Under-Employment and the Trickle-Down of Unemployment—Online Appendix Not for Publication

Regis Barnichon Yanos Zylberberg

March 30, 2018

Section 1 contains the proofs of Propositions 1 to 3 pertaining to the static model as well as a graphical representation of the model in general equilibrium. Section 2 contains the proof of Proposition 4 pertaining to the dynamic model. Section 3 generalizes the static model to $N = 3$ islands.

1 Complements for the static model

1.1 Proof of Proposition 1: partial equilibrium with under-employment

Consider first the problem of type-h workers. A type-h worker has two choices, he can (i) look for a job in island H, or (ii) look for a job in island L, i.e., move down the occupation ladder. We now consider these two possibilities.

When a type-h worker looks for a job in island H, he faces two possible outcomes: (a) with probability $e^{-q_H(1-x_h)}$, he is the only applicant and receives $\beta\varphi_{hH}$, or (b), with probability $1-e^{-q_H(1-x_h)}$, he is in competition with other workers and receives 0 (regardless of whether he ends up employed or unemployed). The expected payoff of a worker type-h who searches for a job in island H, $E\omega_{hH}$, is thus

$$E\omega_{hH} = \beta e^{-q_H(1-x_h)}\varphi_{hH}.$$

The expected wage is increasing in x_h. When a lot of type-h workers descend to island L, it becomes easier for the ones who stayed in island H to be the only applicant to a job and receive a high wage.

When a type-h worker looks for a job in island L, he faces three possible outcomes: (a) with probability $e^{-q_L x_h n_h}e^{-q_L}$, he is the only applicant and receives $\beta\varphi_{hL}$. Note that he produces less than in his "home" island and thus receives a lower wage than would have been the case if he had been the only applicant to a type-h firm, (b) with probability $1 - e^{-q_L x_h n_h}$, he is in competition with other type-h workers and receives 0 (regardless of whether he ends up employed or unemployed), and (c) with probability $e^{-q_L x_h n_h}(1 - e^{-q_L})$, he is in competition with type ℓ workers only and receives $\beta(\varphi_{hL} - \varphi_{\ell L})$.[1] The expected payoff of a worker type-h who searches for a job in island L, ω_{hL}, is thus

$$E\omega_{hL} = \beta e^{-q_L x_h n_h} e^{-q_L} \varphi_{hL} + \beta e^{-q_L x_h n_h} (\varphi_{hL} - \varphi_{\ell L}) \left[1 - e^{-q_L}\right].$$

The expected wage in island L is decreasing in x_h: when there are fewer type-h workers in island L, there is less competition in island L, and type-h workers can expect a higher wage.

In order to ensure under-employment in equilibrium, we need to assume that a high-skill worker would have a higher expected wage in the low-tech island when all high-skill workers remain in the high-tech island. Specifically, the condition writes $E\omega_{hH}(x_h = 0) < E\omega_{hL}(x_h = 0)$, i.e.,

$$e^{-q_H} \varphi_{hH} < \varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L} \tag{A1}$$

Under condition (A1), there is some under-employment in equilibrium and a type-h worker must be indifferent between looking for a job in island H or in island L. The arbitrage condition, $A(x_h)$, determines, x_h, the equilibrium allocation of workers

$$A(x_h) = -e^{-q_H(1-x_h)} \varphi_{hH} + e^{-q_L x_h n_h} e^{-q_L} \varphi_{hL} + e^{-q_L x_h n_h} (\varphi_{hL} - \varphi_{\ell L}) \left[1 - e^{-q_L}\right] = 0 \tag{A2}$$

We now show that when there is under-employment of type-h workers, type ℓ workers remain in island L and do not search in island H.

When a type ℓ worker looks for a job in island L, he faces two possible outcomes: (a) with probability $e^{-q_L(1+n_h x_h)}$, he is the only applicant and receives $\beta\varphi_{\ell L}$, or (b), with probability $1 - e^{-q_L(1+n_h x_h)}$, he is in competition with other workers and receives 0:

$$E\omega_{\ell L} = \beta e^{-q_L(1+n_h x_h)} \varphi_{\ell L}.$$

Type-ℓ workers will not search for a job in island H as long as there are type-h workers in island L. Indeed, as long as type-h workers are indifferent between their "home" island and the island below, type-ℓ workers will always prefer to remain in island L. In island H, type-ℓ workers would only receive a positive wage when

[1] As noted earlier, despite the presence of competing applicants, a single type-h applicant can extract some of the surplus thanks to due to his productivity advantage over the other applicants.

not competing with any other applicants, i.e., $E\omega_{\ell H} = \beta e^{-q_H(1-x_h)}\varphi_{\ell H}$ and Equation (A2) implies:

$$E\omega_{\ell H} = E\omega_{hH}\frac{\varphi_{\ell H}}{\varphi_{hH}} = e^{-q_L x_h n_h}\frac{\varphi_{\ell H}}{\varphi_{hH}}(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L}e^{-q_L}).$$

As a consequence,

$$E\omega_{\ell H} = \frac{\varphi_{\ell H}}{\varphi_{hH}}\frac{\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L}e^{-q_L}}{\varphi_{\ell L}e^{-q_L}}E\omega_{\ell L},$$

Log super-modularity, $\frac{\varphi_{\ell H}}{\varphi_{hH}} < \frac{\varphi_{\ell L}}{\varphi_{hL}}$, implies that

$$\frac{\varphi_{\ell H}}{\varphi_{hH}}\frac{\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L}e^{-q_L}}{\varphi_{\ell L}e^{-q_L}} < 1$$

and thus $E\omega_{\ell H} < E\omega_{\ell L}$.

To see that the equilibrium is unique, note that the trade-off faced by type-h workers is monotonic, i.e., as they apply more in island L, their relative gain of doing so is strictly decreasing. Since the relative gain for a type-h worker of searching in island L is initially positive for $x_h = 0$ (condition A1) but is negative when $x_h = 1$ (since $\varphi_{hH} > \varphi_{hL}$), it only crosses the x-axis once, and the intersection defines the unique equilibrium.

1.2 General equilibrium with under-employment

Proof of Proposition 2 The workers' no-arbitrage conditions are already derived in Proposition 1. The number of job openings in each island is given by the free entry condition, and we only need to express the firm's expected profit as a function the number of job openings v_L or the "initial" queue length $q_L = \frac{n_\ell}{v_L}$.

Consider first a firm that enters island L. The firm's profit will depend on the number of applications it receives. There are five cases:

1. The firm has no applicant. Profit is zero.
2. The firm has only one applicant. The firm gets a share $1-\beta$ of the generated surplus, i.e., $(1-\beta)\varphi_{\ell L}$ if the applicant is of type-ℓ (which happens with probability $P(a_1 = 1, a_2 = 0) = q_L e^{-q_L}e^{-q_L x_h n_h}$), and $(1-\beta)\varphi_{hL}$ if the applicant is of type-h (which happens with probability $P(a_1 = 0, a_2 = 1) = q_L x_h n_h e^{-q_L x_h n_h}e^{-q_L}$).
3. The firm has more than one applicant of type-ℓ (and no applicant of type-h). The firm gets all the surplus: $\varphi_{\ell L}$.

This happens with probability $e^{-x_h n_h q_L}\left[1 - e^{-q_L} - q_L e^{-q_L}\right]$.

4. The firm has more than one applicants of type-h. The firm gets all the surplus: φ_{hL}. This happens with probability $1 - e^{-x_h n_h q_L} - x_h n_h q_L e^{-x_h n_h q_L}$.

