

ONLINE APPENDIX FOR
“Fiscal Rules and Discretion in a World Economy”

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In this Online Appendix, we provide proofs for the results in Section IV of the paper. Since several of our claims are for the case of log preferences, we begin by restating our problem under this preference structure.

B1. Log Preferences

Take $U(g) = \log(g)$ and $W(x) = \delta \log(\tau + x)$ for $\delta > 0$. Equations (1) and (6) imply

$$(B1) \quad g^f(\theta, R) = \frac{\theta}{\theta + \beta\delta} \left(\tau + \frac{\tau}{R} \right),$$

$$(B2) \quad \tau + x^f(\theta, R) = \frac{\beta\delta}{\theta + \beta\delta} R \left(\tau + \frac{\tau}{R} \right).$$

Equivalently, letting $s^f(\theta) = \frac{\beta\delta}{\theta + \beta\delta}$ be type θ 's savings rate under full flexibility, we have

$$g^f(\theta, R) = \left(1 - s^f(\theta) \right) \left(\tau + \frac{\tau}{R} \right).$$

For a given rule $\theta^* \in [0, \bar{\theta}]$, the aggregate savings rate in the economy is

$$S(\theta^*) = \int_{\underline{\theta}}^{\theta^*} \frac{\beta\delta}{\theta + \beta\delta} f(\theta) d\theta + \int_{\theta^*}^{\bar{\theta}} \frac{\beta\delta}{\theta^* + \beta\delta} f(\theta) d\theta.$$

The coordinated program in (11) can be written as

$$(B3) \quad \max_{\theta^* \in [0, \bar{\theta}]} \left\{ \begin{array}{l} \int_{\underline{\theta}}^{\theta^*} \left(\theta \log \left(\frac{\theta}{\theta + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta + \beta\delta} \right) \right) f(\theta) d\theta \\ + \int_{\theta^*}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta^*}{\theta^* + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta^* + \beta\delta} \right) \right) f(\theta) d\theta \\ - \log(1 - S(\theta^*)) - \delta \log(S(\theta^*)) \end{array} \right\}.$$

The uncoordinated program in (7), on the other hand, reduces to the first two lines of (B3).

As explained in the paper, the difference between the optimal coordinated fiscal rule and the optimal uncoordinated fiscal rule can be expressed as a function of the redistributive and disciplining effects of the interest rate. Under log preferences,

the sum of these two effects is

$$(B4) \quad \rho + \lambda = \frac{1 - \delta R(\theta_c^*)}{R(\theta_c^*) (1 + R(\theta_c^*))}.$$

Equation (B4) is the same as equation (15) but allowing for any $\delta > 0$. This equation shows that the redistributive effect of the interest rate dominates the disciplining effect if and only if $R(\theta_c^*) < 1/\delta$. As discussed in the paper, the redistributive effect is stronger on the margin when interest rates are low.³⁶

B2. Proof of Proposition 3

The first part of the proposition follows from the arguments in the text. We prove the second part by example. Take log preferences. Analogous to the expressions in Section B1 above, given cutoffs $\theta^* \in [0, \bar{\theta}]$ and $\theta^{**} \in [0, \theta^*]$, the aggregate savings rate in the economy is

$$S(\theta^*, \theta^{**}) = \int_{\underline{\theta}}^{\theta^{**}} \frac{\beta\delta}{\theta^{**} + \beta\delta} f(\theta) d\theta + \int_{\theta^{**}}^{\theta^*} \frac{\beta\delta}{\theta + \beta\delta} f(\theta) d\theta + \int_{\theta^*}^{\bar{\theta}} \frac{\beta\delta}{\theta^* + \beta\delta} f(\theta) d\theta,$$

and the coordinated program can be written as

$$(B5) \quad \max_{\theta^* \in [0, \bar{\theta}], \theta^{**} \in [0, \theta^*]} \left\{ \begin{array}{l} \int_{\underline{\theta}}^{\theta^{**}} \left(\theta \log \left(\frac{\theta^{**}}{\theta^{**} + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta^{**} + \beta\delta} \right) \right) f(\theta) d\theta \\ + \int_{\theta^{**}}^{\theta^*} \left(\theta \log \left(\frac{\theta}{\theta + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta + \beta\delta} \right) \right) f(\theta) d\theta \\ + \int_{\theta^*}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta^*}{\theta^* + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta^* + \beta\delta} \right) \right) f(\theta) d\theta \\ - \log(1 - S(\theta^*, \theta^{**})) - \delta \log(S(\theta^*, \theta^{**})) \end{array} \right\}.$$

