

Capital Flows and Foreign Exchange Intervention “Online Appendix”

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A The Household’s Problem

The representative Home household maximizes

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\frac{C^{1-\omega}}{1-\omega} - \frac{L^{1+\varphi}}{1+\varphi} \right) dt \right]$$

subject to

$$dA = (WL - PC) dt + A(\mu_A dt + \sigma_A dZ)$$

with $A(\mu_A dt + \sigma_A dZ) = A i dt + \mathcal{E}F[(i^* - i + \mu_\mathcal{E}) dt + \sigma_\mathcal{E} dZ] + dM + dR + dT$, where

$$C \equiv \left[(1-\alpha)^{\frac{1}{\eta}} C_H^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C_F^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

and $C_F \equiv \exp \int_0^1 \ln C_i di$. The intratemporal first order conditions yield

$$C_H = (1-\alpha) \left(\frac{P_H}{P} \right)^{-\eta} C \quad C_F = \alpha \left(\frac{P_F}{P} \right)^{-\eta} C \quad C_i = \left(\frac{P_i}{P_F} \right)^{-1} C_F$$

$\forall i \in [0, 1]$. Home’s CPI is defined as $P \equiv \left[(1-\alpha) P_H^{1-\eta} + \alpha P_F^{1-\eta} \right]^{\frac{1}{1-\eta}}$. The price of imported goods is given by $P_F \equiv \exp \int_0^1 \ln P_i di$. The intertemporal household’s problem is:

$$V(A, \mathbf{S}) = \max_{C, L} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\frac{C^{1-\omega}}{1-\omega} - \frac{L^{1+\varphi}}{1+\varphi} \right) dt \right]$$

subject to

$$dA = (WL - PC) dt + A(\mu_A dt + \sigma_A dZ)$$

$$d\mathbf{S} = \mathbf{S} \mu_\mathbf{S} dt + \mathbf{S} \sigma_\mathbf{S} dZ$$

where \mathbf{S} is a generic vector of states and μ_A and σ_A are a function of the states only. The HJB for this problem is

$$\rho V = \sup_{C,L} \left[\frac{C^{1-\omega}}{1-\omega} - \frac{L^{1+\varphi}}{1+\varphi} + \mathcal{A}(V) \right]$$

where

$$\begin{aligned} \mathcal{A}(V) = & V_A(WL - PC + A\mu_A) + \sum_i V_{S_i} S_i \mu_{S_i} \\ & + \frac{1}{2} \left(V_{AA} A^2 \sigma_A^2 + 2 \sum_i A S_i \sigma_A \sigma_{S_i} V_{AS_i} + \sum_i \sum_j V_{S_i S_j} S_i S_j \sigma_{S_i} \sigma_{S_j} \right) \end{aligned}$$

is the infinitesimal generator operator. The first order conditions with respect to C and L are

$$\begin{aligned} C^\omega P &= \frac{1}{V_A} \\ L^\varphi &= V_A W \end{aligned}$$

Derive both sides of the HJB with respect to A to obtain the law of motion of V_A :

$$\frac{dV_A}{V_A} = (\rho - i) dt + \sigma_{V_A} dZ$$

where $\sigma_{V_A} = \frac{V_{AA}}{V_A} A \sigma_A + \sum_S \frac{V_{AS}}{V_A} S \sigma_S$. Finally I apply Ito's Lemma to the first order condition to derive the Euler equation

$$dC = \frac{1}{\omega} \left(i - \mu_P - \rho + \sigma_P^2 + \sigma_C \sigma_P + \frac{1+\omega}{2\omega} \sigma_C^2 \right) dt + \frac{1}{\omega} \sigma_C dZ$$

where $\sigma_C = -\sigma_{V_A} - \sigma_P$.

B Calvo Pricing in Continuous Time

A Producer Price Dynamics

Domestic producer price indexes are defined by

$$P_k(t) \equiv \left[\int_0^1 P_{H,j}(t)^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}}$$

In order to derive its law of motion, let's write it down in discrete time first and then take the limit as the length of the time interval goes to zero. The length of a period is $[t, t + dt)$. During a period, a fraction $1 - e^{-\theta dt}$ of firms receive the Calvo signal that will allow them to

set a new price at time $t + dt$. The remaining fraction $e^{-\theta dt}$ will not be able to change price and will be stuck with the price posted at t . Let $S(t)$ be the set of firms not re-optimizing their posted price at time t . Using the fact that all firms resetting prices will choose an identical price $\bar{P}_H(t)$, I obtain

$$\begin{aligned} P_H(t + dt) &= \left[\int_0^1 P_{H,j}(t + dt)^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}} \\ &= \left[\int_{S(t)} P_{H,j}(t)^{1-\epsilon} dj + (1 - e^{-\theta dt}) \bar{P}_H(t + dt)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \\ &= \left[e^{-\theta dt} P_H(t)^{1-\epsilon} + (1 - e^{-\theta dt}) \bar{P}_H(t + dt)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \end{aligned}$$

where the last equality follows from the fact that the distribution of prices among firms not adjusting at time $t + dt$ corresponds to the distribution of posted prices at time t , though with a total mass reduced to $e^{-\theta dt}$. The equation above can be rewritten as

$$P_H(t + dt)^{1-\epsilon} - P_H(t)^{1-\epsilon} = - (1 - e^{-\theta dt}) P_H(t)^{1-\epsilon} + (1 - e^{-\theta dt}) \bar{P}_H(t + dt)^{1-\epsilon}$$

Now, I cannot take the the limit as $dt \rightarrow 0$ since the last term is not time t measurable. Therefore add and subtract $(1 - e^{-\theta dt}) \bar{P}_H(t)^{1-\epsilon}$ to obtain

$$\begin{aligned} P_H(t + dt)^{1-\epsilon} - P_H(t)^{1-\epsilon} &= (1 - e^{-\theta dt}) \left(\bar{P}_H(t)^{1-\epsilon} - P_H(t)^{1-\epsilon} \right) \\ &\quad + (1 - e^{-\theta \Delta t}) \Delta \bar{P}_H(t)^{1-\epsilon} \end{aligned}$$

Now Taylor expand $e^{-\theta dt}$ around $dt = 0$ and take the limit for $dt \rightarrow 0$ to obtain

$$\pi_H(t) = \frac{dP_H(t)}{P_H(t)} = \frac{\theta}{1-\epsilon} \left[\left(\frac{\bar{P}_H(t)}{P_H(t)} \right)^{1-\epsilon} - 1 \right] dt$$

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B Price Dispersion

The aggregate loss of efficiency induced by price dispersion among firms is $\Delta(t) \equiv \int_0^1 \left[\frac{P_{H,j}(t)}{P_H(t)} \right]^{-\epsilon} dj$. Its law of motion can be derived as in the previous section. Let's write the discrete time analogue and appropriately shrink the length of the period

$$\begin{aligned} \Delta(t + dt) &= \int_0^1 \left[\frac{P_{H,j}(t + dt)}{P_H(t + dt)} \right]^{-\epsilon} dj \\ &= P_H(t + dt)^\epsilon \left[\int_{S(t)} P_{H,j}(t)^{-\epsilon} dj + (1 - e^{-\theta dt}) \bar{P}_{Hk}(t + dt)^{-\epsilon} \right] \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{P_H(t+dt)}{P_H(t)} \right)^\epsilon \int_{S(t)} \left(\frac{P_{H,j}(t)}{P_H(t)} \right)^{-\epsilon} dj \\
&\quad + (1 - e^{-\theta dt}) P_H(t+dt)^\epsilon \bar{P}_H(t+dt)^{-\epsilon} \\
&= \left(\frac{P_H(t+dt)}{P_H(t)} \right)^\epsilon e^{-\theta dt} \Delta(t) + (1 - e^{-\theta dt}) P_H(t+dt)^\epsilon \bar{P}_H(t+dt)^{-\epsilon}
\end{aligned}$$

As before, I add and subtract the last term lagged, Taylor expand the exponential terms and take the limit as $dt \rightarrow 0$ to obtain

$$d\Delta(t) = \left[\theta \left(\frac{\bar{P}_H(t)}{P_H(t)} \right)^{-\epsilon} + \Delta(t) (\epsilon \pi_H(t) - \theta) \right] dt$$

or

$$d\Delta(t) = \left[\theta \left(1 - \frac{\epsilon - 1}{\theta} \pi_H(t) \right)^{\frac{\epsilon}{\epsilon - 1}} + \Delta(t) (\epsilon \pi_H(t) - \theta) \right] dt$$

C Optimal Price Setting

A measure one of monopolistic firms (indexed by $j \in [0, 1]$) engage in infrequent price setting a la Calvo. Each firm re-optimizes its price $P_{H,j}(t)$ only at discrete dates determined by a Poisson process with intensity θ . The time δ between two re-optimizations is distributed according to the exponential density: $\theta e^{-\theta \delta}$. A firm that is allowed to re-optimize its price at time t maximizes the present discounted value of future profits¹

$$\max_{\bar{P}_{H,j}(t)} \mathbb{E} \left[\int_t^\infty \frac{P(t)}{C(t)^{-\omega}} \frac{C(u)^{-\omega}}{P(u)} e^{-(\rho+\theta)(u-t)} \{ \bar{P}_{H,j}(t) Y_j(u|t) - \mathcal{C}_H(Y_j(u|t)) \} du \right]$$

subject to the demand schedule

$$Y_j(u|t) = \left[\frac{\bar{P}_{H,j}(t)}{P_H(u)} \right]^{-\epsilon} Y(u)$$

where $\mathcal{C}(\cdot)$ is the firms nominal cost function. The first-order condition associated with the problem is

$$\mathbb{E} \left[\int_t^\infty \frac{P(t)}{C(t)^{-\omega}} \frac{C(u)^{-\omega}}{P(u)} e^{-(\rho+\theta)(u-t)} Y_j(u|t) \{ \bar{P}_{H,j}(t) - \mathcal{M}MC(Y_j(u|t)) \} du \right] = 0$$

where MC is the nominal marginal cost function and $\mathcal{M} \equiv \frac{\epsilon}{\epsilon - 1}$. Note that in the limiting case of no price rigidities ($\theta \rightarrow \infty$), this condition collapses to the familiar optimal price-setting condition under flexible prices $P_{H,j}(t) = \mathcal{M}MC(Y_j(t))$.

