Ultimatum Bargaining with Rational Inattention: Online Appendix

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A Omitted Proofs

A.1 Proof of Proposition 1

For best responses, the proposition follows from results in Yang (2015), Matějka and McKay (2015), and Denti, Marinacci and Montrucchio (2020). I now take the proposition's statement for best responses as given, and claim that for β to be a credible response, (5) must hold for *all* (v, x). To do so, fix any (v, x), and let $\{(\mu_n, \sigma_n)\}_n$ be a sequence satisfying the desired criteria. Because β is a best response to all elements of the sequence, (5) must hold μ_n-almost everywhere with $\pi_n = \mathbb{E}_{\mu_n}[\beta]$. In particular, it must hold at (v, x) because $\mu_n\{(\mathbf{v}, \mathbf{x}) = (v, x)\} > 0$. As $\sigma_n(\cdot|v)$ converges strongly to $\sigma(\cdot|v)$ for all v, μ_n also converges strongly to μ, meaning $\pi_n = \mathbb{E}_{\mu_n}[\beta] \to \mathbb{E}_{\mu}[\beta] = \pi$. The desired equality follows.

To prove the proposition's conditions are sufficient for credibility, I construct an appropriate sequence $\{(\mu_n, \sigma_n)\}_n$ utilizing the monotonicity of (5)'s right-hand side. I begin with the proposition's third case. Lemma 1 at the end of this section establishes redundancy of the inequalities $\mathbb{E}_{\mu}\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] \geq 1$ and $\mathbb{E}_{\mu}\left[e^{\frac{1}{\kappa}(\mathbf{x}-\mathbf{v})}\right] \geq 1$. In other words, the proposition's case (iii) is equivalent to β satisfying (5) for all (v, x) with $\pi = \mathbb{E}_{\mu}[\beta] \in (0, 1)$. Now, without loss, fix any $x' \leq v'$. For a given $y \in X$, let $\delta_y \in \Delta X$ be the distribution putting probability 1 on y. Notice (5) implies β is strictly decreasing with x' and satisfies $\beta(v', x') \geq \beta(v', v') = \pi$. Clearly, some $\epsilon \geq 0$ and $\alpha \in (0, 1)$ exist such that $\alpha\beta(v', x') + (1-\alpha)\beta(v', v'+\epsilon) = \pi$. Define

$$\sigma_n(\cdot|v) = \begin{cases} \frac{n-1}{n}\sigma(\cdot|v') + \frac{1}{n}\left(\alpha\delta_{x'} + (1-\alpha)\delta_{v'+\epsilon}\right) & \text{if } v = v' \\ \frac{n-1}{n}\sigma(\cdot|v) + \frac{1}{n}\delta_v & \text{otherwise.} \end{cases}$$

By construction, σ_n pointwise converges strongly to σ and $\sigma_n(\{x'\}|v') > 0$.

Moreover, letting μ_n be the corresponding consistent beliefs, we have that β satisfies (5) for $\pi = \mathbb{E}_{\mu_n}[\beta]$; that is, β is a best response to (μ_n, σ_n). Hence, β is a credible response to (μ, σ).

Suppose now that $\pi = 0$ and $\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] \leq 1$. Once again, without loss, fix any $x' < v'$. Then, $\beta(v, y) = 0$ for all y and v. Because the function $y \mapsto e^{\frac{1}{\kappa}(v-y)}$ strictly decreases with y, an $\epsilon \geq 0$ and $\alpha \in (0,1)$ exist such that $\alpha e^{\frac{1}{\kappa}(v'-x')} + (1-\alpha)\, e^{-\frac{1}{\kappa}\epsilon} = 1$. Define

$$\sigma_n(\cdot|v) = \begin{cases} \frac{n-1}{n}\sigma(\cdot|v') + \frac{1}{n}\left(\alpha\delta_{x'} + (1-\alpha)\,\delta_{v'+\epsilon}\right) & \text{if } v = v' \\ \frac{n-1}{n}\sigma(\cdot|v) + \frac{1}{n}\delta_v & \text{otherwise.} \end{cases}$$

By construction, $\sigma_n(\cdot|v)$ converges strongly to $\sigma(\cdot|v)$ for all v and $\sigma_n(\{x'\}|v') > 0$. Moreover, letting μ_n be the corresponding consistent beliefs, we have $\mathbb{E}_{\mu_n}\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] = \frac{n-1}{n}\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] + \frac{1}{n} \leq 1$ for all n, and so $\beta(v, x) = 0$ is a best response to (μ_n, σ_n); that is, β is a credible response to (μ, σ). The proof for $\pi = 1$ is analogous.

To complete the proof, it remains only to prove the lemma below.

Lemma 1. *Suppose $\mu \in \Delta(V \times X)$ and $\pi \in (0,1)$ are such that*

$$\mathbb{E}_\mu\left[\frac{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}}{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})} + 1 - \pi}\right] = \pi.$$

Then, $1 \leq \min\left\{\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right], \mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{x}-\mathbf{v})}\right]\right\}$.

Proof. As a preliminary step, define the function

$$\begin{aligned} g : \mathbb{R}_+ \times (0,1) &\to \mathbb{R}_+ \\ (y, \gamma) &\mapsto \frac{\gamma}{\gamma + (1-\gamma)\, y}. \end{aligned}$$

Observe

$$\frac{\partial g}{\partial y} = \frac{-\gamma(1-\gamma)}{(\gamma + (1-\gamma)\, y)^2} < 0 < \frac{2\gamma(1-\gamma)^2(\gamma + (1-\gamma)\, y)}{(\gamma + (1-\gamma)\, y)^4} = \frac{\partial^2 g}{\partial y^2},$$

meaning that $y \mapsto g(y,\gamma)$ is strictly decreasing and convex for all $\gamma \in (0,1)$. Observe further that $g(1,\gamma) = \gamma$ for all γ. Because $g(\cdot,\gamma)$ is decreasing, $g(y,\gamma) \leq \gamma$ if and only if $y \geq 1$. Observe

$$\pi = \mathbb{E}_\mu\left[\frac{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}}{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}+1-\pi}\right] = \mathbb{E}_\mu\left[g\left(e^{-\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})},\pi\right)\right] \geq g\left(\mathbb{E}_\mu\left[e^{-\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right],\pi\right),$$

where the second inequality follows from convexity. Similarly,

$$\begin{aligned} 1-\pi &= 1-\mathbb{E}_\mu\left[\frac{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}}{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}+1-\pi}\right] = \mathbb{E}_\mu\left[\frac{1-\pi}{\pi e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}+1-\pi}\right] \\ &= \mathbb{E}_\mu\left[g\left(e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})},1-\pi\right)\right] \geq g\left(\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right],1-\pi\right). \end{aligned}$$

Thus, we obtain $1 \leq \mathbb{E}_\mu\left[e^{-\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] = \mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{x}-\mathbf{v})}\right]$ and $1 \leq \mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right]$, as required. □

A.2 Proof of Proposition 2

For $\mathcal{E} = (\mu,\beta,\sigma)$ to be a trading CE, it is necessary and sufficient for (a) μ to be consistent with σ, (b) β be a credible response to (μ,σ) such that $\pi > 0$, and (c) σ be a best response to β. Because these conditions imply that $\mathcal{E}$ is a CE, B must reject S's offer with some probability by Corollary 1. Combined with Proposition 1, we obtain that one can replace (b) with the following requirement: β satisfies (5) for all (v,x) and $\pi = \mathbb{E}_\mathcal{E}[\beta] \in (0,1)$.