5. The firm has one applicant of type-h, and at least one other low-type applicant. The most productive worker is hired and gets a share β of the surplus generated over hiring the second-best applicant. The firm generates a profit $\varphi_{\ell L} + (1 - \beta)(\varphi_{hL} - \varphi_{\ell L})$. This happens with probability $x_h n_h q_L e^{-x_h n_h q_L}(1 - e^{-q_L})$.

The expected profit for a firm with technology L is thus given by

$$\pi_L(x_h, q_L) = \varphi_{hL} - e^{-x_h n_h q_L}\left[\left(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right)(1 + (2-\beta)x_h n_h q_L) + (2-\beta)\varphi_{\ell L} q_L e^{-q_L}\right].$$

Proceeding in a similar fashion, one can show that the expected profit for a firm with technology H is given by

$$\pi_H(x_h, q_H) = \left(1 - e^{-(1-x_h)q_H} - (1-x_h)q_H e^{-(1-x_h)q_H}\right)\varphi_{hH} + (1-x_h)q_H e^{-(1-x_h)q_H}(1-\beta)\varphi_{hH}.$$

Free entry imposes two no-profit conditions in addition to workers' arbitrage equations. The uniqueness of the equilibrium is a direct consequence of Corollaries A1 and A2 that we prove next.

Expected wage curves in general equilibrium As in the Partial Equilibrium (PE) case described in the main text, Figure A2 depicts the equilibrium underemployment rate as the intersection the $E\omega_{hL}$ curve, the expected wage earned in island L, and the $E\omega_{hH}$ curve, the expected wage earned in island H. The following corollary captures formally how the expected wage in island L or H (accounting for free-entry conditions) depends on the share of type-h workers searching in island L.

Corollary A1 (Expected income of type-h workers)**.** *Conditional on the free-entry conditions holding in both islands, the expected income of type-h workers searching in island H, $E\omega_{hH}(x_h, q_H(x_h))$, is independent of x_h. The expected income of type-h workers searching in island L, $E\omega_{hL}(x_h, q_L(x_h))$, is strictly decreasing in x_h with* $\left|\frac{dE\omega_{hL}}{dx_h}\right| < \left|\frac{\partial E\omega_{hL}}{\partial x_h}\right|$.

Proof. First, it is straightforward from the expression of $\pi_H(x_h, q_H)$ that the free entry condition $\pi_H = c_H$ imposes that $q_H(1 - x_h)$ is constant, so that the expected wage in island H, $E\omega_{hH}$, is constant. We can thus restrict our analysis to the arbitrage condition coupled with the free entry condition in island L.

$$\begin{cases} E\omega_{hL} = \left[\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right] e^{-q_L x_h n_h} & (L^S) \\ \varphi_{hL} - c_L = \left[\left(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right)(1 + (2-\beta) q_L x_h n_h) + (2-\beta)\varphi_{\ell L} q_L e^{-q_L}\right] e^{-q_L x_h n_h} & (L_1^D) \end{cases}$$

The (L_1^D) equation, or free-entry condition, defines a job creation function $q_L(x_h)$. As before, we only consider interior solutions, i.e., we impose the condition

$$E\omega_{hL}\left(x_h = 1, q_L(x_h = 1)\right) < E\omega_{hH} < E\omega_{hL}\left(x_h = 0, q_L(x_h = 0)\right) \tag{A3}$$

where $q_L(x_h)$ captures the general equilibrium response of vacancy posting to movements in x_h and is given by the free entry condition.

Under condition (A3), the relative gain of searching for a job in island L is positive for $x_h = 0$ ($E\omega_{hH} < E\omega_{hL}$) and negative for $x_h = 1$ ($E\omega_{hH} > E\omega_{hL}$). More specifically, the condition imposes

$$\left[\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L(x_h=1)}\right] e^{-q_L(x_h=1)n_h} < \left[\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L(x_h=0)}\right].$$

Combining the (L_1^D) and (L^S) equations, it can be shown with a little bit of algebra that:

$$\begin{aligned}
\frac{dE\omega_{hL}}{dx_h} &= \frac{\partial E\omega_{hL}}{\partial x_h} + q_L^{'}(x_h)\frac{\partial E\omega_{hL}}{\partial q_L} \\
&= \frac{(2-\beta)q_L(E\omega_{hL} - E\omega_{\ell L})}{[(2-\beta)q_L x_h n_h - (1-\beta)]E\omega_{hL} + (2-\beta)q_L E\omega_{\ell L}} q_L^{'}(x_h) E\omega_{\ell L} < 0,
\end{aligned}$$

and that $q_L^{'}(x_h)\frac{\partial E\omega_{hL}}{\partial q_L} > 0$. This proves Corollary A1.

Moreover, using that $dE\omega_{hL}/dx_h < 0$ with the fact that $E\omega_{hH}(x_h)$ is constant guarantees the uniqueness of the equilibrium. □

The $E\omega_{hH}$ curve, the expected wage earned in island H, is flat, i.e., the expected income in island H is independent of the number of high-skill workers searching in island L. This result comes from the fact that the equilibrium queue length is independent of the number of job seekers, as we discuss at length in the paper.

Turning to the $E\omega_{hL}$ curve, an important property of the model continues to hold in GE: the wage schedule of high-skill workers looking in island L is decreasing in x_h. This is in stark contrast with random search models with heterogeneous workers, in which the $E\omega_{hL}$ curve would be upward slopping. Indeed, worker heterogeneity gives rise to a general equilibrium effect, in that firms respond to changes in the average productivity of the unemployment pool: As more high-skill workers search in island L, this raises firms' probability to meet high-skill applicants (who generate a higher surplus than low-skill applicants), which raises firms' profits, and leads to more job creation. Thus, in random search model, the job creation effect would lead to an upward slopping wage schedule. This does not happen in our framework, because hiring is not random and the wage schedule $E\omega_{hL}$ is still decreasing in the number of high-skill workers searching in island L.[2] However, the general-equilibrium job creation effect is still present in our model, and the wage schedule is flatter with endogenous firm entry than in PE. As under-employment increases, the average productivity of the unemployment pool increases, which fosters firm entry and limits the increase in congestion generated by the inflow of workers.

Turning to type-ℓ workers, Figure A3 plots the expected wage curve of type-ℓ workers, both in PE and in GE, and the following corollary captures formally how the expected wage in island L depends on the share of type-h workers searching in island L.

Corollary A2 (Expected income of type-ℓ workers)**.** *The expected income of type-ℓ workers searching in island L, $E\omega_{\ell L}(x_h, q_L(x_h))$, is a non-monotonic function of x_h; decreasing over $[0, x_h^*]$ and increasing over $[x_h^*, 1]$ with $x_h^* \in [0, 1]$.*

Proof. Combining the (L_1^D) and (L^S) equations, it can be shown with a little bit of algebra that:

$$\begin{aligned}
\frac{dE\omega_{\ell L}}{dx_h}(x_h, q_L(x_h)) &= \frac{\partial E\omega_{\ell L}}{\partial x_h} + q_L'(x_h)\frac{\partial E\omega_{\ell L}}{\partial q_L} \\
&= \frac{[(2-\beta)q_L x_h n_h - (1-\beta)](E\omega_{\ell L} - E\omega_{hL})}{[(2-\beta)q_L x_h n_h - (1-\beta)]E\omega_{hL} + (2-\beta)q_L E\omega_{\ell L}} q_L'(x_h) E\omega_{\ell L} \gtrless 0
\end{aligned}$$