¹I assume that firms commit to supply whatever quantity demanded at the posted price, even if that implies negative profits

The firm's cost function is $\mathcal{C}(Y_j(u|t)) = Y_j(u|t)(1-\tau)W(u)$, therefore the nominal marginal cost is

$$MC(Y_j(u|t)) = (1-\tau)W(u) \equiv MC(u)$$

The FOC can be rewritten as

$$\bar{P}_{H,j}(t) = P_H(t) \frac{\mathbb{E} \left[\int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - \epsilon \pi_H(s)] ds} \frac{C(u)^{-\omega}}{P(u)} \mathcal{M}MC(u) Y(u) \right\} du \right]}{\mathbb{E} \left[\int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - (\epsilon-1)\pi_H(s)] ds} \frac{C(u)^{-\omega}}{P(u)} P_H(u) Y(u) \right\} du \right]}$$

where I used the result that the dynamics of the price level is locally deterministic. Let

$$\begin{aligned} \mathcal{U}(t) &\equiv \mathbb{E} \left[\int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - \epsilon \pi_H(s)] ds} \frac{C(u)^{-\omega}}{P(u)} \mathcal{M}MC(u) Y(u) \right\} du \right] \\ \mathcal{V}(t) &\equiv \mathbb{E} \left[\int_t^\infty \left\{ e^{-\int_t^u [\rho + \theta - (\epsilon-1)\pi_H(s)] ds} \frac{C(u)^{-\omega}}{P(u)} P_H(u) Y(u) \right\} du \right] \end{aligned}$$

then, the Feynman-Kac representation formula establishes that \mathcal{U} and \mathcal{V} are the unique solutions to the partial differential equations

$$(\rho + \theta - \epsilon \pi_H(t)) \mathcal{U}(t) = \mathcal{A}(\mathcal{U}) + \frac{C(t)^{-\omega}}{P(t)} \mathcal{M}MC(t) Y(t)$$

$$[\rho + \theta - (\epsilon - 1) \pi_H(t)] \mathcal{V}(t) = \mathcal{A}(\mathcal{V}) + \frac{C(t)^{-\omega}}{P(t)} P_H(t) Y(t)$$

where the operator \mathcal{A} is the infinitesimal generator of the stochastic process, defined as $\mathcal{A}f = \mu^x \nabla_x f(x) + \frac{1}{2} \text{tr} \left[\sigma^x H(f) (\sigma^x)^T \right]$. Hence, their laws of motion are

$$\frac{d\mathcal{U}(t)}{\mathcal{U}(t)} = \left[\rho + \theta - \epsilon \pi_H(t) - \frac{C(t)^{-\omega}}{P(t)} \frac{\mathcal{M}MC(t) Y(t)}{\mathcal{U}(t)} \right] dt + \sigma_{\mathcal{U}}(t) dZ(t)$$

$$\frac{d\mathcal{V}(t)}{\mathcal{V}(t)} = \left[\rho + \theta - (\epsilon - 1) \pi_H(t) - \frac{C(t)^{-\omega}}{P(t)} \frac{P_H(t) Y(t)}{\mathcal{V}(t)} \right] dt + \sigma_{\mathcal{V}}(t) dZ(t)$$

Finally, the law of motion of PPI inflation can be derived using $\pi_H(t) = \frac{\theta}{\epsilon-1} \left[1 - \left(\frac{\mathcal{U}(t)}{\mathcal{V}(t)} \right)^{1-\epsilon} \right]$ and the laws of motion for \mathcal{U} and \mathcal{V} :

$$\begin{aligned} d\pi_H(t) &= [(\epsilon - 1) \pi_H(t) - \theta] \left[\pi_H(t) + \frac{C(t)^{-\omega}}{P(t)} Y(t) \left(\frac{\mathcal{M}MC(t)}{\mathcal{U}(t)} - \frac{P_H(t)}{\mathcal{V}(t)} \right) \right] dt \\ &\quad + [(\epsilon - 1) \pi_H(t) - \theta] \left[\left(\epsilon \frac{\sigma_\pi(t)^2}{2} - \sigma_\pi(t) \sigma_{\mathcal{V}}(t) \right) dt + \sigma_\pi(t) dZ(t) \right] \end{aligned}$$

where $\sigma_\pi(t) = \sigma_\nu(t) - \sigma_u(t)$.

C Log-linearization

The deterministic laws of motion that describe the equilibrium dynamics are

$$\begin{aligned}
d\Lambda &= -\gamma\Lambda \left(\hat{A} + \hat{F}_H^* + \hat{X}_H \right) dt \\
d\hat{A} &= \left[(i - \pi_H)\hat{A} - \gamma \left(\hat{A} + \hat{F}_H^* + \hat{X}_H \right) \left((1 - \beta)\hat{X}_H - \beta \left(\hat{A} + \hat{F}_H^* \right) \right) + 1 - \frac{\mathcal{S}C}{\mathcal{Q}Y} \right] dt - \hat{A} \frac{dY}{Y} \\
dY &= Y \frac{dY^*}{Y^*} + \frac{1}{\omega} [Y - \alpha \mathcal{S}^\eta Y^*] \frac{d\Lambda}{\Lambda} + \left[\eta Y + \left(\frac{1}{\omega} - \eta \right) (1 - \alpha) (Y - \alpha \mathcal{S}^\eta Y^*) \left(\frac{\mathcal{S}}{\mathcal{Q}} \right)^{\eta-1} \right] \frac{d\mathcal{S}}{\mathcal{S}} \\
d\mathcal{S} &= \mathcal{S} (i - \rho - \pi_H) dt - \mathcal{S} \omega \frac{dY^*}{Y^*} - \mathcal{S} \frac{d\Lambda}{\Lambda} \\
d\pi_H &= [(\epsilon - 1)\pi_H - \theta] \left[\pi_H + \frac{1}{\mathcal{V}} \left(\mathcal{M} (1 - \tau) \Delta^\varphi Y^{1+\varphi} \left(1 - \frac{\epsilon - 1}{\theta} \pi_H \right)^{-\frac{1}{1-\epsilon}} - Y C^{-\omega} \frac{\mathcal{Q}}{\mathcal{S}} \right) \right] dt \\
d\mathcal{V} &= \left[(\rho + \theta - (\epsilon - 1)\pi_H) \mathcal{V} - C^{-\omega} \frac{P_H Y}{P} \right] dt \\
d\Delta &= \left[\theta \left(1 - \frac{\epsilon - 1}{\theta} \pi_H \right)^{\frac{\epsilon}{\epsilon-1}} + \Delta (\epsilon \pi_H - \theta) \right] dt \\
dY^* &= \frac{1}{\omega} Y^* (i^* - \rho - \pi^*) dt
\end{aligned}$$

with

$$\begin{aligned}
C &= (\Lambda \mathcal{Q})^{\frac{1}{\omega}} C^* \\
\mathcal{Q} &= [\alpha + (1 - \alpha) \mathcal{S}^{\eta-1}]^{\frac{1}{\eta-1}} \\
\mathcal{S} &= \left(\frac{Y}{C^*} \right)^{\frac{1}{\eta}} \left[(1 - \alpha) \Lambda^{\frac{1}{\omega}} \mathcal{Q}^{\frac{1}{\omega} - \eta} + \alpha \right]^{-\frac{1}{\eta}}
\end{aligned}$$

The log linearized versions are

$$\begin{aligned}
d\lambda &= -\gamma \left(\hat{a} + \hat{f}_H^* - \hat{x} \right) dt \\
d\hat{a} &= [\rho \hat{a} - (s - q + c - y)] dt \\
dy &= dy^* + \frac{1 - \alpha}{\omega} d\lambda + \left[\eta + \left(\frac{1}{\omega} - \eta \right) (1 - \alpha)^2 \right] ds \\
ds &= (i - \rho - \pi_H) dt - \omega dy^* - d\lambda \\
d\pi_H &= \rho \pi_H - \theta (\rho + \theta) (\varphi y + \omega c + \alpha s) dt \\
dy^* &= \frac{1}{\omega} (i^* - \rho - \pi^*) dt
\end{aligned}$$

and

$$\begin{aligned}
c &= y^* + \frac{1}{\omega}(\lambda + q) \\
q &= (1 - \alpha)s \\
s &= \frac{1}{\eta}(y - y^*) - \frac{1 - \alpha}{\eta} \left(\frac{1}{\omega}\lambda + \left(\frac{1}{\omega} - \eta \right) q \right)
\end{aligned}$$

where I used the first order approximation $\hat{x}_H = -\hat{x}$. Therefore, I obtain

$$\begin{aligned}
d\lambda &= -\gamma \left(\hat{a} + \hat{f}_H^* - \hat{x} \right) dt \\
d\hat{a} &= \left(\rho\hat{a} - \frac{1}{\omega}\lambda - \frac{1 - \alpha(1 - \omega)}{\omega}s - y^* + y \right) dt \\
dy &= dy^* + \frac{1 - \alpha}{\omega}d\lambda + \left[\eta + \left(\frac{1}{\omega} - \eta \right) (1 - \alpha)^2 \right] ds \\
ds &= (i - \rho - \pi_H) dt - \omega dy^* - d\lambda \\
d\pi_H &= \rho\pi_H - \kappa(\varphi y + \omega y^* + \lambda + s) dt \\
dy^* &= \frac{1}{\omega}(i^* - \rho - \pi^*) dt
\end{aligned}$$

where $\kappa \equiv \theta(\rho + \theta)$, with

$$s = \frac{\omega(y - y^*) - (1 - \alpha)\lambda}{\omega\eta + (1 - \omega\eta)(1 - \alpha)^2}$$

Now assume $\omega = \eta = 1$. Then

$$\begin{aligned}
d\lambda &= -\gamma \left(\hat{a} + \hat{f}_H^* - \hat{x} \right) dt \\
d\hat{a} &= (\rho\hat{a} - \alpha\lambda) dt \\
dy &= (i - \rho - \pi_H) dt - \alpha d\lambda \\
d\pi_H &= \rho\pi_H - \kappa(1 + \varphi)y - \kappa\alpha\lambda dt
\end{aligned}$$

and y^* does not affect equilibrium variables. The welfare function is

$$\mathbb{W} = \int_0^\infty e^{-\rho t} \left[\ln C - \frac{L^{1+\varphi}}{1+\varphi} \right] dt$$

where

$$C = \Lambda S^{1-\alpha} C^* \quad L = \Delta Y$$

and

$$S = \frac{Y_H}{[\alpha + (1 - \alpha)\Lambda] C^*}$$

Log-linearize both C and L to obtain

$$\begin{aligned}
\ln C &= t.i.p. + \ln \Lambda + (1 - \alpha) \ln \mathcal{S} \\
&= t.i.p. + \ln \Lambda - (1 - \alpha) \ln [\alpha + (1 - \alpha) \Lambda] - (1 - \alpha) (1 - \beta) \ln Y \\
&\simeq t.i.p. + \alpha (2 - \alpha) \lambda - \alpha (1 - \alpha)^2 \frac{1}{2} \lambda^2 + (1 - \alpha) y
\end{aligned}$$

$$\frac{L^{1+\varphi}}{1+\varphi} \simeq t.i.p. + (1 - \alpha) (\delta + y) + (1 + \varphi) \frac{1 - \alpha}{2} y^2$$