Suppose β satisfies (5) for all (v,x) and $\mathbb{E}_\mathcal{E}[\beta] \in (0,1)$. It remains to show σ is a best response to β if and only if Proposition 2's Part (ii) and (7) both hold. Towards this end, I recast S's problem as a strictly concave maximization program. Notice that, taking $\pi \in (0,1)$ as given, a v-type S has the same objective as a zero-marginal-cost monopolist facing the demand given by (5). Because this demand curve slopes downward, we can calculate its inverse,

$$\chi(v,b) = v - \kappa \ln\left[\frac{b(1-\pi)}{\pi(1-b)}\right],$$

where $b \in (0,1)$ is the probability B accepts the offer $\chi(v,b)$ when the good's quality is v; that is, $b = \beta(v, \chi(v,b))$. Observe $\beta(x,v)$ converges to 0 as x goes to infinity. Therefore, a v-type S solves

$$\max_{b \in (0, \overline{b}_v]} b\left(v - \kappa \ln\left[\frac{b(1-\pi)}{\pi(1-b)}\right]\right), \tag{8}$$

where $\overline{b}_v := \beta(v,0)$. The program (8) is strictly concave because $p \mapsto p \ln[p/(1-p)]$ is strictly convex. Therefore, (8) admits a unique solution. It follows that S must use a pure strategy.

Next, I show that solving (8) is equivalent to satisfying the program's first-order condition (FOC),

$$\frac{b}{1-b} + \ln\left(\frac{b}{1-b}\right) = \frac{v-\kappa}{\kappa} + \ln\left(\frac{\pi}{1-\pi}\right). \tag{9}$$

Because (8) is a strictly concave program, showing the FOC always admits a solution in $\left(0, \overline{b}_v\right)$ is sufficient. Observe the FOC's left-hand side is continuously increasing in $b \in (0,1)$ and goes to $-\infty$ and ∞, respectively, as b goes to 0 and 1. Therefore, given v, (9) admits a unique solution in $b_v^* \in (0,1)$. I now argue this solution is strictly below $\overline{b}_v$. Because $\beta(v,x)$ is strictly decreasing in x, showing b_v^* is below $\overline{b}_v = \beta(v,0)$ is equivalent to showing that $\chi(v, b_v^*) > 0$. To see that $\chi(v, b_v^*) > 0$, simply rearrange the FOC and substitute in the definition of $\chi(v, b_v^*)$ to get

$$\chi(v, b_v^*) = \kappa\left(1 + \frac{b_v^*}{1-b_v^*}\right). \tag{10}$$

Thus, solving (8) is equivalent to solving the program's FOC, (9).

To complete the proof, I show equivalence of (9), (6), and (7). To prove this equivalence, notice first one can rearrange (10) (which is equivalent to (9)) to get $b_v^* = 1 - \kappa/\chi(v, b_v^*)$; that is, (9) and (7) are equivalent. Second, observe that exponentiating and applying $W(\cdot)$ to S's FOC, (9), gives $\frac{b_v^*}{1-b_v^*} = W\left[\frac{\pi}{1-\pi} e^{\frac{1}{\kappa}(v-\kappa)}\right]$. Substituting back into (10) gives (6). Thus, (9) is equivalent to (6). The proof is now complete.

A.3 Proof of Theorem 1

The main text establishes $\mathbb{E}_\nu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\kappa)}\right] > 1$ is necessary for a trading CE to exist. In what follows, assume $\mathbb{E}_\nu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\kappa)}\right] > 1$. It remains to show two things. First, a trading CE exists. Second, whenever a trading CE exists, it is unique. Towards this goal, define the following three functions

$$\begin{aligned}
\zeta : (0,1) \times V &\to \mathbb{R}_+, \\
(p, v) &\mapsto \kappa\left(1 + W\left(\frac{p}{1-p} e^{\frac{1}{\kappa}(v-\kappa)}\right)\right), \\
f : (0,1) \times V &\to \mathbb{R}_+ \\
(p, v) &\mapsto \frac{e^{\frac{1}{\kappa}(v-\zeta(p,v))}}{1 - p + p e^{\frac{1}{\kappa}(v-\zeta(p,v))}},
\end{aligned}$$

and

$$\begin{aligned}
F : (0,1) &\to \mathbb{R}_+ \\
p &\mapsto \mathbb{E}_\nu\left[f\left(p, \mathbf{v}\right)\right].
\end{aligned}$$

Note that F is continuous due to continuity of f and finiteness of V.

Theorem 1's proof is based on the following lemma.

Lemma 2. *A trading CE exists whenever a $p^* \in (0,1)$ exists such that*

$$F\left(p^*\right) = 1. \tag{11}$$

Conversely, if $\mathcal{E} = (\mu, \beta, \sigma)$ is a trading CE, then (11) holds for $p^* = \pi = \mathbb{E}_{\mathcal{E}}\left[\beta\left(\mathbf{v}, \mathbf{x}\right)\right] \in (0,1)$.

I use the lemma's first part to establish existence of a trading CE. Uniqueness of a trading CE relies on the lemma's second part. The lemma's proof is based on the identity

$$pf\left(p, v\right) = \frac{W\left(\frac{p}{1-p} e^{\frac{1}{\kappa}(v-\kappa)}\right)}{1 + W\left(\frac{p}{1-p} e^{\frac{1}{\kappa}(v-\kappa)}\right)}. \tag{12}$$

To prove this identity, use that $e^{W(x)}/x = 1/W(x)$ to get the following equality chain,[19]

$$pf(p,v) = \frac{pe^{\left[\frac{1}{\kappa}(v-\kappa)-W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)\right]}}{1-p+pe^{\left[\frac{1}{\kappa}(v-\kappa)-W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)\right]}} = \frac{\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}}{e^{W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)}+\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}}$$
$$= \frac{1}{\left[W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)\right]^{-1}+1} = \frac{W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)}{1+W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)}.$$

I now turn to proving Lemma 2.

