Online Appendix
"Does Incomplete Spanning in International Financial Markets Help to Explain Exchange Rates?"

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October 2018*

Section 1 presents the proof of our main general results. Section 2 reports the quantitative implications of our results when the home and foreign log SDFs exhibit different volatilities. Section 3 studies three examples: a simple consumption-based example; a Cox, Ingersoll, and Ross (1985) model with common factors; a consumption-based example with heteroscedasticity. Section 4 reports summary statistics on the exchange rate entropy.

1 Proofs of Main Results

In this section, we gather all the proofs of the main results in the text, in the order they appear there. We distinguish between the propositions and their corollaries, which are model-free findings, and the results, which are model-specific.

Proof of Proposition 1

Proof. We start from the domestic investor's Euler equation for the foreign risk-free asset, and the foreign investor's Euler equation for the domestic risk-free asset respectively:

$$\begin{aligned} E_t\left(\widehat{M}^*_{t+1}\right) &= E_t\left(M_{t+1}\frac{S_{t+1}}{S_t}\right) = E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = 1/R^{f,*}_t, \\ E_t\left(M_{t+1}\right) &= E_t\left(M^*_{t+1}\frac{S_t}{S_{t+1}}\right) = E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = 1/R^{f,}_t. \end{aligned}$$

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By using conditional joint log normality of the foreign SDF and $exp(\eta)$, the first Euler equation implies that:

$$\begin{aligned} E_t\left(\log M^*_{t+1}\right)+\frac{1}{2}Var_t\left(\log M^*_{t+1}\right) &= E_t\left(\log M^*_{t+1}\right)+\mu_{t,\eta}+\frac{1}{2}Var_t\left(\log M^*_{t+1}\right) \\ &+ \frac{1}{2}Var_t\left(\eta_{t+1}\right)+covar_t(\eta_{t+1},\log M^*_{t+1}), \end{aligned}$$

where $\mu_{t,\eta}=E_t\left(\eta_{t+1}\right)$. This implies that $covar_t\left(m^*_{t+1},\eta_{t+1}\right)=-\mu_{t,\eta}-0.5var_t\left(\eta_{t+1}\right)$. We move on to the second equation. The second Euler equation for the domestic risk-free asset implies that:

$$\begin{aligned} E_t\left(\log M_{t+1}\right)+\frac{1}{2}Var_t\left(\log M_{t+1}\right) &= E_t\left(\log M_{t+1}\right)-\mu_{t,\eta}+\frac{1}{2}Var_t\left(\log M_{t+1}\right) \\ &+ (1/2)Var_t\left(\eta_{t+1}\right)-covar_t(\eta_{t+1},\log M_{t+1}). \end{aligned}$$

This implies that $covar_t\left(m_{t+1},\eta_{t+1}\right)=-\mu_{t,\eta}+0.5var_t\left(\eta_{t+1}\right)$.

The inequality restrictions on $\mu_{t,\eta}$ follow directly from the Cauchy-Schwarz inequality for (1) $|covar_t\left(m^*_{t+1},\eta_{t+1}\right)|\le std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right)$ and (2) $|covar_t\left(m_{t+1},\eta_{t+1}\right)|\le std_t\left(m_{t+1}\right)std_t\left(\eta_{t+1}\right)$. Finally, we also impose that (3):

$$|covar_t\left(m^*_{t+1}-m_{t+1},\eta_{t+1}\right)|\le std_t\left(m^*_{t+1}-m_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta}\le-(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$-(\mu_{t,\eta}+\frac{1}{2}var_t\left(\eta_{t+1}\right))\le std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$-(\mu_{t,\eta})\le std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right)+\frac{1}{2}var_t\left(\eta_{t+1}\right)).$$

When $\mu_{t,\eta}\ge-(1/2)var_t\left(\eta_{t+1}\right)$, the first inequality implies that:

$$\mu_{t,\eta}+\frac{1}{2}var_t\left(\eta_{t+1}\right)\le std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta}\le std_t\left(m^*_{t+1}\right)std_t\left(\eta_{t+1}\right)-\frac{1}{2}var_t\left(\eta_{t+1}\right).$$

Next, we turn to the second inequality. When $\mu_{t,\eta}\ge(1/2)var_t\left(\eta_{t+1}\right)$, the second inequality implies that:

$$\mu_{t,\eta}-\frac{1}{2}var_t\left(\eta_{t+1}\right)\le std_t\left(m_{t+1}\right)std_t\left(\eta_{t+1}\right).$$

This in turn implies that:

$$\mu_{t,\eta}\le std_t\left(m_{t+1}\right)std_t\left(\eta_{t+1}\right)+\frac{1}{2}var_t\left(\eta_{t+1}\right).$$

When $\mu_{t,\eta} \leq (1/2) var_t(\eta_{t+1})$, the second inequality implies that:

$$-(\mu_{t,\eta} - \frac{1}{2} var_t(\eta_{t+1})) \leq std_t(m_{t+1})\, std_t(\eta_{t+1}).$$

This in turn implies that:

$$-\mu_{t,\eta} \leq std_t(m_{t+1})\, std_t(\eta_{t+1}) - \frac{1}{2} var_t(\eta_{t+1}).$$

Finally, the third inequality implies that:

$$std_t(\eta_{t+1}) \leq std_t\left(m^*_{t+1} - m_{t+1}\right).$$

□

Proof of Corollary 1

Proof. We start from the definition of log changes in exchange rates: $var_t(\Delta s_{t+1}) = var_t(\eta_{t+1} + m^*_{t+1} - m_{t+1})$. This can be simplified to:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) + var_t(\eta_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- 2cov_t(m_{t+1}, \eta_{t+1}) + 2cov_t(\eta_{t+1}, m^*_{t+1}). \end{aligned}$$

Proposition 1 implies that:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m_{t+1}) + var_t(m^*_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}) \\ &- var_t(\eta_{t+1}) - var_t(\eta_{t+1}) + var_t(\eta_{t+1}), \end{aligned}$$

which establishes the result. Finally, we prove the volatility results. The volatility of the log pricing kernel in the foreign country is given by

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2covar_t(m^*_{t+1}, \eta_{t+1}).$$

The result follows directly from the covariance condition. Note that $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - \frac{1}{2} var_t(\eta_{t+1})$.

$$var_t\left(m^*_{t+1} + \eta_{t+1}\right) = var_t(m^*_{t+1}) + var_t(\eta_{t+1}) + 2(-\mu_{t,\eta} - \frac{1}{2} var_t(\eta_{t+1})).$$

□

Proof of Corollary 2

Proof. The expression for the log risk premium follows because $covar_t\left(m^*_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta} - var_t(\eta_{t+1})/2$. The expression for the risk premium in level follows because $var_t[rx^{FX}_{t+1}]/2 = var_t(\Delta s_{t+1})/2$ which is given by:

$$\frac{1}{2} var_t(m_{t+1}) + \frac{1}{2} var_t(m^*_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) - \frac{1}{2} var_t(\eta_{t+1}).$$

The log risk premium is increased by $\mu_{t,\eta}$ relative to the complete markets case. The foreign investor's log risk premium on domestic currency is naturally the opposite of the one above. The symmetry does not hold in levels because of the usual Jensen term. The foreign investor's risk premium in levels on a long position in domestic currency is given by:

$$E_t[rx_{t+1}^{FX}]+\frac{1}{2}var_t[rx_{t+1}^{FX}] = cov_t(m_{t+1}^*, \Delta s_{t+1}) = var_t\left(m_{t+1}^*\right) - covar_t\left(m_{t+1}^*, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - \mu_{t,\eta}.$$

$\square$

Proof of Corollary 3

Proof. This result follows immediately from Proposition 1. We subtract the second $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta}+0.5var_t\left(\eta_{t+1}\right)$ from the first covariance condition $covar_t\left(m_{t+1}, \eta_{t+1}\right) = -\mu_{t,\eta}+0.5var_t\left(\eta_{t+1}\right)$. That delivers the results. $\square$

Proof of Proposition 2

Proof. By definition, the conditional entropy of a random variable X_{t+1} is equal to:

$$L_t(X_{t+1}) = \log E_t(X_{t+1}) - E_t(\log X_{t+1})$$

We assume here that both investors have access to risk-free rates. Let us start again from the Euler equation of the foreign investor:

$$\frac{1}{R_t^{f,*}} = E_t\left(M_{t+1}^* \exp(\eta_{t+1})\right)$$

Taking logs leads to:

$$-r_t^{f,*} = \log E\left(M_{t+1}^* \exp(\eta_{t+1})\right) = L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) + E_t\left(\log M_{t+1}^*\right) + E_t(\eta_{t+1}).$$

But the risk-free rate also satisfies the Euler equation $E\left(M_{t+1}^* R_t^{f,*}\right) = 1$. Taking logs again leads to:

$$\log E\left(M_{t+1}^* R_t^{f,*}\right) = L\left(M_{t+1}^* R_t^{f,*}\right) + E_t\left(\log M_{t+1}^*\right) + r_t^{f,*} = 0$$