We can see that $E\omega_{\ell L}(x_h, q_L(x_h))$ is not monotonically decreasing, implying that a larger number of high-skilled workers does not necessarily imply lower expected income for low-skilled workers. For $\beta < 1$, $E\omega_{\ell L}(x_h, q_L(x_h))$ is initially decreasing and then increases once $(2-\beta)q_L x_h n_h > (1-\beta)$. This proves Corollary A2. □

As in the PE case, the expected income of low-skill workers declines with the share of high-skill workers looking in island L, at least for x_h low enough. This property of the model is again in contrast with a random search model, in which an increase in the quality of the unemployment pool leads to more job creation, which raises the job finding rate of *all* job seekers. This general-equilibrium job creation effect is present in this model, but it is dominated, at least for low values of x_h, by the effect of skill-biased job competition. Because high-skill workers are systematically hired over competing low-skill applicants, an increase in x_h implies a lower expected income for low-skill workers. However, as x_h increases and the pool

[2] With non-random hiring and skill-biased job competition, the wage schedule of high-skill workers is downward sloping, because the expected income of high skill workers is driven by their uniqueness, as it determines both their ability to find a job easily (by being preferably hired over low-skill workers) and to obtain a wage premium over low-skill workers. As the number of high-skill workers increase, they become less unique, leading to a lower job finding rate (as they face more competition from their peers) and a lower wage premium.

becomes more homogeneous (i.e., becomes dominated by high-skill workers), the degree of heterogeneity in the unemployment pool diminishes and the skill-biased job competition effect becomes weaker. This explains why, for large values of x_h, an increase in the number of type-h workers can raise low skill workers' labor market prospects.

1.3 Proof of Proposition 3: Constrained optimal allocation

In this section, we prove Proposition 3, and we discuss the inefficiency in more details.

Denote by $\mathcal{Y}$ the aggregate output of the economy. The maximization program of the central planner can be written as

$$\max_{x_h, q_L, q_H} \{\mathcal{Y}\},$$

subject to the free-entry conditions

$$\begin{cases} \pi_H(x_h, q_H) = c_H \\ \pi_L(x_h, q_L) = c_L \end{cases}.$$

With free entry, the profit of firms (net of vacancy posting costs) is zero, so that maximizing output is equivalent to maximizing the wage bill of workers, and the planner's problem can be written as

$$\begin{cases} \max_{x_h} \Omega \\ \Omega = (1 - x_h) n_h E\omega_{hH} (x_h, q_H(x_h)) + n_h x_h E\omega_{hL} (x_h, q_L(x_h)) + E\omega_{\ell L} (x_h, q_L(x_h)) \end{cases} \tag{A4}$$

with $q_H(x_h)$ and $q_L(x_h)$ given by firms' free entry conditions

$$\begin{cases} \pi_H(x_h, q_H(x_h)) = c_H \\ \pi_L(x_h, q_L(x_h)) = c_L \end{cases}.$$

Using the expression $\pi_H(x_h, q_H)$, we can immediately see that free entry ensures that movements in x_h do not affect $(1 - x_h)q_H$. When workers are homogeneous, as in island H, the number of job seekers has no effect on the ratio of job openings to job seekers: the firm responds to the addition of one more worker by creating more vacancies to keep the queue length q constant. As a result, we have, as shown in Corollary A1,

$$\frac{dE\omega_{hH}(x_h, q_H(x_h))}{dx_h} = 0,$$

so that a high-skilled exerts no externality on the high-tech island.

The problem then simplifies to

$$\max_{x_h, q_L} \left\{(1-x_h)n_h E\omega_{hH} + n_h x_h e^{-q_L n_h x_h}\left[\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right] + e^{-q_L n_h x_h - q_L}\varphi_{\ell L}\right\},$$

subject to

$$\pi_L(x_h, q_L) = \varphi_{hL} - e^{-x_h n_h q_L}\left[\left(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right)(1 + (2-\beta)x_h n_h q_L) + (2-\beta)\varphi_{\ell L} q_L e^{-q_L}\right] = c_L.$$

Denote by $E\omega_L$ the expected income for a job seeker searching in island L and by $E\pi_L$ the expected profit (gross of vacancy posting cost) of a firm in island L. The planner's first-order condition with respect to x_h gives

$$A(x_h, q_L) - \frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} - \lambda \frac{\partial E\pi_L}{\partial x_h} = 0, \tag{A5}$$

where

$$\begin{cases} \frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} & = \frac{\partial E\omega_{\ell L}}{\partial x_h} + x_h n_h \frac{\partial E\omega_{hL}}{\partial x_h} \\ & = q_L n_h e^{-q_L n_h x_h}\left[x_h n_h \left(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right) + \varphi_{\ell L} e^{-q_L}\right] \\ \frac{\partial E\pi_1}{\partial x_h} & = q_L n_h e^{-q_L n_h x_h}\left[((2-\beta)q_L x_h n_h - (1-\beta))\left(\varphi_{hL} - \varphi_{\ell L} + \varphi_{\ell L} e^{-q_L}\right) + (2-\beta)\varphi_{\ell L} e^{-q_L}\right] \end{cases}$$

and $A(x_h, q_L)$ corresponds to the no-arbitrage condition in the decentralized allocation. Note that $\frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} = \frac{\partial E\omega_L}{\partial x_h} - n_h E\omega_{hL}$ is the effect of a marginal high-skill job seeker in island L on the expected wage of job seekers in island L net of $n_h E\omega_{hL}$, the compositional change that a high-skill worker triggers (but internalizes) when he moves to the low-tech island.

The first-order condition in q_L gives:

$$-\frac{\partial E\omega_L}{\partial q_L} - \lambda \frac{\partial \pi_L}{\partial q_L}(x_h, q_L) = 0, \tag{A6}$$

where

$$\begin{cases} \frac{\partial E\omega_L}{\partial q_L} = x_h/q_L B_{x_h}(x_h, q_L) + (1 + x_h n_h)\varphi_{\ell L} e^{-q_L x_h n_h - q_L} \\ \frac{\partial E\pi_L}{\partial q_L} = x_h/q_L \frac{\partial E\pi_L}{\partial x_h} + \left[q_L(2-\beta)(1+x_h n_h) - (1-\beta)\right]\varphi_{\ell L} e^{-q_L x_h n_h - q_L} \end{cases}$$

Combining (A5) and (A6) gives

$$A(x_h, q_L) = \frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} - \frac{\partial \pi_L}{\partial x_h} / \frac{\partial \pi_L}{\partial q_L} \frac{\partial E\omega_L}{\partial q_L}. \tag{A7}$$

The impact of a marginal under-employed job seeker on the average wages of workers in island L can be decomposed into (i) a direct (negative) impact through higher congestion, $\frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h}$, and (ii) an indirect (positive) impact through job creation, $-\frac{\partial \pi_L}{\partial x_h} / \frac{\partial \pi_L}{\partial q_L} \frac{\partial E\omega_L}{\partial q_L}$, as firms' expected profit increases with the share of high-skill job seekers.

Developing and simplifying Equation (A7) gives the expression stated in Proposition 3:

$$A(x_h, q_L) = \frac{(1-\beta)\, n_h q_L \varphi_{\ell L} e^{-2q_L x_h n_h - q_L} \left(\varphi_{hL} - \varphi_{\ell L}\right)}{\frac{\partial E\pi_L}{\partial q_L}} > 0. \tag{A8}$$

With $A(x_h, q_L) > 0$, we have $E\omega_{hL} > E\omega_{hH}$, so that there is too much under-employment in the decentralized allocation.