Take $\delta = 1$ and $F(\theta)$ to be exponential with parameter 0.0785 and set $[\underline{\theta}, \bar{\theta}]$ given by $[0.05, 2]$. The parameter and truncation we choose ensure that $\mathbb{E}[\theta] = 1$. For a range of β , Figure B1 depicts the cutoff θ_u^* in the optimal uncoordinated rule and the cutoffs θ_c^* and θ_c^{**} in the optimal coordinated rule, as a function of β . Recall that $\theta_u^{**} \leq \underline{\theta}$ always holds. Hence, as shown in the figure, we find that there exist $(U(\cdot), W(\cdot), F(\theta), \tau, \beta)$ such that $\theta_c^* > \theta_u^*$, $\theta_c^* > \underline{\theta}$, and $\theta_c^{**} > \underline{\theta} \geq \theta_u^{**}$.

B3. Proof of Proposition 4

To prove the first part of the proposition, we follow analogous steps as in the proof of Proposition 2 for the case of $\beta \leq \underline{\theta}$. We show that setting a coordinated cutoff $\theta^* = \underline{\theta}$ is not optimal. Note that by Proposition 2, if $\psi = 0$, then $\theta^* = \underline{\theta}$

³⁶The relevant threshold for $R(\theta_c^*)$ depends on δ because a reduction in δ has a similar effect as a reduction in $R(\theta_c^*)$: all types shift spending to the present when δ declines.

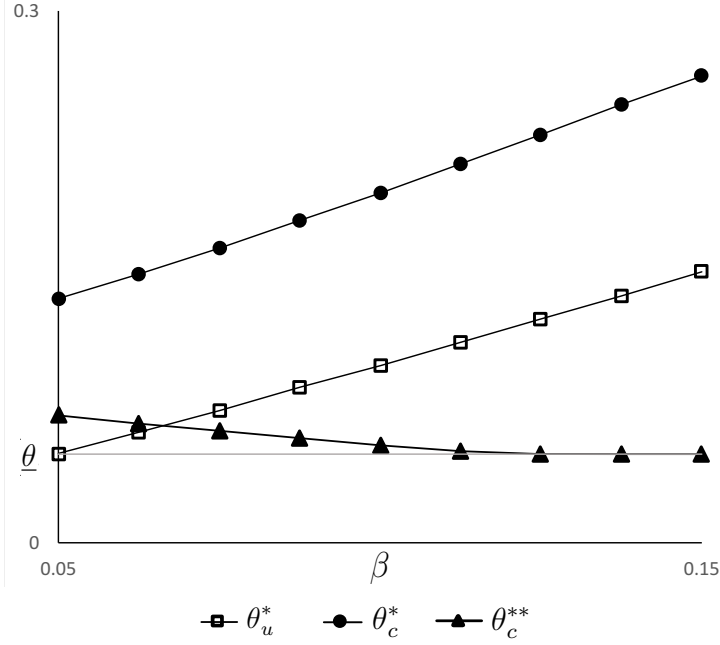


FIGURE B1. MAXIMUM SURPLUS LIMITS.

is indeed not optimal. Moreover, since the objective function of the coordinated problem is continuous in θ^* and ψ , it follows that for $\psi = \varepsilon$, $\varepsilon > 0$ arbitrarily small, $\theta^* = \underline{\theta}$ is not optimal either. Therefore, given $\beta \leq \underline{\theta}$, there exists $\bar{\psi} \in (0, 1)$ such that if $\psi \leq \bar{\psi}$, then $\theta_c^* > \theta_u^*$ and $\theta_c^* > \underline{\theta}$.

