Is Inflation Default? The Role of Information in Debt Crises

Online Appendix

Marco Bassetto and Carlo Galli

May 21, 2019

Proposition 5 (Belief Stochastic Dominance)**.** In each period, agents' posterior beliefs over θ are increasing in their private signal in the sense of first-order stochastic dominance. Whenever prices do not fully reveal the value of θ, this monotonicity property is strict.

Proof of Proposition 5. Denote with $F(\theta|x_{i,2}, q_2, \rho)$ the cumulative distribution function (cdf) of the posterior beliefs on θ for a second-period agent with private signal $x_{i,2}$, after observing a signal ρ of the primary-market price and when the secondary-market price is q_2. Similarly, let $h(x|\theta, q_2, \rho)$ be the probability density function of the second-period idiosyncratic signal conditional on (θ, q_2, ρ), and $G(\theta|q_2, \rho)$ be the conditional cdf of θ given q_2 and ρ. By Bayes' rule,

$$F(\theta|x, q_2, \rho) = \frac{\int_{-\infty}^{\theta} h(x|y, q_2, \rho) dG(y|q_2, \rho)}{\int_{-\infty}^{+\infty} h(x|y, q_2, \rho) dG(y|q_2, \rho)}. \tag{OA1}$$

To prove the proposition, we show that, if $x_2 < \hat{x}_2$, then $\frac{F(\theta|x_2, q_2, \rho)}{F(\theta|\hat{x}_2, q_2, \rho)} > 1$ whenever the two cumulative distribution functions are strictly between 0 and 1.[1] First, note that the ratio converges

[1]Since h is a normal density (with unbounded support), equation (OA1) implies that $F(\cdot|x_2, q_2, \rho)$ and $F(\cdot|\hat{x}_2, q_2, \rho)$ are absolutely continuous with respect to each other, for any values of x_2 and $\hat{x}_2$; hence, the sets on which they are 0 and 1 coincide.

to 1 as $\theta \to +\infty$. We obtain

$$\frac{F(\theta|x_2,q_2,\rho)}{F(\theta|\hat{x}_2,q_2,\rho)} = \frac{\int_{-\infty}^{\theta} h(x_2|y,q_2,\rho)dG(y|q_2,\rho)}{\int_{-\infty}^{\theta} h(\hat{x}_2|y,q_2,\rho)dG(y|q_2,\rho)} \cdot \frac{\int_{-\infty}^{+\infty} h(\hat{x}_2|y,q_2,\rho)dG(y|q_2,\rho)}{\int_{-\infty}^{+\infty} h(x_2|y,q_2,\rho)dG(y|q_2,\rho)}.$$

The second fraction on the right-hand side is independent of θ. $h(\cdot|\theta,q_2,\rho)$ is independent of (q_2,ρ) and normally distributed, so that $\frac{h(x_2|y)}{h(\hat{x}_2|y)} > \frac{h(x_2|\theta)}{h(\hat{x}_2|\theta)}$ for all $y < \theta$. We next prove that $W(\theta) := \frac{\int_{-\infty}^{\theta} h(x_2|y,q_2,\rho)dG(y|q_2,\rho)}{\int_{-\infty}^{\theta} h(\hat{x}_2|y,q_2,\rho)dG(y|q_2,\rho)}$ is decreasing in θ, and strictly so in regions of positive probability. This completes the proof, since we know that $\frac{F(\theta|x_2,q_2,\rho)}{F(\theta|\hat{x}_2,q_2,\rho)}$ converges to 1 in the limit. Let $\theta_2 > \theta_1$ such that $G(\theta_1|q_2,\rho) > 0$,[2] then

$$W(\theta_2) - W(\theta_1) = \frac{\int_{y\le\theta_1} h(x_2|y)dG(y|q_2,\rho) + \int_{\theta_1}^{\theta_2} h(x_2|y)dG(y|q_2,\rho)}{\int_{y\le\theta_1} h(\hat{x}_2|y)dG(y|q_2,\rho) + \int_{\theta_1}^{\theta_2} h(\hat{x}_2|y)dG(y|q_2,\rho)} - \frac{\int_{y\le\theta_1} h(x_2|y)dG(y|q_2,\rho)}{\int_{y\le\theta_1} h(\hat{x}_2|y)dG(y|q_2,\rho)} =$$