To get some intuition behind this inefficiency result, notice that the planner's optimal allocation (A7) can be written as

$$A(x_h, q_L) = \underbrace{\frac{\partial E\omega_L}{\partial q_L}}_{<0} \left(\frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} / \frac{\partial E\omega_L}{\partial q_L} - \frac{\partial E\pi_L}{\partial x_h} / \frac{\partial E\pi_L}{\partial q_L} \right) > 0$$

which implies that the inefficiency comes from the fact that

$$\frac{\partial E\pi_L}{\partial x_h} / \frac{\partial E\pi_L}{\partial q_L} > \frac{\tilde{\partial} E\omega_L}{\tilde{\partial} x_h} / \frac{\partial E\omega_L}{\partial q_L}. \tag{A9}$$

The left-hand side captures how much (in expectation) firms benefit from having one more high-skilled job seeker in the unemployment pool (an increase in x_h) relative to having one more average job seeker (an increase in q_L). The right-hand side captures the same ratio from the perspective of an average job seeker in island L. Expression (A9) states that firms benefit more than workers from a marginal high-skill worker (relative to a marginal average job seeker), that is under-employment distorts the surplus sharing rule between firms and workers by raising firms' share of the surplus. This leads firms to post too many vacancies, attract too many high-skill job seekers, and generates a too high under-employment rate.

To see how wage bargaining drives this result, it is helpful to think in terms of surplus sharing between firms and workers.

Recall from Equation (3) in the main text that the firm and its first-best applicant (with productivity φ_{1st}) share the surplus as follows: the firm captures all the surplus generated by the second-best applicant (with productivity φ_{2nd}) and splits with the first-best applicant the residual surplus ($\varphi_{1st} - \varphi_{2nd}$) in shares $1-\beta$ and β. Thus, firms' expected profit and workers' expected wage can be compactly written as

$$\begin{cases} E^{(w)}\omega_L = \beta(\varphi - E^{(w)}\varphi_{2nd}) \\ E^{(f)}\pi_L = E^{(f)}\varphi_{2nd} + (1-\beta)E^{(f)}(\varphi_{1st} - \varphi_{2nd}) \end{cases}$$

where $E^{(w)}$ denotes expectation for a worker in island L with productivity φ and $E^{(f)}$ denotes expectation for a firm in island L (with the conventions that $\varphi_{2nd} = 0$ if a worker is the only applicant and that a worker receives zero if he is not the first-best applicant).[3][4] With these expressions and a little bit of algebra, we can re-write (A9) as

$$\frac{\partial E^{(f)}\varphi_{1st}}{\partial x_h} / \frac{\partial E^{(f)}\varphi_{2nd}}{\partial x_h} > \frac{\partial E^{(f)}\varphi_{1st}}{\partial q_L} / \frac{\partial E^{(f)}\varphi_{2nd}}{\partial q_L}. \tag{A10}$$

Expression (A10) states that the inefficiency comes from the following property of the model: compared to adding one more average job seeker in island L (an increase in q_L), adding one more high-skill worker (an increase in x_h) raises firms' expected productivity of the first-best applicant by more than it raises firms' expected productivity of the second-best applicant.

In turn, this property stems from two key aspects of the model: (i) the ranking advantage enjoyed by higher-skilled workers, and (ii) the fact that the firm gets a share of the surplus of the first-best applicant over the second-best.[5] In other words, the inefficiency depends on the sharing rule between the firm and the first-best applicant in the presence of a second-best applicant. When the marginal high-skill worker gets paid his marginal surplus (over the second-best applicant) and thus does

---

[3]The expected productivity of the second-best applicant from the viewpoint of workers verifies:

$$E^{(w)}\varphi_{2nd} = P(0,1^+)\varphi_{\ell L} + x_h n_h \left[P(0,1^+)\varphi_{\ell L} + P(1^+,0^+)\varphi_{hL}\right]$$

where $P(a_h, a_\ell)$ denotes the probability of a wage bargaining configuration with a_h type-h applicants and a_ℓ type-ℓ applicants and with $P(0,1^+) = e^{-q_L n_h x_h}(1-e^{-q_L})$ and $P(1^+,0^+) = 1 - e^{-q_L n_h x_h}$.

[4]The expected productivity of the first- and second-best applicants from the viewpoint of firms verifies:

$$\begin{cases} E^{(f)}\varphi_{1st} = P(0,1^+)\varphi_{\ell L} + P(1^+,0^+)\varphi_{hL} \\ E^{(f)}\varphi_{2nd} = P(2^+,0^+)\varphi_{hL} + \left(P(\ell L^+) + P(0,2^+)\right)\varphi_{\ell L} \end{cases}$$

with $P(2^+,0^+) = 1 - e^{-q_L n_h x_h} - q_L n_h x_h e^{-q_L n_h x_h}$, $P(\ell L^+) = q_L n_h x_h e^{-q_L n_h x_h}(1-e^{-q_L})$, and $P(0,2^+) = e^{-q_L n_h x_h}(1 - e^{-q_L} - q_L e^{-q_L})$.

[5]To see that, note that, because high-skill workers jump the queue in front of low-skill workers (i), they are more often first-best applicants than second-best applicants. From the perspective of the firm, a high-skill job seeker will thus have a stronger effect on $E^{(f)}\varphi_{1st}$ than on $E^{(f)}\varphi_{2nd}$ (compared to an average job seeker), which means that an additional high-skill job seeker raises the marginal surplus produced by the first-best applicant. Since the firm gets a share of that marginal surplus (ii), a marginal high-skill job seeker in island L raises the share of the surplus going to the firm.

not affect the firm surplus—as in job auctions (Shimer, 1999; Julien, Kennes and King, 2000) or when $\beta = 1$ in our setup—, the decentralized allocation is constrained efficient.

1.4 Graphical representation of the model in general equilibrium

The General Equilibrium (GE) allocation is the triple (x_h, q_L, q_H) verifying firms' free entry conditions in islands L and H, and the arbitrage equation between islands L and H for type-h workers. In this subsection, we show that one can represent this equilibrium allocation in a graphical manner, in a similar fashion to the standard Diamond-Mortensen-Pissarides model.

We start with the characterization of the worker's and firm's best responses in island H, which is an homogeneous island only populated by type-h workers.

With free entry, the equilibrium queue length $q_H(1-x_h)$ is independent of the supply of type-h workers. This result comes from the fact that the equilibrium queue length is independent of the number of job seekers, exactly as in a standard Diamond-Mortensen-Pissarides model with homogeneous workers. Recall that island H is an homogeneous island only populated by type-h workers. With free entry, it is easy to see from the firm's no profit condition that the equilibrium queue length $q_H(1-x_h)$ is independent of the supply of type-h workers.[6] Specifically, free entry, or (L_H^D) in the main text, pins down the equilibrium queue length in island H—$q_H(1-x_h)$—, regardless of the number of type-h workers (i.e., regardless of $1-x_h$). This is similar to what happens in standard search and matching models (Pissarides, 1985) where the supply of (homogeneous) labor has no effect on the equilibrium queue length. Even though a higher number of type-h workers improves the matching probability of a firm, free entry ensures that more firms enter the market in order to keep profits constant.[7]

This result points to a more general property of our model in GE: matching with ranking reduces to the canonical random matching model when workers are homogeneous.[8]

[6] Recall that the queue length is number of job seekers over the number of job openings and that the number of job seekers in island H is given by $L_2(1-x_h)$.

[7] In a search and matching model, at a given vacancy level, an increase in the number of job seekers (coming from say out of the labor force, as in Pissarides (2000), Chapter 5) raises firms matching probability, i.e., reduces hiring costs, and leads more firms to enter the market, keeping profit and thus the queue length unchanged.