To prove the second part of the proposition, take log preferences and assume θ_c^* is a unique and interior global optimum with $\theta_c^* > \theta_u^*$. We consider the program that solves for the optimal coordinated fiscal rule taking into account that a fraction ψ of governments choose θ_u^* . Analogous to the analysis in Section B1 above, given a rule $\theta^* > \theta_u^*$, the aggregate savings rate in the economy is (we allow here for any $\delta > 0$; the statement of Proposition 4 takes $\delta = 1$):

$$\begin{aligned}
 S(\theta^*, \psi) = & (1 - \psi) \left(\int_{\underline{\theta}}^{\theta^*} \frac{\beta\delta}{\theta + \beta\delta} f(\theta) d\theta + \int_{\theta^*}^{\bar{\theta}} \frac{\beta\delta}{\theta^* + \beta\delta} f(\theta) d\theta \right) \\
 & + \psi \left(\int_{\underline{\theta}}^{\theta_u^*} \frac{\beta\delta}{\theta + \beta\delta} f(\theta) d\theta + \int_{\theta_u^*}^{\bar{\theta}} \frac{\beta\delta}{\theta_u^* + \beta\delta} f(\theta) d\theta \right).
 \end{aligned}
 \tag{B6}$$

Note that $dS(\theta^*, \psi)/d\psi > 0$ for $\theta_u^* < \theta^*$. The coordinated program, taking the

heterogeneity into account, can be written as

$$\max_{\theta^* \in [0, \bar{\theta}]} \left\{ \begin{aligned} & (1 - \psi) \left[\int_{\frac{\underline{\theta}}{\bar{\theta}}}^{\theta^*} \left(\theta \log \left(\frac{\theta}{\theta + \beta \delta} \right) + \delta \log \left(\frac{\beta \delta}{\theta + \beta \delta} \right) \right) f(\theta) d\theta \right. \\ & \quad \left. + \int_{\theta^*}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta^*}{\theta^* + \beta \delta} \right) + \delta \log \left(\frac{\beta \delta}{\theta^* + \beta \delta} \right) \right) f(\theta) d\theta \right] \\ & + \psi \left[\int_{\frac{\underline{\theta}}{\bar{\theta}}}^{\theta_u^*} \left(\theta \log \left(\frac{\theta}{\theta + \beta \delta} \right) + \delta \log \left(\frac{\beta \delta}{\theta + \beta \delta} \right) \right) f(\theta) d\theta \right. \\ & \quad \left. + \int_{\theta_u^*}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta_u^*}{\theta_u^* + \beta \delta} \right) + \delta \log \left(\frac{\beta \delta}{\theta_u^* + \beta \delta} \right) \right) f(\theta) d\theta \right] \\ & - \log(1 - S(\theta^*, \psi)) - \delta \log(S(\theta^*, \psi)) \end{aligned} \right\}$$

subject to (9).

The first-order condition, assuming an interior optimum, is
(B7)

$$\int_{\theta_c^*}^{\bar{\theta}} \left(\frac{\theta}{\theta_c^*} - \frac{\theta + \delta}{\theta_c^* + \beta \delta} \right) f(\theta) d\theta - \left(\int_{\theta_c^*}^{\bar{\theta}} \frac{\beta \delta}{(\theta_c^* + \beta \delta)^2} f(\theta) d\theta \right) \left(\frac{1}{1 - S(\theta_c^*, \psi)} - \delta \frac{1}{S(\theta_c^*, \psi)} \right) = 0.$$

Since θ_c^* is the unique global optimum, we can determine its comparative statics with respect to ψ by implicit differentiation of (B7). Since the program is locally concave, the derivative of the left-hand side of (B7) with respect to θ_c^* is negative. If we can establish that the derivative of the left-hand side of (B7) with respect to ψ is negative, then this implies that θ_c^* is locally decreasing in ψ . We find that this is indeed the case: the derivative of the left-hand side of (B7) with respect to ψ is

$$- \left(\int_{\theta_c^*}^{\bar{\theta}} \frac{\beta \delta}{(\theta_c^* + \beta \delta)^2} f(\theta) d\theta \right) \frac{dS(\theta_c^*, \psi)}{d\psi} \left(\frac{1}{(1 - S(\theta_c^*, \psi))^2} + \delta \frac{1}{(S(\theta_c^*, \psi))^2} \right) < 0,$$

where we have taken into account that $dS(\theta_c^*, \psi)/d\psi > 0$ since $\theta_u^* < \theta_c^*$.