Proof of Lemma 2. Suppose (11) holds for $p \in (0,1)$. Let $\mathcal{E} = (\mu, \beta, \sigma)$ be the assessment in which $\sigma(\mathbf{x} = \zeta(p^*, v)|v) = 1$, β satisfies (5) but with p^* replacing π, and μ is consistent with σ. I show $\mathcal{E}$ satisfies the conditions of Proposition 2. To do so, observe

$$\pi = \mathbb{E}_{\mathcal{E}}[\beta(\mathbf{v},\mathbf{x})] = \mathbb{E}_{\nu}[p^* f(p^*,\mathbf{v})] = p^* F(p^*) = p^*,$$

meaning $\mathcal{E}$ satisfies the proposition's Parts 1 and 2 (with $z_v = \zeta(p^*, v)$). Note β satisfies case (iii) of Proposition 1. Finally, use (12) to obtain,

$$\beta(v, z_v) = p^* f(p^*, v) = \frac{W\left(\frac{p^*}{1-p^*}e^{\frac{1}{\kappa}(v-\kappa)}\right)}{W\left(\frac{p^*}{1-p^*}e^{\frac{1}{\kappa}(v-\kappa)}\right)+1}$$
$$= 1 - \frac{1}{1+W\left(\frac{p^*}{1-p^*}e^{\frac{1}{\kappa}(v-\kappa)}\right)} = 1 - \frac{\kappa}{z_v},$$

establishing Proposition 2's Part 3. Hence, $\mathcal{E}$ is a trading CE.

For the converse, let $\mathcal{E} = (\mu, \beta, \sigma)$ be a trading CE. Proposition 2 immediately delivers that $(0,1) \ni \pi = \mathbb{E}_{\mathcal{E}}[\beta(\mathbf{v},\mathbf{x})] = \mathbb{E}_{\nu}[\pi f(\pi,\mathbf{v})]$. Dividing both sides of said equality by π completes the proof. □

A key to using Lemma 2 involves showing $F(p) < 1$ for all p sufficiently

[19] To see that $e^{W(x)}/x = 1/W(x)$, observe $W(x) = W\left(W(x)e^{W(x)}\right)$ for all $x \in \mathbb{R}_+$, and so $x = W(x)e^{W(x)}$. That $1/W(x) = e^{W(x)}/x$ holds for all $x > 0$ follows.

close to 1. I establish this result in the following lemma.

Lemma 3. *A $\bar{p} < 1$ exists such that $F(p) < 1$ for all $p \in [\bar{p}, 1)$.*

Proof. Because $\zeta(p, v)$ increases with p and satisfies $\lim_{p\to 1} \zeta(p, v) = \infty$, it follows that a $\bar{p} < 1$ exists such that $e^{\frac{1}{\kappa}(\zeta(\bar{p},v)-v)} > 1$ for all $v \in V$ and all $p \geq \bar{p}$. Therefore,

$$F(p) = \mathbb{E}_\nu \left[\frac{e^{\frac{1}{\kappa}(v-\zeta(p,v))}}{1-p+pe^{\frac{1}{\kappa}(v-\zeta(p,v))}} \right] = \mathbb{E}_\nu \left[\left((1-p)\, e^{\frac{1}{\kappa}(\zeta(p,v)-v)} + p \right)^{-1} \right] < 1$$

holds for all $p \geq \bar{p}$, as required. □

Existence. I now show existence of a trading CE when $\mathbb{E}_\nu \left[e^{\frac{1}{\kappa}(\mathbf{v}-\kappa)} \right] > 1$. By Lemma 2, it is sufficient to show a $p^* \in (0, 1)$ exists such that $F(p^*) = 1$. I show this existence using the Intermediate Value Theorem. To do so, notice

$$\lim_{p\to 0} f(p, v) = \lim_{p\to 0} \frac{e^{\frac{1}{\kappa}(v-\zeta(p,v))}}{1-p+pe^{\frac{1}{\kappa}(v-\zeta(p,v))}} = e^{\frac{1}{\kappa}(v-\kappa)},$$

where the second equality follows from $\lim_{p\to 0} \zeta(p, v) = \kappa$. Thus, given a strictly positive $\epsilon < \mathbb{E}_\nu \left[e^{\frac{1}{\kappa}(\mathbf{v}-\kappa)} - 1 \right]$, a $\underline{p} \in (0, 1)$ exists such that

$$F\left(\underline{p}\right) = \mathbb{E}_\nu \left[f\left(\underline{p}, \mathbf{v}\right) \right] \geq \mathbb{E}_\nu \left[e^{\frac{1}{\kappa}(\mathbf{v}-\kappa)} - \epsilon \right] > 1.$$

Moreover, $F(\bar{p}) < 1$ by Lemma 3. Thus, the Intermediate Value Theorem delivers a $p^* \in \operatorname{co}\left\{\underline{\mathrm{p}}, \bar{\mathrm{p}}\right\}$ such that $F(p^*) = 1$. Existence of a trading CE follows from Lemma 2. □

Uniqueness. Next, I prove the trading CE is unique. To do so, I claim below that a unique p^* exists such that $F(p^*) = 1$, which is sufficient for showing uniqueness: if $F(p^*) = 1$ for only one $p^* \in (0, 1)$, then $\pi = \mathbb{E}_\mathcal{E} [\beta(\mathbf{v}, \mathbf{x})] = p^*$ holds for any trading CE, $\mathcal{E}$. Uniqueness then follows from Proposition 2.