$$\frac{\int_{\theta_1}^{\theta_2} h(x_2|y)dG(y|q_2,\rho)\int_{y\le\theta_1} h(\hat{x}_2|y)dG(y|q_2,\rho) - \int_{y\le\theta_1} h(x_2|y)dG(y|q_2,\rho)\int_{\theta_1}^{\theta_2} h(\hat{x}_2|y)dG(y|q_2,\rho)}{\int_{y\le\theta_2} h(\hat{x}_2|y)dG(y|q_2,\rho)\int_{y\le\theta_1} h(\hat{x}_2|y)dG(y|q_2,\rho)} \le$$

$$\frac{h(\hat{x}_2|\theta_1)}{h(x_2|\theta_1)\int_{y\le\theta_2} h(\hat{x}_2|y)dG(y|q_2,\rho)\int_{y\le\theta_1} h(\hat{x}_2|y)dG(y|q_2,\rho)}\cdot$$

$$\left[\int_{\theta_1}^{\theta_2} h(x_2|y)dG(y|q_2,\rho)\int_{y\le\theta_1} h(x_2|y)dG(y|q_2,\rho) - \int_{\theta_1}^{\theta_2} h(x_2|y)dG(y|q_2,\rho)\int_{y\le\theta_1} h(x_2|y)dG(y|q_2,\rho)\right] = 0,$$

where the inequality is strict if G has positive mass on $(\theta_1,\theta_2]$.

The posterior beliefs on θ of a first-period trader with private signal $x_{i,1}$ are given by $F(\theta|x_{i,1},q_1)$. Proving these are increasing in $x_{i,1}$ in the sense of first-order stochastic dominance follows the same steps used above for second-period beliefs. □

Proposition 6 (Informational Equivalence of z and q in the case of debt payoff (Section III) and no recall ($\tau_\rho = 0$))**.** Let $\pi(\theta)$ be the debt payoff in equation (20). Assume that in equilibrium the first-period price q_1 is a continuous function of (θ,ϵ_1) and the second-period price q_2 is a continuous function of (θ,ϵ_2). Let Σ_1 be the σ-algebra generated by the π-system $\{q\in\mathbb{R} : q_1 \le q\}$ and $\hat{\Sigma}_1$ by $\{z\in\mathbb{R} : z_1 \le z\}$, with z_1 as defined in (5). Similarly, let Σ_2 be the σ-algebra generated by the π-system $\{q\in\mathbb{R} : q_2 \le q\}$ and $\hat{\Sigma}_2$ by $\{z\in\mathbb{R} : z_2 \le z\}$, with z_2 as defined in (3). Then $\Sigma_1 = \hat{\Sigma}_1$ and $\Sigma_2 = \hat{\Sigma}_2$.

[2] If $G(\theta_1|q_2,\rho) = 0$, then $F(\theta_1|x,q_2,\rho) = 0$ for all x.

Proof of Proposition 6. First, note that equation (3) follows directly from Proposition 5 and risk neutrality. Second, note that the function $\hat{x}_2(q_2)$ is defined via the indifference condition

$$\delta + (1-\delta)\text{Prob}(\theta \geq \bar{\theta} | x_{i,2} = \hat{x}_2, q_2) = q_2. \tag{OA2}$$

Consider interior prices $q_2 \in (\delta, 1)$. Since conditional repayment probabilities are strictly increasing in the private signal $\hat{x}_2$, it follows that $\hat{x}_2(q_2)$ exists and is unique.[3] Then the market clearing condition (3) is a single-valued mapping from the price q_2 to the linear combination of shocks $z_2 := \theta + \epsilon_2/\sqrt{\beta_2\psi_2} = \hat{x}_2(q_2)$.

Next, we use the property above to prove that corner prices cannot arise with positive probability in equilibria in which the price is continuous in (θ, ϵ_2). Suppose by contradiction that a positive-probability set H can be found for which q_2 is equal to δ.[4] Since H has positive probability, we can find two pairs (θ^A, ϵ_2^A) and (θ^B, ϵ_2^B) that correspond to two different values of z_2: z_2^A and z_2^B. Next, consider the price as a function of θ moving along the two lines $\theta + \epsilon_2/\sqrt{\beta_2\psi_2} = z_2^A$ and $\theta + \epsilon_2/\sqrt{\beta_2\psi_2} = z_2^B$. As θ increases along the lines, the price will eventually have to increase, since a price of δ implies that H must lie below $\bar{\theta}$ almost surely. Since q_2 is continuous, there must be two points $(\tilde{\theta}^A, \tilde{\epsilon}_2^A)$ and $(\tilde{\theta}^B, \tilde{\epsilon}_2^B)$ on the two lines where the price is interior and the same. This contradicts what we have proved, since we showed that, whenever the price is interior, $z_2 = \hat{x}_2(q_2)$, with $\hat{x}_2$ being single valued.