Thus

$$\mathbb{W} = \int_0^\infty e^{-\rho t} \left[\begin{array}{l} t.i.p. + \alpha (2 - \alpha) \lambda - \frac{\alpha(1-\alpha)^2}{2} \lambda^2 \\ - (1 - \alpha) \frac{\epsilon}{2\kappa} \pi_H^2 - (1 - \alpha) \frac{1+\varphi}{2} y^2 \end{array} \right] dt$$

Now use the following second order approximation to the budget constraint

$$\begin{aligned}
\int_0^\infty e^{-\rho t} [\alpha \lambda + \gamma (1 - \beta) \hat{x}^2] dt &= 0 \\
y &= \gamma (1 - \beta) \frac{2 - \alpha}{1 - \alpha} \hat{x}^2
\end{aligned}$$

to replace the linear term in \mathbb{W} and obtain

$$\mathbb{W} = \int_0^\infty e^{-\rho t} \left[\begin{array}{l} t.i.p. - \gamma (2 - \alpha) (1 - \beta) \hat{x}^2 - \frac{\alpha(1-\alpha)^2}{2} \lambda^2 \\ - (1 - \alpha) \frac{\epsilon}{2\kappa} \pi_H^2 - (1 - \alpha) \frac{1+\varphi}{2} y^2 \end{array} \right] dt$$

Finally, the loss function is

$$\mathbb{L} = \frac{1}{2} \int_0^\infty e^{-\rho t} (\phi_x \hat{x}^2 + \phi_\lambda \lambda^2 + \phi_\pi \pi_H^2 + \phi_y y^2) dt$$

where

$$\begin{aligned}
\phi_x &= 2\gamma (2 - \alpha) (1 - \beta) \\
\phi_\lambda &= \alpha (1 - \alpha)^2 \\
\phi_\pi &= \frac{\epsilon}{\kappa} (1 - \alpha) \\
\phi_y &= (1 - \alpha) (1 + \varphi)
\end{aligned}$$

D Proofs

Proof of Proposition 1

PROOF Under flexible prices, the deterministic planner problem is

$$\max_{\hat{X}_H, C, Y, \mathcal{S}, \mathcal{Q}} \int_0^\infty e^{-\rho t} \left(\frac{C^{1-\omega}}{1-\omega} - \frac{Y^{1+\varphi}}{1+\varphi} \right) dt$$

subject to

$$\begin{aligned} C &= (\Lambda \mathcal{Q})^{\frac{1}{\omega}} C^* \\ Y &= C^* \mathcal{S}^\eta \left[(1-\alpha) \Lambda^{\frac{1}{\omega}} \mathcal{Q}^{\frac{1}{\omega}-\eta} + \alpha \right] \\ \mathcal{Q} &= [\alpha + (1-\alpha) \mathcal{S}^{\eta-1}]^{\frac{1}{\eta-1}} \end{aligned}$$

and the laws of motion

$$\begin{aligned} d\Lambda &= -\gamma \Lambda (\hat{A} + \hat{X}_H) dt \\ d\hat{A} &= \hat{A} \left\{ i - \pi_H - \frac{1}{\omega} \left[\omega \eta + (1-\omega \eta) (1-\alpha) \left(1 - \alpha \mathcal{S}^\eta \frac{Y^*}{Y} \right) \left(\frac{\mathcal{S}}{\mathcal{Q}} \right)^{\eta-1} \right] (i - \rho - \pi_H) \right\} dt \\ &\quad \left\{ \left(1 - \frac{\mathcal{S} C}{\mathcal{Q} Y} \right) - \gamma (\hat{A} + \hat{X}_H) \left[(1-\beta) \hat{X}_H + \hat{A} (\eta - \beta) \right] \right\} dt \\ &\quad - \frac{\gamma}{\omega} \hat{A} (\hat{A} + \hat{X}_H) \left((1-\omega \eta) (1-\alpha) \left(\frac{\mathcal{S}}{\mathcal{Q}} \right)^{\eta-1} - 1 \right) \left(1 - \alpha \mathcal{S}^\eta \frac{Y^*}{Y} \right) dt \end{aligned}$$

Notice that in steady state it must be the case that $\hat{X}_H = \hat{A}$. This implies that, in steady state, the central bank uses FX intervention only to correct the financial friction and not to permanently alter the terms of trades or the real exchange rate. Therefore, in a symmetric steady state $\hat{X}_H = \hat{A} = 0$ and $\Lambda = \mathcal{Q} = \mathcal{S} = 1$. The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{C^{1-\omega}}{1-\omega} - \frac{Y^{1+\varphi}}{1+\varphi} + \lambda_C \left[C - (\Lambda \mathcal{Q})^{\frac{1}{\omega}} C^* \right] + \lambda_{\mathcal{Q}} \left\{ \mathcal{Q} - [\alpha + (1-\alpha) \mathcal{S}^{\eta-1}]^{\frac{1}{\eta-1}} \right\} \\ &\quad + \lambda_Y \left[Y - C^* \mathcal{S}^\eta \left[(1-\alpha) \Lambda^{\frac{1}{\omega}} \mathcal{Q}^{\frac{1}{\omega}-\eta} + \alpha \right] \right] + \lambda_A \mu_A + \lambda_\Lambda \mu_\Lambda \end{aligned}$$

The FOCs evaluated at the symmetric steady state are

$$\begin{aligned} 0 &= Y^{1-\omega} + Y \lambda_C - \lambda_A \\ 0 &= -Y^{1+\varphi} + Y \lambda_Y + \lambda_A \\ 0 &= -Y \lambda_C \frac{1}{\omega} + \lambda_Y (\alpha - 1) Y \left(\frac{1}{\omega} - \eta \right) + \lambda_{\mathcal{Q}} + \lambda_A \end{aligned}$$

$$\begin{aligned} 0 &= -Y\lambda_Y\eta - \lambda_Q(1-\alpha) - \lambda_A \\ 0 &= -\lambda_\Lambda\gamma\Lambda \end{aligned}$$

and

$$\begin{aligned} d\lambda_A &= \left\{ \lambda_A \left\{ \left[i - \pi_H - \frac{1}{\omega} \left[\omega\eta + (1-\omega\eta)(1-\alpha)^2 \right] (i - \rho - \pi_H) \right] \right\} - \gamma\Lambda\lambda_\Lambda \right\} dt \\ d\lambda_\Lambda &= \left\{ \rho\lambda_\Lambda + \lambda_C \frac{1}{\omega} Y + \lambda_Y Y (1-\alpha) \frac{1}{\omega} \right\} dt \end{aligned}$$

Therefore, steady state output is given by

$$Y^{\omega+\varphi} = \frac{(2-\alpha)\eta-1}{(2-\alpha)\eta-1+\alpha}$$

When prices are flexible

$$P_H = (1-\tau) \frac{\epsilon}{\epsilon-1} W = (1-\tau) \frac{\epsilon}{\epsilon-1} Y^{\omega+\varphi} P$$

Hence, the optimal labor subsidy is

$$\tau = 1 - \frac{\epsilon-1}{\epsilon} \frac{(2-\alpha)\eta-1+\alpha}{(2-\alpha)\eta-1}$$

Proof of Proposition 2

PROOF When prices are flexible and the central bank does not intervene in the asset market, the allocation solves

$$\begin{aligned} d\lambda &= -\gamma(\hat{a} + \hat{f}_H^*) dt \\ d\hat{a} &= (\rho\hat{a} - \alpha\lambda) dt \\ d\hat{f}_H^* &= -\varrho\hat{f}_H^* \end{aligned}$$

given the initial conditions $\hat{f}_H^*(0) = \varepsilon$, $\hat{a}(0) = 0$ and the terminal condition $\lim_{t \rightarrow \infty} e^{-\rho t} \hat{a}(t) = 0$.

Let $\mathbf{z}^\top = \left[\lambda \quad \hat{a} \quad \hat{f}_H^* \right]$. The natural allocation solves the system of differential equations $d\mathbf{z} = \mathbf{A}\mathbf{x}dt$, where

$$\mathbf{A} = \begin{bmatrix} 0 & -\gamma & -\gamma \\ -\alpha & \rho & 0 \\ 0 & 0 & -\varrho \end{bmatrix}$$

