi} \equiv \mathrm{E}_{\hat{t}}\left[\hat{\chi}\right] = \exp(-\hat{\nu}_{\chi}\hat{t}) + \hat{\bar{\chi}}\left(1-\exp(-\hat{\nu}_{\chi}\hat{t})\right)$ and

$\hat{\mu}_\lambda \equiv \mathrm{E}_{\hat{t}}\left[\hat{\lambda}\right] = \exp(-\hat{\nu}_\lambda \hat{t}) + \hat{\bar{\lambda}}\left(1-\exp(-\hat{\nu}_\lambda \hat{t})\right)$, and we note that $\hat{\mu}_\lambda = 1$ (*Assumption III*).[12]

We begin by considering terms that only involve capital stock uncertainty, which are evaluated without approximation. The probability density function for time $\hat{s}$, but with the expectation operator evaluated at time $\hat{t}$, is

$$f_k = \frac{1}{\sqrt{2\pi\hat{\sigma}_K^2 \Delta\hat{s}}} \exp\left(-\frac{1}{2}\left(\frac{(\hat{k}-\hat{\alpha}_k \hat{s})^2}{\hat{\sigma}_K^2 \Delta\hat{s}}\right)\right), \tag{D2.1}$$

where $\hat{\alpha}_k = \hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2$. Combining with the discount factor $\exp\left(-\hat{r}_\Omega \Delta\hat{s}\right)$ in (C3.18) and an additional factor accounting for the decay of the atmospheric carbon stock, we have without further approximation

$$\begin{gathered} \mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\exp\left(-\left(\hat{r}_\Omega + \theta_{ET}\hat{\varphi}\right)\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{1-\eta} \exp(-\hat{r}^{\star}\Delta\hat{s}) \quad \text{and} \\ \mathrm{E}_{\hat{t}}\left[\hat{K}^{2-\eta}\right]\exp\left(-(\hat{r}_\Omega + \hat{g}^{(0)} + (\theta_{ET}-1)\hat{\varphi})\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{2-\eta} \exp(-\hat{r}^{\star\star}\Delta\hat{s}), \end{gathered} \tag{D2.2}$$

where $\hat{r}^{\star} \equiv \hat{r}^{*} + (1+\theta_{ET})\hat{\varphi} = \hat{r}^{(0)} - \hat{g}^{(0)} + (1+\theta_{ET})\hat{\varphi}$ and $\hat{r}^{\star\star} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} - (1-\eta)\hat{\sigma}_K^2 + \theta_{ET}\hat{\varphi} \;\; + \hat{r}^{\star} - (1-\eta)\hat{\sigma}_K^2 - \hat{\varphi}$. We use alternative star symbols $\star$ as superscripts to denote rates corrected for atmospheric carbon stock decay. To leading order, we have for the terms involving the carbon stock:

$$\begin{gathered} \mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4), \\ \mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}-1}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\left[1+\frac{1}{2}\left(\theta_{ET}-1\right)\left(\theta_{ET}-2\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4), \\ \mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}-2}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-2}\left[1+\frac{1}{2}\left(\theta_{ET}-2\right)\left(\theta_{ET}-3\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4), \end{gathered} \tag{D2.3}$$

[12] Consistent with the other non-dimensional variables, $\hat{\mu}_\chi \equiv \mu_\chi / \chi_0$ and $\hat{\mu}_\lambda \equiv \mu_\lambda / \lambda_0$.

where we let the subscript on Σ^2 denote the relevant elements of the covariance matrix $\mathbf{\Sigma}$ (C2.35) and we have ignored any contributions to uncertainty from new emissions through their dependence on uncertain future GDP (*Assumption I*). Making *Assumption II* more precise, we retain terms up to second order in a perturbation expansion in Σ.[13] The following terms also make a contribution to the forcing (C3.17)-(C3.18): $\hat{\mathrm{X}}\hat{\Lambda}, \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}$, $\left(\hat{\chi}-\hat{\mu}_{\chi}\right)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}, \hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}, \left(\hat{\lambda}-\hat{\mu}_{\lambda}\right)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}, \hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}$ and $\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}$. Keeping only those terms contributing to the leading-order effect of climatic uncertainty, we have

(D2.4a)
$$\mathrm{E}_{\hat{t}}\left[\hat{\mathrm{X}}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[1+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$
$$\mathrm{E}_{\hat{t}}\left[\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{\theta_{\chi T}}\left[\theta_{\chi T}+1+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}(\theta_{\chi T}-1)\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$

(D2.4b)
$$\mathrm{E}_{\hat{t}}\left[\left(\hat{\chi}-\hat{\mu}_{\chi}\right)\hat{\mathrm{X}}_{\hat{\chi}}\left(\hat{\chi}\right)\right]=\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$
$$\mathrm{E}_{\hat{t}}\left[\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{\theta_{\chi T}-1}\left[(\theta_{\chi T}+1)\theta_{\chi T}\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^2),$$

(D2.5a)
$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}\left[1+\frac{1}{2}(\theta_{\lambda}+1)\theta_{\lambda}\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$
$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{\theta_{\lambda}}\left[(\theta_{\lambda}+1)+\frac{1}{2}(\theta_{\lambda}+1)\theta_{\lambda}(\theta_{\lambda}-1)\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$

(D2.5b)
$$\mathrm{E}_{\hat{t}}\left[\left(\hat{\lambda}-\hat{\mu}_{\lambda}\right)\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}\left[(\theta_{\lambda}+1)\theta_{\lambda}\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$
$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{\theta_{\lambda}-1}\left[(\theta_{\lambda}+1)\theta_{\lambda}\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^2).$$

Using (D2.2)-(D2.5), we now consider the terms in the forcing (C3.17) consecutively and let the subscript indices correspond to the sequence of terms in (C3.17) (left to right).

[13] We also retain the term proportional to $\hat{\Sigma}_{\chi}^2\hat{\Sigma}_{\lambda}^2$, which is fourth order, although this is inconsistent from a perturbation theory perspective. We know from comparison to Result A, which we can evaluate exactly numerically (see Appendix F), that this term is the largest higher-order term (notably, in the case of highly convex damages) we otherwise ignore. We thus increase the accuracy of Results 1 and 2 by a few percent (see Appendix F).