I begin by proving F is strictly convex, for which it is sufficient to show

that $p \mapsto f(p, v)$ is strictly convex for all v. Let

$$\phi(p) = p\left(1 + 1/W\left(\frac{p}{1-p}e^{\frac{1}{\kappa}(v-\kappa)}\right)\right).$$

By (12), $f(p, v) = 1/\phi(p)$. Therefore, ϕ being strictly concave is sufficient for $p \mapsto f(p, v)$ to be strictly convex. Using the Implicit Function Theorem on the identity $W(y)\, e^{W(y)} = y$, one obtains

$$W'(y) = \left[e^{W(y)}\left(1 + W(y)\right)\right]^{-1}.$$

Therefore, the first derivative of ϕ is

$$\frac{\mathrm{d}\phi}{\mathrm{d}p} = 1 + \frac{1}{W\left(\frac{p}{1-p}e^{\frac{v-\kappa}{\kappa}}\right)} - \frac{1}{(1-p)\, W\left(\frac{p}{1-p}e^{\frac{v-\kappa}{\kappa}}\right)\left(1 + W\left(\frac{p}{1-p}e^{\frac{v-\kappa}{\kappa}}\right)\right)},$$

whereas the second derivative of ϕ is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}p^2} = -\left(\frac{W\left(\frac{p}{1-p}e^{\frac{v-\kappa}{\kappa}}\right)}{(1-p)^2\, p\left(1 + W\left(\frac{p}{1-p}e^{\frac{v-\kappa}{\kappa}}\right)\right)^3}\right) < 0$$

for all $p > 0$. Hence, ϕ is strictly concave, meaning $p \mapsto f(p, v)$ is strictly convex. That F is strictly convex follows.

I now prove $F(p^*) = 1$ can hold for at most one $p^* \in (0, 1)$. To do so, suppose a p and p' in $(0, 1)$ exist such that $p < p'$ and $F(p) = F(p') = 1$. By Lemma 3, $p' < \bar{p}$ and $F(\bar{p}) < 1$. Because F is strictly convex, it is also strictly quasiconvex, meaning

$$1 = F(p') < \max\left\{F(\bar{p}), F(p)\right\} = 1,$$

which is impossible. □

A.4 Proof of Corollary 2

Observe first v and z_v are strictly co-increasing by (6). By (7), both $\beta(v, z_v)$ and

$$\mathbb{E}_{\mathcal{E}}[U_S|\mathbf{v} = v] = z_v\beta(v, z_v) = z_v - \kappa$$

strictly increase in z_v. Moreover, (5) implies $v - x$ and $\beta(v, x)$ are strictly co-increasing when $\pi \in (0, 1)$, meaning $v - z_v$ strictly co-increases with v. Finally, observe $\beta(v_l, z_{v_l}) < \mathbb{E}_{\mathcal{E}}[\beta] < \beta(v_h, z_{v_h})$ because β is not constant μ-almost surely and $\beta(v_l, z_{v_l})$ ($\beta(v_h, z_{v_h})$) minimizes (maximizes) β across $\operatorname{supp}\mu$. That $v_l - z_{v_l} < 0$ ($0 < v_h - z_{v_h}$) follows from equivalence of $\beta(v, x) < \pi = \mathbb{E}_{\mathcal{E}}[\beta]$ ($\beta(v, x) > \pi$) and $v < x$ ($v > x$), which follows from (5).

B Unrefined Equilibria

The current section characterizes the UE set, and compares this set to the game's CE. For $\pi \in [0, 1]$, define $\psi_\pi : V \times X \to [0, 1]$ via (5); that is,

$$\psi_\pi(v, x) := \frac{\pi e^{\frac{1}{\kappa}(v-x)}}{1 - \pi + \pi e^{\frac{1}{\kappa}(v-x)}}.$$

Moreover, for $x \in X$, let $\delta_x \in \Delta X$ be the distribution that puts a unit mass on x. The following proposition characterizes the UE set of the one-period game.

Proposition 3. *Let (μ, σ, β) be a consistent assessment. Then, (μ, σ, β) is a UE if and only if it falls into one of the following three cases:*

Case 1. $\beta(v, x) = 0$ *μ-a.s. and for all $x \in (0, \infty)$, and* $\int e^{\frac{1}{\kappa}(v-x)}\mathrm{d}\mu \leq 1$.

Case 2. $\sigma(\cdot|v) = \delta_{z_v}$ *for some* $z \in X^V$, $\beta(v, z_v) = 1$, $\int e^{-\frac{1}{\kappa}(v-z_v)}\mathrm{d}\mu \leq 1$, *and* $x\beta(v, x) \leq z_v$ *for all* (v, x).

Case 3. $y, z \in X^V$, $\alpha_v \in [0, 1]^V$, *and* $\pi_0 \in (0, 1)$ *exist such that:*

(a) $\sigma(\cdot|v) = \alpha_v\delta_{z_v} + (1 - \alpha_v)\delta_{y_v}$;

(b) $z_v\psi_{\pi_0}(v, z_v) = y_v\psi_{\pi_0}(v, y_v) \geq \beta(v, x)\, x$ *for all* (v, x);

(c) π_0 *solves* $\sum_{v \in V} \nu(v) \left[\alpha_v \psi_{\pi_0}(v, z_v) + (1 - \alpha_v) \psi_{\pi_0}(v, y_v)\right] = \pi_0$*;*

(d) For $x \in \{y_v, z_v\}$*:* $\beta(v, x) = \psi_{\pi_0}(v, x)$*.*

Proposition 3 shows UEs come in three types. In the first type, B automatically rejects all of S's positive offers, and S makes offers that justify B's behavior. In the second type of UE, the parties trade for sure. Trade fails with some probability for all other UE types, meaning only the second UE type is efficient. In this type of UE, S uses a pure strategy in which he makes attractive offers, namely, $\int e^{\frac{1}{\kappa}(z_v - v)} \mathrm{d}\mu \leq 1$. B accepts these offers for sure, and rejects other offers frequently enough to deter S from making them. Notice the second type of UE includes the unrefined equilibria constructed in section 3. As such, this type of UE allows for any division of the efficient surplus between the players.

In the third type of UE, trade happens with an interior probability, $\pi_0 \in (0, 1)$. Given v, S randomizes over at most two offers. S cannot randomize over more than two offers, because S's profit given v from every on-path offer x must be equal to $x\psi_{\pi_0}(v, x)$. When π_0 is interior, $x\psi_{\pi_0}(v, x)$ is strictly log concave, meaning S can be indifferent between (and so randomize over) at most two offers .

It is easy to see B's strategy is credible in the first type of UE if and only if she automatically rejects the offer $x = 0$ as well. In other words, in the first type of UE, B's strategy is credible only if trade fails for sure.

In the second type of UE, B accepts for sure all of S's on-path offers, but rejects off-path offers with positive probability. As explained in the main text, B's threat to reject some off-equilibrium offers is non-credible given such on-path behavior, meaning the second type of UE cannot be a CE.