Plugging the implied value of the log risk-free rate in the first equation above delivers the result, noting that $L_t(a_t X_{t+1}) = L_t(X_{t+1})$ for any variable a_t known at date t:

$$L\left(M_{t+1}^*\right) + E_t\left(\log M_{t+1}^*\right) = L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) + E_t\left(\log M_{t+1}^*\right) + E_t(\eta_{t+1}),$$

which simplifies to:

$$L_t\left(M_{t+1}^* \exp(\eta_{t+1})\right) = L\left(M_{t+1}^*\right) - E_t(\eta_{t+1}).$$

Likewise, one can show that:

$$L_t\left(M_{t+1} \exp(-\eta_{t+1})\right) = L\left(M_{t+1}\right) + E_t(\eta_{t+1}).$$

Finally, we derive restrictions the set of feasible $\mu_{t,\eta}$ from non-negativity of $L_t\left(M_{t+1}\exp(-\eta_{t+1})\right)$, $L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right)$ and $L_t\left(\frac{S_{t+1}}{S_t}\right)$. To start, note that:

$$L_t\left(M_{t+1}\exp(-\eta_{t+1})\right) = \log E_t\left(M_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right) + E_t(\eta_{t+1}) \geq 0$$

$$L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) = \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right) - E_t(\eta_{t+1}) \geq 0$$

This implies that the following restrictions need to be satisfied:

$$-\mu_{t\eta} \leq \log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right).$$

$$\mu_{t\eta} \leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right),$$

which in turn implies that:

$$-\left(\log E_t\left(M_{t+1}\exp(-\eta_{t+1})\right) - E_t\log\left(M_{t+1}\right)\right) \leq \mu_{t\eta} \leq \log E_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - E_t\log\left(M^*_{t+1}\right)$$

Finally, we also know that

$$L_t\left(\frac{S_{t+1}}{S_t}\right) = -E_t(\eta_{t+1}) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t\log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq 0$$

This, in turn, implies that:

$$\log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) - E_t\log\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \geq \mu_{t\eta}$$

□

Proof of Corollary 4

Proof. Note that the entropy of the ratio of two random variables is:

$$\begin{aligned}
L_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) &= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) - E_t(\log X_{t+1}) + E_t(\log Y_{t+1}) \\
&= \log E_t\left(\frac{X_{t+1}}{Y_{t+1}}\right) + L_t(X_{t+1}) - \log E_t(X_{t+1}) - L(Y_{t+1}) + \log E_t(Y_{t+1}).
\end{aligned}$$

By applying this fomula to the following expression with $X_{t+1} = M^*_{t+1}/M_{t+1}$ and $Y_{t+1} = M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]$, we obtain

$$\begin{aligned}
L_t\left(e^{-\eta_{t+1}}\right) &= L_t\left(\frac{M^*_{t+1}/M_{t+1}}{M^*_{t+1}/[M_{t+1}e^{-\eta_{t+1}}]}\right) \\
&= L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) + \log E_t\left(\frac{M^*_{t+1}}{M_{t+1}e^{-\eta_{t+1}}}\right),
\end{aligned}$$

This last step leads to the result in the text as the second term is the entropy of the change in exchange rates.

□

Proof of Corollary 5

Proof. The first result just follows from the definition of the log change in the exchange rate and the definition of the risk-free rate at home and abroad. The second result follows immediately because $E_t[rx_{t+1}^{FX}] + L_t(rx_{t+1}^{FX}) = E_t[rx_{t+1}^{FX}] + L_t(S_{t+1}/S_t)$; only S_{t+1}/S_t is random.

$$\begin{aligned}
\Delta RP &= RP^{IM} - RP^{CM} = -L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) + \mu_{t,\eta} + L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) \\
&= -L_t\left(e^{-\eta_{t+1}}\right) + \log E_t\left(e^{-\eta_{t+1}}\right) - \log E_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) + \log E_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) + \mu_{t,\eta} \\
&= -\log E_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) + \log E_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) = \Delta L + \mu_{t,\eta}.
\end{aligned}$$

The second line uses the entropy of a ratio of two random variables. □

Proof of Result 1

Proof. We start from the complete market benchmark. The conditional entropy of the pricing kernel M_{t+1} is equal to:

$$\begin{aligned}
L_t(M_{t+1}) &= L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}}\right) \\
&= \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi\theta.
\end{aligned}$$

The entropy of the jump component is presented in Equation (24), page 1981 of Backus, Chernov, and Zin (2011) and derived in their Appendix A. The entropy of the 'complete spanning' exchange rate is given by:

$$\begin{aligned}
L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) &= L_t\left(e^{-\gamma(\Delta c_{t+1}^* - \Delta c_{t+1})}\right) = L_t\left(e^{-\gamma w_{t+1}^*}\right) + L_t\left(e^{-\gamma z_{t+1}^* + \gamma z_{t+1}}\right) + L_t\left(e^{\gamma w_{t+1}}\right), \\
&= \frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta^* + \gamma\theta - \gamma\gamma^*\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right) + \gamma^*\varpi\theta^* - \gamma\varpi\theta.
\end{aligned}$$

The log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned}
E_t\left[rx_{t+1}^{FX}\right] &= -L_t(M_{t+1}^*) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c_{t+1}^*}) + L_t(e^{-\gamma\Delta c_{t+1}}), \\
&= -L_t(e^{-\gamma w_{t+1}^*}) - L_t(e^{-\gamma z_{t+1}^*}) + L_t(e^{-\gamma w_{t+1}}) + L_t(e^{-\gamma z_{t+1}}), \\
&= -\frac{\gamma^{2,*}\sigma^{*,2}}{2} - \varpi\left(e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} - 1\right) \\
&+ \frac{\gamma^2\sigma^2}{2} + \varpi\left(e^{-\gamma\theta + (\gamma\delta)^2/2} - 1\right) - (\gamma^*\varpi\theta^* - \gamma\varpi\theta).
\end{aligned}$$

Hence, the foreign currency risk premium in levels is given by:

$$\begin{aligned}
E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right] &= \gamma^2\sigma^2 + \varpi\left(e^{-\gamma\theta + (\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta^* + (\gamma\delta^*)^2/2} - 1\right) \\
&+ \varpi\left(e^{-\gamma\theta^* + \gamma\theta - 2\gamma\gamma^*\rho_{z,z^*}\delta\delta^* + (\gamma\delta)^2/2 + (\gamma\delta^*)^2/2} - 1\right).
\end{aligned}$$

Next, we introduce incomplete spanning as described in the main text. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned} L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c_{t+1}-\gamma e_{t+1}}\right) = L_t\left(e^{-\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right), \\ &= \gamma^2\sigma^2/2 + \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi(\theta+\theta_d) \end{aligned}$$

The entropy of the sum of two Poisson mixtures ($L_t\left(e^{-\gamma z_{t+1}-\gamma d_{t+1}}\right)$ above) is a generalization of the result presented in Backus, Chernov, and Zin (2011). The co-entropy condition in Proposition 2, $\mu_{t,\eta} = L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right)$, implies here that:

$$\begin{aligned} \gamma\varpi\theta_d &= L_t\left(M_{t+1}e^{-\eta_{t+1}}\right) - L\left(M_{t+1}\right) \\ &= \varpi\left(e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,e}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - 1\right) - \varpi\left(e^{-\gamma\theta+(\gamma\delta)^2/2} - 1\right) + \gamma\varpi\theta_d. \end{aligned}$$

Simplifying, we obtain:

$$0 = e^{-\gamma(\theta+\theta_d)+\gamma^2\delta\delta_d\rho_{z,d}+(\gamma\delta_d)^2/2+(\gamma\delta)^2/2} - e^{-\gamma\theta+(\gamma\delta)^2/2}.$$

This leads to:

$$-\gamma(\theta+\theta_d) + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 + (\gamma\delta)^2/2 = -\gamma\theta + (\gamma\delta)^2/2.$$

This is equivalent to the following restriction on the wedge:

$$-\gamma\theta_d + \gamma^2\delta\delta_d\rho_{z,d} + (\gamma\delta_d)^2/2 = 0.$$

Next, we turn to the foreign pricing kernel. The conditional entropy of the perturbed pricing kernel is equal to:

$$\begin{aligned} L_t\left(M^*_{t+1}e^{\eta_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c^*_{t+1}+\gamma d_{t+1}}\right) = L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t(e^{-\gamma z_{t+1}+\gamma d_{t+1}}) \\ &= \gamma^2\sigma^{2,*}/2 + \varpi^*\left(e^{-\gamma(\theta^*-\theta^*_e)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+(\gamma\delta^*_d)^2/2+(\gamma\delta^*)^2/2} - 1\right) + \gamma\varpi(\theta^*) - \gamma\varpi(\theta_d) \end{aligned}$$

The co-entropy condition in Proposition 2, $-\mu_{t,\eta} = L_t\left(M^*_{t+1}\exp(\eta_{t+1})\right) - L\left(M^*_{t+1}\right)$, implies here that:

$$\left[1 - e^{\gamma\theta_d-\gamma^2\delta\delta_d\rho_{z^*,d}+(\gamma\delta_d)^2/2)}\right]\varpi e^{-\gamma\theta^*+(\gamma\delta^*)^2/2} = 0.$$

This is equivalent to the following condition:

$$\gamma\theta_d - \gamma^2\delta^*\delta_d\rho_{z^*,d} + (\gamma\delta_d)^2/2 = 0.$$

Collecting all of the no-arbitrage restrictions, we obtain the conditions first described in Result 1:

$$\begin{aligned} -\gamma\theta_e + \gamma^2\delta\delta_e\rho_{z,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma\theta_e - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0 \\ \gamma^2\delta\delta_e\rho_{z,e} - \gamma^2\delta^*\delta_e\rho_{z^*,e} + (\gamma\delta_e)^2/2 &= 0. \end{aligned}$$

The third condition is implied by the first two conditions.