To consider the covariance terms in the forcing (C3.17), we also expand in $\Delta\hat{k} \equiv \hat{k} - \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2\right)\hat{t}$ and only consider deviations from the zeroth-order mean consistent with our search for leading-order terms only. The following terms arise:

(D2.6)
$$
\begin{aligned}
\mathrm{E}_{\hat{t}}\left[\Gamma_1\right] = (1+\theta_{ET})\hat{\varphi}\Bigg[& 1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}} \\
& + \frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} + \frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda} \\
& + \frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^{\ 2}\hat{\mu}_\lambda^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda} \\
& + (1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}} + (1-\eta)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} \\
& + (1-\eta)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda} + \theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi} \\
& + \theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda} \\
& + \left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg]\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\ 1+\theta_{\chi T}}\hat{\mu}_\lambda^{\ 1+\theta_\lambda},
\end{aligned}
$$

(D2.7)
$$
\begin{aligned}
\mathrm{E}_{\hat{t}}\left[\Gamma_2\right] = \hat{\nu}_\chi(1+\theta_{\chi T})\Bigg(& \theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} \\
& + \frac{1}{2}\theta_{\chi T}\theta_\lambda(1+\theta_\lambda)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^{\ 2}\hat{\mu}_\lambda^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda} \\
& (1-\eta)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} + \theta_{ET}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi} \\
& + \left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg)\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\ 1+\theta_{\chi T}}\hat{\mu}_\lambda^{\ 1+\theta_\lambda}
\end{aligned}
$$

$$+\hat{\nu}_\chi(1+\theta_{\chi T})\left(\hat{\chi}(\hat{t})-\hat{\bar{\chi}}\right)e^{-\hat{\nu}_\chi\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}+\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^{\ 2}}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\left(1-\eta\right)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+\left(1-\eta\right)\theta_{\chi T}\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}$$

$$+\left(1-\eta\right)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}+\theta_{ET}\theta_{\chi T}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi}$$

$$+\theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda}$$

$$\left.+\theta_{\chi T}\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\right]\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.8)

$$\mathrm{E}_{\hat{t}}\left[\Gamma_3\right]=\hat{\nu}_\lambda\left(1+\theta_\lambda\right)\left(\theta_\lambda\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}\right.$$

$$+\frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\theta_\lambda\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$\left(1-\eta\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}+\theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda}$$

$$\left.\left(1+\theta_{\chi T}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\right)\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda}$$

$$+\hat{\nu}_\lambda(1+\theta_\lambda)\left(\hat{\lambda}(\hat{t})-\hat{\bar{\lambda}}\right)e^{-\hat{\nu}_\lambda\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}+\frac{1}{2}\theta_\lambda\left(\theta_\lambda-1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(\theta_{\lambda}-1\right)\frac{\hat{\sigma}_{\lambda}^{2}\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\chi}\Delta\hat{s}\right)}{2\hat{v}_{\chi}}\frac{1-\exp\left(-2\hat{v}_{\lambda}\Delta\hat{s}\right)}{2\hat{v}_{\lambda}}$$

$$+(1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+(1-\eta)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}}{\hat{\mu}_{\chi}}\frac{1-\exp\left(-\hat{v}_{\chi}\Delta\hat{s}\right)}{\hat{v}_{\chi}}$$

$$+(1-\eta)\theta_{\lambda}\frac{\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\hat{v}_{\lambda}\Delta\hat{s}\right)}{\hat{v}_{\lambda}}+\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_{\chi}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\chi}}$$

$$+\theta_{ET}\theta_{\lambda}\frac{\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\lambda}}$$

$$\left.+\left(1+\theta_{\chi T}\right)\theta_{\lambda}\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{v}_{\chi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{v}_{\chi}+\hat{v}_{\lambda}}\right]\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.9)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{4}\right]=\left[-\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\chi}^{2}\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\lambda}\Delta\hat{s}\right)}{2\hat{v}_{\lambda}}\right]$$
$$\times\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.10)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{5}\right]=\left[-\frac{1}{2}\left(1+\theta_{\lambda}\right)\theta_{\lambda}\frac{\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\lambda}^{2}}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\chi}^{2}\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\chi}\Delta\hat{s}\right)}{2\hat{v}_{\chi}}\right]$$
$$\times\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.11) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{6}\right]=-(1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.12) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{7}\right]=-(1-\eta)\left(1+\theta_{\lambda}\right)\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{\theta_{\lambda}},$$

(D2.13) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{8}\right]=-\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{\theta_{\lambda}},$$

(D2.14) $$\mathrm{E}_{\hat{t}}\left[\Gamma_9\right]=-\theta_{ET}\hat{\mu}^*\Big[1+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}$$

$$+\frac{1}{2}\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}+\frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\left(2-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}+\left(2-\eta\right)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}$$

$$+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg]\mathrm{E}_{\hat{t}}\left[\hat{K}^{2-\eta}\right]e^{-\hat{g}^{(0)}\Delta\hat{s}}\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\left(\hat{s}\right)\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.15) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{10}\right]=-\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\hat{\sigma}_E^2\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-2}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.16) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{11}\right]=-(1-\eta)\theta_{ET}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.17) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{12}\right]=-\theta_{ET}(1+\theta_{\chi T})\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.18) $$\mathrm{E}_{\hat{t}}\left[\Gamma_{13}\right]=-\theta_{ET}(1+\theta_\lambda)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{\theta_\lambda},$$

where elements of the covariance matrix have been substituted from (C2.5).

D3. Leading-order solution (Results 1 and 2)

Combining all the leading-order terms in the forcing equation (D2.6)-(D2.18) and substituting into (C3.19), further assuming $\lambda_0=\bar{\lambda}$, so that $\hat{\mu}_\lambda=1$ (*Assumption III*), we obtain Result 2 after considerable manipulation (including integrating by parts).

Result 2: The optimal risk-adjusted SCC is

(D3.1)
$$\hat{P}=\frac{\epsilon\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\Big|_{\hat{P}=0}}{\hat{r}^\star}\times$$
$$\left(1+\theta_{ET}\hat{\mu}^*\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^{\star\star}}\Upsilon_{\theta_{ET}\neq 0}+(1+\theta_{\chi T})\frac{\hat{\nu}_\chi}{\hat{r}^\star}\frac{\hat{\bar{\chi}}-\hat{\chi}}{\hat{\chi}}\Upsilon_{\chi_0\neq\bar{\chi}}+\Delta_{EE}+\Delta_{\chi\chi}+\Delta_{\lambda\lambda}+\Delta_{\chi\times\lambda}+\Delta_{CK}+\Delta_{CC}\right),$$

where we distinguish six so-called 'risk adjustments' denoted by the symbol Δ with subscripts denoting the state variable(s) from which the risk originates.