Thus, the game's trading CE belongs to the third type of UE. Among the many UEs of this type, the game's trading CE is one where $\beta = \psi_{\pi_0}$, and S never randomizes. The reason is that $x \mapsto x\psi_{\pi_0}(v, x)$ is strictly log concave, meaning if a v-type S is indifferent between two different offers, neither offer can maximize $x\psi_{\pi_0}(v, x)$. Consequently, if S ever mixes in a UE of the third type, β must be different from ψ_{π_0} for some offers. Such a response cannot be

credible, because credibility requires $\beta = \psi_{\pi_0}$. As such, S must be using a pure strategy in every UE of this type in which B's response is credible. Moreover, this pure strategy must be such that S's offer, z_v, is the unique maximizer of $x\psi_{\pi_0}(v, x)$. As explained in the text, these requirements pin down a unique candidate equilibrium. I now prove Proposition 3.

Proof of Proposition 3. Suppose first that (μ, β, σ) is a consistent assessment that falls into one of the above cases. Note that in each of these cases, S's strategy is clearly optimal given β. Moreover, it is easy to verify that β satisfies the conditions of Proposition 1 for $\pi_0 = \mathbb{E}_\mu[\beta]$. As such, β is a best response for B, σ is a best response for S, and μ is consistent; that is, (μ, β, σ) is a UE.

Suppose now that (μ, β, σ) is a UE. Let $\pi = \mathbb{E}_\mu[\beta]$. Because β is a best response to (μ, σ), whenever $\pi = 0$, $\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{v}-\mathbf{x})}\right] \leq 1$ must hold by Proposition 1. S's expected utility is zero, because his equilibrium offers are always rejected. S-optimality then requires that $\beta(v, x) = 0$ for all $x > 0$. Hence, we are in Case 1.

Suppose $\pi = 1$. Then, $\mathbb{E}_\mu\left[e^{\frac{1}{\kappa}(\mathbf{x}-\mathbf{v})}\right] \leq 1$ by Proposition 1. Moreover, on path, S's offers are surely accepted. Therefore, S cannot be randomizing. If he were, B must be accepting all of S's on-path offers with probability 1, in which case S strictly prefers making his highest on-path offers over any lower on-path offer. Hence, $\sigma(\cdot|v) = \delta_{z_v}$ for some $z_v \in X$. Letting z_v be S's equilibrium offer given v, one then has $\beta(v, x)\, x \leq z_v$ for all x by S-optimality. We are therefore in Case 2.

Finally, suppose $\pi \in (0, 1)$. Let s_v be the equilibrium payoff of a v-type S. By Proposition 1 and S optimality, $x\beta(v, x) \leq s_v$ for all x, and $x\beta(v, x) = x\psi_\pi(v, x) = s_v$ on a $\sigma(\cdot|v)$-almost sure set. Because π is interior, ψ_π has a logit form, meaning $x\psi_\pi(v, x)$ is strictly log-concave. As such, for any s_v and v, at most two x values exist such that $x\psi_\pi(v, x) = s_v$. Hence, $\operatorname{supp}\sigma$ has at most two elements; that is, (a) holds. (b) is a restatement of S optimality, whereas Proposition 1 implies (c) and (d). □

C Other Cost Functions

In this section, I show my results hold under cost functions other than entropy reduction. In particular, I prove trade is inefficient, and B's surplus is positive whenever trade occurs and quality is uncertain.

I assume B uses a recommendation strategy. For such strategies to be sufficient, the general cost of information needs to satisfy rather mild properties. In particular, recommendation strategies are sufficient whenever attention costs are convex and respect the Blackwell order. From a revealed preference perspective, these assumptions are essentially without loss; see De Oliveira et al. (2017) for a more detailed discussion.

Over the domain of recommendation strategies, one can generalize mutual information by generalizing (1). For a strictly convex function $\varphi : [0,1] \to \mathbb{R}$, consider the class of cost functions defined by[20]

$$C_{\varphi}(\beta, \mu) = \mathbb{E}_{\mu}[\varphi(\beta(\mathbf{v}, \mathbf{x}))] - \varphi(\mathbb{E}_{\mu}[\beta(\mathbf{v}, \mathbf{x})]).$$

Note the above specializes to mutual information when $\varphi(y) = y \ln y + (1-y)\ln(1-y)$. I assume φ is twice continuously differentiable. The function $y \ln y + (1-y)\ln(1-y) = -\mathbf{H}(y)$ satisfies this requirement except at 0 and 1, where its derivative goes to $-\infty$ and ∞, respectively. Although allowing for unbounded derivatives at the edges does not alter the presented results, I only present the analysis of the bounded derivative case. I also assume the resulting cost function, C_{φ}, is convex in β, and strictly convex over mixtures of non-constant and constant recommendation strategies; that is, for all $r, t \in (0,1)$ and all β such that $\mu\{\beta = \mathbb{E}_{\mu}[\beta]\} < 1$, we have

$$C_{\varphi}(r\beta + (1-r)t\mathbf{1}, \mu) < rC_{\varphi}(\beta, \mu) + (1-r)C_{\varphi}(t\mathbf{1}, \mu) = rC_{\varphi}(\beta, \mu),$$

where $\mathbf{1}$ is the constant unit function $(v, x) \mapsto 1$.[21]

The following theorem generalizes my main conclusions to the case in which

[20] Similar cost functions appear in Fosgerau et al. (2017) and Yang (2020).

[21] One can show that any convex quadratic polynomial—that is a function of the form $\varphi(y) = ay^2 + by + c$ for $a > 0$—satisfies these properties.

B's attention costs are given by C_φ.

Theorem 2. *Suppose B's attention costs are given by C_φ as described above. Then, in every CE, $\mathcal{E} = (\mu, \beta, \sigma)$,*

1. *Trade is inefficient, $\mathbb{E}_{\mathcal{E}}[\beta] < 1$.*
2. *If quality is uncertain, $|V| \geq 2$, and if trade occurs, $\mathbb{E}_{\mathcal{E}}[\beta] > 0$, B's expected utility is strictly positive, $\mathbb{E}_{\mathcal{E}}[U_B] > 0$.*

The first step in proving the theorem is to show trade must be inefficient. The broad argument goes as follows. For trade to be efficient, B must accept all of S's on-path offers for sure. For B to do so credibly, she must accept for sure all offers that give her a strictly positive surplus. As such, no *strictly* positive surplus offer can be optimal for S; that is, S's offer must leave B with a weakly negative surplus. However, such offers are always rejected with some positive probability when B's response is credible, a contradiction. This argument is somewhat different from what happens with mutual information. With mutual information, B credibly accepts all on-path offers for sure only if she accepts for sure *all* offers, which cannot happen in equilibrium. The reason the two arguments differ is that $\varphi(x) = x \ln x + (1-x)\ln(1-x)$ has an unbounded derivative at 0 and 1. When the derivative of φ is bounded (as assumed in Theorem 2), a more subtle argument is needed.