We turn now to the entropy of the exchange rate. When markets are incomplete, the exchange rate's entropy is given by:

$$
\begin{aligned}
L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) &= L_t\left(e^{-\gamma\Delta c^*_{t+1}+\gamma d_{t+1}+\gamma\Delta c_{t+1}}\right), \\
&= L_t\left(e^{-\gamma w^*_{t+1}}\right) + L_t\left(e^{\gamma w_{t+1}}\right) + L_t\left(e^{-\gamma z^*_{t+1}+\gamma z_{t+1}+\gamma d_{t+1}}\right), \\
&= \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} + \gamma^*\varpi^*\theta^* - \gamma\varpi\theta - \gamma\varpi\theta_d \\
&\quad +\varpi\left(e^{\gamma(\theta+\theta_d-\theta^*)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta_d)^2}{2}+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right).
\end{aligned}
$$

The entropy gap between the complete and incomplete spanning exchange rate is thus:

$$
\begin{aligned}
L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= \varpi\left(e^{\gamma(\theta+\theta_d-\theta^*)-\gamma^2\delta^*\delta_d\rho_{z^*,d}+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta_d)^2}{2}+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right) \\
&- \gamma\varpi\theta_d - \varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+(\gamma\delta)^2/2+(\gamma\delta^*)^2/2}-1\right)
\end{aligned}
$$

Using the no-arbitrage condition on the wedges $\gamma\theta_d = \gamma^2\delta^*\delta_d\rho_{z^*,d} - (\gamma\delta_d)^2/2 = 0$, we obtain the following result:

$$
\begin{aligned}
L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) &= \varpi\left(e^{\gamma(\theta-\theta^*)+\gamma^2\delta\delta_d\rho_{z,d}-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right) \\
&- \gamma\varpi\theta_d\varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}-1\right).
\end{aligned}
$$

This can be restated as :

$$
\begin{aligned}
\Delta L_t = L_t^{IM} - L_t^{CM} &= L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M^*_{t+1}}{M_{t+1}}\right) \\
&= -\gamma\varpi\theta_d + \varpi\left(e^{-\gamma\theta^*+\gamma\theta-\gamma^2\rho_{z,z^*}\delta\delta^*+\frac{(\gamma\delta)^2}{2}+\frac{(\gamma\delta^*)^2}{2}}\right)(e^{\gamma^2\delta\delta_d\rho_{z,d}}-1).
\end{aligned}
$$

This is the second part of Result 1. Taking into account the no-arbitrage conditions on the wedge, when the wedge does not have a drift ($\theta_d = 0$) and the two countries share the same parameters ($\theta = \theta^*$, $\delta = \delta^*$), we obtain:

$$
\Delta L_t = \varpi\left(e^{-\gamma^2\rho_{z,z^*}\delta^2+(\gamma\delta)^2}\right)(e^{(-\gamma^2\delta_e^2}-1) < 0.
$$

Finally, we turn to the risk premium in levels on a long position in foreign currency, which is given by :

$$
E_t\left[rx^{FX}_{t+1}\right] + L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right) - L_t\left(M^*_{t+1}\right) + \mu_{t,\eta} + L_t\left(\frac{M^*_{t+1}e^{\eta_{t+1}}}{M_{t+1}}\right).
$$

Hence, the change in the risk premium from complete to incomplete spanning is given by the change in entropy, $L_t^{IM} - L_t^{CM}$, plus the drift term: $\gamma\varpi\theta_d$. As a result, the change in the risk premium is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \varpi \left(e^{-\gamma\theta^* + \gamma\theta - \gamma^2 \rho_{z,z^*} \delta\delta^* + \frac{(\gamma\delta)^2}{2} + \frac{(\gamma\delta^*)^2}{2}} \right) (e^{\gamma^2 \delta\delta_d \rho_{z,d}} - 1).$$

This is the third part of Result 1. $\square$

Proof of Result 2

Proof. We need to implement the following conditions:

$$\begin{aligned} covar_t\left(m^*_{t+1}, \eta_{t+1}\right) &= -\mu_{t,\eta} - \frac{1}{2} var_t\left(\eta_{t+1}\right), \\ covar_t\left(m_{t+1}, \eta_{t+1}\right) &= -\mu_{t,\eta} + \frac{1}{2} var_t\left(\eta_{t+1}\right), \end{aligned}$$

Using the expression for the SDF, we obtain the following conditions:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)} z_t^* &= -(\psi z_t + \psi^* z_t^*) - \frac{1}{2}\left((\gamma - \kappa) z_t + (\gamma^* - \kappa^*) z_t^*\right), \\ +\sqrt{\gamma}\sqrt{(\gamma - \lambda)} z_t &= -(\psi z_t + \psi^* z_t^*) + \frac{1}{2}\left((\gamma - \kappa) z_t + (\gamma^* - \kappa^*) z_t^*\right). \end{aligned}$$

These conditions imply that:

$$\begin{aligned} \psi^* &= \frac{1}{2}(\gamma^* - \kappa^*), \\ \psi &= -\frac{1}{2}(\gamma - \kappa). \end{aligned}$$

as well as:

$$\begin{aligned} -\sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)} &= -\psi^* - \frac{1}{2}\left(\gamma^* - \kappa^*\right) = -\left(\gamma^* - \kappa^*\right), \\ +\sqrt{\gamma}\sqrt{(\gamma - \lambda)} &= -\psi + \frac{1}{2}\left(\gamma - \kappa\right) = \left(\gamma - \kappa\right), \end{aligned}$$

where we have used the expressions for the ψ's. This delivers the following end result:

$$\begin{aligned} \gamma^* - \sqrt{\gamma^*}\sqrt{(\gamma^* - \lambda^*)} &= \kappa^*, \\ \gamma - \sqrt{\gamma}\sqrt{(\gamma - \lambda)} &= \kappa. \end{aligned}$$

$\square$

Proof of Result 3

Proof. The risk premium in logs on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] &= r^{f,*}_t - r^f_t + E_t(\Delta s_{t+1}) = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1} + \eta_{t+1}\right)\right] \\ &= \frac{1}{2}\left[(\gamma + 2\psi)z_t - (\gamma^* - 2\psi^*)z^*_t\right]. \\ &= \frac{1}{2}\left[(\gamma - (\gamma - \kappa))z_t - (\gamma^* - (\gamma^* - \kappa^*))z^*_t\right] \\ &= \frac{1}{2}\left[\kappa z_t - \kappa^* z^*_t\right] \end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned} E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(m_{t+1}, \Delta s_{t+1}) \\ &= \frac{1}{2}\left[(\kappa + \kappa)z_t - (\kappa^* - \kappa^*)z^*_t\right] \\ &= \kappa z_t \end{aligned}$$

Recall that the short rate is given by: $r_t = \alpha + (\chi - \frac{1}{2}\gamma)z_t$. Hence, the regression slope coefficient on $r_t - r^*_t$ is

$$\frac{cov(rx^{FX}_{t+1}, r_t - r^*_t)}{var(r_t - r^*_t)} = \frac{.5\kappa(\chi - \frac{1}{2}\gamma) + .5\kappa^*(\chi - \frac{1}{2}\gamma^*)}{(\chi - \frac{1}{2}\gamma)^2 + (\chi - \frac{1}{2}\gamma^*)^2}$$

Hence, in the symmetric case, we end up with:

$$\frac{.5\kappa}{(\chi - \frac{1}{2}\gamma)}$$

□

2 Quantitative Implications in Asymmetric Models

In this section, we study the case of asymmetric models, where the volatilities of the log home and foreign SDFs differ. In the main text, we assume that $std_t(m_{t+1}) = std_t(m^*_{t+1}) = 0.5$. In this appendix, we assume that $std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$. Since the average volatility of the two SDFs is the same as in the benchmark case, the volatility of the wedge needed to match the empirical volatility of the exchange rates is also the same as in the main text. We thus focus on the currency risk premium and the exchange rate cyclicality.

Figure 2 plots the theoretical currency risk premium in logs and levels and its empirical counterpart. The parameters are identical to those in Figure 1 in the main text, matching an exchange rate volatility of 11%. The currency risk premia are plotted against the first moment of the wedge, $E_t(\eta_{t+1})$. The key difference with the main text is that the complete market model delivers a large currency risk premium, since in that case

$$E_t[rx^{FX}_{t+1}] = \frac{1}{2}\left[var_t\left(m_{t+1}\right) - var_t\left(m^*_{t+1}\right)\right] = 4\%.$$

One does not need to add any exchange rate predictability (through the first moment of the wedge) in order to match the currency risk premium in our sample. Matching a larger currency risk premium would call for more asymmetry in the volatilities of m_{t+1} and m^*_{t+1} because the range of permissible drifts is limited.