[8] A technical difference between our framework and Pissarides (1985) is that, in our set-up, an increase in the supply of workers also improves the bargaining position of the firm (as workers compete against each other when negotiated the wage). This difference has no consequence on the equilibrium queue length, because the bargaining position is also solely a function of the queue length $q_H(1-x_h)$. As a result, no matter the level of $1-x_h$, free entry ensures that the queue

Since free entry in island H fixes $q_H(1-x_h)$, characterizing the equilibrium allocation reduces to finding the pair (x_h, q_L) that satisfies (i) firms' free entry condition in island L and (ii) type-h worker's arbitrage condition. Although one could depict the equilibrium in the (x_h, q_L) space, we prefer to depict it in the (x_h, v_L) space (recall that $v_L = n_\ell/q_L$ with n_ℓ fixed), since it corresponds to the (U, V) space representation used in standard search and matching models.

As shown in Figure A1, the equilibrium is then determined by the intersection of two curves: a "labor demand curve", (L_L^D), given by firms' free entry condition (also called job creation condition) as in search and matching models, and a "labor supply curve", (L^S), characterizing the number of type-h workers in island L and given by the arbitrage condition of type-h workers between islands L and H.

The labor demand curve is upward sloping and non-linear. To understand the shape of the labor demand curve (L_L^D), it is again useful to go back to the standard Diamond-Mortensen-Pissarides (DMP) model, in which workers are homogeneous. Recall that the total number of job seekers in island L is given by $n_\ell(1 + x_h n_h/n_\ell)$. We can thus represent the labor demand curve, or job creation curve, in a similar fashion to DMP models by plotting the job creation curve in (U, V) space. Starting from a world with only type-ℓ workers and $x_h = 0$ (i.e., being to the left of the y-axis in Figure A1), all workers are homogeneous and, as in the DMP model, increasing the number of type-ℓ (increasing n_ℓ) does not affect the equilibrium queue length v_L/n_ℓ. As a result, the labor demand curve (dashed blue line) crosses the origin at 0. Now, consider the case where one adds type-h workers and $x_h > 0$. Because firms generate a higher profit when hiring type-h workers than when hiring type-ℓ workers, an increase in x_h generates a disproportionate increase in the number of firms in island L, and the equilibrium queue length $\frac{v_L}{n_\ell(1+x_h n_h/n_\ell)}$ increases. In other words, the slope of the labor demand curve is initially increasing with x_h. This portion of the labor demand curve can be seen as capturing a general equilibrium job creation effect: as the share of type-h workers in island L increases, the quality (i.e., skill level) of the average applicant improves, and this leads to a disproportionate increase in job creation. Then, as the number of type-h workers becomes large relative to the number of type-ℓ, the labor market in island L resembles more to more to that of an homogeneous market with only type-h workers, in which the queue length is independent of the number of type-h and the slope of (L_L^D) is again independent of x_h (dashed red line).

The labor supply curve is capturing how x_h depends on v_L and is also upward slopping: the larger the number of job openings, the less competition type-h workers

length adjusts to keep profits (including the fix cost) nil.

will face when searching in island L, and the higher their expected wage. As a result, an increase in v_L raises the incentive of type-h to move down to island L and increases x_h.

2 Complements for the dynamic model

2.1 Proof of Proposition 4

Value functions We start by deriving the expressions for the value of unemployment and the value for a firm to open a vacancy. We focus on equilibria with a positive under-employment rate for type-$\underline{h}$ workers. In such equilibria with non-zero under-employment of type-$\underline{h}$, we will see that (i) type-ℓ workers only look for jobs in island L, (ii) workers of type-$\overline{h}$ only look for jobs in island H.
First, the values of unemployment for the three types verify

$$
\begin{aligned}
U_{\ell L,t} &= b+\delta EU_{\ell L,t+1}+\delta\nu_L e^{-\nu_L q_{L,t}}e^{-\nu_L q_{L,t}x_{h,t}n_h}\left(EW_{\ell L,t+1}(1,0)-EU_{\ell L,t+1}\right)\\
U_{\underline{h}L,t} &= b+\delta EU_{\underline{h}L,t+1}+\delta\nu_L e^{-\nu_L q_{L,t}x_{h,t}n_h}[e^{-\nu_L q_L}(EW_{\underline{h}L,t+1}(0,1)-EU_{\underline{h}L,t+1})\\
&\quad +(1-e^{-\nu_L q_{L,t}})(EW_{\underline{h}L,t+1}(1^+,1)-EU_{\underline{h}L,t+1})]\\
U_{\underline{h}H,t} &= b+\delta EU_{\underline{h}H,t+1}+\delta\nu_H e^{-\nu_H q_{H,t}(1-x_{h,t})}e^{-\nu_H q_{H,t}\eta_h}\left(EW_{\underline{h}H,t+1}(1,0)-EU_{\underline{h}H,t+1}\right)\\
U_{\overline{h}H,t} &= b+\delta EU_{\overline{h}H,t+1}+\delta\nu_H e^{-\nu_H q_{H,t}\eta_h}[e^{-\nu_H q_{H,t}(1-x_{h,t})}(EW_{\overline{h}H,t+1}(0,1)-EU_{\overline{h}H,t+1})\\
&\quad +(1-e^{-\nu_H q_{H,t}(1-x_{h,t})})(EW_{\overline{h}H,t+1}(1^+,1)-EU_{\overline{h}H,t+1})]
\end{aligned}
$$

where $EW_{ij}(m)$ with $m=(a,b)$ denotes a type-i worker's expected value of being employed in island j with initial bargaining configuration m with (including himself) a higher-type applicants and b lower-type applicants, with the convention that a^+ (resp. b^+) denotes a bargaining configuration with *at least* a (resp. b) applicants. For instance, $EW_{\ell L}(1,0)$ is the expected employment value for a type-ℓ matched with a firm in island L who is the only applicant during wage bargaining.
Second, the value for a firm to open a vacancy in island L verifies,

$$
\begin{aligned}
\Pi^o_L(x_{h,t},q_{L,t}) &= \delta e^{-\nu_L q_{L,t}x_{h,t}n_h}\nu_L q_{L,t}x_{h,t}n_h(1-e^{-\nu_L q_{L,t}})E_t\Pi_{L,t+1}(1^+,1)\\
&+ \delta e^{-\nu_L q_{L,t}x_{h,t}n_h}e^{-\nu_L q_{L,t}}\nu_L q_{L,t}E_t\Pi_{L,t+1}(1,0)\\
&+ \delta e^{-\nu_L q_{L,t}x_{h,t}n_h}(1-e^{-\nu_L q_{L,t}}-e^{-\nu_L q_{L,t}}\nu_L q_{L,t})E_t\Pi_{L,t+1}(2^+,0)\\
&+ \delta(1-e^{-\nu_L q_{L,t}x_{h,t}n_h}-e^{-\nu_L q_{L,t}x_{h,t}n_h}\nu_L q_{L,t}x_{h,t}n_h)E_t\Pi_{L,t+1}(0^+,2^+)\\
&+ \delta e^{-\nu_L q_{L,t}x_{h,t}n_h}e^{-\nu_L q_{L,t}}\nu_L q_{L,t}x_{h,t}n_h E_t\Pi_{L,t+1}(0,1)
\end{aligned}
$$