D3.1. Risk adjustments

The risk adjustments for atmospheric carbon stock uncertainty (Δ_{EE}), climate sensitivity uncertainty ($\Delta_{\chi\chi}$), damage ratio uncertainty ($\Delta_{\lambda\lambda}$), the interaction of climate sensitivity and damage ratio uncertainty ($\Delta_{\chi\times\lambda}$), the correlation between economic risk and all three climatic risks (Δ_{CK}), and the correlation between the three climatic risks themselves (Δ_{CC}) are respectively

(D3.2)

$$\Delta_{EE} \equiv \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^\star - 2\hat{\varphi}}\Upsilon_{EE}, \quad \Delta_{\chi\chi} \equiv \frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1}{\hat{r}^\star + 2\hat{\nu}_\chi}\Upsilon_{\chi\chi},$$

$$\Delta_{\lambda\lambda} \equiv \frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{r}^\star + 2\hat{\nu}_\lambda}\Upsilon_{\lambda\lambda}, \ \Delta_{\chi\times\lambda} \equiv \frac{1}{16}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\nu}_\chi\hat{\nu}_\lambda\hat{\mu}_\chi^2}\frac{\hat{r}^\star}{\hat{r}^\star - (1+\theta_{ET})\hat{\varphi}}$$

$$\times\left(\frac{(1+\theta_{ET})\hat{\varphi}+2\hat{\nu}_\chi}{\hat{r}^\star + 2\hat{\nu}_\chi} + \frac{(1+\theta_{ET})\hat{\varphi}+2\hat{\nu}_\lambda}{\hat{r}^\star + 2\hat{\nu}_\lambda} - \frac{(1+\theta_{ET})\hat{\varphi}+2(\hat{\nu}_\chi+\hat{\nu}_\lambda)}{\hat{r}^\star + 2\hat{\nu}_\lambda + 2\hat{\nu}_\chi} - \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{r}^\star}\right)\Upsilon_{\chi\times\lambda},$$

$$\Delta_{CK} \equiv -(\eta-1)\hat{\sigma}_K\left(\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_E}{\hat{E}\left(\hat{r}^\star-\hat{\varphi}\right)}\Upsilon_{KE} + \left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}}{\hat{r}^\star+\hat{\nu}_\chi}\frac{\hat{\sigma}_\chi}{\hat{\mu}_\chi}\Upsilon_{K\chi} + \left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_\lambda}{\hat{r}^\star+\hat{\nu}_\lambda}\Upsilon_{K\lambda}\right),$$

$$\Delta_{CC} \equiv \theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}\hat{\mu}_\chi}\frac{\hat{r}^\star}{\hat{r}^\star-\hat{\varphi}}\frac{1}{\hat{r}^\star+\hat{\nu}_\chi}\Upsilon_{E\chi}$$

$$+\left(1+\theta_\lambda\right)\left(\left(1+\theta_{\chi T}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{r}^\star+\hat{\nu}_\lambda+\hat{\nu}_\lambda}\frac{1}{\hat{\mu}_\chi}\Upsilon_{\chi\lambda} + \theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{E}}\frac{\hat{r}^\star}{\hat{r}^\star-\hat{\varphi}}\frac{1}{\hat{r}^\star+\hat{\nu}_\lambda}\Upsilon_{E\lambda}\right).$$

D3.2. Correction factors

In addition to 'risk adjustments', we distinguish two types of so-called 'correction factors', denoted by the symbol Υ with subscripts again denoting the state variable(s) from which the risk originates: for $\theta_{ET} \neq 0$ and for $\chi_0 \neq \bar{\chi}$. In equation (D3.1), the correction factors $\Upsilon_{\theta_{ET}\neq 0}$ and $\Upsilon_{\chi_0\neq\bar{\chi}}$ are deterministic corrections for $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$, respectively. The remaining correction factors (Υ) in (D.3.2) multiply a risk adjustment (Δ) and must be linearly combined with unity, so that, for example,

$\Upsilon_{\chi\chi} \equiv 1 + \Upsilon_{\chi\chi,\theta_{ET}\neq 0} + \Upsilon_{\chi\chi,\chi_0\neq\bar{\chi}}$. These combined correction factors are equal to unity if $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$ (e.g., $\Upsilon_{\chi\chi} \equiv 1$). We give the correction factors in terms of dimensional quantities below (using the definitions in Appendix A.2.1), so that they can be used directly in Result 2 given dimensionally in Appendix A.4.

The correction factors for $\theta_{ET} \neq 0$ are

(D3.4)

$$\Upsilon_{\theta_{ET}\neq 0} = \frac{r^{\star\star}}{1-(1+\theta_{ET})\varphi/r^{\star}} \int_0^\infty \left(e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T}} d\Delta s,$$

$$\Upsilon_{Ki,\theta_{ET}\neq 0} = \theta_{ET}\mu^* \frac{K(t)}{E(t)} \frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \Bigg[\left(1 + \frac{(1+\theta_{ET})\varphi}{\nu_i} \right) \int_0^\infty e^{-(\nu_i + r^{\star} - \varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i)} d\Delta s$$

$$-(1+\theta_{ET}) \frac{\varphi}{r^{\star}} \frac{r^{\star}+\nu_i}{\nu_i} \int_0^\infty e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i)} d\Delta s +$$

$$\frac{2-\eta}{1-\eta} \frac{\nu_i + r^{\star}}{\nu_i} \int_0^\infty \left(e^{-r^{\star\star}\Delta\hat{s}} - e^{-(r^{\star\star}+\nu_i)\Delta s} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i)} d\Delta s \Bigg] \text{ for } i = \chi, \lambda,$$

$$\Upsilon_{ij,\theta_{ET}\neq 0} = \theta_{ET}\mu^* \frac{K}{E} \frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \Bigg[\left(1 + \frac{(1+\theta_{ET})\varphi}{\nu_i + \nu_j} \right) \int_0^\infty e^{-(r^{\star}+\nu_i+\nu_j-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1}$$

$$\times \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i) - \mathrm{I}_\chi(j)} d\Delta s - \frac{(1+\theta_{ET})\varphi}{\nu_i+\nu_j} \frac{r^{\star}+\nu_i+\nu_j}{r^{\star}} \int_0^\infty e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i) - \mathrm{I}_\chi(j)} d\Delta s$$