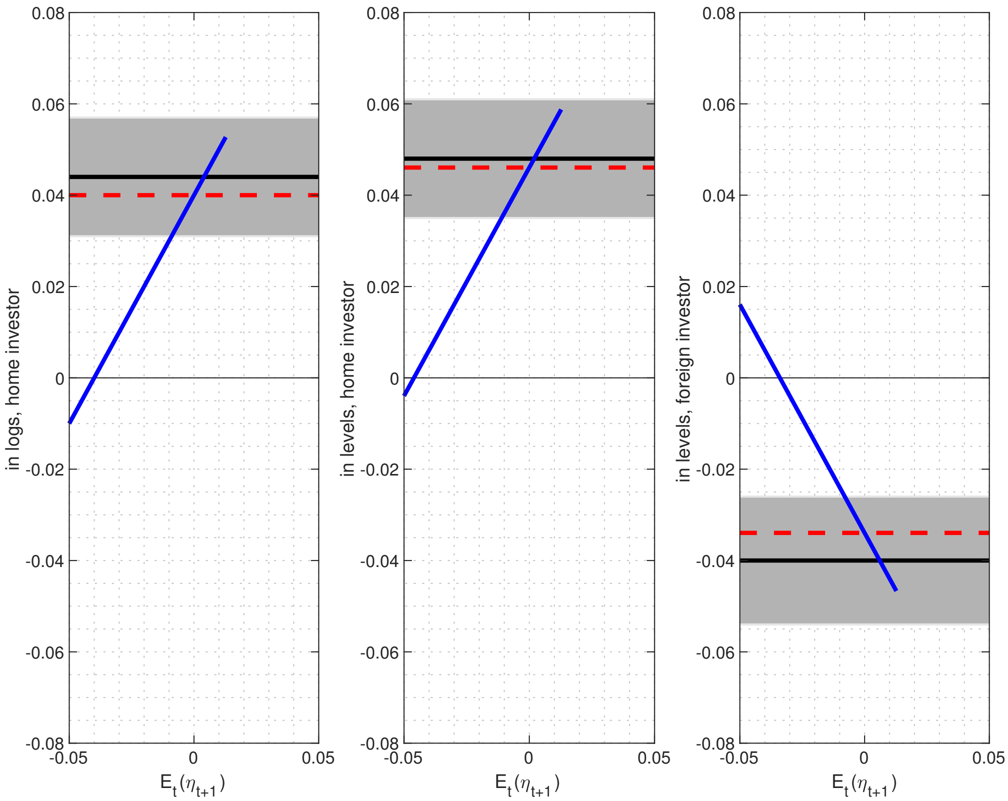

Figure 1: Currency Risk Premia — Asymmetric Case: The figure reports the foreign currency risk premium in logs (left panel), as well as in levels, from the perspective of the home investor (center panel) or foreign investor (right panel), against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The figure is drawn assuming a maximum Sharpe ratio of 0.54 and 0.46 in the home and foreign countries ($std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$). The volatility of the wedge, $std_t(\eta_{t+1})$, is chosen to match the empirical volatility of the exchange rate changes (11%). The red dotted line shows each moment in a complete market model with the same SDF volatilities. The gray area indicates the value of the average carry trade excess return in the data: it is centered around the mean log excess return (4.4%, left panel) or the mean excess return from the perspective of the home and foreign investor (4.8% and -4.0% in the center and right panels); the area represents one standard error (1.3%) above and below the mean.

Figure 2 plots different measures of exchange rate cyclicality against the drift of the wedge. The parameters are the same as for Figure 2, where the volatility of the wedge is chosen to match the volatility of the exchange rate changes. The difference with Figure 3 in the main text is twofold. First, when markets are complete, the correlation between the home SDF and the change in exchange rates is even more negative than before: it is now close to -0.8, implying a strong appreciation of the home currency in bad times at home. Second, even when introducing a large drift in the wedge, this correlation is still less than 0.1, and thus never imply a strong depreciation of the home currency in bad times at home. Our key cyclicality result, the cyclicality slope of one, is naturally unchanged.

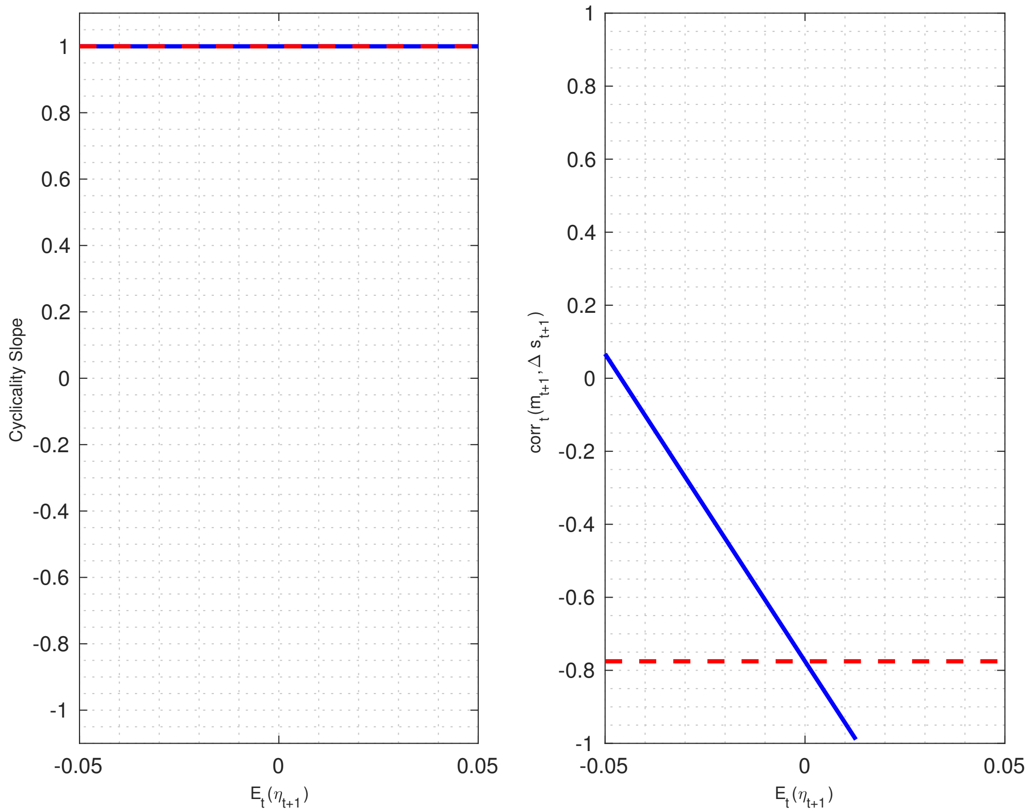

Figure 2: Exchange Rate Cyclicality — Asymmetric Case: The figure reports the slope coefficient in a regression of the difference in log SDFs, $m^*_{t+1} - m_{t+1}$ on the log change in exchange rates (left panel) and the correlation between the log home SDF and the change in the exchange rates, $corr_t(\Delta s_{t+1}, m_{t+1})$, (left panel) against the first moment of the incomplete market wedge, denoted $E_t(\eta_{t+1})$. The red dotted line shows the values of these three moments when markets are complete. The figure is drawn assuming a maximum Sharpe ratio of 0.54 and 0.46 in the home and foreign countries ($std_t(m_{t+1}) = 0.54$ and $std_t(m^*_{t+1}) = 0.46$).

3 Three Examples

This section presents three examples: a simple consumption-based example; a Cox, Ingersoll, and Ross (1985) model with common factors; a consumption-based example with heteroscedasticity.[1]

3.1 A Simple Consumption-Based Example

In this section, we study in detail the consumption-based example that is mentioned rapidly in the main text.

Complete Markets We start from the complete market benchmark. The model is described in the main text.

Result 4. *The complete markets foreign currency risk premium in levels (defined from the perspective of the home investor) is given by:*

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[rx_{t+1}^{FX}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

The proof of Result 4 is as follows.