The eigenvalues of \mathbf{A} are $\left[-\varrho \quad -\nu \quad \rho + \nu \right]$, where $\nu = \frac{-\rho + \sqrt{\rho^2 + 4\alpha\gamma}}{2} > 0$. The first two

eigenvectors of \mathbf{A} are

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{\gamma(\rho+\varrho)}{(\nu-\varrho)(\rho+\varrho+\nu)} \\ -\frac{\nu(\rho+\nu)}{(\nu-\varrho)(\rho+\varrho+\nu)} \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} \frac{\rho+\nu}{\alpha} \\ 1 \\ 0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} \lambda \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix} = \zeta_1 \begin{bmatrix} -\frac{\gamma(\rho+\varrho)}{(\nu-\varrho)(\rho+\varrho+\nu)} \\ -\frac{\nu(\rho+\nu)}{(\nu-\varrho)(\rho+\varrho+\nu)} \\ 1 \end{bmatrix} e^{-\varrho t} + \zeta_2 \begin{bmatrix} \frac{\rho+\nu}{\alpha} \\ 1 \\ 0 \end{bmatrix} e^{-\nu t}$$

where the parameters ζ_1 and ζ_2 are determined by the initial conditions $\hat{a}(0) = 0$ and $\hat{f}_H^*(0) = \varepsilon$. Thus, I obtain

$$\lambda = \begin{bmatrix} \frac{\gamma}{\nu} \\ \frac{\gamma}{\rho+\varrho+\nu} \end{bmatrix}^\top \begin{bmatrix} \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

where the states evolve as follows

$$\begin{bmatrix} d\hat{a} \\ d\hat{f}_H^* \end{bmatrix} = \begin{bmatrix} -\nu & -\nu\frac{\rho+\nu}{\rho+\varrho+\nu} \\ 0 & -\varrho \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{f}_H^* \end{bmatrix} dt$$

with $\begin{bmatrix} \hat{a}(0) & \hat{f}_H^*(0) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \end{bmatrix}$. The solution for λ can be rewritten as

$$\lambda = \gamma\varepsilon \frac{(\rho+\nu)e^{-\nu t} - (\rho+\varrho)e^{-\varrho t}}{(\nu-\varrho)(\rho+\varrho+\nu)}$$

and thus obtain

$$\begin{aligned} \mathbb{L} &= \frac{\phi}{2} \frac{(\gamma\varepsilon)^2}{\rho+2\varrho} \frac{\rho(\rho+\nu+\varrho)+2\varrho\nu}{(\rho+\varrho+\nu)^3(\rho+2\nu)} \\ &\quad \frac{\phi}{2} \frac{(\gamma\varepsilon\varrho)^2}{\rho+2\varrho} \frac{\rho(\rho+\nu+\varrho)+2\varrho\nu}{(\rho+\varrho+\nu)^3(\rho+2\nu)} \end{aligned}$$

where $\phi = \phi_F = \alpha(1-\alpha)\frac{1+\varphi-\alpha\varphi}{1+\varphi}$. Finally, I take derivatives to obtain

$$\frac{\partial \mathbb{L}}{\partial \gamma} = \frac{2\mathbb{L}}{\gamma} \frac{2\varrho(\nu^3+5\nu^2\rho+4\nu\rho^2+\rho^3)(\nu+\rho)+\varrho^2(2\nu+\rho)^3+\rho(\nu+\rho)^4}{(\rho+\varrho+\nu)(\rho+2\nu)^2(\rho(\rho+\nu+\varrho)+2\varrho\nu)} > 0$$

$$\frac{\partial \mathbb{L}}{\partial \varrho} \frac{\varrho}{\mathbb{L}} = 2 \frac{\rho^2(\nu+\rho)^2 - \varrho^3(2\nu+\rho) + 4\nu\varrho^2(\nu+\rho) + 2\rho\varrho(\nu+\rho)(2\nu+\rho)}{[\rho(\rho+\nu+\varrho)+2\varrho\nu](\rho+\varrho+\nu)(\rho+2\varrho)}$$

$$\frac{\partial \mathbb{L}}{\partial \alpha} \frac{\alpha}{\mathbb{L}} = \frac{1 - 2\alpha + (1 - 3\alpha)(1 - \alpha)\varphi}{(1 - \alpha)(1 + \varphi - \alpha\varphi)} - 2\nu(\rho + \nu) \frac{3\nu^2(\rho + 2\varrho) + \nu\rho(5\rho + 6\varrho) + 2\rho^2(\rho + \varrho)}{[\rho(\rho + \nu + \varrho) + 2\varrho\nu](\rho + \varrho + \nu)(\rho + 2\nu)^2}$$

It is easy to show that

$$\begin{aligned} \lim_{\varrho \downarrow 0} \mathbb{L} &= 0 & \lim_{\varrho \downarrow 0} \frac{\partial \mathbb{L}}{\partial \varrho} \frac{\varrho}{\mathbb{L}} &> 0 \\ \lim_{\varrho \uparrow \infty} \mathbb{L} &= 0 & \lim_{\varrho \uparrow \infty} \frac{\partial \mathbb{L}}{\partial \varrho} \frac{\varrho}{\mathbb{L}} &< 0 \end{aligned}$$

and the discriminant of the cubic polynomial at the numerator of $\frac{\partial \mathbb{L}}{\partial \varrho}$ is negative. Hence, $\exists! \tilde{\varrho} \in (0, \infty)$ such that $\frac{\partial \mathbb{L}}{\partial \varrho} > 0$ for $\varrho < \tilde{\varrho}$, and $\frac{\partial \mathbb{L}}{\partial \varrho} < 0$ for $\varrho > \tilde{\varrho}$. Similarly

$$\begin{aligned} \lim_{\alpha \downarrow 0} \mathbb{L} &= 0 & \lim_{\alpha \downarrow 0} \frac{\partial \mathbb{L}}{\partial \alpha} \frac{\alpha}{\mathbb{L}} &> 0 \\ \lim_{\alpha \uparrow 1} \mathbb{L} &= 0 & \lim_{\alpha \uparrow 1} \frac{\partial \mathbb{L}}{\partial \alpha} \frac{\alpha}{\mathbb{L}} &< 0 \end{aligned}$$

The first term in the expression for $\frac{\partial \mathbb{L}}{\partial \alpha} \frac{\alpha}{\mathbb{L}}$ is decreasing in α , going from 1 to $-\infty$, while the second is increasing, going from 0 to a positive number. Hence, $\exists! \tilde{\alpha} \in (0, 1)$ such that $\frac{\partial \mathbb{L}}{\partial \alpha} > 0$ for $\alpha < \tilde{\alpha}$, and $\frac{\partial \mathbb{L}}{\partial \alpha} < 0$ for $\alpha > \tilde{\alpha}$.

Proof of Propositions 3

PROOF The planner's problem is

$$\min_{\hat{x}} \frac{1}{2} \int_0^{\infty} e^{-\rho t} (\hat{x} + \phi \lambda^2) dt$$

subject to

$$\begin{aligned} d\lambda &= -\gamma (\hat{a} + \hat{f}_H^* - \hat{x}) dt \\ d\hat{a} &= (\rho \hat{a} - \alpha \lambda) dt \\ d\hat{f}_H^* &= -\varrho \hat{f}_H^* dt \end{aligned}$$

given the initial conditions $\hat{f}_H^*(0) = \varepsilon$, $\hat{a}(0) = 0$ and the terminal condition $\lim_{t \rightarrow \infty} e^{-\rho t} \hat{a}(t) = 0$. The Hamiltonian associated with this problem is

$$H = \frac{\phi}{2} \lambda^2 + \frac{1}{2} \hat{x}^2 + \mu^\lambda \left(-\gamma \hat{a} + \gamma \hat{x} - \gamma \hat{f}_H^* \right) + \mu^a (\rho \hat{a} - \alpha \lambda)$$

The FOC is $\hat{x} = -\gamma \mu^\lambda$, while the laws of motion of the costates are

$$d\mu^\lambda = \rho \mu^\lambda - \phi \lambda + \alpha \mu^a$$

$$d\mu^a = \gamma\mu^\lambda$$

subject to the initial condition $\mu^\lambda(0) = 0$. The first order conditions can be used to replace μ^λ and $d\mu^a$ and obtain a second order differential equation that the optimal foreign exchange intervention must satisfy:

$$dd\hat{x} = \rho d\hat{x} + (\alpha\gamma + \phi\gamma^2)\hat{x} - \phi\gamma^2(\hat{a} + \hat{f}_H^*)$$

with $\hat{x}(0) = 0$. Let $\mathbf{z}^\top = \begin{bmatrix} d\hat{x} & \hat{x} & \lambda & \hat{a} & \hat{f}_H^* \end{bmatrix}$, then the system of differential equations that must be solved is $d\mathbf{z} = \mathbf{A}\mathbf{z}dt$, where

$$\mathbf{A} = \begin{bmatrix} \rho & \alpha\gamma + \phi\gamma^2 & 0 & -\phi\gamma^2 & -\phi\gamma^2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -\gamma & -\gamma \\ 0 & 0 & -\alpha & \rho & 0 \\ 0 & 0 & 0 & 0 & -\varrho \end{bmatrix}$$

The eigenvalues of \mathbf{A} are $\begin{bmatrix} -\varrho & -\underline{\nu} & -\bar{\nu} & \rho + \underline{\nu} & \rho + \bar{\nu} \end{bmatrix}$, where

$$\underline{\nu} \equiv \frac{-\rho + \sqrt{\rho^2 + 2\gamma(2\alpha + \phi\gamma - \sqrt{\phi\gamma(4\alpha + \phi\gamma)})}}{2}$$

$$\bar{\nu} \equiv \frac{-\rho + \sqrt{\rho^2 + 2\gamma(2\alpha + \phi\gamma + \sqrt{\phi\gamma(4\alpha + \phi\gamma)})}}{2}$$

The solution can be written as

$$\begin{bmatrix} d\hat{x} \\ \hat{x} \\ \lambda \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix} = \zeta_1 \begin{bmatrix} \frac{\varrho^2(\rho+\varrho)\phi\gamma^2}{(\bar{\nu}-\varrho)(\rho+\varrho+\bar{\nu})(\underline{\nu}-\varrho)(\rho+\varrho+\underline{\nu})} \\ \frac{-\varrho(\rho+\varrho)\phi\gamma^2}{(\bar{\nu}-\varrho)(\rho+\varrho+\bar{\nu})(\underline{\nu}-\varrho)(\rho+\varrho+\underline{\nu})} \\ \frac{\gamma(\rho+\varrho)(\varrho(\rho+\varrho)-\alpha\gamma)}{(\bar{\nu}-\varrho)(\rho+\varrho+\bar{\nu})(\underline{\nu}-\varrho)(\rho+\varrho+\underline{\nu})} \\ \frac{\alpha\gamma(\varrho(\rho+\varrho)-\alpha\gamma)}{(\bar{\nu}-\varrho)(\rho+\varrho+\bar{\nu})(\underline{\nu}-\varrho)(\rho+\varrho+\underline{\nu})} \\ 1 \end{bmatrix} e^{-\varrho t} + \zeta_2 \begin{bmatrix} \frac{\underline{\nu}(\rho+\underline{\nu})-\alpha\gamma}{\alpha\gamma} \\ \frac{\underline{\nu}(\rho+\underline{\nu})-\alpha\gamma}{\alpha\gamma} \\ \frac{\rho+\underline{\nu}}{\alpha} \\ 1 \\ 0 \end{bmatrix} e^{-\underline{\nu}t} + \zeta_3 \begin{bmatrix} \frac{\bar{\nu}(\rho+\bar{\nu})-\alpha\gamma}{\alpha\gamma} \\ \frac{\bar{\nu}(\rho+\bar{\nu})-\alpha\gamma}{\alpha\gamma} \\ \frac{\rho+\bar{\nu}}{\alpha} \\ 1 \\ 0 \end{bmatrix} e^{-\bar{\nu}t}$$