The second step is to show B's expected utility is strictly positive whenever trade occurs and quality is uncertain. The argument here follows similar lines to the argument with mutual information. If B's expected utility is zero, B must be accepting all offers with the same probability. Because trade happens but is inefficient, B accepts all offers with the same probability only if S only makes B zero-surplus offers, which turns out to contradict S optimality. That B's surplus is positive follows.

I now turn to formally proving the theorem. I begin by deriving various properties that β must satisfy in a CE. These properties all hinge on B's first-order conditions for B's (not necessarily credible) best responses when her costs are C_φ. Finding B's best responses means finding the β in $L^1(\mu)$ that

solves

$$\max_{\beta(\cdot)\in L^1(\mu)} \mathbb{E}_\mu\left[(\mathbf{v}-\mathbf{x})\,\beta\left(\mathbf{v},\mathbf{x}\right)-\varphi\left(\beta\left(\mathbf{v},\mathbf{x}\right)\right)\right]+\varphi\left(\mathbb{E}_\mu\left[\beta\right]\right)$$
$$\text{s.t. } \beta\left(\cdot\right)\geq 0,\ 1-\beta\left(\cdot\right)\geq 0.$$

Observe $C_\varphi\left(\cdot,\mu\right)$ is continuous, convex, and differentiable (the latter follows from $\varphi'\left(1\right)<\infty$), meaning B maximizes a concave differentiable objective over a convex domain. Letting $\mathcal{P}\left(\mu\right)=\left\{(v,x):\mu\left\{(\mathbf{v},\mathbf{x})=(v,x)\right\}>0\right\}$ be the set of quality-offer pairs to which μ assigns a strictly positive probability, the convex multiplier rule (Pourciau, 1980) delivers two positive functions, $\lambda:\mathcal{P}\left(\mu\right)\to\mathbb{R}_+$ and $\gamma:\mathcal{P}\left(\mu\right)\to\mathbb{R}_+$, such that $\lambda\left(v,x\right)\beta\left(v,x\right)=0$, $\left(1-\beta\left(v,x\right)\right)\gamma\left(v,x\right)=0$, and

$$v-x-\varphi'\left(\beta\left(v,x\right)\right)+\varphi'\left(\mathbb{E}_\mu\left[\beta\right]\right)+\lambda\left(v,x\right)-\gamma\left(v,x\right)=0$$

all hold for all $(v,x)\in\mathcal{P}\left(\mu\right)$. Because φ is strictly convex, φ' is strictly increasing, and therefore invertible. As such, we can rearrange the above first-order condition to obtain the following lemma.

Lemma 4. *β is optimal given (μ,σ) only if two positive functions $\lambda:\mathcal{P}\left(\mu\right)\to\mathbb{R}_+$, and $\gamma:\mathcal{P}\left(\mu\right)\to\mathbb{R}_+$ exist such that for which the following holds for all $(v,x)\in\mathcal{P}\left(\mu\right)$:*

$$\beta\left(v,x\right)=\left(\varphi'\right)^{-1}\left(v-x+\varphi'\left(\mathbb{E}_\mu\left[\beta\right]\right)+\lambda\left(v,x\right)-\gamma\left(v,x\right)\right), \tag{13}$$
$$0=\beta\left(v,x\right)\lambda\left(v,x\right), \tag{14}$$
$$0=\left(1-\beta\left(v,x\right)\right)\gamma\left(v,x\right). \tag{15}$$

I now use Lemma 4 to reason about properties of B's credible responses.

Lemma 5. *Suppose β is a credible response to (μ,σ). Then,*

1. *$\beta\left(v,x\right)=1$ only if $v\geq x$. Moreover, if $\mathbb{E}_\mu\left[\beta\right]=1$, then $\beta\left(v,x\right)=1$ whenever $v\geq x$.*

2. $\beta(v,x) = 0$ *only if* $v \leq x$. *Moreover, if* $\mathbb{E}_\mu[\beta] = 0$, *then* $\beta(v,x) = 0$ *whenever* $v \leq x$.

Proof. We only prove Part 1 here. The proof for Part 2 is analogous. Suppose $\beta(v,x) = 1$. Because β is a credible response, a sequence strongly converging to μ, $\mu_n \xrightarrow{s} \mu$, exists such that for all n, β is a best response to μ_n and $\mu_n\{(\mathbf{v},\mathbf{x}) = (v,x)\} > 0$. Let λ_n and γ_n be the multipliers from Lemma 4 for the n-th element of the sequence. By (14), $\lambda_n(v,x) = 0$. Applying φ' to both sides of (13) and rearranging delivers

$$v - x = \varphi'(1) - \varphi'(\mathbb{E}_{\mu_n}[\beta]) + \gamma_n(v,x).$$

We therefore have that

$$v - x \geq \varphi'(1) - \varphi'(\mathbb{E}_{\mu_n}[\beta]) \to \varphi'(1) - \varphi'(\mathbb{E}_{\mu}[\beta]) \geq 0, \tag{16}$$

where the second inequality follows from φ being strictly convex. The conclusion follows.

Now suppose $\mathbb{E}_\mu[\beta] = 1$, and let (v,x) be such that $v \geq x$. Suppose for a contradiction that $\beta(v,x) < 1$. Because β is a credible response, a sequence strongly converging to μ, $\mu_n \xrightarrow{s} \mu$, exists such that for all n, β is a best response to μ_n and $\mu_n\{(\mathbf{v},\mathbf{x}) = (v,x)\} > 0$. Take λ_n and γ_n to be the multipliers corresponding to β and μ_n. Observe that $\gamma_n(v,x) = 0$ for all n by (15). Therefore, (13) implies

$$\begin{aligned} v - x &= \varphi'(\beta(v,x)) - \varphi'(\mathbb{E}_{\mu_n}[\beta]) - \lambda_n(v,x) \\ &\leq \varphi'(\beta(v,x)) - \varphi'(\mathbb{E}_{\mu_n}[\beta]) \to \varphi'(\beta(v,x)) - \varphi'(1) < 0, \end{aligned}$$

a contradiction. □

Lemma 6. *Suppose* β *is a credible response to* (μ,σ). *Then,* $\mathbb{E}_\mu[\beta] = 1$ *if and only if* $\mu\{\mathbf{v} > \mathbf{x}\} = 1$.