B4. Proof of Proposition 5

To prove the first part of the proposition, we follow the same steps as in the proof of Proposition 2 for the case of $\beta \leq \underline{\theta}$, taking into account that (4) is now replaced by (16). Suppose $L \leq 0$. Then note that any rule with $\theta_c^* \in [\theta_u^*, \underline{\theta}]$ is weakly dominated by a rule with $\theta_c^* = \underline{\theta}$, as an increase in θ_c^* to $\underline{\theta}$ changes the allocation only through its positive effect on the interest rate, and this improves welfare given $L \leq 0$. Therefore, to prove the first part of the proposition for $L \leq 0$, it suffices to show that $\theta_c^* = \underline{\theta}$ is not optimal. This is what we prove next.

Note that $R'(\theta^*)$ continues to satisfy (A2), and it satisfies (A9) when $\theta^* = \underline{\theta}$. The first-order condition of the coordinated problem must therefore satisfy equation (A8). If $\theta^* = \underline{\theta}$, then $g^f(\theta^*, R) = \tau + L$ and $x^f(\theta^*, R) = -RL$, so that (A8) becomes

$$-R'(\underline{\theta}) \left(\int_{\underline{\theta}}^{\bar{\theta}} W'(x^f(\theta^*, R)) f(\theta) d\theta \right) L.$$

Recall that $R'(\underline{\theta}) > 0$. Thus, if $L < 0$, the expression above is strictly positive, implying that $\theta^* = \underline{\theta}$ is not optimal as an increase in θ^* would increase welfare. If instead $L = 0$, then by the proof of Proposition 2, $\theta^* = \underline{\theta}$ is not optimal either. Hence, given $\beta \leq \underline{\theta}$, we obtain $\theta_c^* > \theta_u^*$ and $\theta_c^* > \underline{\theta}$ for $L \leq 0$.

Finally, since the objective function of the coordinated problem is continuous in θ^* and L , it follows that for $L = \varepsilon$, $\varepsilon > 0$ arbitrarily small, the result holds as well. Therefore, given $\beta \leq \underline{\theta}$, there exists $\bar{L} > 0$ such that if $L \leq \bar{L}$, then $\theta_c^* > \theta_u^*$ and $\theta_c^* > \underline{\theta}$.

To prove the second part of the proposition, we consider the problem under log preferences as in Section B1, but with (4) now replaced by (16). The program in (B3) becomes (we allow here for any $\delta > 0$; the statement of Proposition 5 takes $\delta = 1$):

$$\max_{\theta^* \in [0, \bar{\theta}]} \left\{ \begin{array}{l} \int_{\underline{\theta}}^{\theta^*} \left(\theta \log \left(\frac{\theta}{\theta + \beta\delta} \right) + \log \left(\frac{\beta\delta}{\theta + \beta\delta} \right) \right) f(\theta) d\theta \\ + \int_{\theta^*}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta^*}{\theta^* + \beta\delta} \right) + \delta \log \left(\frac{\beta\delta}{\theta^* + \beta\delta} \right) \right) f(\theta) d\theta \\ - \log(1 - S(\theta^*)) - \delta \log(S(\theta^*) + L/\tau) \end{array} \right\}.$$

The first-order condition, assuming an interior optimum, is (B8)

$$\int_{\theta_c^*}^{\bar{\theta}} \left(\frac{\theta}{\theta_c^*} - \frac{\theta + \delta}{\theta_c^* + \beta\delta} \right) f(\theta) d\theta - \left(\int_{\theta_c^*}^{\bar{\theta}} \frac{\beta\delta}{(\theta_c^* + \beta\delta)^2} f(\theta) d\theta \right) \left(\frac{1}{1 - S(\theta^*)} - \delta \frac{1}{S(\theta^*) + L/\tau} \right) = 0.$$