where the coefficients $\begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}$ are determined using the initial conditions $\hat{a}(0) = 0$, $\hat{x}(0) = 0$ and $\hat{f}_H^*(0) = \varepsilon$. Thus, I obtain

$$\lambda = \begin{bmatrix} -\frac{\gamma}{\rho+\underline{\nu}+\bar{\nu}} \\ \frac{\xi}{\bar{\nu}}\gamma + \frac{1-\xi}{\underline{\nu}}\gamma \\ \frac{\xi\gamma}{\rho+\varrho+\bar{\nu}} + \frac{(1-\xi)\gamma}{\rho+\varrho+\underline{\nu}} \end{bmatrix}^\top \begin{bmatrix} \hat{x} \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

with $\xi \equiv \frac{\alpha\gamma}{\alpha\gamma + \underline{\nu}(\rho + \underline{\nu})}$. The states $\begin{bmatrix} \hat{x} & \hat{a} & \hat{f}_H^* \end{bmatrix}$ evolve as

$$\begin{bmatrix} d\hat{x} \\ d\hat{a} \\ d\hat{f}_H^* \end{bmatrix} = \begin{bmatrix} -\xi\bar{\nu} - (1-\xi)\underline{\nu} & \frac{\alpha\gamma}{\rho + \underline{\nu} + \bar{\nu}} & 0 \\ \frac{\phi\gamma^2}{\rho + \underline{\nu} + \bar{\nu}} & -\xi\underline{\nu} - (1-\xi)\bar{\nu} & 0 \\ \phi\gamma^2 \frac{\rho + \varrho + \frac{\underline{\nu}\bar{\nu}}{\rho + \underline{\nu} + \bar{\nu}}}{(\rho + \varrho + \bar{\nu})(\rho + \varrho + \underline{\nu})} & -\frac{\alpha\gamma\xi}{\rho + \varrho + \bar{\nu}} - \frac{\alpha\gamma(1-\xi)}{\rho + \varrho + \underline{\nu}} & -\varrho \end{bmatrix}^\top \begin{bmatrix} \hat{x} \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

with $\begin{bmatrix} \hat{x}(0) & \hat{a}(0) & \hat{f}_H^*(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \varepsilon \end{bmatrix}$.

Proof of Lemma 1

PROOF The optimal intervention rule can be derived by manipulating the closed form solutions of Proposition 3. The explicit expressions of the derivatives with respect to ϱ are

$$\begin{aligned} \frac{\partial |\psi_\lambda|}{\partial \varrho} &= \frac{\phi\gamma}{\rho + \bar{\nu} + \underline{\nu}} \frac{\xi\underline{\nu}(\rho + \underline{\nu}) + (1-\xi)\bar{\nu}(\rho + \bar{\nu})}{[\rho + \varrho + \xi\underline{\nu} + (1-\xi)\bar{\nu}]^2} > 0 \\ \frac{\partial |\psi_a|}{\partial \varrho} &= \frac{\phi\gamma}{\alpha} \left(\rho + \frac{\bar{\nu}\underline{\nu}}{\rho + \bar{\nu} + \underline{\nu}} \right) \frac{\xi\underline{\nu} + (1-\xi)\bar{\nu}}{[\rho + \varrho + \xi\underline{\nu} + (1-\xi)\bar{\nu}]^2} > 0 \\ \frac{\partial |\psi_x|}{\partial \varrho} &= -\frac{(1-\xi)\xi(\bar{\nu} - \underline{\nu})^2}{[\rho + \varrho + \xi\underline{\nu} + (1-\xi)\bar{\nu}]^2} < 0 \end{aligned}$$

The explicit expressions of the derivatives with respect to ϕ and γ have been derived using the symbolic toolkit available in Wolfram Mathematica. Their expressions are too big to be reported here and are available upon request.

Optimal Joint FX and Monetary Policies The planner's problem is

$$\min_{\hat{x}, i} \frac{1}{2} \int_0^\infty e^{-\rho t} [\hat{x} + \phi_\lambda \lambda^2 + \phi_\pi \pi_H^2 + \phi_y y^2] dt$$

subject to

$$\begin{aligned} d\lambda &= -\gamma (\hat{a} + \hat{f}_H^* - \hat{x}) dt \\ d\hat{a} &= (\rho\hat{a} - \alpha\lambda) dt \\ dy &= \left[i - \rho - \pi_H + \alpha\gamma (\hat{a} + \hat{f}_H^* - \hat{x}) \right] dt \\ d\pi_H &= [\rho\pi_H - \kappa(1 + \varphi)y - \alpha\kappa\lambda] dt \\ d\hat{f}_H^* &= -\varrho\hat{f}_H^* \end{aligned}$$

given the initial conditions $\hat{f}_H^*(0) = \varepsilon$, $\hat{a}(0) = 0$ and the terminal condition $\lim_{t \rightarrow \infty} e^{-\rho t} \hat{a}(t) = 0$. The Hamiltonian associated with this problem is

$$H = \frac{1}{2} \hat{x}^2 + \frac{\phi_\lambda}{2} \lambda^2 + \frac{\phi_\pi}{2} \pi_H^2 + \frac{\phi_y}{2} y^2 + \mu^\lambda \left(-\gamma \hat{a} - \gamma \hat{f}_H^* + \gamma \hat{x} \right) + \mu^a (\rho \hat{a} - \alpha \lambda) \\ + \mu^y \left[i - \rho - \pi_H + \alpha \gamma \left(\hat{a} + \hat{f}_H^* - \hat{x} \right) \right] + \mu^{\pi_H} [\rho \pi_H - \kappa (1 + \varphi) y - \alpha \kappa \lambda]$$

The FOCs are $\mu^y = 0$ and $\hat{x} = \alpha \gamma \mu^y - \gamma \mu^\lambda$, while the laws of motion of the costates are

$$\begin{aligned} d\mu^\lambda &= \rho \mu^\lambda - \phi_\lambda \lambda + \alpha \mu^a + \alpha \kappa \mu^{\pi_H} \\ d\mu^a &= \gamma \mu^\lambda - \alpha \gamma \mu^y \\ d\mu^y &= \rho \mu^y - \phi_y y + \kappa (1 + \varphi) \mu^{\pi_H} \\ d\mu^{\pi_H} &= -\phi_\pi \pi_H + \mu^y \end{aligned}$$

The optimal monetary policy is

$$i - \rho = \left[1 - \kappa (1 + \varphi) \frac{\phi_\pi}{\phi_y} \right] \pi_H dt - \alpha \gamma \left(\hat{a} + \hat{f}_H^* - \hat{x} \right)$$

while the optimal foreign intervention satisfies

$$d\hat{x} = \rho d\hat{x} + (\alpha \gamma + \phi_\lambda \gamma^2) \hat{x} - \phi_\lambda \gamma^2 \hat{a} - \phi_\lambda \gamma^2 \hat{f}_H^* + \gamma \alpha \kappa \phi_\pi \pi_H$$

Let $\mathbf{z}^\top = \left[y \quad \pi_H \quad d\hat{x} \quad \hat{x} \quad \lambda \quad \hat{a} \quad \hat{f}_H^* \right]$, then the system of differential equations that must be solved is $d\mathbf{z} = \mathbf{A} \mathbf{z} dt$ where

$$\mathbf{A} = \begin{bmatrix} 0 & -\kappa (1 + \varphi) \frac{\phi_\pi}{\phi_y} & 0 & 0 & 0 & 0 & 0 \\ -\kappa (1 + \varphi) & \rho & 0 & 0 & -\alpha \kappa & 0 & 0 \\ 0 & \kappa \alpha \gamma \phi_\pi & \rho & \alpha \gamma + \gamma^2 \phi_\lambda & 0 & -\phi_\lambda \gamma^2 & -\phi_\lambda \gamma^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & -\gamma & -\gamma \\ 0 & 0 & 0 & 0 & -\alpha & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varrho \end{bmatrix}$$

The solution has the following form

$$\mathbf{z} = \zeta_1 \mathbf{v}_1 e^{\nu_1 t} + \zeta_2 \mathbf{v}_2 e^{\nu_2 t} + \zeta_3 \mathbf{v}_3 e^{\nu_3 t} + \zeta_4 \mathbf{v}_4 e^{\nu_4 t} + \zeta_5 \mathbf{v}_5 e^{\nu_5 t} + \zeta_6 \mathbf{v}_6 e^{\nu_6 t} + \zeta_7 \mathbf{v}_7 e^{\nu_7 t}$$

where ν_j and \mathbf{v}_j are the eigenvalues and associated eigenvectors of \mathbf{A} . Unfortunately, the eigenvalues of \mathbf{A} cannot be derived analytically, therefore the system can only be solved numerically.

Optimal Monetary Policy

PROOF The problem is similar to the one solved before, but this time $\hat{x} = 0$. Let $\mathbf{z}^\top = \begin{bmatrix} y & \pi_H & \lambda & \hat{a} & \hat{f}_H^* \end{bmatrix}$, then the system of differential equations that must be solved is $d\mathbf{z} = \mathbf{A}\mathbf{z}dt$ where

$$\mathbf{A} = \begin{bmatrix} 0 & -\kappa(1+\varphi)\frac{\phi_\pi}{\phi_y} & 0 & 0 & 0 \\ -\kappa(1+\varphi) & \rho & -\alpha\kappa & 0 & 0 \\ 0 & 0 & 0 & -\gamma & -\gamma \\ 0 & 0 & -\alpha & \rho & 0 \\ 0 & 0 & 0 & 0 & -\varrho \end{bmatrix}$$

given the initial conditions $\hat{f}_H^*(0) = \varepsilon$, $\hat{a}(0) = y_H(0) = 0$ and the terminal condition $\lim_{t \rightarrow \infty} e^{-\rho t} \hat{a}(t) = 0$. The eigenvalues of \mathbf{A} are $\begin{bmatrix} -\varrho & -\nu & -\iota & \rho + \nu & \rho + \iota \end{bmatrix}$ where $\iota = \frac{-\rho + \sqrt{\rho^2 + 4\kappa^2(1+\varphi)^2 \frac{\phi_\pi}{\phi_y}}}{2}$. The solution is

$$\begin{bmatrix} y \\ \pi_H \\ \lambda \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix} = \zeta_1 \begin{bmatrix} -\frac{\nu(\rho+\nu)(\rho+\varrho)(\rho+\iota)}{(\varrho-\iota)(\rho+\varrho+\iota)} \\ -\kappa \frac{\nu(\rho+\nu)\varrho(\rho+\varrho)}{(\varrho-\iota)(\rho+\varrho+\iota)} \\ -\gamma(\rho+\varrho) \\ -\nu(\rho+\nu) \\ (\nu-\varrho)(\rho+\varrho+\nu) \end{bmatrix} e^{-\varrho t} + \zeta_2 \begin{bmatrix} \frac{(\rho+\nu)\iota(\rho+\iota)}{(\nu-\iota)(\rho+\nu+\iota)} \\ \kappa \frac{\nu(\rho+\nu)}{(\nu-\iota)(\rho+\nu+\iota)} \\ \frac{\gamma}{\nu} \\ 1 \\ 0 \end{bmatrix} e^{-\nu t} + \zeta_3 \begin{bmatrix} \frac{\rho+\iota}{\kappa} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-\iota t}$$