Having established that the price is almost surely interior, notice that z_2 is continuous in (θ, ϵ_2) by construction and so is q_2 by assumption. Hence, the mapping from q_2 to z_2 that exists from the arguments in the previous paragraphs must be continuous and thus measurable. This then implies that z_2 is also Σ_2-measurable.

We next prove that q_2 is $\hat{\Sigma}_2$-measurable. This proof follows the arguments of Pálvölgyi and Venter (2015). By contradiction, suppose that (on a set of positive measure) there are two vectors $(\theta^C, \epsilon_2^C) \neq (\theta^D, \epsilon_2^D)$ that lie on the same straight line indexed by z_2 but that correspond

[3]Existence follows because, when $q_2 \in (\delta, 1)$, the price does not reveal fully whether $\theta \geq \bar{\theta}$. Bayes' rule then implies that the left-hand side converges to δ as $\hat{x}_2 \to -\infty$ and to 1 as $\hat{x}_2 \to \infty$.

[4]The same logic applies to the case in which $q_2 = 1$.

to different prices q_2^C and q_2^D, i.e. such that

$$\theta^C + \epsilon_2^C/\sqrt{\beta_2\psi_2} = z_2, \text{ and } q_2(\theta^C, \epsilon_2^C) = q_2^C$$
$$\theta^D + \epsilon_2^D/\sqrt{\beta_2\psi_2} = z_2, \text{ and } q_2(\theta^D, \epsilon_2^D) = q_2^D.$$

Since q_2 is continuous, the intermediate value theorem ensures that, for any curve that connects (θ^C, ϵ_2^C) to (θ^D, ϵ_2^D), there must be at least one point (θ, ϵ_2) such that $q_2(\theta, \epsilon_2) = \frac{q_2^C + q_2^D}{2}$. First we apply the theorem to the curve represented by the straight line connecting (θ^C, ϵ_2^C) to (θ^D, ϵ_2^D), and denote with $(\hat{\theta}, \hat{\epsilon}_2)$ the point on such line such that $q_2(\hat{\theta}, \hat{\epsilon}_2) = (q_2^C + q_2^D)/2$. Along this line z_2 remains constant. Second, we apply the theorem to any other curve which intersects our straight line z_2 only at (θ^C, ϵ_2^C) and (θ^D, ϵ_2^D), again such that $(\tilde{\theta}, \tilde{\epsilon}_2)$ lies on the curve and $q_2(\tilde{\theta}, \tilde{\epsilon}_2) = (q_2^C + q_2^D)/2$. It follows that we have found two different points, $(\hat{\theta}, \hat{\epsilon}_2)$ and $(\tilde{\theta}, \tilde{\epsilon}_2)$, that correspond to the same price but are such that $\hat{\theta} + \hat{\epsilon}_2/\sqrt{\beta_2\psi_2} \neq \tilde{\theta} + \tilde{\epsilon}_2/\sqrt{\beta_2\psi_2}$. This contradicts the necessary market clearing condition (3).

The proof for the first period repeats the same steps as above.

It is possible to generalize the proposition to the generic increasing payoff function $\pi(\theta)$ of Section II, but the proof is considerably more involved, so here we choose to focus on the debt application. □

Proposition 7. Assume that neither q_1 nor q_2 fully reveals the state of the economy. In any equilibrium in which q_2 and z_2 convey the same information given ρ, q_2 is a strictly increasing function of z_2. Furthermore, the expected resale price for a first-period trader is strictly increasing in her private signal.

Proof of Proposition 7. Proposition 5 proves that $\mathbb{E}[\pi(\theta)|x_{i,2}, z_2, \rho]$ is strictly increasing in the private signal. Repeating the same steps, we can also prove that it is increasing in the market signal z_2, since z_2 also satisfies the monotone likelihood ratio property. Combining the two facts, the expected value perceived by the marginal trader, $\mathbb{E}[\pi(\theta)|x_{i,2} = z_2, z_2, \rho]$, is strictly increasing in z_2, which implies from equation (4) that q_2 is also increasing in z_2.