the value for a firm to open a vacancy in island H verifies

$$\begin{aligned}
\Pi_H^o(x_{h,t}, q_{H,t}) &= \delta e^{-\nu_H q_{H,t}\eta_h} \nu_H q_{H,t}\eta_h (1 - e^{-\nu_H q_{H,t}(1-x_{h,t})}) E_t \Pi_{H,t+1}(1^+, 1) \\
&+ \delta e^{-\nu_H q_{H,t}\eta_h} e^{-\nu_H q_{H,t}(1-x_{h,t})} \nu_H q_{H,t}(1 - x_{h,t}) E_t \Pi_{H,t+1}(1, 0) \\
&+ \delta e^{-\nu_H q_{H,t}\eta_h} \left(1 - e^{-\nu_H q_{H,t}(1-x_{h,t})}(1 + \nu_H q_{H,t}(1 - x_{h,t}))\right) E_t \Pi_{H,t+1}(2^+, 0) \\
&+ \delta(1 - e^{-\nu_H q_{H,t}\eta_h} - e^{-\nu_H q_{H,t}\eta_h} \nu_H q_{H,t}\eta_h) E_t \Pi_{H,t+1}(0^+, 2^+) \\
&+ \delta e^{-\nu_H q_{H,t}\eta_h} e^{-\nu_H q_{H,t}(1-x_{h,t})} \nu_H q_{H,t}\eta_h E_t \Pi_{H,t+1}(0, 1)
\end{aligned}$$

where $E\Pi_j(m)$ with $m = (a, b)$ is the firm's expected value of match in island j with initial bargaining configuration m with a lower-type applicants and b higher-type applicants. For instance, $E\pi_L(0^+, 2^+)$ is the firm's expected profit of a match in island L that initially bargained with at least two type-h applicants (and any number of type-ℓ applicants).

Steady-state We now study the equilibrium in steady-state. First, we show that (i) higher-types are always preferred by firms upon matching, and (ii) type-ℓ workers only look for jobs in island L and type-$\overline{h}$ workers only look for jobs in island H when there is under-employment with some type-$\underline{h}$ searching in island L.

As a preliminary, we note two results that will be useful. First, super-modularity $\varphi_{\ell H}/\varphi_{\ell L} < \varphi_{hH}/\varphi_{hL}$ implies $Y_{\ell H}/Y_{\ell L} < Y_{hH}/Y_{hL}$, given that Y is the present discounted output of a match with $Y_{ij,t} = \varphi_{ij,t} + \delta(1 - s)E_t Y_{ij,t+1}$.

Second, the probability to be in a bargaining configuration m is not type-specific. Indeed, the ranking mechanism in the model is an endogenous outcome of the wage bargaining, it does not come from different contact probabilities for different types. For instance, in island L, a worker's probability to be bargaining alone with the firm is $p_L(0, 0) = e^{-q_L} e^{-q_L x_h n_h}$ regardless of one's type. A worker's probability to be in competition with no type-h but at least one type-ℓ is $p_L(1^+, 0) = (1 - e^{-q_L})e^{-q_L x_h n_h}$ again regardless of one's type.

We first show (i): higher-types are always preferred by firms upon matching. Since a type-ℓ worker only receives a non-zero share of a match surplus when he is the only applicant, his steady-state value of unemployment $U_{\ell L}$ is given by

$$(1 - \delta)U_{\ell L} = b + \delta p_L(0, 0) \left(EW_{\ell L}(0, 0) - U_{\ell L}\right).$$

Using the surplus-sharing rule that gives the worker a share β of the match surplus, we get $EW_{\ell L}(0,0) - U_{\ell L} = \beta(Y_{\ell L} - \frac{(1-\delta)}{1-\delta(1-s)}U_{\ell L})$, so that we can express $U_{\ell L}$ as

$$\left(1 - \delta + \frac{\beta\delta(1-\delta)}{1-\delta(1-s)}p_L(0,0)\right)U_{\ell L} = b + \delta p_L(0,0)Y_{\ell L}. \tag{A11}$$

We can proceed similarly and write $U_{\underline{h}L}$, the steady-state value of unemployment for a type-$\underline{h}$ worker in island L as

$$\left(1 - \delta + \frac{\beta\delta(1-\delta)}{1-\delta(1-s)}p_L(0,0)\right)U_{\underline{h}L} = b + \delta p_L(0,0)Y_{\underline{h}L} + \delta p_L(1^+,0)(Y_{\underline{h}L} - U_{\underline{h}L} - Y_{\ell L} + U_{\ell L}). \tag{A12}$$

Subtracting Equations (A11) and (A12) to get rid of b, we get

$$\left(1 - \delta + \frac{\beta\delta(1-\delta)}{1-\delta(1-s)}\right)(U_{\underline{h}L} - U_{\ell L}) = \delta(Y_{\underline{h}L} - Y_{\ell L}). \tag{A13}$$

To prove that firms in island L always prefer higher-type workers, we need to show that the surplus generated by higher-type workers is always the highest, i.e., that $\left(Y_{\underline{h}L} - Y_{\ell L} - \frac{\delta(1-\delta)}{1-\delta(1-s)}(U_{hL} - U_{\ell L})\right) > 0$. One can verify that this is indeed the case by using Equation (A13). The same reasoning works with any two types in any island, which guarantees (i).

We now show (ii), i.e., that type-ℓ workers only look for jobs in island L and type $\overline{h}$ workers only look for jobs in island H.

To see that, assume that all type-ℓ workers are looking for a job in island L, and consider a type ℓ worker who would choose to move up the occupation ladder and search for a job in island H. In island H, this type-ℓ worker would only receive a positive wage when not competing with any other applicants, i.e., his unemployment value in island H is $U_{\ell H} = b + p_H(0,0)(W_{\ell H}(0,0) - U_{\ell H}) + \delta U_{\ell H}$ with $p_H(0,0)$ the probability of facing no other type-ℓ or type-h applicants. Writing up the value of unemployment of a type-$\underline{h}$ worker searching island H

$$U_{\underline{h}H} = b + p_H(0,0)\left(W_{\underline{h}H}(0,0) - U_{\underline{h}H}\right) + \delta U_{\underline{h}H},$$

and noting again that the contact probability is independent of a worker's type, we can combine the two employment values to write

$$(1-\delta)U_{\ell H} - b = ((1-\delta)U_{\underline{h}H} - b)\frac{W_{\ell H}(0,0) - U_{\ell H}}{W_{\underline{h}H}(0,0) - U_{\underline{h}H}} = ((1-\delta)U_{\underline{h}L} - b)\frac{W_{\ell H}(0,0) - U_{\ell H}}{W_{\underline{h}H}(0,0) - U_{\underline{h}H}}$$

where we used the no-arbitrage condition $U_{\underline{h}L} = U_{\underline{h}H}$ for the last equality.

Substituting the steady-state expression for $U_{\ell H}$, we get

$$(1-\delta)U_{\ell H} - b = \left[f_0^1(W^0_{\underline{h}L} - U_{\underline{h}L}) + f_1^1(W^0_{\underline{h}L} - U_{\underline{h}L} - (W^0_{\ell L} - U_{\ell L}))\right] \frac{W^0_{\ell H} - U_{\ell H}}{W^0_{\underline{h}H} - U_{\underline{h}H}}. \tag{A14}$$

where $f_0^1 = p_H(0,0)$ and $f_1^1 = p_H(0,1)$.