Thus, I obtain

$$\begin{bmatrix} \pi_H \\ \lambda \end{bmatrix} = \begin{bmatrix} \kappa \frac{1+\varphi}{\rho+\iota} & 0 \\ \kappa \frac{\rho+\nu}{\rho+\nu+\iota} & \frac{\gamma}{\nu} \\ \frac{\kappa\alpha\gamma(1+\frac{\rho+\nu}{\varrho-\iota})}{(\rho+\varrho+\iota)(\rho+\varrho+\nu)} & \frac{\gamma}{\rho+\varrho+\nu} \end{bmatrix}^\top \begin{bmatrix} y \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

$$\pi_H = \frac{\kappa\alpha\gamma(1+\frac{\rho+\nu}{\varrho-\iota})}{(\rho+\varrho+\iota)(\rho+\varrho+\nu)} \hat{f}_H^* + \kappa \frac{\rho+\nu}{\rho+\nu+\iota} \hat{a} + \kappa \frac{(1+\varphi)}{\rho+\iota} y$$

with

$$\begin{bmatrix} dy \\ d\hat{a} \\ d\hat{f}_H^* \end{bmatrix} = \begin{bmatrix} -\iota & 0 & 0 \\ \frac{-\iota(\rho+\iota)(\rho+\nu)}{(1+\varphi)(\rho+\nu+\iota)} & -\nu & 0 \\ \frac{-\iota^2(\rho+\iota)\nu(\rho+\nu)}{(1+\varphi)(\rho+\varrho+\nu)(\rho+\varrho+\iota)(\rho+\nu+\iota)} & -\nu \frac{\rho+\nu}{\rho+\varrho+\nu} & -\varrho \end{bmatrix}^\top \begin{bmatrix} y \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

and the initial conditions $\begin{bmatrix} y(0) & \hat{a}(0) & \hat{f}_H^*(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \varepsilon \end{bmatrix}$.

Proof of Proposition 4

PROOF The planner solves

$$\min_i \frac{1}{2} \int_0^\infty e^{-\rho t} (\phi_\pi \pi_H^2 + \phi_y y^2) dt$$

subject to

$$\begin{aligned} dy &= \left[i - \rho - \pi_H + \alpha\gamma (\hat{a} + \hat{f}_H^* - \hat{x}) \right] dt \\ d\pi_H &= (\rho\pi_H - \kappa y - \alpha\kappa\lambda) dt \\ d\hat{f}_H^* &= -\varrho \hat{f}_H^* \end{aligned}$$

The optimal monetary policy is still

$$i - \rho = \left(1 - \kappa \frac{\phi_\pi}{\phi_y} \right) \pi_H dt + \alpha d\lambda$$

while the foreign exchange intervention satisfies

$$d\hat{x} = \rho d\hat{x} + (\alpha\gamma + \phi\gamma^2) \hat{x} - \phi\gamma^2 (\hat{a} + \hat{f}_H^*)$$

Let $\mathbf{z}^\top = \left[y \ \pi_H \ d\hat{x} \ \hat{x} \ \lambda \ \hat{a} \ \hat{f}_H^* \right]$, then the system of differential equations that must be solved is $d\mathbf{z} = \mathbf{A}\mathbf{z}dt$ where

$$\mathbf{A} = \begin{bmatrix} 0 & -\kappa(1+\varphi) \frac{\phi_\pi}{\phi_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\kappa(1+\varphi) & \rho & 0 & 0 & -\alpha\kappa & 0 & 0 & 0 \\ 0 & 0 & \rho & \alpha\gamma + \phi\gamma^2 & 0 & -\phi\gamma^2 & -\phi\gamma^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & -\gamma & -\gamma & 0 \\ 0 & 0 & 0 & 0 & -\alpha & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varrho \end{bmatrix}$$

given the initial conditions $\hat{f}_H^*(0) = \varepsilon$, $\hat{a}(0) = \hat{x}(0) = y(0) = 0$ and the terminal condition $\lim_{t \rightarrow \infty} e^{-\rho t} \hat{a}(t) = 0$. The eigenvalues of \mathbf{A} are $\left[-\varrho \ -\underline{\nu} \ -\bar{\nu} \ -\iota \ \rho + \underline{\nu} \ \rho + \bar{\nu} \ \rho + \iota \right]$. Using the same approach as before I obtain

$$\begin{bmatrix} \pi_H \\ \lambda \end{bmatrix} = \begin{bmatrix} \frac{\kappa \frac{1+\varphi}{\rho+\iota}}{\frac{(\rho+\underline{\nu}+\bar{\nu})(\rho+\iota+\underline{\nu})(\rho+\iota+\bar{\nu})}{-\alpha\gamma\iota\kappa}} & 0 \\ \kappa\xi \frac{\rho+\underline{\nu}}{\rho+\iota+\underline{\nu}} + \kappa(1-\xi) \frac{\rho+\bar{\nu}}{\rho+\iota+\bar{\nu}} & \frac{\gamma}{\rho+\underline{\nu}+\bar{\nu}} \\ \frac{\alpha\gamma}{\bar{\nu}-\varrho} \frac{\rho\varrho+\varrho^2-\alpha\gamma}{\rho+\varrho+\bar{\nu}} - \frac{\kappa}{\underline{\nu}-\varrho} \frac{\rho+\iota+\varrho}{\rho+\varrho+\underline{\nu}} + & \frac{\xi}{\bar{\nu}}\gamma + \frac{1-\xi}{\underline{\nu}}\gamma \\ \frac{1-\xi}{\underline{\nu}-\varrho} \frac{\alpha\gamma\kappa}{\rho+\iota+\underline{\nu}} + \frac{\xi}{\bar{\nu}-\varrho} \frac{\alpha\gamma\kappa}{\rho+\iota+\bar{\nu}} & \frac{\xi\gamma}{\rho+\varrho+\bar{\nu}} + \frac{(1-\xi)\gamma}{\rho+\varrho+\underline{\nu}} \end{bmatrix}^\top \begin{bmatrix} y \\ \hat{x} \\ \hat{a} \\ \hat{f}_H^* \end{bmatrix}$$

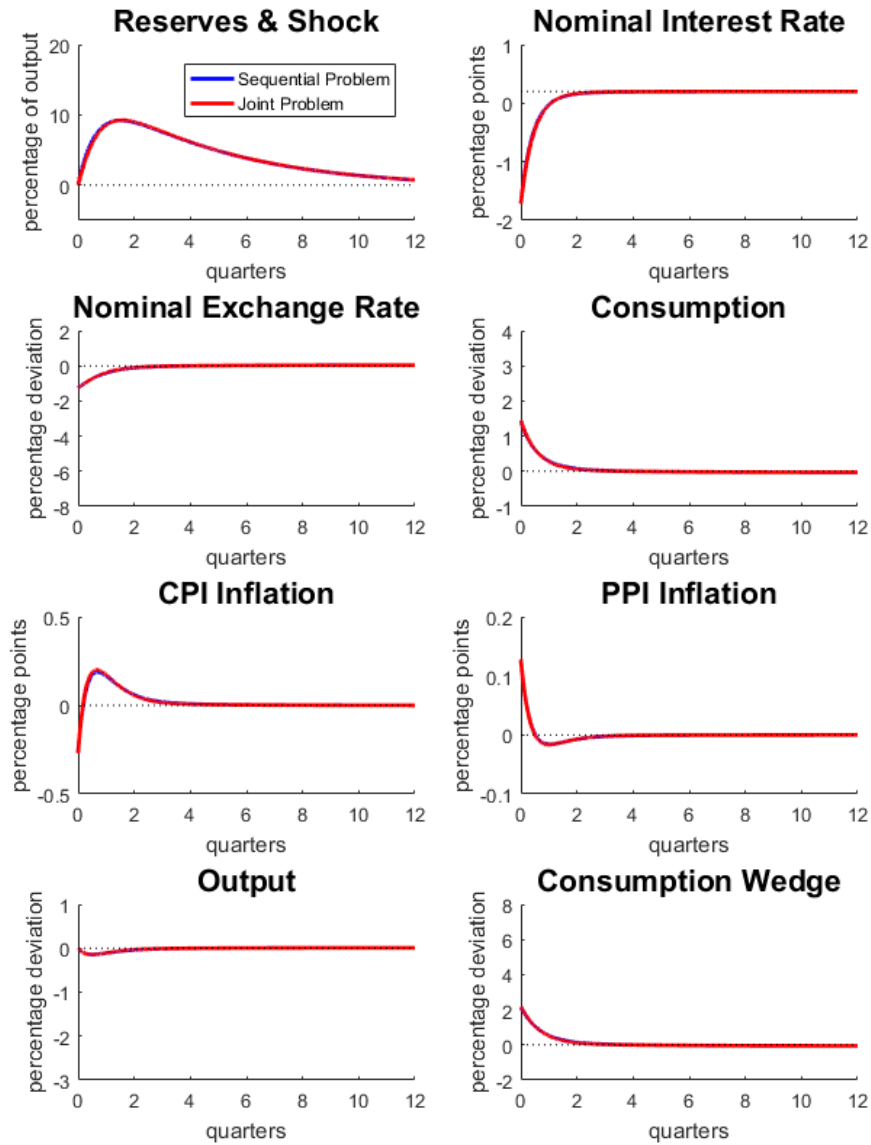
while the states $\mathbf{s} = \left[y \quad \hat{x} \quad \hat{a} \quad \hat{f}_H^* \right]^\top$ evolve as $d\mathbf{s} = -\mathbf{M}sdt$ where