Proof. The entropy of the domestic pricing kernel is given by:

$$L_t(M_{t+1}) = L_t\left(e^{-\gamma\Delta c_{t+1}}\right) = \frac{\gamma^2\sigma^2}{2}.$$

As a result, the entropy of the exchange rate is:

$$L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = L_t(e^{-\gamma w_{t+1}^* + \gamma w_{t+1}}) = \frac{\gamma^2\sigma^{*2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

When markets are complete, the log currency risk premium is given by the difference in the entropy of the domestic and the foreign pricing kernels:

$$\begin{aligned} E_t\left[rx_{t+1}^{FX}\right] &= -L_t\left(M_{t+1}^*\right) + L_t(M_{t+1}) = -L_t(e^{-\gamma\Delta c_{t+1}^*}) + L_t(e^{-\gamma\Delta c_{t+1}}) \\ &= -L_t(e^{-\gamma w_{t+1}^*}) + L_t(e^{-\gamma w_{t+1}}) = -\frac{\gamma^{2,*}\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2}. \end{aligned}$$

As a result, the currency risk premium in levels (defined from the perspective of the home investor) is given by:

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_{t+1}}{S_t}\right] = \gamma^2\sigma^2 - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

Likewise, the currency risk premium in levels (defined from the perspective of the foreign investor) is given by:

$$-E_t\left[rx_{t+1}^{FX}\right] + L_t\left[\frac{S_t}{S_{t+1}}\right] = \gamma^2\sigma^{*2} - \gamma^2\rho_{w,w^*}\sigma\sigma^*.$$

□

[1] Other examples of multi-country term structure models that rely on the complete market assumption to address the carry trade and forward premium puzzle include Frachot (1996), Hodrick and Vassalou (2002), Brennan and Xia (2006), Graveline and Joslin (2011), Sarno, Schneider and Wagner (2012), and Lustig, Roussanov and Verdelhan (2011, 2014).

Incomplete markets Next, we introduce incomplete spanning. Assume that the wedge takes the form $\eta_{t+1} = \gamma d_{t+1}$, where $d \sim N(\mu_d, \sigma_d^2)$.

Result 5. *The wedge has to satisfy the following conditions:*

$$\begin{aligned} \mu_d &= \frac{\gamma^2 \sigma_d^2}{2} + \rho_{w,d} \gamma^2 \sigma \sigma_d, \\ -\mu_d &= \frac{\gamma^2 \sigma_d^2}{2} - \rho_{w^*,d} \gamma^2 \sigma^* \sigma_d. \end{aligned}$$

The change in exchange rate variance from complete to incomplete spanning is given by:

$$\Delta Var_t = Var_t^{IM} - Var_t^{CM} = -\gamma^2 \sigma_d^2.$$

The change in the currency risk premium (defined from the perspective of the home investor) from complete to incomplete spanning is given by:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \rho_{w,d} \gamma^2 \sigma \sigma_d.$$

The change in the currency risk premium (defined from the perspective of the foreign investor) from complete to incomplete spanning is:

$$\Delta RP_t^* = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d} \gamma^2 \sigma \sigma_d.$$

Result 5 implies that in the symmetric case (when the drift of the wedge is zero), the change in the currency risk premium in level is $\Delta RP_t = \Delta RP_t^* = -.5\gamma^2 \sigma_d^2$. In that case, introducing a wedge decreases the currency risk premium from the perspective of both domestic and foreign agents. The Sharpe ratio declines as well:

$$SR_t^{FX} = \frac{\gamma}{\sqrt{2}} \sqrt{\sigma^2 (1-\rho) - \frac{\sigma_d^2}{2}}.$$

The proof of Result 5 is as follows:

Proof. The conditional entropy of the perturbed home pricing kernel is given by:

$$L_t \left(M_{t+1} e^{-\eta_{t+1}} \right) = L_t (e^{-\gamma \Delta c_{t+1} - \gamma d_{t+1}}) = L_t (e^{-\gamma w_{t+1} - \gamma d_{t+1}}) = \frac{\gamma^2 \sigma^2}{2} + \frac{\gamma^2 \sigma_d^2}{2} + \rho_{w,d} \gamma^2 \sigma \sigma_d.$$

Applying Proposition 5, it then implies that the drift of the wedge satisfies:

$$\mu_e = \frac{\gamma^2 \sigma_d^2}{2} + \rho_{w,d} \gamma^2 \sigma \sigma_d.$$

The conditional entropy of the perturbed foreign pricing kernel is equal to:

$$L_t \left(M_{t+1}^* e^{\eta_{t+1}} \right) = L_t (e^{-\gamma \Delta c_{t+1}^* + \gamma d_{t+1}}) = L_t (e^{-\gamma w_{t+1}^* + \gamma d_{t+1}}) = \frac{\gamma^2 \sigma^{2,*}}{2} + \frac{\gamma^2 \sigma_d^2}{2} - \rho_{w^*,d} \gamma^2 \sigma \sigma_d.$$

Proposition 2 then implies that the drift of the wedge satisfies:

$$-\mu_d = \frac{\gamma^2\sigma_d^2}{2} - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

When markets are incomplete, the entropy of the 'incomplete spanning' exchange rate is given by:

$$\begin{aligned} L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) &= L_t(e^{-\gamma\Delta c_{t+1}^* + \gamma d_{t+1} + \gamma\Delta c_{t+1}}) = L_t(e^{-\gamma w_{t+1}^* + \gamma w_{t+1} + \gamma d_{t+1}}) \\ &= \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\sigma^*\sigma_d\rho_{w^*,d} + \gamma^2\sigma\sigma_d\rho_{w,d} - \gamma^2\rho_{w,w^*}\sigma\sigma^* + \frac{\gamma^2\sigma_d^2}{2}. \end{aligned}$$

By summing the two conditions that define the drift of the wedge, one obtains that:

$$0 = \gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d - \rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

The entropy of the 'incomplete spanning' exchange rate is thus simply:

$$L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) = \frac{\gamma^2\sigma^{*,2}}{2} + \frac{\gamma^2\sigma^2}{2} - \gamma^2\rho_{w,w^*}\sigma\sigma^* - \frac{\gamma^2\sigma_d^2}{2}.$$

The entropy gap between the complete and incomplete spanning exchange rate is then:

$$\Delta L_t = L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = -\frac{\gamma^2\sigma_d^2}{2}.$$

According to Proposition 5, the risk premium in levels on a long position in foreign currency is given by :

$$E_t\left[rx_{t+1}^{FX}\right] + L_t\left(\frac{S_{t+1}}{S_t}\right) = L_t\left(M_{t+1}\right) - L_t\left(M_{t+1}^*\right) + \mu_{t,\eta} + L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right).$$

The change in the risk premium from complete to incomplete spanning is thus given by the change in entropy, $-.5\gamma^2\sigma_d^2$, plus the drift term, $\mu_d = 0.5\gamma^2\sigma_d^2 + \rho_{w,d}\gamma^2\sigma\sigma_d$. The difference between the risk premium in incomplete and complete markets is:

$$\Delta RP_t = RP_t^{IM} - RP_t^{CM} = \mu_{t,\eta} + L_t\left(\frac{M_{t+1}^* e^{\eta_{t+1}}}{M_{t+1}}\right) - L_t\left(\frac{M_{t+1}^*}{M_{t+1}}\right) = \rho_{w,d}\gamma^2\sigma\sigma_d.$$

Similarly, the foreign risk premium in levels on a long position in foreign currency is given by :

$$-E_t\left[rx_{t+1}^{FX}\right] + L_t\left(\frac{S_t}{S_{t+1}}\right) = -L_t\left(M_{t+1}\right) + L_t\left(M_{t+1}^*\right) - \mu_{t,\eta} + L_t\left(\frac{M_{t+1}}{M_{t+1}^* e^{\eta_{t+1}}}\right).$$

The change in the foreign risk premium from complete to incomplete spanning is given by:

$$\Delta RP_t^* = RP_t^{*IM} - RP_t^{*CM} = -\rho_{w^*,d}\gamma^2\sigma\sigma_d.$$

Proposition 2 implies that the restrictions on the wedges are given by:

$$\begin{aligned} \mu_d &= \gamma^2\sigma_d^2/2 + \rho_{w,d}\gamma^2\sigma\sigma_d, \\ -\mu_d &= \gamma^2\sigma_e^2/2 - \rho_{w^*,d}\gamma^2\sigma^*\sigma_d. \end{aligned}$$

□

3.2 A Cox, Ingersoll, and Ross (1985) Example with Common Factors

The stylized model in the main text rules out correlation of interest rates across countries. However, the key insights carry over to a setting with correlated interest rates. To show this result, we use a CIR model with common factors. The Cox, Ingersoll and Ross (1985) model (denoted CIR) is defined by the following two equations:

$$\begin{aligned} -\log M_{t+1} &= \alpha + \chi z_t + \varphi z_t^* + \sqrt{\gamma z_t} u_{t+1} + \sqrt{\delta z_t^*} u_{t+1}^*, \qquad (1) \\ z_{t+1} &= (1-\phi)\theta + \phi z_t - \sigma \sqrt{z_t} u_{t+1}, \\ z_{t+1}^* &= (1-\phi)\theta + \phi z_t - \sigma \sqrt{z_t^*} u_{t+1}^*, \qquad (2) \end{aligned}$$

where $u_{t+1} \sim \mathbb{N}(0,1)$ and $u_{t+1}^* \sim \mathbb{N}(0,1)$ are i.i.d. The foreign pricing kernel is specified as in Equation (1) with the same parameters. However, the foreign country has different loadings:

$$-\log M_{t+1} = \alpha + \chi^* z_t + \varphi^* z_t^* + \sqrt{\gamma^* z_t} u_{t+1} + \sqrt{\delta^* z_t^*} u_{t+1}^*.$$

To give content to the notion that z_t is a domestic factor and z_t^* is a foreign factor, we assume that $\gamma \geq \gamma^*$ and that $\delta \leq \delta^*$: the domestic (foreign) pricing kernel is more exposed to the domestic (foreign) shock than the foreign (domestic) pricing kernel. We assume that investors can trade the domestic risk-free and at least two risky domestic assets[2], but they can only trade the foreign risk-free asset. The squared maximum SRs at home and abroad are, respectively, $var_t(m_{t+1}) = \gamma z_t + \delta z_t^*$, and $var_t(m_{t+1}^*) = \gamma^* z_t + \delta^* z_t^*$.