Since by assumption θ_c^* is the unique global optimum given L , we can determine its comparative statics with respect to L by implicit differentiation of (B8). Since the program is locally concave, the derivative of the left-hand side of (B8) with respect to θ_c^* is negative. If we can establish that the derivative of the left-hand side of (B8) with respect to L is negative, then this implies that θ_c^* is locally decreasing in L . We find that this is indeed the case: the derivative of the left-hand side of (B8) with respect to L is

$$- \left(\int_{\theta_c^*}^{\bar{\theta}} \frac{\beta\delta}{(\theta_c^* + \beta\delta)^2} f(\theta) d\theta \right) \left(\frac{\delta/\tau}{(S(\theta^*) + L/\tau)^2} \right) < 0.$$

B5. *Proof of Proposition 6*

Define γ_{ui}^* as the optimal uncoordinated rule for country group i with parameters $\{f_i, \beta_i\}$, and let γ_c^* be the optimal coordinated rule for both country groups, given $\{f_N, \beta_N, f_S, \beta_S, \psi\}$. The first part of the proposition ($\beta \geq \bar{\beta}$) follows from analogous reasoning as in the proof of the first part of Proposition 2: if $\beta_S = \beta_N = 1$, then

$$(B9) \quad \gamma_c^* < \min \{\gamma_{uN}^*, \gamma_{uS}^*\}.$$

To prove the second part of the proposition ($\beta \leq \bar{\beta}$), take $\beta_i \leq \bar{\theta}_i$ for $i = N, S$. By Proposition 1, $\theta_{ui}^* \leq \bar{\theta}_i$ for $i = N, S$, implying $\gamma_{ui}^* \leq \underline{\gamma}$. Note that any rule $\gamma_c^* \leq \underline{\gamma}$ would yield the same allocation and hence the same welfare as a rule $\gamma_c^* = \underline{\gamma}$. Therefore, to prove the proposition, it suffices to show that $\gamma_c^* = \underline{\gamma}$ is not optimal. To prove this, consider a fiscal rule $\gamma^* = \underline{\gamma}$ with associated interest rate $R = R(\gamma^*)$. Welfare under this rule is given by (18). The first derivative with respect to γ^* is

$$\begin{aligned} & \frac{\partial g^f(\gamma^*, R)}{\partial \gamma^*} \int_{\gamma^*}^{\bar{\gamma}} \left(\theta(\gamma) U' \left(g^f(\gamma^*, R) \right) - RW' \left(x^f(\gamma^*, R) \right) \right) h(\gamma) d\gamma \\ & + R'(\gamma^*) \left(\int_{\gamma^*}^{\gamma^*} \frac{dg^f(\gamma, R)}{dR} \left(\theta(\gamma) U' \left(g^f(\gamma, R) \right) - RW' \left(x^f(\gamma, R) \right) \right) h(\gamma) d\gamma \right. \\ & \quad \left. + \int_{\gamma^*}^{\bar{\gamma}} \frac{dg^f(\gamma^*, R)}{dR} \left(\theta(\gamma) U' \left(g^f(\gamma^*, R) \right) - RW' \left(x^f(\gamma^*, R) \right) \right) h(\gamma) d\gamma \right) \\ & + R'(\gamma^*) \left(\int_{\gamma^*}^{\gamma^*} W' \left(x^f(\gamma, R) \right) (\tau - g^f(\gamma, R)) h(\gamma) d\gamma \right. \\ & \quad \left. + \int_{\gamma^*}^{\bar{\gamma}} W' \left(x^f(\gamma^*, R) \right) (\tau - g^f(\gamma^*, R)) h(\gamma) d\gamma \right). \end{aligned}$$

The rest of the proof proceeds in the same way as the proof of our main result for homogeneous countries in Proposition 2 and is thus omitted.