$$\mathbf{M} = \begin{bmatrix} \iota & 0 & 0 & 0 \\ \frac{-\alpha\gamma\iota^2(\rho+\iota)}{(1+\varphi)(\rho+\underline{\nu}+\bar{\nu})(\rho+\iota+\underline{\nu})(\rho+\iota+\bar{\nu})} & \xi\bar{\nu} + (1-\xi)\underline{\nu} & -\frac{\alpha\gamma}{\rho+\underline{\nu}+\bar{\nu}} & 0 \\ \iota\frac{\rho+\iota}{1+\varphi} \left[\xi\frac{\rho+\underline{\nu}}{\rho+\iota+\underline{\nu}} + (1-\xi)\frac{\rho+\bar{\nu}}{\rho+\iota+\bar{\nu}} \right] & -\frac{\phi\gamma^2}{\rho+\underline{\nu}+\bar{\nu}} & \xi\underline{\nu} + (1-\xi)\bar{\nu} & 0 \\ \frac{\iota\alpha\gamma}{\bar{\nu}-\varrho} \frac{\rho\varrho+\varrho^2-\alpha\gamma}{1+\varphi} \frac{\rho+\iota}{\rho+\varrho+\bar{\nu}} \frac{\rho+\varrho}{\underline{\nu}-\varrho} \frac{\rho+\varrho}{\rho+\varrho+\underline{\nu}} + & \frac{-\phi\gamma^2}{\rho+\varrho+\bar{\nu}} \frac{\rho+\varrho+\frac{\underline{\nu}\bar{\nu}}{\rho+\underline{\nu}+\bar{\nu}}}{\rho+\varrho+\underline{\nu}} & \frac{\alpha\gamma\xi}{\rho+\varrho+\bar{\nu}} + \frac{\alpha\gamma(1-\xi)}{\rho+\varrho+\underline{\nu}} & \varrho \\ \frac{\rho+\iota}{\underline{\nu}-\varrho} \frac{\iota\alpha\gamma(1-\xi)}{(1+\varphi)(\rho+\varrho+\underline{\nu})} \frac{\rho+\underline{\nu}}{\rho+\iota+\underline{\nu}} + \frac{\rho+\iota}{\bar{\nu}-\varrho} \frac{\iota\alpha\gamma\xi}{(1+\varphi)(\rho+\varrho+\bar{\nu})} \frac{\rho+\bar{\nu}}{\rho+\iota+\bar{\nu}} & & & \end{bmatrix}^\top$$

with $\left[y(0) \quad \hat{x}(0) \quad \hat{a}(0) \quad \hat{f}_H^*(0) \right] = \left[0 \quad 0 \quad 0 \quad \varepsilon \right]$.

Figure D.1 plots the allocation implemented by the solution of the sequential problem, blue line, and the solution of the joint problem, red line. The two allocations are almost indistinguishable from each other. In light of the discussion developed in the main body of the paper on the relationship between the two tools, this is not a surprising result. The planner wants to stabilize the path of λ for two reasons. First, because it enters directly into the loss function with weight ϕ_λ . Second, because it shifts the Phillips curve and improves the trade-off between the output gap and domestic inflation. The sequential problem does not ignore the second channel, but rather it collapses both channels into the choice of a single parameter, ϕ . In fact, the optimal ϕ that minimizes total welfare in Proposition 4 is strictly greater than ϕ_λ . The remaining difference between the two solutions is only due to the relative dynamics of λ , y , and π_H , and is therefore negligible.

Figure D.1: Sequential Problem vs Joint Problem



Impulse responses for the solution to the joint problem (blue line) and to the sequential problem (red line).

E Details of the Empirical Analysis and Robustness Checks

Discrete-Time Model

The equations of the model used to estimate the parameters γ and χ are the discrete-time versions of the log-linearized equations in Section C. The complete system is

$$\begin{aligned}\Delta q_{t+1} &= r_t - \omega \Delta y_{t+1}^* + \gamma (\hat{a}_t + \chi vix_t^*) \\ \Delta \hat{a}_t &= \rho \hat{a}_{t-1} + \frac{\alpha}{1-\alpha} \left[y_t^* - y_t + \left(\frac{2-\alpha}{1-\alpha} \eta - 1 \right) q_t \right] \\ \Delta y_{t+1} &= \alpha \Delta y_{t+1}^* + \frac{1-\alpha}{\omega} r_t + \alpha \eta \frac{2-\alpha}{1-\alpha} \Delta q_{t+1}\end{aligned}$$

where r is the domestic real interest rate

$$r_t = i_t - \pi_{H,t+1} - \frac{\alpha}{1-\alpha} \Delta q_{t+1}$$

and

$$\begin{aligned}\Delta \pi_{H,t+1} &= \rho \pi_{H,t} - \frac{\alpha \kappa}{1-\alpha} \left[\frac{\omega + \varphi(1-\alpha)}{\alpha} y_t - \omega y_t^* + \left(1 - \omega \eta \frac{2-\alpha}{1-\alpha} \right) q_t \right] \\ i_t - \rho &= \psi_\pi \left(\pi_{H,t} + \frac{\alpha}{1-\alpha} \Delta q_t \right) + \psi_y y_t\end{aligned}$$

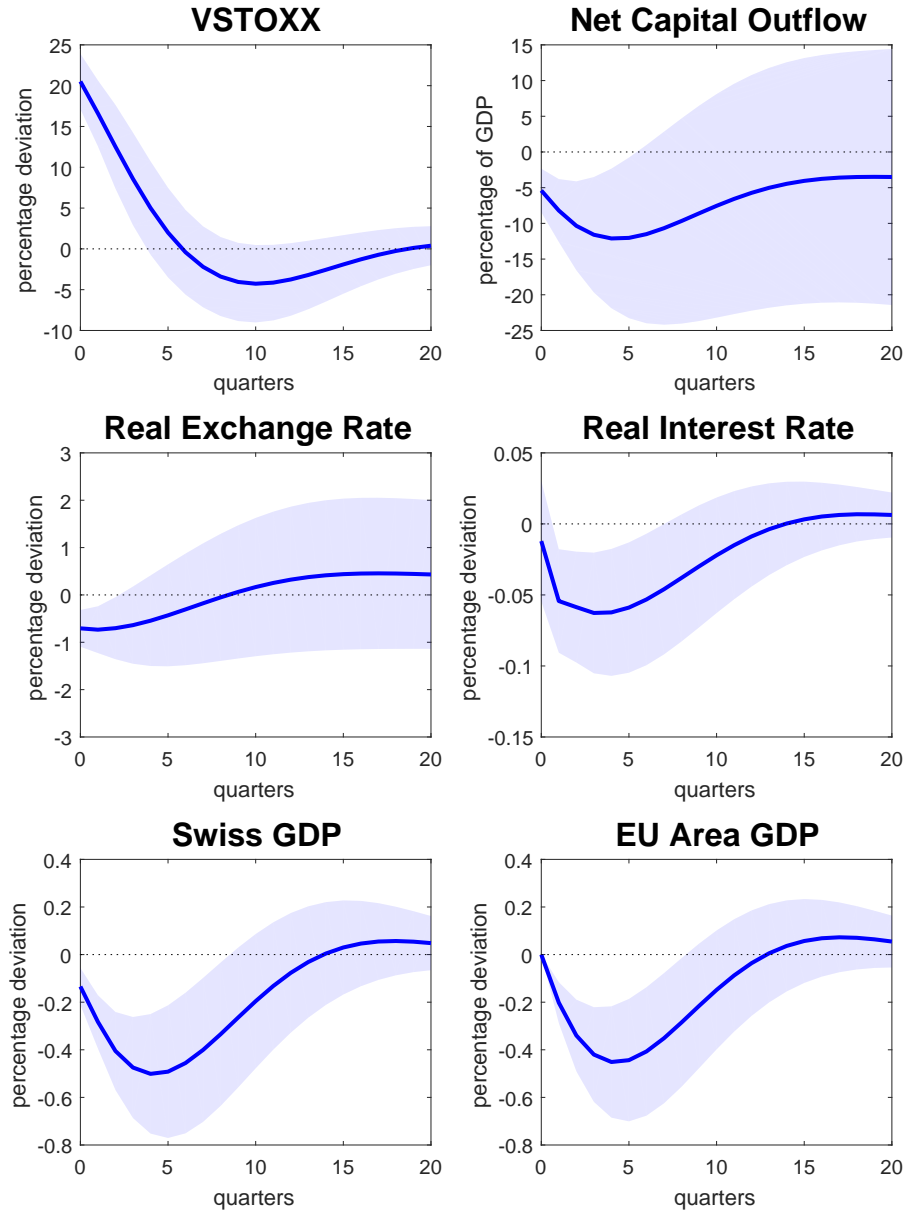
with $\kappa = \frac{1-\theta}{\theta} (1 + \rho - \theta)$. The exogenous variables vix_t^* and y_t^* follow a VAR(1) process as estimated on the observed data.

Robustness Checks

To gauge the robustness of the empirical analysis I perform four robustness exercises. First, I re-estimate the VAR model including the Swiss real interest rate. The real interest rate is computed as the difference between the 3-month interbank rate and the one quarter ahead realized core CPI inflation, where the core CPI index excludes food, beverage, tobacco, and energy prices. The impulse responses plotted in Figure E.1 show that an increase in the foreign demand for domestic assets is associated with a fall in the domestic real interest rate of up to 0.6 percentage points. This strengthens the case for using foreign exchange intervention in response to capital flow shocks, as monetary policy alone is unable to stabilize the real interest rate. Second, I re-estimate both the VAR model and the economic model using a mixture of EU and US data. Foreign output and global risk aversion are obtained as weighted averages of EU and US data, with weights of 0.7 and 0.3 respectively. US output is measured by real GDP figures published by the U.S. Bureau of Economic Analysis, while US

global risk aversion is proxied by the CBOE Volatility Index (VIX). Results of the estimation are reported in Figure E.2. The empirical impulse responses are very similar to those obtained with the baseline specification. The main difference is the smaller appreciation of the Swiss franc real exchange rate, which is also reflected in the slightly lower estimate obtained for γ . Both, however, are still statistically significant. This is unsurprising and is due to the fact that the dollar is itself a reserve currency and tends to appreciate in times of global uncertainty. The third robustness check involves the identification of the global risk-aversion shock. I re-estimate the VAR and the model by ordering *via** first, to allow foreign output to respond contemporaneously to innovation in the VSTOXX. Results are reported in Figure E.3. Again, the empirical impulse responses are quite similar to the baseline specification. Interestingly enough, the estimation of the structural parameters shifts some weight from χ to γ . The estimates for γ is 50 percent higher than in the baseline. By inspecting the empirical impulse responses, it becomes apparent that this result is driven by the larger response of capital outflow. Finally, in the fourth robustness check, I relax the assumption of unitary elasticities of intertemporal and intratemporal substitution. Following Baurle and Menz (2008) and Baurle and Kaufmann (2014), I set ω equal to 1.2 and η equal to 1.5. Results are reported in Figure E.4. The estimate for γ is remarkably close to the baseline estimate, while the estimate for χ is higher. With this calibration, however, the model does a better job at matching the empirical impulse responses. While the theoretical impulse response for the real exchange rate is barely affected, the impulse response for net capital outflow is closer to its empirical counterpart and lies entirely within the 90 percent confidence interval.