$$+ \frac{r^{\star}+\nu_i+\nu_j}{\nu_i+\nu_j} \int_0^\infty \left(e^{-r^{\star\star}\Delta s} - e^{-(r^{\star\star}+\nu_i+\nu_j)\Delta\hat{s}} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T} - \mathrm{I}_\chi(i) - \mathrm{I}_\chi(j)} d\Delta s \Bigg] \text{ for } i, j = \chi, \lambda,$$

$$\Upsilon_{EE,\theta_{ET}\neq 0} = -\frac{\left(r^{\star}-2\varphi\right)}{2\varphi} \frac{\mu^* \dfrac{K}{E}(\theta_{ET}-2)}{1-(1+\theta_{ET})\varphi/r^{\star}} \Bigg[\int_0^\infty \Bigg(\frac{(\theta_{ET}-1)\varphi}{r^{\star}-2\varphi} e^{-(r^{\star}-3\varphi)\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s}$$

$$-e^{-(r^{\star\star}-2\varphi)\Delta s} + e^{-r^{\star\star}\Delta s} \Bigg) \left(e(\Delta s)\right)^{\theta_{ET}-3} \left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)} \right)^{1+\theta_{\chi T}} ds,$$

$$\Upsilon_{\chi\times\lambda,\theta_{ET}\neq 0} = \frac{\theta_{ET}\mu^* K(t)/E(t)}{\frac{(1+\theta_{ET})\varphi}{r^\star} - \frac{(1+\theta_{ET})\varphi+2\nu_\chi}{r^\star+2\nu_\chi} - \frac{(1+\theta_{ET})\varphi+2\nu_\lambda}{r^\star+2\nu_\lambda} + \frac{(1+\theta_{ET})\varphi+2(\nu_\chi+\nu_\lambda)}{r^\star+2(\nu_\lambda+\nu_\chi)}}$$

$$\times\int_0^\infty\left(\frac{(1+\theta_{ET})\varphi}{r^\star}e^{-(r^\star-\varphi)\Delta s} - e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi+2\nu_\chi}{r^\star+2\nu_\chi}e^{-(r^\star+2\nu_\chi-\varphi)\Delta s}\right.$$

$$+e^{-(r^{\star\star}+2\nu_\chi)\Delta s} - \frac{(1+\theta_{ET})\varphi+2\nu_\lambda}{r^\star+2\nu_\lambda}e^{-(r^\star+2\nu_\lambda-\varphi)\Delta s} + e^{-(r^{\star\star}+2\nu_\lambda)\Delta s}$$

$$\left.\frac{(1+\theta_{ET})\varphi+2(\nu_\chi+\nu_\lambda)}{r^\star+2\nu_\lambda+2\nu_\chi}e^{-(r^\star+2\nu_\lambda+2\nu_\chi-\varphi)\Delta s} - e^{-(r^{\star\star}+2\nu_\chi+2\nu_\lambda)\Delta s}\right)\times\left(e(\Delta s)\right)^{\theta_{ET}-1}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-1} d\Delta s,$$

where $\mathrm{I}_\chi(i)=1$ for $i=\chi$ and $\mathrm{I}_\chi(i)=0$ for $i\neq\chi$ (cf. indicator function), the function that takes into account future changes to the mean carbon stock $e(\Delta s)=1+(\mu^* K(t)/E(t))(\exp(\varphi\Delta s)-1)/\varphi$, and the time-varying mean climate sensitivity $\mu_\chi(\Delta s)=\mu_\chi(t)\exp(-\nu_\chi\Delta s)+\bar{\mu}_\chi(1-\exp(-\nu_\chi\Delta s))$.

The correction factors for $\chi_0\neq\bar{\chi}$ are:

(D3.5)

$$\Upsilon_{\chi_0\neq\bar{\chi}} = r^\star\int_0^\infty\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}} e^{-(r^\star+\nu_\chi)\Delta s} d\Delta s,$$

$$\Upsilon_{ij,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}-\mathrm{I}_\chi(i)-\mathrm{I}_\chi(j)\right)\nu_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+\nu_i+\nu_j}{\nu_i+\nu_j}\int_0^\infty\left(e^{-(r^\star+\nu_\chi)\Delta s} - \frac{r^\star}{r^\star+\nu_i+\nu_j}e^{-(r^\star+\nu_\chi+\nu_i+\nu_j)\Delta s}\right)$$

$$\times\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-\mathrm{I}_\chi(i)-\mathrm{I}_\chi(j)} d\Delta\hat{s},$$

$$\Upsilon_{Ki,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}-\mathrm{I}_\chi(i)\right)\nu_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+\nu_i}{\nu_i}\int_0^\infty\left(e^{-(r^\star+\nu_\chi)\Delta s} - \frac{r^\star}{r^\star+\nu_\chi}e^{-(\nu_\chi+\nu_i)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-\mathrm{I}_\chi(i)} d\Delta s,$$

$$\Upsilon_{EE,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}\right)\nu_\chi\frac{\left(\bar{\chi}-\mu_\chi(t)\right)}{\mu_\chi(t)}\int_0^\infty\left(\frac{r^\star}{2\varphi}e^{-(r^\star+\nu_\chi-2\varphi)\Delta s} - \frac{r^\star-2\varphi}{2\varphi}e^{-(r^\star+\nu_\chi)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}-2}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}} d\Delta s,$$

$$\Upsilon_{\chi\times\lambda,\chi_0\neq\bar{\chi}} = \frac{\left(1-\frac{(1+\theta_{ET})\varphi}{r^\star}\right)\left(\theta_{\chi T}-1\right)\nu_\chi \frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}}{-\frac{(1+\theta_{ET})\varphi}{r^\star}+\frac{(1+\theta_{ET})\varphi+2\nu_\chi}{r^\star+2\nu_\chi}+\frac{(1+\theta_{ET})\varphi+2\nu_\lambda}{r^\star+2\nu_\lambda}-\frac{(1+\theta_{ET})\varphi+2(\nu_\chi+\nu_\lambda)}{r^\star+2(\nu_\lambda+\nu_\chi)}}$$

$$\int_0^\infty\left(1-\frac{r^\star}{r^\star+2\nu_\chi}e^{-2\nu_\chi\Delta s}-\frac{r^\star}{r^\star+2\nu_\lambda}e^{-2\nu_\lambda\Delta s}+\frac{r^\star}{r^\star+2\nu_\lambda+2\nu_\chi}e^{-2(\nu_\lambda+\nu_\chi)\Delta s}\right)$$

$$\times e^{-(r^\star+\nu_\chi)\Delta s}\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-2} d\Delta s.$$

We do not explicitly give the correction factors for the correlation terms involving carbon stock uncertainty. Equation (D3.1) together with (D3.2)-(D3.5) gives the optimal SCC according to Result 2.