The expected resale price for a first-period trader who received a private signal $x_{i,1}$ is given by

$$\begin{aligned}\mathbb{E}[q_2|x_{i,1},q_1] &= \mathbb{E}[\mathbb{E}[\pi(\theta)|x_{i,2}=z_2,z_2,\rho]|x_{i,1},q_1] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{E}[\pi(\theta)|x_{i,2}=z_2,z_2,\rho]|x_{i,1},q_1,\theta]|x_{i,1},q_1] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{E}[\pi(\theta)|x_{i,2}=z_2,z_2,\rho]|q_1,\theta]|x_{i,1},q_1].\end{aligned} \tag{OA3}$$

In the equation above, the last step follows from the fact that $x_{i,1}$, ρ, and z_2 are independent of each other conditional on θ. ρ is a noisy public signal of the first-period price observed by second-period agents. As such, conditional on the actual first-period price q_1, it is independent of fundamentals and of the private signal $x_{i,1}$. Hence, the beliefs of the first-period trader about ρ are independent of $x_{i,1}$. The distribution of z_2 conditional on θ, ρ is equal to the distribution conditional on θ alone and it is strictly increasing in θ in the sense of first-order stochastic dominance. It follows that $\mathbb{E}[\mathbb{E}[\pi(\theta)|x_{i,2}=z_2,z_2,\rho]|q_1,\theta]$ is strictly increasing in θ. Repeating the steps of Proposition 5, the distribution of θ conditional on $x_{i,1}, q_1$ is strictly increasing in $x_{i,1}$ and therefore $\mathbb{E}[q_2|x_{i,1},q_1]$ is strictly increasing in $x_{i,1}$. □

Derivation of q_1 in equation (13)

We start from equation (7):

$$q_1(z_1) = \mathbb{E}[q_2(z_2,\rho)|x_{i,1}=z_1,z_1].$$

We then use the expression for q_2 from (10), and the distributions of z_2 and ρ conditional on z_1 given by (12):

$$q_1(z_1) = \int\int\int \pi(\theta) d\Phi\left(\frac{\theta-(1-w_\rho-w_{z_2})\mu_0-w_\rho\rho-w_{z_2}z_2}{\sigma_2}\right) d\Phi\left(\frac{z_2-(1-w_1)\mu_0-w_1z_1}{\sigma_{2|1}}\right) d\Phi\left(\frac{\rho-z_1}{\sigma_\eta}\right).$$

Defining $y := (z_2 - (1-w_1)\mu_0 - w_1 z_1)/\sigma_{2|1}$ and changing the variables of integration we get

$$q_1(z_1) =$$
$$\int \pi(\theta) \int\int \frac{1}{\sigma_2}\phi\left(\frac{\theta - \mu_0(1-w_\rho - w_{z_2}w_1) - z_1(w_\rho + w_{z_2}w_1) - \eta_1(w_\rho\sigma_\eta) - y(w_{z_2}\sigma_{2|1})}{\sigma_2}\right) d\Phi(y)d\Phi(\eta_1)d\theta$$
$$= \int \pi(\theta) \int \frac{1}{\sqrt{w_{z_2}^2\sigma_{2|1}^2 + \sigma_2^2}}\phi\left(\frac{\theta - \mu_0(1-w_\rho - w_{z_2}w_1) - z_1(w_\rho + w_{z_2}w_1) - \eta_1(w_\rho\sigma_\eta)}{\sqrt{w_{z_2}^2\sigma_{2|1}^2 + \sigma_2^2}}\right)\phi(\eta_1)d\eta_1\ d\theta$$
$$= \int \pi(\theta)\frac{1}{\sqrt{w_{z_2}^2\sigma_{2|1}^2 + \sigma_2^2 + w_\rho^2\sigma_\eta^2}}\phi\left(\frac{\theta - \mu_0(1-w_\rho - w_{z_2}w_1) - z_1(w_\rho + w_{z_2}w_1)}{\sqrt{w_{z_2}^2\sigma_{2|1}^2 + \sigma_2^2 + w_\rho^2\sigma_\eta^2}}\right) d\theta.$$

This shows that q_1 exists and is unique for all $z_1 \in \mathbb{R}$ and yields the expression (13) in the main text.

References

Pálvölgyi, Dömötör, and Gyuri Venter. 2015. "Multiple Equilibria in Noisy Rational Expectations Economies." *http://dx.doi.org/10.2139/ssrn.2524105*.