We denote the target volatility of the incomplete markets exchange rate can be stated as: $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. We can compute the implied volatility of the incomplete markets exchange rate process using our formula:

$$var_t(\Delta s_{t+1}) = (\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}) z_t + (\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}) z_t^* - var_t(\eta_{t+1}).$$

Then we simply choose the volatility of the noise to be equal to: $var_t(\eta_{t+1}) = (\gamma + \gamma^* - \kappa) z_t + (\delta + \delta^* - \kappa^*) z_t^*$.

Result 6. *In the CIR model with country-specific factors, we can define an exchange rate process S_t that satisfies $\Delta s_{t+1} = \eta_{t+1} + m_{t+1}^* - m_{t+1}$ with variance $var_t(\Delta s_{t+1}) = \kappa z_t + \kappa^* z_t^*$. where η_t follows:*

$$\begin{aligned} \eta_{t+1} &= \beta + \psi z_t + \psi^* z_t^* - \sqrt{(\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda) z_t} u_{t+1} + \sqrt{(\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*) z_t^*} u_{t+1}^* \\ &+ \sqrt{(\lambda - \kappa) z_t} \epsilon_{t+1} + \sqrt{(\lambda^* - \kappa^*) z_t^*} \epsilon_{t+1}^*, \end{aligned}$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^* \leq \lambda^* \leq \delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*}$. The drift imputed to exchange rates is given by $\mu_{t,\eta} = \beta + \psi z_t + \psi^* z_t^*$. where ϵ_{t+1} and ϵ_{t+1}^* are

[2] If they can trade two different longer maturity bonds, then the domestically traded assets span all of the shocks.

$\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ *and* $\kappa^* \leq \lambda^* \leq \gamma^*$ *satisfies:*

$$
\begin{aligned}
\kappa &= -(\sqrt{\gamma}+\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda))}+\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}, \\
\kappa^* &= -(\sqrt{\delta}+\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*))}+\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}, \\
\psi &= -(1/2)(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}, \\
\psi^* &= -(1/2)(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}.
\end{aligned}
$$

If we allowed domestic investors to trade two foreign risky assets, then the wedges disappear. The additional covariance restrictions in Equation (13) imply that $\kappa=\lambda=\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}$ and $\kappa^*=\lambda^*=\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}$, because the log returns are affine in the shocks. This in turn implies that the wedges are zero ($\eta=0$).

The proof of Result 6 is as follows:

Proof. Hence, we can write a square root process for η:

$$
\begin{aligned}
\eta_{t+1} &= \beta+\psi z_t+\psi^* z_t^*-\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)z_t}u_{t+1}+\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)z_t^*}u_{t+1}^* \\
&+ \sqrt{(\lambda-\kappa)z_t}\epsilon_{t+1}+\sqrt{(\lambda^*-\kappa^*)z_t^*}\epsilon_{t+1}^*,
\end{aligned}
$$

where ϵ_{t+1} and ϵ_{t+1}^* are $\sim N(0,1)$, $\kappa \leq \lambda \leq \gamma$ and $\kappa^* \leq \lambda^* \leq \gamma^*$. The drift imputed to exchange rates is given by $\mu_{t,\eta}=\beta+\psi z_t+\psi^* z_t^*$.

To ensure that the Euler equations for the risk-free are satisfied, we also need to implement the following conditions:

$$
\begin{aligned}
covar_t\left(m_{t+1}^*,\eta_{t+1}\right) &= -\mu_{t,\eta}-\frac{1}{2}var_t\left(\eta_{t+1}\right), \\
covar_t\left(m_{t+1},\eta_{t+1}\right) &= -\mu_{t,\eta}+\frac{1}{2}var_t\left(\eta_{t+1}\right).
\end{aligned}
$$

Using the expressions for the log SDFs and η, these expressions can be restated as follows:

$$
\begin{aligned}
&- \sqrt{\gamma^*}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}z_t-\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}z_t^* \\
&= -(\psi z_t+\psi^* z_t^*)-\frac{1}{2}\left((\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa)z_t+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*)z_t^*\right), \\
&+ \sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}z_t+\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda)}z_t^* \\
&= -(\psi z_t+\psi^* z_t^*)+\frac{1}{2}\left((\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa)z_t+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*)z_t^*\right).
\end{aligned}
$$

By collecting the terms in z_t and z_t^*, we obtain the following four equations that need to be solved for 4 unknowns:

$$
\begin{aligned}
-\sqrt{\gamma^*}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= -(\psi)-\frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa), \\
-\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} &= -(\psi^*)-\frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

$$
\begin{aligned}
+\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= -(\psi)+\frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa) \\
+\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa)} &= -(\psi^*)+\frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

By adding the 1st and 3rd, and the 2nd and 4th equation, we obtain the following expression for the drift terms:

$$
\begin{aligned}
(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda} &= -2\psi), \\
(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*} &= -2\psi^*.
\end{aligned}
$$

By substituting for ψ and ψ^* in the original four equations, we obtain the following conditions:

$$
\begin{aligned}
+\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)} &= +\frac{1}{2}(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+\frac{1}{2}(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\kappa), \\
-\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)} &= +\frac{1}{2}(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+\frac{1}{2}(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\kappa^*).
\end{aligned}
$$

These conditions can be solved for κ and κ^*:

$$
\begin{aligned}
\kappa &= -2\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}), \\
\kappa^* &= +2\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}).
\end{aligned}
$$

These conditions imply that:

$$
\begin{aligned}
\kappa &= -(\sqrt{\gamma}+\sqrt{\gamma^*})\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda))}+\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}, \\
\kappa^* &= -(\sqrt{\delta}+\sqrt{\delta^*})\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*))}+\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}.
\end{aligned}
$$

□

Result 7. *The risk premium in logs on a long position in foreign currency is:*

$$
\begin{aligned}
E_t[rx^{FX}_{t+1}] &= \frac{1}{2}\left[\gamma-\gamma^*-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z^*_t
\end{aligned}
$$

The risk premium in levels on a long position in foreign currency is given by:

$$
\begin{aligned}
E_t[rx^{FX}_{t+1}]+\frac{1}{2}var_t[rx^{FX}_{t+1}] &= \left[\gamma-\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}-\sqrt{\gamma}\sqrt{\gamma^*}\right]z_t \\
&+ \left[\delta-\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}-\sqrt{\delta}\sqrt{\delta^*}\right]z^*_t.
\end{aligned}
$$

These expressions can readily be compared to the complete markets log currency risk premium, $\frac{1}{2}\left[(\gamma-\gamma^*)z_t+(\delta-\delta^*)z_t^*\right]$, and the complete markets risk premium in levels, $(\gamma-\sqrt{\gamma}\sqrt{\gamma^*})z_t+(\delta-\sqrt{\delta}\sqrt{\delta^*})z_t^*$. Clearly, this establishes that the incomplete markets risk premium in levels is always smaller than the complete markets risk premium. In addition, there is less return predictability as well.

The proof of Result 7 is as follows:

Proof. Note that the risk premium in logs is given by

$$\begin{aligned}
E_t[rx_{t+1}^{FX}] &= r_t^{f,*}-r_t^f+E_t(\Delta s_{t+1})=\frac{1}{2}\left[var_t\left(m_{t+1}\right)-var_t\left(m_{t+1}^*+\eta_{t+1}\right)\right] \\
&= \frac{1}{2}\left[(\gamma-\gamma^*+2\psi)z_t+(\delta-\delta^*+2\psi^*)z_t^*)\right] \\
&= \frac{1}{2}\left[\gamma-\gamma^*-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z_t^*
\end{aligned}$$

The risk premium in levels on a long position in foreign currency is given by:

$$\begin{aligned}
E_t[rx_{t+1}^{FX}]+\frac{1}{2}var_t[rx_{t+1}^{FX}] &= -cov_t(m_{t+1},\Delta s_{t+1}) \\
&= \frac{1}{2}\left[(\gamma-\gamma^*+2\psi+\kappa)z_t+(\delta-\delta^*+2\psi^*+\kappa^*)z_t^*)\right] \\
&= \frac{1}{2}\left[\gamma-\gamma^*+\kappa-(\sqrt{\gamma}-\sqrt{\gamma^*})\sqrt{\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda}\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*+\kappa^*-(\sqrt{\delta}-\sqrt{\delta^*})\sqrt{\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*}\right]z_t^*, \\
&= \frac{1}{2}\left[\gamma-\gamma^*-2\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}+(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*})\right]z_t \\
&+ \frac{1}{2}\left[\delta-\delta^*+2\sqrt{\delta^*}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}+(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*})\right]z_t^* \\
&= \left[\gamma-\sqrt{\gamma}\sqrt{(\gamma+\gamma^*-2\sqrt{\gamma}\sqrt{\gamma^*}-\lambda)}-\sqrt{\gamma}\sqrt{\gamma^*}\right]z_t \\
&+ \left[\delta-\sqrt{\delta}\sqrt{(\delta+\delta^*-2\sqrt{\delta}\sqrt{\delta^*}-\lambda^*)}-\sqrt{\delta}\sqrt{\delta^*}\right]z_t^*
\end{aligned}$$