Appendix E: Calibration (For Online Publication)

E1. Asset returns, risk aversion and intertemporal substitution

We follow the calibration of Pindyck and Wang (2013), but ignore the effect of catastrophic shocks.[14,15] Using monthly asset data from the S&P 500 for the period 1947-2008, we obtain an annual return on assets (capital gains plus dividends) of $r^{(0)} =$ 7.2%/year with annual volatility of $\sigma_K = 12\%$. For a return on safe assets of 0.80%/year based on the annualized monthly return on 3-months T-bills, we obtain a risk premium of $\Delta r^{(0)} \equiv r^{(0)} - r_{\mathrm{rf}}^{(0)} = 6.4\%$/year and calibrate the coefficient of relative risk aversion as $\eta =$ 4.3 (cf. $\Delta r^{(0)} = \eta \sigma_K^2$). Taking the growth rate to be equal to the historical growth rate of $g^{(0)} =$ 2.0%/year, the equation $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta\sigma_K^2/2$ (cf. (B9)) defines the combinations of ρ and γ that are consistent with historical asset returns. Setting the coefficient of elasticity of intertemporal substitution $\mathrm{EIS} = 2/3$, we obtain $\gamma = \mathrm{EIS}^{-1} = 1.5$ and thus a rate of time preference is ρ = 5.8%/year. In section V.A we also consider an alternative calibration where EIS = 1.5 (larger than one as is assumed in asset pricing theory) and adjust ρ = 4.8%/year, so that the same risky and risk-free financial returns are matched.

E2. Productivity, fossil fuel, adjustment costs and the depreciation rate

To calibrate total factor productivity, we consider the production function in the absence of climate damage that can be obtained by setting $P = 0$ (i.e. at zeroth order), namely $Y^{(0)} = A^* K$ with $A^* = A^{1/\alpha}\left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$. Pindyck and Wang (2013) use empirical estimates of the physical, human and intangible capital stocks and find $A^* = 0.113$ /year, which we adopt. Based on emissions of $F_0^{(0)} = 9.1$ GtC/year in 2015, energy costs making up a share $1-\alpha = 4.3\%$ of world GDP at PPP in 2015 of \$116 trillion/year, we estimate the

[14] Pindyck and Wang (2013) use Poisson shocks to capture small risks of large disasters (cf. Barro, 2016) and thus match skewness and kurtosis of asset returns. These shocks are responsible for approximately 1%-point of the risk premium.

[15] The alternative is to calibrate our AK-model to the observed volatility of consumption or output (cf. Gollier, 2012), which are generally much less volatile than capital (asset returns). Because the volatilities of capital, consumption and output are equal to the volatility of capital in an AK-model, this alternative calibration gives a much lower volatility and, consequently, a higher coefficient of relative risk aversion to match the equity premium (see also the discussion in Pindyck and Wang, 2013). Historical data for the growth rate of world GDP for 1961-2015 imply a volatility of σ_K = 1.5%/year$^{1/2}$ and thus a much higher value of risk aversion of $\eta = 2.8 \times 10^2$ for an equity premium of 6.4%/year. Kocherlota (1996) obtains σ_K = 3.6%/year$^{1/2}$ from US annual consumption growth during 1889-1978, which gives η = 49. We use σ_K = 1.5%/year$^{1/2}$, but not the corresponding high values of risk aversion.

fossil fuel cost to be $b = Y_0^{(0)}(1-\alpha)/F_0^{(0)} = \$5.4\times10^2$/tC.[16] The gross marginal productivity of capital is thus $Y_K^{(0)}\big|_{t=0} = \alpha A^* = 0.11$/year.[17] Using Pindyck and Wang's (2013) consumption-investment ratio $c^{(0)}/i^{(0)} = 2.84$ and the identity $\alpha A^* = c^{(0)} + i^{(0)}$, we obtain initial values of $c^{(0)} =$ 8.0%/year and $i^{(0)} = 2.8$%/year. Using $q^{(0)} = c^{(0)}/(r^{(0)} - g^{(0)}) =$ 1.5 and $q^{(0)} = (1-\omega i^{(0)})^{-1}$, we get the adjustment-cost parameter ω = 12.5 year. Finally, we find the depreciation rate that is consistent with the assumed rate of economic growth: $\delta = i^{(0)} - \omega(i^{(0)})^2/2 - g^{(0)} = 0.33$%/year.

E3. Atmospheric carbon stock and uncertainty

Here we calibrate our carbon stock model (4) to the Law Dome Ice Core 2000-year data set and historical emissions. The first column of Fig. E1 shows maximum-likelihood estimates, from which it is evident that estimates displaying a certain linear relationship between φ and μ are of comparable likelihood.[18]

These loci of maximum likelihood are shown separately in Fig. E2, with the overall maximum denoted by a red circle and corresponding values given in Table E1. The remaining columns in Fig. E1 show the predicted and observed rate of change of the atmospheric carbon stock (second column), the predicted and observed atmospheric carbon stock (third column) and the remaining variability (fourth column).[19]

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[16] We estimate the share of energy costs from data for energy use and energy costs from BP Statistical Review of World Energy 2017. Data for emissions are obtained from the same source available online at https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html. Our estimate of energy costs as a percentage of GDP is in good agreement with data from the U.S. Energy Information Administration available online at https://www.eia.gov/totalenergy/data/annual/showtext.php?t=ptb0105.

[17] This is in line with Caselli and Feyrer (2007), who estimate annual marginal products of capital of 8.5% for rich countries and 6.9% for poor countries, and an observed annual risk premium of 5-7%. They use a depreciation rate of 6.0% to calculate the capital stock from investment, include the share of reproducible capital rather than the share of total capital, account for differences in prices between capital and consumption goods and correct for inflation.

[18] Annual data from the Law Dome firn and ice core records and the Cape Grim record are available online at ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt.This data is based on spline fits to different dataset with different spline windows across time reflecting changes in the temporal resolution of the data. The discrete nature of the fitted data is evident for the early years. Annual carbon emissions from fossil fuel consumption and cement production are available online at http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html.