Proof. Because $\beta(v,x) \leq 1$ for all (v,x), $\mathbb{E}_\mu[\beta] = 1$ is equivalent to $\beta(\mathbf{v},\mathbf{x}) = 1$ μ-almost surely. That $\mathbb{E}_\mu[\beta] = 1$ holds whenever $\mu\{\mathbf{v} > \mathbf{x}\} = 1$ follows

from optimality of credible responses. For the converse, observe Lemma 5 implies $\beta(v,x)=1$ holds only if $v \geq x$; that is, only if $\{(v,x):\beta(v,x)=1\} \subseteq \{(v,x):v \geq x\}$. Therefore, $\mu\{\mathbf{v} \geq \mathbf{x}\} \geq \mu\{\beta(\mathbf{v},\mathbf{x})=1\}=1$. Thus, showing $\mu\{\mathbf{v}=\mathbf{x}\}=0$ holds is sufficient for proving the lemma. Suppose for a contradiction that $\mu\{\mathbf{v}=\mathbf{x}\}>0$. Let v be such that $\mu\{\mathbf{v}=\mathbf{x}=v\}>0$, and take some $x>v$. Because β is credible, a sequence $\mu_n \xrightarrow{s} \mu$ exists such that for all n, β is a best response to μ_n and $\mu_n\{(\mathbf{v},\mathbf{x})=(v,x)\}>0$. Let γ_n and λ_n be the corresponding multipliers. By Lemma 5, $\beta(v,x)<1$, and therefore $\mathbb{E}_{\mu_n}[\beta]<1$. Because $\beta(v,v)=1$, (14) implies $\lambda_n(v,v)=0$. Examining (13) at $(v,v)\in\mathcal{P}(\mu_n)$, applying φ' to both sides and rearranging gives

$$0=\varphi'(1)-\varphi'(\mathbb{E}_{\mu_n}[\beta])+\gamma_n(v,v)>0,$$

a contradiction. □

I now show all credible equilibria are inefficient.

Proof that every CE is inefficient. Let $\mathcal{E}=(\mu,\beta,\sigma)$ be a CE satisfying $\mathbb{E}_\mu[\beta]=1$. We show $\mu\{\mathbf{v}=\mathbf{x}\}=1$. By Lemma 5, $\beta(v,x)=1$ whenever $v \geq x$. It is therefore never optimal for a v-type S to offer anything below v. Said differently, $\sigma\{\mathbf{x} \geq v|v\}=1$, meaning $\mu\{\mathbf{x} \geq \mathbf{v}\}=1$. Because $\mathbb{E}_\mu[\beta]=1$, $\beta(v,x)=1$ μ-almost surely, meaning, by Lemma 5, $\mathbf{v} \geq \mathbf{x}$ μ-almost surely as well. Hence, we have obtained that $\mu\{\mathbf{x} \geq \mathbf{v}\}=\mu\{\mathbf{x} \leq \mathbf{v}\}=1$, or $\mu\{\mathbf{x}=\mathbf{v}\}=1$. Therefore, $\mu\{\mathbf{v}>\mathbf{x}\}=0$, which contradicts $\mu\{\mathbf{v}>\mathbf{x}\}=1$ from Lemma 6. The conclusion follows. □

Next, I show if trade occurs and quality is uncertain, B's trade surplus is strictly positive. I begin with the following lemma.

Lemma 7. *Suppose β is a credible response to (μ,σ), $\mu\{\mathbf{v}=\mathbf{x}\}>0$, and $\mathbb{E}_\mu[\beta]\in(0,1)$. Then,*

1. *For all $v\in V$, $\beta(v,v)=\mathbb{E}_\mu[\beta]$.*
2. *Two functions $\lambda: V\times X\to\mathbb{R}_+$ and $\gamma: V\times X\to\mathbb{R}_+$ exist that satisfy (13), (14), (15).*

3. An $\epsilon > 0$ exists such that for all $|v - x| \leq \epsilon$,

$$\beta(v, x) = (\varphi')^{-1}(v - x + \varphi'(\mathbb{E}_\mu[\beta])). \tag{17}$$

Proof of Part 1. Fix some $v \in V$. Because β is a credible response, a sequence $\mu_n \xrightarrow{s} \mu$ exists such that for all n, β is a best response to μ_n and $\mu_n\{(v, v)\} > 0$. Let γ_n and λ_n be the corresponding multipliers. Because $\mu_n \xrightarrow{s} \mu$, it must be that $\mathbb{E}_{\mu_n}[\beta] \to \mathbb{E}_\mu[\beta]$. As such, $\mathbb{E}_{\mu_n}[\beta] \in (0, 1)$ must hold for all sufficiently large n. Now, consider (13) specialized for the case in which $x = v$:

$$\beta(v, v) = (\varphi')^{-1}(\varphi'(\mathbb{E}_{\mu_n}[\beta]) - \gamma_n(v, v) + \lambda_n(v, v)).$$

Because φ' is strictly increasing, its inverse is strictly increasing as well. As such, $\beta(v, v) > \mathbb{E}_{\mu_n}[\beta]$ holds only if $\lambda_n(v, v) > 0$. But by (14), $\lambda_n(v, v) > 0$ holds only if $\beta(v, v) = 0 < \mathbb{E}_{\mu_n}[\beta]$, a contradiction. Hence, $\beta(v, v) \leq \mathbb{E}_{\mu_n}[\beta]$. An analogous argument establishes $\beta(v, v) \geq \mathbb{E}_{\mu_n}[\beta]$, and so $\beta(v, v) = \mathbb{E}_{\mu_n}[\beta] \to \mathbb{E}_\mu[\beta]$, as required. □

Proof of Part 2. Pick any (v', x'). Because β is a credible response, a sequence $\mu_n \xrightarrow{s} \mu$ exists such that for all n, β is a best response to μ_n and $\mu_n\{(v', x')\} > 0$. Because $\mu_n \xrightarrow{s} \mu$ and $\mu\{x = v\} > 0$, a v exists such that for all large n, $\mu_n\{(v, v)\} > 0$. The same argument from Part 1 then establishes that $\mathbb{E}_{\mu_n}[\beta] = \beta(v, v) = \mathbb{E}_\mu[\beta]$ for all n large. Fixing such n and letting γ_n and λ_n be the multipliers from the sequence, we can therefore set $\gamma(v', x') = \gamma_n(v', x')$ and $\lambda(v', x') = \lambda_n(v', x')$ to obtain $\gamma(v', x')$ and $\lambda(v', x')$ that satisfy the desired first-order conditions for (v', x'). Repeating this process for all (v', x') yields the desired functions. □