B6. *Infinite Horizon*

Consider an infinite horizon version of our model, with periods $t \in \{0, 1, \dots, T\}$, $T \rightarrow \infty$, and discount factor $\delta \in (0, 1)$. The government's welfare at t before the realization of its type θ_t is

$$(B10) \quad \mathbb{E} \left[\theta_t U(g_t) + \sum_{k=1}^{\infty} \delta^k \theta_{t+k} U(g_{t+k}) \right].$$

The government's welfare at t after the realization of θ_t , when choosing spending g_t , is

$$(B11) \quad \theta_t U(g_t) + \beta \mathbb{E} \left[\sum_{k=1}^{\infty} \delta^k \theta_{t+k} U(g_{t+k}) \right].$$

Spending g_t satisfies the government's dynamic budget constraint:

$$(B12) \quad g_t + \frac{x_{t+1}}{R_{t+1}} = \tau + x_t,$$

where x_t is the level of assets with which the government enters period t and we set $x_0 = 0$. The sum of total assets across all governments must be zero in each period. We assume that θ_t is i.i.d. across countries and time with an expected value $\mathbb{E}[\theta_t] = 1$. Because there are no aggregate shocks, it follows that the sequence of interest rates $\{R_t\}_{t=0}^{\infty}$ is deterministic, with $R_0 = 1$. We focus on fiscal rules at t which depend only on payoff-relevant variables: x_t and the sequence of future interest rates $\{R_{t+k}\}_{k=1}^{\infty}$.³⁷ We can then define

$$(B13) \quad W_{t+1}(x_{t+1}) = \mathbb{E} \left[\sum_{k=1}^{\infty} \delta^k \theta_{t+k} U(g_{t+k}) \right]$$

as the continuation welfare at $t+1$ associated with assets x_{t+1} and the continuation sequence of interest rates and fiscal rules. Taking this continuation welfare as given, a fiscal rule at t can be represented as a cutoff type θ^* , where the government has full flexibility if $\theta_t \leq \theta^*$ and no flexibility if $\theta_t > \theta^*$. An individual government's optimal choice of fiscal rule is analogous to that in the two-period setting:

PROPOSITION 7: *In an infinite horizon economy with i.i.d. shocks, the optimal uncoordinated fiscal rule is a time-invariant cutoff θ_u^* satisfying (9).*

PROOF:

Given a deterministic sequence of interest rates, an uncoordinated fiscal rule can be represented as a cutoff sequence $\theta_u^*(t, x_t)$, which depends on time t and the assets x_t with which a government enters the period. The dependence of the rule on time captures the fact that time indexes the future path of interest rates. Moreover, with some abuse of notation, we can let $g^f(\theta_t, t, x_t)$ correspond to type θ_t 's flexible level of spending given time t and assets x_t . The government's

³⁷If countries do not coordinate their rules, then rules of this form are optimal under i.i.d. shocks. See Halac and Yared (2014) for a discussion.

uncoordinated problem can be written recursively as:

$$\max_{\theta_u^*(t, x_t) \in [0, \bar{\theta}]} \left\{ \int_{\underline{\theta}}^{\theta_u^*(t, x_t)} \left(\theta_t U(g^f(\theta_t, t, x_t)) + W_{t+1}(x_{t+1}^f(\theta_t, t, x_t)) \right) f(\theta_t) d\theta_t \right. \\ \left. + \int_{\theta_u^*(t, x_t)}^{\bar{\theta}} \left(\theta_t U(g^f(\theta_u^*(t, x_t), t, x_t)) + W_{t+1}(x_{t+1}^f(\theta_u^*(t, x_t), t, x_t)) \right) f(\theta_t) d\theta_t \right\} \\ \text{subject to (B12) and} \\ g^f(\theta_t, t, x_t) = \arg \max_g \{ \theta_t U(g) + W_{t+1}(R_{t+1}(\tau + x_t - g)) \}.$$

Standard arguments imply that W_{t+1} is a concave and continuously differentiable function of x_{t+1} . Hence, this problem is isomorphic to that of the two-period model, and by Proposition 1 the optimal choice of $\theta_u^*(t, x_t)$ satisfies (9).

We next study the implications of a time-invariant coordinated rule θ^* for the interest rate.