[19] By setting φ = 0, we can estimate the fraction μ of emissions that stays in the atmosphere forever, whilst the remainder is instantaneously absorbed by the oceans and other carbon sinks. Calibrating to this data, we find μ = 0.68, 0.64, 0.56 and 0.43 for the periods 1750-2004, 1800-2004, 1900-2004 and 1959-2004, respectively. Performing a similar analysis, Le Quéré et al. (2009) find that, between 1959 and 2008, 43% of each year's CO_2 emissions remained in the atmosphere on average.

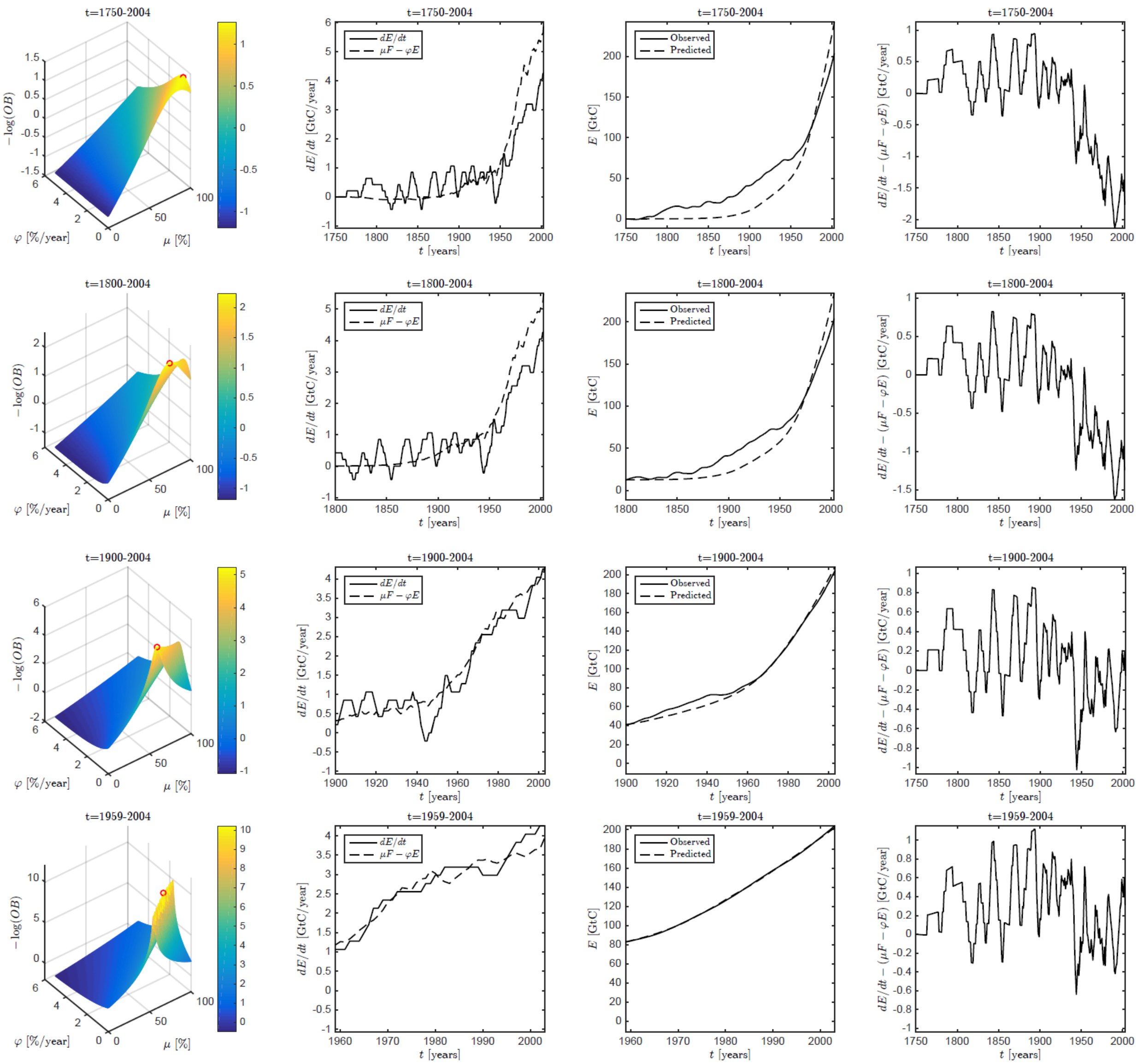


FIGURE E1. HISTORICAL ATMOSPHERIC CARBON STOCK CALIBRATION

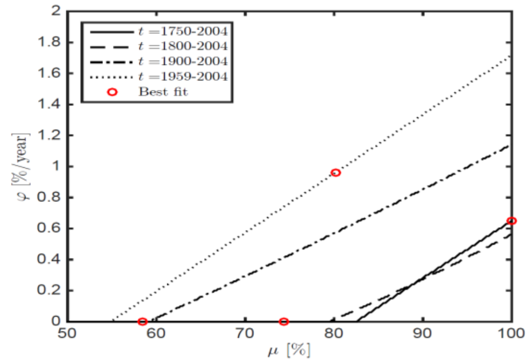


FIGURE E2. LOCI OF BEST FIT OF ATMOSPHERIC STOCK CALIBRATION

Fig. E1 indicates that our model (4) captures the observed historical variations in the atmospheric carbon stock reasonably well, including for very long time periods. The final column in Table F1 shows volatility as percentage of the initial carbon stock, from which we note that the stochastic carbon stock adjustment to the optimal SCC will be tiny if estimated from historical emissions.

TABLE E1. ATMOSPHERIC CARBON STOCK CALIBRATION

Time	μ	φ [%/year]	σ_E [GtC/year$^{1/2}$]	σ_E/S_0 [%/year$^{1/2}$]	σ_E/E_0 [%/year$^{1/2}$]
1750-2004	1.0	0.66	0.31	0.036	0.12
1800-2004	0.75	0.00	0.26	0.029	0.10
1900-2004	0.59	0.00	0.21	0.025	0.081
1959-2004	0.79	0.91	0.23	0.027	0.089

E4. Calibration of the curvature of the temperature-carbon stock relationship

The curvature of our temperature relationship (5), $T(E,\chi)=\chi^{1+\theta_\chi}(E/S_{\mathrm{PI}})^{1+\theta_E}$, is constant: $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$. The radiative law for global mean temperature,

$T \propto \ln(S/S_{\mathrm{PI}})/\ln(2) \propto \ln\left((E+S_{\mathrm{PI}})/S_{\mathrm{PI}}\right)/\ln(2)$ (Arrhenius, 1896)[20] gives $\theta_E = -E/(E+S_{PI})$. If we evaluate the temperature relationship at double (quadruple) the pre-industrial stock $E = S_{\mathrm{PI}}$ ($E = 3S_{\mathrm{PI}}$), we obtain $\theta_E = -0.50$ (or $\theta_E = -0.75$).[21] For $S_0 =$ 0.854 TtC or $E_0 = 0.258$ TtC (Fiven $S_{\mathrm{PI}} = 0.596$ TtC), we get $\theta_E = -0.30$. We set $\theta_E = -0.36$ for our base case calibration.