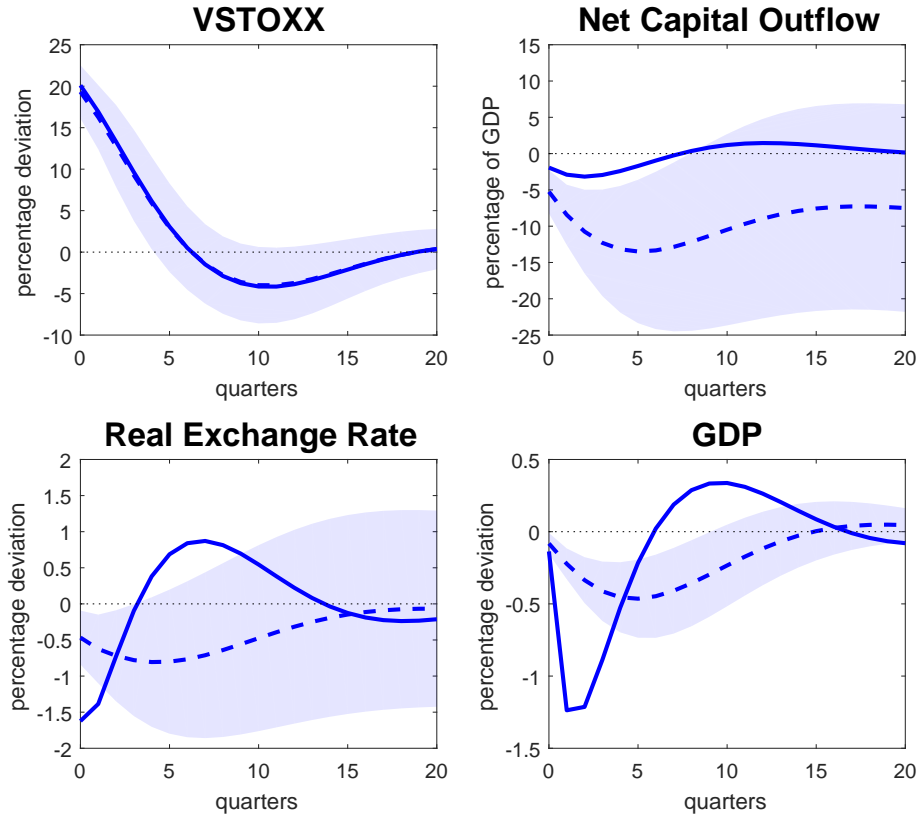
Figure E.1: VAR-based Impulse Responses (Augmented VAR)



VAR-based impulse responses (solid line) and 90 percent confidence intervals (shaded areas). Capital outflow is expressed as cumulative deviation from the unshocked path in percentage of GDP. All other variables are expressed as percentage deviation from their unshocked path.

Figure E.2: Estimation Results and Impulse Responses (EU and US Data)

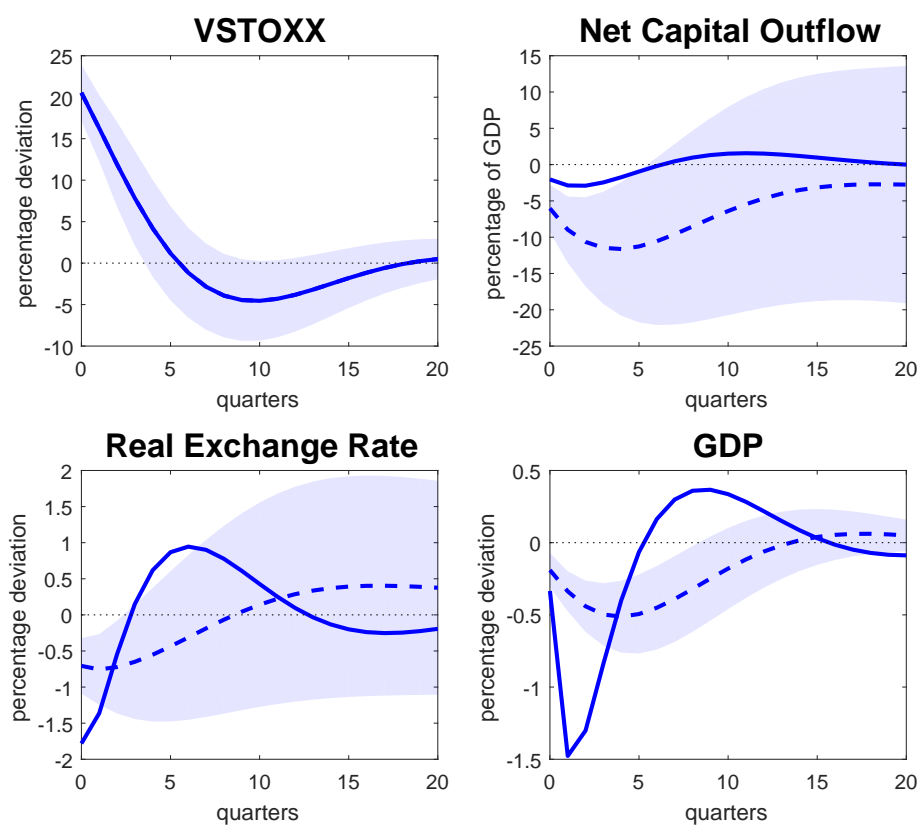
Parameter	Description	Value	S. E.
γ	Financial sector inverse risk-bearing capacity	0.171	0.062
χ	Proportionality between vix and investors' demand	0.756	0.313



Model-based impulse responses (solid line), VAR-based impulse responses (dashed line) and 90 percent confidence intervals (shaded areas). Net capital outflow is expressed as cumulative deviation from the unshocked path in percentage of GDP. All other variables are expressed as percentage deviation from their unshocked path.

Figure E.3: Estimation Results and Impulse Responses (Alternative Identification)

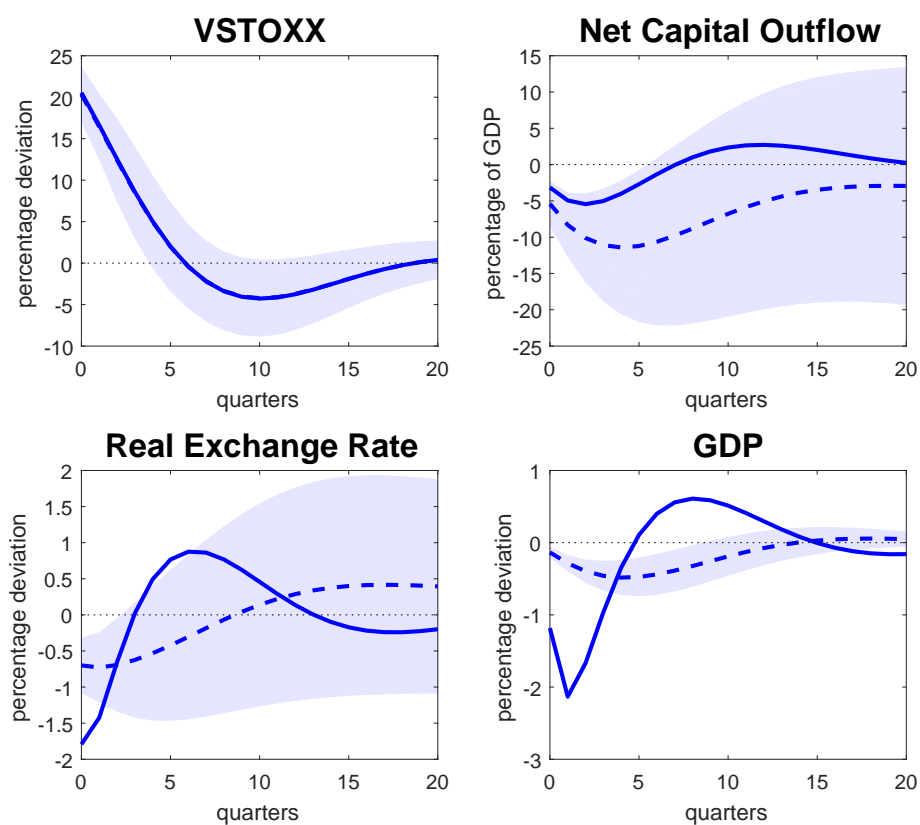
Parameter	Description	Value	S. E.
γ	Financial sector inverse risk-bearing capacity	0.296	0.125
χ	Proportionality between vix and investors' demand	0.573	0.245



Model-based impulse responses (solid line), VAR-based impulse responses (dashed line) and 90 percent confidence intervals (shaded areas). Net capital outflow is expressed as cumulative deviation from the unshocked path in percentage of GDP. All other variables are expressed as percentage deviation from their unshocked path.

Figure E.4: Estimation Results and Impulse Responses (Alternative Calibration)

Parameter	Description	Value	S. E.
γ	Financial sector inverse risk-bearing capacity	0.145	0.072
χ	Proportionality between vix and investors' demand	1.236	0.613



Model-based impulse responses (solid line), VAR-based impulse responses (dashed line) and 90 percent confidence intervals (shaded areas). Net capital outflow is expressed as cumulative deviation from the unshocked path in percentage of GDP. All other variables are expressed as percentage deviation from their unshocked path.

References

- Bäurle, Gregor, and Daniel Kaufmann.** 2014. “Exchange rate and price dynamics in a small open economy – The role of the zero lower bound and monetary policy regimes.” *SNB Working Paper No. 2014-10*.
- Bäurle, Gregor, and Tobias Menz.** 2008. “Monetary policy in a small open economy model: A DSGE-VAR approach for Switzerland.” *Study Center Gerzensee, Working Paper No. 08.03*.