LEMMA 3: *Consider an infinite horizon economy with i.i.d. shocks and $U(g_t) = \log(g_t)$. If all countries are subject to a time-invariant rule θ^* in each period, the interest rate R_t is constant over time and satisfies*

(B14)

$$R_t = R(\theta^*) = \left[\int_{\underline{\theta}}^{\theta^*} \frac{\beta\delta/(1-\delta)}{\theta + \beta\delta/(1-\delta)} f(\theta) d\theta + \int_{\theta^*}^{\bar{\theta}} \frac{\beta\delta/(1-\delta)}{\theta^* + \beta\delta/(1-\delta)} f(\theta) d\theta \right]^{-1}.$$

PROOF:

Under log preferences, (B10) can be written as

(B15)

$$\mathbb{E} \left[\theta_t \log(1 - s_t) + \frac{\delta}{1-\delta} \log(s_t) + \sum_{k=1}^{\infty} \delta^k \left(\theta_{t+k} \log(1 - s_{t+k}) + \frac{\delta}{1-\delta} \log(s_{t+k}) \right) \right] + \chi(\theta_t, t, x_t),$$

where s_t is a savings rate satisfying

$$g_t = (1 - s_t) \left(\tau + \sum_{k=1}^{\infty} \frac{\tau}{\prod_{l=1}^k R_{t+l}} + x_t \right)$$

and, using the above expression, $\chi(\theta_t, t, x_t)$ satisfies

$$(B16) \quad \chi(\theta_t, t, x_t) = \theta_t \log \left(\frac{g_t}{1 - s_t} \right) + \frac{\delta}{1-\delta} \log \left(\frac{g_t}{1 - s_t} \right) + \sum_{k=1}^{\infty} \frac{\delta^k}{1-\delta} \log(R_{t+k}).$$

Analogously, (B11) can be written as
(B17)

$$\theta_t \log(1 - s_t) + \beta \left\{ \frac{\delta}{1 - \delta} \log(s_t) + \mathbb{E} \left[\sum_{k=1}^{\infty} \delta^k \left(\theta_{t+k} \log(1 - s_{t+k}) + \frac{\delta}{1 - \delta} \log(s_{t+k}) \right) \right] \right\} + \omega_t(x_t),$$

where $\omega(x_t)$ satisfies

$$\omega(x_t) = \theta_t \log\left(\frac{g_t}{1 - s_t}\right) + \beta \left(\frac{\delta}{1 - \delta} \log\left(\frac{g_t}{1 - s_t}\right) + \sum_{k=1}^{\infty} \frac{\delta^k}{1 - \delta} \log(R_{t+k}) \right).$$

Denote the flexible savings rate in period t by

$$s^f(\theta_t) = \frac{\beta\delta}{\theta_t + \beta\delta},$$

which is a function of θ_t and does not depend on future interest rates or current assets. Now consider a time-invariant fiscal rule θ^* in a T -period economy. The analog of (B17) in a finite horizon setting implies that at date $T - 1$, a country chooses its flexible savings rate if $\theta_{T-1} \leq \theta^*$ and the flexible savings rate that would correspond to type θ^* if $\theta_{T-1} > \theta^*$. It then follows by backward induction that $s(\theta_t, t, x_t) = \max\{s^f(\theta_t), s^f(\theta^*)\}$ at each $t \in \{0, \dots, T - 1\}$. Taking the limit of the T -period economy as $T \rightarrow \infty$, the global resource constraint at t can therefore be written as

$$\left[\int_{\underline{\theta}}^{\theta^*} \frac{\theta_t}{\theta_t + \beta\delta/(1 - \delta)} f(\theta_t) d\theta_t + \int_{\theta^*}^{\bar{\theta}} \frac{\theta^*}{\theta^* + \beta\delta/(1 - \delta)} f(\theta_t) d\theta_t \right] \left(\tau + \sum_{k=1}^{\infty} \frac{\tau}{\prod_{l=1}^k R_{t+l}} \right) = \tau,$$

where we have taken into account that savings rates are independent of assets and the sum of assets across countries is zero in each period. The fact that this equation holds for all periods t implies (B14).