E.5. Climate sensitivity and uncertainty

If climate sensitivity parameter χ is normally distributed with mean μ_χ and standard deviation Σ_χ, the climate sensitivity $T_2 = \chi^{1+\theta_\chi}$ is described by the probability density function

$$\text{(E1)}\quad f_{T_2}\left(T_2;\mu_\chi,\Sigma_\chi,\theta_\chi\right) = \frac{1}{\sqrt{2\pi}\Sigma_\chi\left(1+\theta_\chi\right)} T_2^{-\frac{\theta_\chi}{1+\theta_\chi}} \exp\left(-\left(T_2^{\frac{1}{1+\theta_\chi}} - \mu_\chi\right)^2 / 2\Sigma_\chi^2\right).$$

Unlike for fat-tailed distributions, which typically have algebraically decaying tails, all moments of (E1) are defined due to its exponential tail (for $\theta_\chi \geq -1$), so that Weitzman's (2009) 'dismal theorem' does not apply. Positive values of θ_χ result in a positively skewed (non-Gaussian) distribution with more probability mass at high temperatures. Leading-order central moments of climate sensitivity can be obtained from performing Taylor-series expansions of $T_2 = \chi^{1+\theta_\chi}$ about its mean μ_χ:

$$\text{(E2a)}\quad E\left[T_2\right] = \mu_\chi^{1+\theta_\chi}\left(1+\frac{1}{2}\theta_\chi(1+\theta_\chi)(\Sigma_\chi/\mu_\chi)^2\right) + O(\Sigma_\chi^4),$$

$$\text{(E2b)}\quad \operatorname{var}\left[T_2\right] \equiv E\left[\left(T_2 - E\left[T_2\right]\right)^2\right] = (1+\theta_\chi)^2 \mu_\chi^{2(1+\theta_\chi)} (\Sigma_\chi/\mu_\chi)^2 + O(\Sigma_\chi^4),$$

[20] In their table 6.2, IPCC (2001) propose a logarithmic relationship for radiative forcing as a function CO_2, also given in IPCC (1990, chapter 2, where original sources are cited), among two other non-logarithmic, but generally concave parametrizations. IPCC (1990, chapter 2, page 51) note that for "low/moderate/high concentrations, the form is well approximated by a linear/square-root/logarithmic dependence", where the limit of validity of the logarithmic calibration is said to be 1000 ppm. For other greenhouse gases alternative parametrizations are proposed: a square-root dependence for methane and a linear dependence for halocarbons.

[21] Whereas the normalized curvature of Arrhenius's (1896) logarithmic radiative law with respect to the atmospheric carbon stock S, namely $ST_{SS}(S)/T_S(S)$ is constant and equal to -1, this limit is only reached for large carbon stock in our case, in which $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$.

$$\text{(E2c)}\quad \operatorname{skew}[T_2] \equiv E\left[\left(T_2 - E[T_2]\right)^3\right] = 3\theta_\chi (1+\theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)} (\Sigma_\chi / \mu_\chi)^4 + O(\Sigma_\chi^6),$$

$$\text{(E2d)}\quad \operatorname{skew}^*[T_2] \equiv \operatorname{skew}[T_2] / \left(\operatorname{var}[T_2]\right)^{3/2} = 3\theta_\chi (\Sigma_\chi / \mu_\chi) + O(\Sigma_\chi^3).$$

Our calibration of the distribution of the climate sensitivity are based on a wide range of distributions reported and used by the IPCC (2014, AR5) (see Fig. 2 in section IV.B). Combining (E1) with the expected carbon stock dynamics in our model, Fig. E3 shows the exceedance probability of temperature in our model as a function of time. The rapid broadening of the distribution with time reflects our calibration to the TCR for short time and the ECS for long time (see section IV.B).

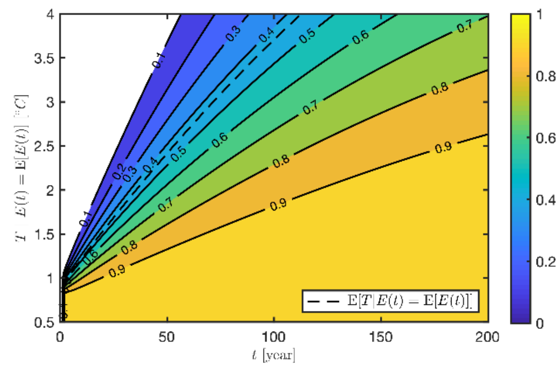
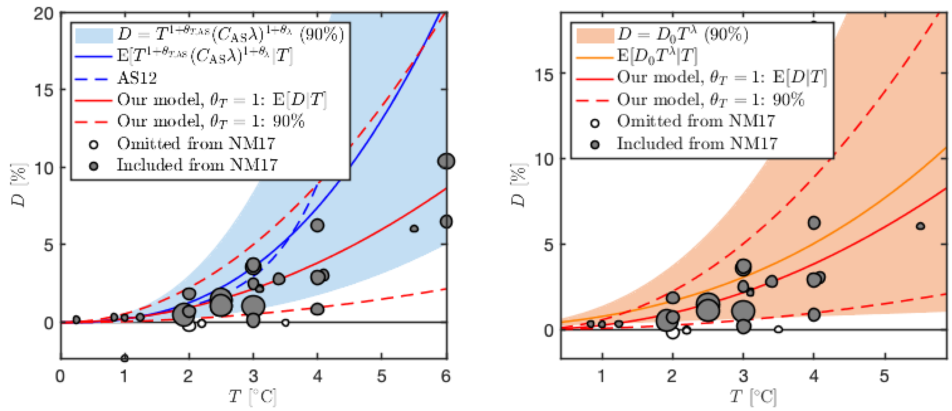


FIGURE E3. CONDITIONAL EXCEEDANCE PROBABILITY OF TEMPERATURE IN OUR MODEL

The skewness of the temperature distribution is evident from the expected temperature (dashed line) being greater than the median temperature, which is shown by the contour with an exceedance probability of 0.5.