A type-ℓ worker will search in island H if $U_{\ell H} > U_{\ell L}$, which is verified if

$$\frac{W^0_{\ell H} - U_{\ell H}}{W^0_{\underline{h}H} - U_{\underline{h}H}} \frac{f_0^1(W^0_{\underline{h}L} - U_{\underline{h}L}) + f_1^1(W^0_{\underline{h}L} - U_{\underline{h}L} - (W^0_{\ell L} - U_{\ell L}))}{f_0^1(W^0_{\ell L} - U_{\ell L})} > 1 \tag{A15}$$

using Equation (A14) and $(1-\delta)U_{\ell L} - b = f_0^1(W^0_{\ell L} - U_{\ell L})$.

However, (log) super-modularity $Y_{\ell H}/Y_{\underline{h}H} < Y_{\ell L}/Y_{\underline{h}L}$ implies:[9]

$$\frac{W_{\ell H}(0,0) - U_{\ell H}}{W_{\underline{h}H}(0,0) - U_{\underline{h}H}} < \frac{W_{\ell L}(0,0) - U_{\ell L}}{W_{\underline{h}L}(0,0) - U_{\underline{h}L}}$$

so that Equation (A15) cannot hold, which ensures that type-ℓ workers do not search in island H.

We can then proceed in the exact same manner to show the second part of (ii); type-$\overline{h}$ only look for jobs in island H.

Next, to guarantee the existence of under-employment in equilibrium, we impose, similar to condition (A3) in the static model, that

$$U_{\underline{h}L}\left(x_h = 1, q_L(x_h = 1)\right) < U_{\underline{h}H}(x_h, q_H(x_h)) < U_{\underline{h}L}\left(x_h = 0, q_L(x_h = 0)\right) \tag{A16}$$

where $q_i(x_h)$ captures the general equilibrium response of vacancy posting in island i to movements in x_h and is given by the free entry conditions. Intuitively, the condition ensures that the relative cost to open a vacancy in island H versus island L is such that island H is sufficiently congested compared to island L (so that $U_{\underline{h}H} < U_{\underline{h}L}$ when $x_h = 0$).

Under this condition, a steady-state equilibrium is uniquely characterized by the arbitrage condition $U_{\underline{h}H} = U_{\underline{h}L}$, and the two free-entry conditions. Uniqueness is guaranteed by the fact that, given $\Pi^o_H(x_h, q_H) = c_H$, the (q_L, x_h) schedule defined by the arbitrage equation is steeper (in absolute value) than the (q_L, x_h) schedule defined by the free-entry condition $\Pi^o_L(x_h, q_L) = c_L$. We leave out the details of the derivations, since the logic of the proof is the exact same as with Corollary A1 for the static general equilibrium model.

[9] This can be seen by proceeding as in (i).

Equilibrium An equilibrium is an allocation of job seekers $\{x_{h,t}\}$ and vacancies $\{q_{L,t}, q_{H,t}\}$ which verify in each period:

$$A(x_{h,t}, q_{L,t}, q_{H,t}) = -U_{\underline{h}H,t}(x_{h,t}, q_{H,t}) + U_{\underline{h}L,t}(x_{h,t}, q_{L,t}) = 0 \qquad (L^S)$$

and the firms' free entry conditions:

$$\begin{cases} \Pi^o_L(x_{h,t}, q_{L,t}) = c_L & (L^D_L) \\ \Pi^o_H(x_{h,t}, q_{H,t}) = c_H & (L^D_H) \end{cases}$$

The equilibrium is a fixed point where firms respond optimally to the choices of workers, and workers respond optimally to vacancy openings. Since vacancy posting is a jump variable, firms' decisions only depend on the contemporary choice of workers and the current realization of productivity, and the model can be written (and solved) as a standard dynamic programming problem for the worker subject to the policy functions $q_L(x_{h,t}, \varepsilon_t)$ and $q_H(x_{h,t}, \varepsilon_t)$ and the no-arbitrage condition.

2.2 Under-employment rate: flow versus stock

In a dynamic setting, there is a difference between $x_{h,t}$, the (unobserved) fraction of type-$\underline{h}$ workers *searching* in the low-tech island, and the under-employment rate, UE_t, which is the (observed) fraction of type-h workers *employed* in the low-tech island and is given by

$$UE_t = \frac{N_{\underline{h}L,t}}{N_{\underline{h}L,t} + N_{\underline{h}H,t} + N_{\overline{h}H,t}} \tag{A17}$$

where $N_{\underline{h}L,t}$ is the number of type-$\underline{h}$ workers employed in island L at time t, $N_{\underline{h}H,t}$ is the number of type-$\underline{h}$ workers employed in island H at time t and $N_{\overline{h}H,t}$ is the number of type-$\overline{h}$ workers employed in island H at time t.

To link $x_{h,t}$ and UE_t, note that $N_{\overline{h}H,t}$, the number of type-$\overline{h}$ workers employed in island H, $N_{\underline{h}H,t}$, the number of type-$\underline{h}$ workers employed in island H, $N_{\underline{h}L,t}$, the number of type-$\underline{h}$ workers employed in island L, and $N_{\ell L,t}$, the number of low-skill workers employed in island L, evolve according to

$$\begin{cases} N_{\overline{h}H,t+1} = N_{\overline{h}H,t} + \left(n_h \eta_h - N_{\overline{h}H,t}\right) f_{\overline{h}H,t} - sN_{\overline{h}H,t} \\ N_{\underline{h}H,t+1} = N_{\underline{h}H,t} + \left(n_h - N_{\underline{h}H,t} - N_{\underline{h}L,t}\right)(1 - x_{h,t}) f_{\underline{h}H,t} - sN_{\underline{h}H,t} \\ N_{\underline{h}L,t+1} = N_{\underline{h}L,t} + \left(n_h - N_{\underline{h}H,t} - N_{\underline{h}L,t}\right) x_{h,t} f_{\underline{h}L,t} - sN_{\underline{h}L,t} \\ N_{\ell L,t+1} = N_{\ell L,t} + \left(n_\ell - N_{\ell L,t}\right) f_{\ell L,t} - sN_{\ell L,t} \end{cases} \tag{A18}$$

With the observed under-employment rate UE_t given by Equation (A17), we can combine Equation (A18) with expressions for the job finding rates, and solve the system of difference equation to express UE_t as a function of $x_{h,t}$, $q_{H,t}$, $q_{L,t}$, UE_{t-1} and the parameters of the model.

3 A model of under-employment with three worker types and three firm types

In this section, we show that the under-employment model of Section 2 (in the text) can be easily characterized in a $N > 2$ context. To illustrate this point, we study the Partial and General equilibrium allocations in the $N = 3$ case, i.e., in an economy with three worker types and three firm types. As in $N = 2$ case, we first derive the conditions to endure the existence of under-employment in equilibrium, then characterize the partial equilibrium allocation, provide some intuition and comparative statics, then describe the general equilibrium allocation, and finally study the optimal allocation.

With $N > 2$, it becomes easier to index islands and workers with numerical indices, and we will consider type-i workers and type-j islands for $i, j = 1, 2, .., N$ with the convention that higher numbers correspond to higher productivity workers/islands.

The $N = 2$ case is a good benchmark to understand how workers decide on which island to search. However, it has no "propagation mechanism", in the sense that type-1 workers cannot respond to the competition of type-2 workers by moving further down the occupation ladder. To capture this possibility, we now study an economy with 3 islands and 3 worker types. For the sake of clarity, we limit our analysis to $N = 3$, but the mechanisms are general and would be present with more islands or worker types.