Consider now the class of rules $\theta^*(t)$ which are possibly time-varying but apply to all countries symmetrically, independently of their assets. We show that there is an optimal coordinated fiscal rule within this class which is time-invariant. Moreover, this rule satisfies our results in Proposition 2.

PROPOSITION 8: *Consider an infinite horizon economy with i.i.d. shocks and $U(g_t) = \log(g_t)$, and take fiscal rules that apply symmetrically to all countries. There exists an optimal coordinated fiscal rule θ_c^* that is time-invariant. Moreover, there exist $\bar{\beta}, \underline{\beta} \in [\underline{\theta}, 1]$, $\bar{\beta} > \underline{\beta}$, such that if $\beta \geq \bar{\beta}$, then $\theta_c^* < \theta_u^*$, whereas if $\beta \leq \underline{\beta}$, then $\theta_c^* > \theta_u^*$ and $\theta_c^* > \underline{\theta}$.*

PROOF:

Using the same arguments as in the proof of Lemma 3, $s(\theta_t, t, x_t) = \max \{s^f(\theta_t), s^f(\theta^*(t))\}$ under a rule $\theta^*(t)$. Define

$$S(\theta^*(t)) = \left[\int_{\underline{\theta}}^{\theta^*(t)} \frac{\beta\delta/(1-\delta)}{\theta + \beta\delta/(1-\delta)} f(\theta) d\theta + \int_{\theta^*(t)}^{\bar{\theta}} \frac{\beta\delta/(1-\delta)}{\theta^*(t) + \beta\delta/(1-\delta)} f(\theta) d\theta \right].$$

Because savings rates are independent of assets, we can write the global resource constraint at t as

$$(B18) \quad (1 - S(\theta^*(t))) \left(\prod_{m=0}^{t-1} S(\theta^*(m)) \right) \left(\prod_{m=0}^t R_m \right) \left(\tau + \sum_{k=1}^{\infty} \frac{\tau}{\prod_{l=1}^k R_l} \right) = \tau,$$

where $R_0 = 1$. Substituting (B18) in (B16) yields

$$\begin{aligned} \chi(\theta_0, 0, 0) &= -\theta_0 \log(1 - S(\theta^*(0))) - \frac{\delta}{1-\delta} \log(S(\theta^*(0))) \\ &\quad - \sum_{t=1}^{\infty} \delta^t \log(1 - S(\theta^*(t))) - \sum_{t=1}^{\infty} \frac{\delta^t}{1-\delta} \log(S(\theta^*(t))) + \left(\theta_0 + \frac{\delta}{1-\delta} \right) \log \tau. \end{aligned}$$

Given (B15), we can write welfare at date 0 as a function of the rule $\theta^*(t)$ as

$$\sum_{t=0}^{\infty} \delta^t \left[\int_{\underline{\theta}}^{\theta^*(t)} \left(\theta \log \left(\frac{\theta}{\theta + \beta\delta/(1-\delta)} \right) + \frac{\delta}{1-\delta} \log \left(\frac{\beta\delta/(1-\delta)}{\theta + \beta\delta/(1-\delta)} \right) \right) f(\theta) d\theta \right. \\ \left. + \int_{\theta^*(t)}^{\bar{\theta}} \left(\theta \log \left(\frac{\theta^*(t)}{\theta^*(t) + \beta\delta/(1-\delta)} \right) + \frac{\delta}{1-\delta} \log \left(\frac{\beta\delta/(1-\delta)}{\theta^*(t) + \beta\delta/(1-\delta)} \right) \right) f(\theta) d\theta \right. \\ \left. - \log(1 - S(\theta^*(t))) - \frac{\delta}{1-\delta} \log(S(\theta^*(t))) + \log \tau \right].$$

Note that the term in the bracket is the same for every t , which implies that there exists a solution with a time-invariant cutoff $\theta^*(t) = \theta_c^*$. Moreover, this bracket is identical to the two-period program in (B3) except that δ is replaced with $\frac{\delta}{1-\delta}$ (and there is the last term which is a constant). The results therefore follow from Proposition 2.