E6. Climate damage uncertainty

In addition to the two calibrations of our model in Fig. 3, two additional calibrations have been considered in footnotes 60 and 61: a calibration based on Ackerman and Stanton (2012) that is of form $D = T^{1+\theta_{T,AS}} (C_{AS}\lambda)^{1+\theta_\lambda}$ and a calibration that is of the form $D = D_0 T^\lambda$ with $\lambda \sim N(\mu_\lambda, \Sigma_\lambda^2)$ (see footnotes 60 and 61 for details). Fig. E4 illustrates these two alternative calibrations with the continuous lines corresponding to expected damages, the shaded areas to the 90% confidence intervals, and the blue dashed line labelled AS12 to the original damage function of Ackerman and Stanton (2012). Also shown are the expected damages for the convex damage case of our model as continuous red lines and the corresponding 90% confidence bands as dashed red lines (cf. Fig. 3b).



(a) Based on Ackerman and Stanton (2012) (b) Based on parameter uncertainty ($D = D_0 T^\lambda$)

FIGURE E4. ALTERNATIVE DAMAGE FUNCTION CALIBRATIONS

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Appendix F: Accuracy of Results 1 and 2 (For Online Publication)

Result A is evaluated numerically by discretization in time before evaluating the expectation operator numerically exactly and summing up the discounted contributions of every time step. Whereas the stochastic processes for χ and λ are autonomous, the stochastic process for K remains autonomous in Result 1, and all three have (independent) probability distributions available in closed form, the probability distribution of E at any time period in the future must combine all uncertain emissions (proportional to K) before that time. As the time integral of a Geometric Brownian motion does not have a closed-form solution, we update the probability distribution function of E every time step with the stochastic emissions and the decay in that period according to the differential equation for E and project on a fixed grid for E to enable transfer of the probability density function between time periods. Of course, the validity of Result A itself still relies on the parameter ϵ being small. Consistent with our perturbation scheme, all our optimal risk-adjusted carbon prices in Results A, 1 and 2 are evaluated along the business-as-usual path for which $P=0$. We assess the accuracy of Results 1 and 2 for some of the calibrations examined in section V. By choosing the grid size sufficiently small and the grid sufficiently large in each case, we ensure that discretization errors for Result A are negligible.

TABLE F1. ACCURACY OF RESULT 1 OR 2 COMPARED TO RESULT A

Impatience ρ [/year]	5.8%	1.5%	0.1%	0.1%	0.1%
Economic volatility σ_K [/year$^{1/2}$]	12%	12%	1.5%	1.5%	1.5%
Damages	Proportional	Proportional	Proportional	Convex	Highly convex (AS12)
Total error in risk-adjusted SCC	-0.02%	-2.0%	0.73%	1.9%	-1.3%

Two factors determine the accuracy of using Result 1 or 2 instead of Result A. First, in Results 1 and 2 we ignore any uncertainty in the atmospheric carbon stock that arises because of the uncertain nature of future economic growth and thus of future emissions (*Assumption I*) For our base case calibration with *proportional* damages ($\theta_{ET}=0$) (*Assumption IV*), the stochastic nature of E does not lead to a change in the SCC. Second, in Results 1 and 2 we only consider leading-order terms in the climate sensitivity uncertainty (*Assumption II*). We can confirm from Table F1 that the combined effect of these two errors is sufficiently small to be ignored for all practical purposes. As expected, it is larger for low discount rates, higher economic volatility, and *convex* damages.

Appendix G: Carbon pricing with some common calibrations

In Table G1, we evaluate the optimal risk-adjusted SCC for different calibrations in the literature. Golosov et al. (2014) use proportional damages, logarithmic utility (IIA = RRA = 1), and ρ = 1.5% per year, which gives a risk-adjusted discount rate $r^{(0)}$ of 3.5% per year. With logarithmic utility, neither the expected rate of growth nor uncertainty about the future rate of growth influences the optimal SCC. Gollier (2012) uses RRA = IIA = 2 and ρ = 0 and calibrates to GDP volatility, which gives a risk-adjusted discount rate $r^{(0)}$ of 4% per year and a risk-adjusted SCC of \$18.5/tCO_2. If the model were to be calibrated to asset return volatility, the risk-adjusted discounted rate drops to 2.5% per year and the risk-adjusted SCC rises to \$62.6/tCO_2. The discount rate is only substantially lowered for asset return uncertainty; asset return uncertainty depresses the discount rate and increases the risk-adjusted SCC as IIA >1 in this calibration.

TABLE G1. ESTIMATES OF THE SCC: COMPARISON WITH OTHER CALIBRATIONS

Model	Base	Golosov et al. (2014)	Gollier (2012)		Stern (2007) +AS12
Volatility based on	asset returns	-	asset returns	GDP	GDP
Deterministic SCC (\$/tCO_2)	11.5	19.0	14.4	14.4	51.6
Risk-adjusted SCC (\$/tCO_2)	**39.8**	**24.6**	**62.6**	**18.5**	**102.9**
Economic risk mark-up	163%	0%	225%	1.1%	0.3%
Carbon stock risk mark-up	0%	0%	0%	0%	-1.1%
Climate sensitivity risk mark-up	41%	13%	57%	12%	66%
Damage ratio mark-up	43%	16%	54%	16%	21%
Total risk mark-up	247%	29%	336%	29%	90%
Discount rate $r^{(0)}$ (per year)	2.9%	3.5%	2.5%	4.0%	3.0%

Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$), except for the final column, which assumes highly convex AS12 damages. The base case is for ρ = 1.5%/year (*ethics-based* calibration).

Our analytical results can also be used for stochastic carbon pricing with very convex damages, i.e., those used in Ackerman and Stanton (2012). The last column of Table 8 uses IIA = RRA = 1.45 and a very low rate of time preference of ρ = 0.1%/year corresponding to a discount rate $r^{(0)}$ of 2.5% per year (for GDP-based economic volatility). These choices reflect the low discount rate and convexity of damages used by Stern (2007). This gives a very high deterministic SCC of \$52 and an even higher risk-adjusted SCC of \$103/tCO_2.

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