Proof of Part 3. Notice Part 1 implies the desired equality for $x = v$. I now show an $\epsilon > 0$ exists such that (17) holds for all v and all $x \in [v - \epsilon, v)$. The argument for $x \in (v, v + \epsilon]$ is analogous, and so is omitted. Suppose no such $\epsilon > 0$ exists. Then, a sequence $\{(v_n, x_n)\}_{n \geq 0}$ exists such that $x_n \in [v_n - \frac{1}{n}, v_n)$ and for which (17) fails for all n. Let γ and λ be the functions from Part 2. A necessary condition for (17) to fail is for either $\gamma(v_n, x_n)$ or $\lambda(v_n, x_n)$ to be

strictly positive. Because $x_n < v_n$, $\beta(v_n, x_n) > 0$ by Lemma 5. As such, by (14), $\lambda(v_n, x_n) = 0$ must hold for all n. Thus, $\gamma(v_n, x_n) > 0$ must hold for all n, and therefore $\beta(v_n, x_n) = 1$ for all n by (15). These facts combined with (13) then yield the contradiction

$$\begin{aligned}\varphi'(1) &= v_n - x_n + \varphi'(\mathbb{E}_\mu[\beta]) - \gamma(v_n, x_n) \\ &\leq \frac{1}{n} + \varphi'(\mathbb{E}_\mu[\beta]) \to \varphi'(\mathbb{E}_\mu[\beta]) < \varphi'(1),\end{aligned}$$

where the last inequality follows from $\mathbb{E}_\mu[\beta] < 1$ and from strict monotonicity of φ'. □

Lemma 8. *Suppose β is a credible response to (μ, σ) that is constant μ-almost surely, and that $\mathbb{E}_\mu[\beta] \in (0, 1)$. Then, $\mathbf{v} = \mathbf{x}$ μ-almost surely.*

Proof. Suppose β satisfies the lemma's premise. It is sufficient to show $\beta(v, x) \neq \mathbb{E}_\mu[\beta]$ whenever $v \neq x$. Suppose for a contradiction $\beta(v, x) = \mathbb{E}_\mu[\beta]$ for some $v < x$ (the proof for $v > x$ is analogous). Because β is credible, a sequence $\mu_n \xrightarrow{s} \mu$ exists such that for all n, β is a best response to μ_n and $\mu_n\{(v, x)\} > 0$. Let λ_n and γ_n be the corresponding multipliers from Lemma 5. Because $\beta(v, x) = \mathbb{E}_\mu[\beta] \in (0, 1)$, (14) and (15) imply $\lambda_n(v, x) = \gamma_n(v, x) = 0$. Substituting $\beta(v, x) = \mathbb{E}_\mu[\beta]$ into (13) and applying φ' delivers

$$\varphi'(\mathbb{E}_\mu[\beta]) = v - x + \varphi'(\mathbb{E}_{\mu_n}[\beta]) \to v - x + \varphi'(\mathbb{E}_\mu[\beta]) < \varphi'(\mathbb{E}_\mu[\beta]),$$

where convergence follows from $\mu_n \xrightarrow{s} \mu$. Hence, we have a contradiction. □

Lemma 9. *Suppose β is a best response to (μ, σ), that $\mathbb{E}_\mu[\beta] > 0$, and that B's expected utility is zero. Then, β is μ-almost surely constant.*

Proof. Suppose otherwise. Consider B's utility restricted to the line segment $y\beta + (1 - y)\mathbf{0}$, where $\mathbf{0}$ is the recommendation strategy that rejects all offers for sure,

$$\begin{aligned}\tilde{U} : [0, 1] &\to \mathbb{R}, \\ y &\mapsto y\mathbb{E}_\mu[(\mathbf{v} - \mathbf{x})\beta(\mathbf{v}, \mathbf{x})] - C_\varphi(y\beta + (1 - y)\mathbf{0}, \mu).\end{aligned}$$

Because β is a best response to (μ, σ), $\tilde{U}$ is maximized. Moreover, $\tilde{U}$ is strictly concave, because β not being a μ-almost sure constant means C_φ is strictly convex on the line segment $\operatorname{co}\{\beta, \mathbf{0}\}$. As such, $y = 1$ is a strict maximizer of $\tilde{U}$. Therefore, $\tilde{U}(1) > \tilde{U}(0) = 0$. Because $\tilde{U}(1)$ is B's expected utility from using β, we have a contradiction. □

We are now ready to prove the second part of the theorem.

Proof of $|V| \geq 2$ ***and*** $\int \beta \mathrm{d}\mu > 0$ ***only if*** $\mathbb{E}(U_B) > 0$. Proof is by contradiction. Suppose $|V| \geq 2$, and a CE, $\mathcal{E} = (\mu, \beta, \sigma)$, exists in which $\mathbb{E}_{\mathcal{E}}[\beta] > 0$ but B's expected utility is zero. By the theorem's previous part, trade is inefficient, and so $\mathbb{E}_\mu[\beta] \in (0, 1)$. Because $\mathbb{E}_{\mathcal{E}}[U_B] = 0$, Lemma 9 delivers that β is constant μ-almost surely. Combined with $\mathbb{E}_{\mathcal{E}}[\beta] \in (0, 1)$, Lemma 8 delivers $\mu\{\mathbf{v} = \mathbf{x}\} = 1$. As such, by Lemma 7, an $\epsilon > 0$ exists such that (17) holds whenever $|v - x| \leq \epsilon$. Consider the problem of a v-type S. Because $\mu\{\mathbf{v} = \mathbf{x}\} = 1$, we have that $\sigma(\{v\}|v) = 1$ for all v, and so offering v is optimal for a v-type S. As such, offering v is also optimal for a v-type S if she were restricted only to offers within ϵ of v. That is, v solves

$$\max_{x \in [v-\epsilon, v+\epsilon]} x\left[(\varphi')^{-1}(v - x + \varphi'(\mathbb{E}_\mu[\beta]))\right].$$

By assumption, φ' is increasing and continuously differentiable, and so the above objective is differentiable. Thus, being an interior optimum, $x = v$ must satisfy S's first-order condition,

$$\begin{aligned} 0 &= \left((\varphi')^{-1}(v - x + \varphi'(\mathbb{E}_\mu[\beta])) - \frac{x}{\varphi''\left((\varphi')^{-1}(v - x + \varphi'(\mathbb{E}_\mu[\beta]))\right)}\right)\Bigg|_{x=v} \\ &= \left(\int \beta \mathrm{d}\mu - \frac{\mathrm{v}}{\varphi''(\mathbb{E}_\mu[\beta])}\right), \end{aligned}$$

which can be rearranged to give $v = (\mathbb{E}_\mu[\beta])\,\varphi''(\mathbb{E}_\mu[\beta])$. Hence, v is a μ-almost sure constant, a contradiction to the assumption that $|V| \geq 2$. □

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