□

Result 8. *The Fama slope coefficient in a regression of log currency excess returns on* $f_t-s_t=$

$r_t - r_t^*$ *is*

$$
\begin{aligned}
& \frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)} \\
& = \frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2} \\
+ & \quad \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}
\end{aligned}
$$

In the relevant region of the parameter space, $(\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*) < 0$ and $(\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*) > 0$. Then the interest rate spread $r_t - r_t^*$ decreases (increases) when z_t increases (z_t^* decreases) –the precautionary motive dominates. This is needed to account for U.I.P. deviations in the data. As a benchmark, we note that the complete markets slope coefficient is given by:

$$
= \frac{.5\,(\gamma)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right) + .5\,(\delta)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}
$$

Recall that $\gamma \geq \gamma^*$ and $\delta \leq \delta^*$. As a result, the first term now decreases in absolute value relative to the complete markets case. The second term decreases as well in absolute value. Even in the model with common factors, the slope coefficients in the predictability regression are pushed closer to zero by the incomplete spanning and we get closer to U.I.P.

The proof of Result 8 is as follows:

Proof. Recall that the short rate is given by: $r_t = \alpha + (\chi - \frac{1}{2}\gamma)z_t + (\phi - \frac{1}{2}\delta)z_t^*$. Hence, the regression slope coefficient on $f_t - s_t = r_t - r_t^*$ is

$$
\begin{aligned}
& \frac{cov(rx_{t+1}^{FX}, f_t - s_t)}{var(f_t - s_t)} = \\
& \frac{.5\left(\gamma - \gamma^* - (\sqrt{\gamma} - \sqrt{\gamma^*})\sqrt{\gamma + \gamma^* - 2\sqrt{\gamma}\sqrt{\gamma^*} - \lambda}\right)\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2} \\
+ & \quad \frac{.5\left(\delta - \delta^* - (\sqrt{\delta} - \sqrt{\delta^*})\sqrt{\delta + \delta^* - 2\sqrt{\delta}\sqrt{\delta^*} - \lambda^*}\right)\left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)}{\left((\chi - \frac{1}{2}\gamma) - (\chi^* - \frac{1}{2}\gamma^*)\right)^2 + \left((\phi - \frac{1}{2}\delta) - (\phi^* - \frac{1}{2}\delta^*)\right)^2}
\end{aligned}
$$

□

3.3 A Consumption-Based Example with Heteroscedasticity

To develop some economic intuition for the dynamics of these wedges, we look at a version of the two-country Lucas (1982) model with heteroskedastic consumption growth. This model produces time-varying risk premia. We use δ to denote the rate of time preference and γ to denote the coefficient of relative risk aversion. The real stochastic discount factor is thus given by:

$$
\begin{aligned}
-\log M_{t+1} &= -(\log\delta - \gamma\mu_g) + \gamma\sigma_{g,t}e_{t+1}, \\
\sigma^2_{g,t} &= (1-\phi)\theta + \phi\sigma^2_{g,t} - \sigma_{g,t}e_{t+1}, \\
-\log M^*_{t+1} &= -(\log\delta - \gamma\mu_g) + \gamma\sigma^*_{g,t}e^*_{t+1}, \\
\sigma^{2,*}_{g,t} &= (1-\phi)\theta + \phi\sigma^{2,*}_{g,t} - \sigma^*_{g,t}e^*_{t+1}.
\end{aligned}
$$

where $\Delta c_{t+1} = \mu_g + \sigma_{g,t}e_{t+1}$, and $\Delta c_{t+1} = \mu_g + \sigma^*_{g,t}e^*_{t+1}$. The consumption growth innovations $e_{t+1} \sim \mathbb{N}(0,1)$ and $e^*_{t+1} \sim \mathbb{N}(0,1)$ are i.i.d. as well as uncorrelated across countries. When markets are complete, the exchange rate variance is thus $var_t(\Delta s_{t+1}) = \gamma^2\sigma^2_{g,t} + \gamma^{2,*}\sigma^{2,*}_{g,t}$. Domestic investors can invest in the domestic risk-free and at least one domestic risky asset (e.g. a longer maturity real zero-coupon bond), and the foreign risk-free, but they cannot invest in foreign risky assets. Hence, only the domestic shocks are spanned.

In this model, we can back out the dynamic process for the wedges that satisfy the necessary conditions of Proposition 1. It turns out that all the wedges take the form:

$$
\begin{aligned}
\eta_{t+1} = \psi\sigma_{g,t} + \psi^*\sigma^*_{g,t} &- \sqrt{(\gamma^2-\lambda)}\sigma_{g,t}e_{t+1} + \sqrt{(\gamma^2-\lambda^*)}\sigma^*_{g,t}e^*_{t+1} \\
&+ \sqrt{(\lambda-\kappa)}\sigma_{g,t}\epsilon_{t+1} + \sqrt{(\lambda-\kappa^*)}\sigma^*_{g,t}\epsilon^*_{t+1}.
\end{aligned}
$$

where ϵ_{t+1} and ϵ^*_{t+1} are standard i.i.d. Gaussian shocks uncorrelated with the consumption growth innovations e_{t+1} and e^*_{t+1}. These shocks are the unspanned component of the exchange rate changes. The parameters κ and κ^* govern the volatility of the exchange rate when markets are incomplete: $var_t(\Delta s_{t+1}) = \kappa\sigma^2_{g,t} + \kappa^*\sigma^{2,*}_{g,t}$. These wedges only affect exchange rates, and as a result, the returns on foreign investments. The returns on domestic investments remain unchanged.

The parameters κ and κ^* are the only two degrees of freedom in the law of motion of the wedge. The other parameters that describe the wedge are implied. The drift term (denoted $\mu_{t,\eta}$ in Proposition 1 and here equal to $\psi\sigma_{g,t} + \psi^*\sigma^*_{g,t}$) is governed by the consumption growth volatilities; it is determined by the no-arbitrage conditions, which imply that $\psi = -\frac{1}{2}(\gamma^2-\kappa)$, and $\psi^* = \frac{1}{2}(\gamma^2-\kappa^*)$. The unexpected component of the wedge depends on the parameters λ and λ^*, which have to satisfy the following restrictions: $\kappa \leq \lambda \leq \gamma^2$ and $\kappa^* \leq \lambda^* \leq \gamma^2$, and are implicitly defined by the following conditions: $\kappa = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2-\lambda}$, $\kappa^* = \gamma^2 - \sqrt{\gamma^2}\sqrt{\gamma^2-\lambda^*}$.

In this example, the domestic investor cannot invest in any foreign risky asset. If we allow the foreign investor to do so, then we need to impose the additional covariance restrictions in condition (13). These conditions imply that η is orthogonal to e_{t+1} and e^*_{t+1}, because the log return on the domestic (foreign) risky asset is affine in the domestic (foreign) innovation, which in turn implies $\kappa = \lambda = \gamma^2$ and $\kappa^* = \lambda^* = \gamma^{*,2}$. We are back in the case of complete markets: $\eta_{t+1} = 0$.

In the two-country Lucas (1982) model, incomplete spanning reduces the exchange rate's exposure to the consumption growth innovations. Instead, the exchange rates are now exposed to shocks uncorrelated with aggregate consumption growth in either country. In the following sections, we study the impact of incomplete spanning on each of the key three exchange rate puzzles without restricting ourselves to the Lucas (1982) model. We start with the Brandt, Cochrane and Santa-Clara (2006) puzzle.

The two-country Lucas (1982) model with heteroskedastic consumption growth provides a simple laboratory for understanding the effects of incompleteness. In that model, the complete markets risk premium in logs on a long position in foreign currency is: $E_t[rx_{t+1}^{FX}] = \frac{1}{2}\gamma^2\left[\sigma_{g,t}^2 - \sigma_{g,t}^{2,*}\right]$, while the complete markets risk premium in levels is given by: $E_t[rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}] = \gamma^2\sigma_{g,t}^2$. In the incomplete spanning economy, the risk premium in logs on a long position in foreign currency is $E_t[rx_{t+1}^{FX}] = \frac{1}{2}\kappa\left[\sigma_{g,t}^2 - \sigma_{g,t}^{2,*}\right]$, while the risk premium in levels on a long position in foreign currency is given by: $E_t[rx_{t+1}^{FX}] + \frac{1}{2}var_t[rx_{t+1}^{FX}] = \kappa\sigma_{g,t}^2$. The incomplete markets model behaves as if risk aversion γ was effectively reduced to $\sqrt{\kappa}$. There is also less return predictability in the incomplete spanning economy. The Fama slope coefficient in a regression of log currency excess returns on $f_t - s_t = r_t - r_t^*$ is $-2\kappa/\gamma^2$. Hence, the slope coefficient falls below 2, its complete markets value, in absolute value. The percentage reduction in the slope coefficient is twice the percentage reduction in volatility $2\log(\sqrt{\frac{\kappa}{\gamma^2}})$.

4 Exchange Rate Entropy

Table 1 reports summary statistics on exchange rate entropy. At the quarterly frequency, the entropy and half-volatility are essentially the same, as if exchange rate changes were normally distributed.

Table 1: Exchange Rate Entropy

	Cross-country Mean	Cross-country Std	Cross-country Min	Cross-country Max
$L(\Delta s)$	0.64	0.15	0.19	0.80
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma_{\Delta s}^2$	0.64	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$L(\Delta q)$	0.63	0.15	0.19	0.81
	(0.05)	(0.02)	(0.03)	(0.08)
$\frac{1}{2}\sigma_{\Delta q}^2$	0.63	0.16	0.19	0.82
	(0.05)	(0.02)	(0.03)	(0.08)

Notes: The table reports summary statistics on exchange rate entropy and volatility. The entropy, denoted $L(\Delta s)$, is measured as the log of the mean change in exchange rate minus the mean of the log change in exchange rate: $L(\Delta s) = logE\left(e^{\Delta s}\right) - E(log\Delta s)$. The volatility is measured as half the variance of the log change in exchange rates. Similar moments are defined for real exchange rates. The table presents the cross-country mean of the bilateral nominal and real exchange rate volatilities, along with the cross-country standard deviation of the bilateral exchange rate volatilities and the corresponding minimum and maximum values across countries. Similar statistics are reported for entropies. Moments are annualized (multiplied by 4) and reported in percentages. Data are quarterly, over the 1973.IV – 2014.IV period. The panel consists of 15 countries: Australia, Belgium, Canada, Denmark, France, Germany, Italy, Japan, Netherlands, New Zealand, Norway, Sweden, Switzerland, U.K., and U.S. The standard errors (reported between brackets) were generated by block-bootstrapping 10,000 samples, each block containing 2 quarters.

5 Projection Arguments

We can project the respective SDFs on the space of traded assets at home and abroad. The space of internationally traded assets only includes the domestic and the foreign risk-free. We can recover our results, including the risk premium, using the projection of the SDFs onto the space of traded payoffs.[3] The usual intuition is that one can add some noise that is unspanned to the SDFs without changing the pricing implications. That intuition is false in this setting, because the noise itself changes the space of traded payoffs through its effect on exchange rates.

5.1 Projection Argument with log SDFs

We use lowercases to denote logs. When projecting the log domestic SDF onto a constant and the innovation in the log exchange rate, we get the following expression:

$$\lambda_{t+1} = proj(m_{t+1}|X) = E_t(m) + \beta\left(\Delta s_{t+1} - E(\Delta s_{t+1})\right).$$

As before, we introduce a wedge in the spot exchange rate η:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}.$$

Hence, the projection slope coefficient is given by:

$$\beta(\eta) = \frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})}.$$

Similarly, when projecting the log foreign SDF onto the space of internationally traded assets, we get the following result:

$$\lambda^*_{t+1} = proj(m^*_{t+1}|X) = E_t(m^*) + \beta^*\left(-\Delta s_{t+1} + E(\Delta s_{t+1})\right).$$

Hence, the foreign projection coefficient is given by:

$$\beta^*(\eta) = -\frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})}.$$

After some algebra, we obtain that the domestic projection coefficient satisfies:

$$\begin{aligned}
\beta(\eta) &= \frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})}, \\
&= \frac{cov_t(m^*_{t+1} - m_{t+1}, m_{t+1})}{var_t(\Delta s_{t+1})} + \frac{-\mu_{t,\eta} + \frac{1}{2}var_t(\eta_{t+1})}{var_t(\Delta s_{t+1})},
\end{aligned}$$

where we have used our second condition in Proposition 1. Note that $\beta(\eta) \leq 0$. The wedge η does not drop out when we project the SDF onto the space of traded assets. Instead, the

[3] The authors acknowledge helpful conversations with John Cochrane, Bob Hodrick and Ben Hebert on this topic.

wedge determines the slope coefficient in the projection. Similarly, the foreign projection slope coefficient is given by:

$$\begin{aligned}\beta^*(\eta) &= -\frac{cov_t(m^*_{t+1} - m_{t+1} + \eta_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})}, \\ &= -\frac{cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1})}{var_t(\Delta s_{t+1})} - \frac{-\mu_{t,\eta} - \frac{1}{2}var_t\left(\eta_{t+1}\right)}{var_t(\Delta s_{t+1})},\end{aligned}$$

where we have used our first condition in Proposition 1. The wedge η does not drop out when we project the foreign SDF on the space of foreign traded assets. Note that $\beta^*(\eta) \leq 0$. Also, note that $\beta^*(\eta) + \beta^*(\eta) = -1$.

When we use these projections of the SDFs, our results are unchanged. The foreign currency risk premium for the home investor is the same as before:

$$\begin{aligned}E_t[rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(\lambda_{t+1}, \Delta s_{t+1}) = -\beta(\eta)var_t(\Delta s_{t+1}), \\ &= var_t\left(m_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) + E_t\left(\eta_{t+1}\right),\end{aligned}$$

where we have used the expression for λ_{t+1}. Similarly, the currency risk premium for the foreign investor is

$$\begin{aligned}E_t[-rx^{FX}_{t+1}] + \frac{1}{2}var_t[rx^{FX}_{t+1}] &= -cov_t(\lambda^*_{t+1}, \Delta s_{t+1}) = -\beta^*(\eta)var_t(\Delta s_{t+1}), \\ &= var_t\left(m^*_{t+1}\right) - covar_t\left(m^*_{t+1}, m_{t+1}\right) - \frac{1}{2}var_t\left(\eta_{t+1}\right) - E_t\left(\eta_{t+1}\right).\end{aligned}$$

Hence, the expressions for the risk premia are identical when we use the projections. It is not the case that the incompleteness wedge disappears from the risk premium expression.

5.2 Projection Argument with level SDFs

Brandt, Cochrane and Santa-Clara (2006) argue that market incompleteness cannot help to resolve the volatility puzzle. This section explains why our results differ from those in Brandt, Cochrane and Santa-Clara (2006). When projecting the domestic SDF onto the space of internationally traded assets, we get the following (projecting on ones and $\frac{S_{t+1}}{S_t}$):

$$\Lambda_{t+1} = proj(M_{t+1}|X) = E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right).$$

On page 675, Brandt, Cochrane and Santa-Clara (2006) define the following expression for the foreign projected SDF:

$$\Lambda^*_{t+1} = \Lambda_{t+1}\frac{S_{t+1}}{S_t},$$

which implies:

$$\Lambda^*_{t+1} = \left[E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right)\right]\frac{S_{t+1}}{S_t},$$

since

$$\Lambda_{t+1} = proj(M_{t+1}|X) = E(M) + \beta\left(\frac{S_{t+1}}{S_t} - E(\frac{S_{t+1}}{S_t})\right).$$

This Λ^*_{t+1} satisfies the foreign investors' Euler equations; it is a valid SDF but Λ^*_{t+1} is not in the foreign payoff space (see quadratic exchange rate terms), and note the projection onto foreign payoff space yields no quadratic terms:

$$proj(M^*_{t+1}|X^*) = E(M^*) + \beta^* \left(\frac{S_t}{S_{t+1}} - E(\frac{S_t}{S_{t+1}}) \right).$$

Λ^*_{t+1} cannot be the minimum variance SDF for the foreign investor. This explains why Brandt, Cochrane and Santa-Clara (2006)'s argument for market incompleteness irrelevance does not apply here: their foreign pricing kernel is no longer in the space of traded payoffs (see Maurer and Tran, 2016, for a related argument).

Further, consider the domestic investors' Euler equations for investing in the risk-free note at home and abroad, evaluated with the projection:

$$\begin{aligned} E_t\left(\Lambda_{t+1} R^f_t\right) &= 1 \\ E_t\left(\Lambda_{t+1} \frac{S_{t+1}}{S_t} R^{f,*}_t\right) &= 1 \end{aligned}$$

The first Euler equation holds by construction. The second Euler equation implies that the multiplicative risk premium on FX is given by:

$$\frac{R^{f,*}_t E_t(\frac{S_{t+1}}{S_t}) - R^f_t}{R^f_t} = -R^{f,*}_t \beta var_t\left(\frac{S_{t+1}}{S_t}\right)$$

The projection argument implies that the incomplete markets risk premium is determined by the sensitivity of the SDF to exchange rate shocks β, as one would expect. As the variance of the exchange rates decreases, UIP is restored and the multiplicative risk premium is zero. Clearly, tthere is nothing that keeps us from pushing the variance of the exchange rates to zero if we adjust $E_t(\frac{S_{t+1}}{S_t})$, holding interest rates fixed; our paper shows how to do this in a log-normal setting.

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