When workers can respond to the presence of higher-skill individuals, the equilibrium level of under-employment is determined by the interaction of two forces, instead of just one in the $N = 2$ case: (i) a force that "pushes" workers down the occupation ladder: as high-skill workers invade the island below, they push lower-skill individuals further down the ladder, exactly as in the $N = 2$ case, and (ii) a force that "pulls" workers down the ladder: as low-skill workers move down the ladder, they free up space in their island, which pulls the higher-skill individuals down the ladder.

Conditions to ensure under-employment in equilibrium First, and as in the $N = 2$ case, we derive here the conditions that ensure that the equilibrium we consider is an under-employment equilibrium. Our conditions boil down to ensuring that the equilibrium is not at a corner solution in which either everyone or no one is under-employed.

In the $N = 3$ case, we impose conditions guaranteeing (i) there is some under-employment of types 3 and 2, (ii) not all type-2 workers search in island 1 and (iii) type-3 workers do not search in island 1. These conditions ensure that at most 2 types co-exist in a given island.

First, a positive fraction of type-3 and type-2 workers are under-employed as long as:

$$\begin{cases} e^{-q_3}\varphi_{3,3} < (\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2}) & (C_3^p) \\ e^{-q_2}\varphi_{2,2} < (\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}) & (C_2^p) \end{cases}$$

Second, a positive fraction of type-2 workers search in island 2 as long as:

$$e^{-q_2 h_3}\varphi_{2,2} > e^{-q_1 h_2}\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right) \quad (D_2^p)$$

This condition implies that, even with all type-3 workers in island 2, type-2 workers would not all descend to island 1. Finally, we derive the condition under which type-3 workers have no incentives to search in island 1. Consider the equilibrium allocation verifying $A_3(x_3, x_2) = 0$ and $A_2(x_2, x_1) = 0$. The expected wage of a type-3 worker searching in island 1 would be

$$E\omega_{3,1} = (\varphi_{3,1} - \varphi_{2,1}) + e^{-q_1 x_2 h_2}\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right]$$

Since $e^{-q_1 x_2 h_2}\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right] = e^{-q_2(1-x_2)-q_2 x_3 h_3}\varphi_{2,2}$,

$$E\omega_{3,1} = (\varphi_{3,1} - \varphi_{2,1}) + e^{-q_2(1-x_2)-q_2 x_3 h_3}\varphi_{2,2}$$

By contrast, the expected wage of a type-3 worker searching in island 2 is:

$$e^{-q_2 x_3 h_3}\left[\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right]$$

It is then immediate that no type-3 workers have the incentives to descend to island 1 as long as:

$$e^{-h_3 q_2} > \tfrac{\varphi_{3,1}}{\varphi_{3,2}} \quad (D_3^p).$$

Finally, we impose the same conditions on the φ's in order to ensure that, as long as type n workers are indifferent between islands n and $n-1$, type $n-1$ workers will never move up to island n.

Partial equilibrium with exogenous labor demand We now characterize the equilibrium allocation and then present some comparative statics exercises to illustrate the mechanisms underlying the equilibrium.

Proposition A1. *With $N = 3$, there is a unique equilibrium allocation of workers satisfying*

- *type-3 workers are indifferent between islands* 3 *and* 2*, and x_3, the share of type* 3 *workers searching in island* 2*, is given by the arbitrage condition*

$$A_3(x_3, x_2) = -E\omega_{3,3} + E\omega_{3,2} = 0 \quad (A_3)$$

with

$$\begin{cases} E\omega_{3,3} = \beta e^{-q_3(1-x_3)}\varphi_{3,3} \\ E\omega_{3,2} = \beta e^{-q_2 x_3 h_3} e^{-q_2(1-x_2)}\varphi_{3,2} + \beta e^{-q_2 x_3 h_3}\left[1 - e^{-q_2(1-x_2)}\right](\varphi_{3,2} - \varphi_{2,2}) \end{cases}$$

- *type-2 workers are indifferent between islands* 2 *and* 1*, and x_2, the share of type* 2 *workers searching in island* 1*, is given by the arbitrage condition*

$$A(x_3, x_2) = -E\omega_{2,2} + E\omega_{2,1} = 0 \quad (A_2)$$

with

$$\begin{cases} E\omega_{2,2} = \beta e^{-q_2(1-x_2)-q_2 x_3 h_3}\varphi_{2,2} \\ E\omega_{2,1} = \beta e^{-q_1 x_2 h_2} e^{-q_1}\varphi_{2,1} + \beta e^{-q_1 x_2 h_2}(\varphi_{2,1} - \varphi_{1,1})\left[1 - e^{-q_1}\right] \end{cases}$$

- *Type* 1 *workers only look for jobs in island* 1.

Proof. We directly consider the general problem of a worker $n \in \{1, \dots, N\}$ who can decide to look for a job in his home island, or instead move down the occupation ladder to look for a job. In the spirit of the $N = 2$ case, we can exclude the possibility that workers look for jobs in higher technology islands or that they descend to lower levels than the one immediately below. The intuition is the same as the one developed in the 2 islands case: as long as a particular type is indifferent between two islands, the more (resp. less) skilled types will always prefer the island above (resp. below). The reason lies in the fact that the relative rent extracted between island $n-1$ and n is increasing in the skills of agents.

A type n worker has two choices, he can (i) look for a job in island n, his "home island", or (ii) look for a job in island -1, i.e., move down the occupation ladder. As in Proposition 1, we consider these two possibilities, and the only difference with the $N = 2$ case is that workers now have to take into account the fact that some higher type workers may be looking for work in their home island.

When a type n worker looks for a job in island n, he faces two possible outcomes: (a) with probability $e^{-q_n(1-x_n)}e^{-q_n x_{n+1}h_{n+1}}$, he is the only applicant and receives $\beta\varphi_{n,n}$, or (b), with probability $1 - e^{-q_n(1-x_n)}e^{-q_n x_{n+1}h_{n+1}}$, he is in competition with other workers (either from his own island n or from island $n+1$) and receives 0 (regardless of whether he ends up employed or unemployed). The expected payoff of a worker type n who searches for a job in island n, $\omega_{n,n}$, is thus

$$E\omega_{n,n} = \beta e^{-q_n(1-x_n)}e^{-q_n x_{n+1}h_{n+1}}\varphi_{n,n}$$

Consider now the case in which worker type n moves down to island $n-1$. There are 3 possibilities: (a) with probability $e^{-q_{n-1}x_n h_n}e^{-q_{n-1}(1-x_{n-1})}$, he is the only applicant and receives $\beta\varphi_{n,n-1}$, (b) with probability $1 - e^{-q_{n-1}x_n h_n}$, he is in competition with type n workers coming, like him, from the island above, and he receives 0 (regardless of whether he ends up employed or unemployed), and (c), with probability $e^{-q_{n-1}x_n h_n}\left(1 - e^{-q_{n-1}(1-x_{n-1})}\right)$, he is in competition with type $n-1$ workers only and receives $\beta(\varphi_{n,n-1} - \varphi_{n-1,n-1})$.[10] The expected payoff of a worker type n who searches for a job in island $n-1$, $\omega_{n,n-1}$, is thus

$$E\omega_{n,n-1} = \beta e^{-q_{n-1}x_n h_n}e^{-q_{n-1}(1-x_{n-1})}\varphi_{n,n-1} + \beta e^{-q_{n-1}x_n h_n}\left[1 - e^{-q_{n-1}(1-x_{n-1})}\right](\varphi_{n,n-1} - \varphi_{n-1,n-1})$$

In equilibrium, a type n worker must be indifferent between staying in island n or moving down to island $n-1$, which implies the arbitrage equation

$$\begin{aligned} &A_n(x_{n+1}, x_n) = -e^{-q_n(1-x_n)}e^{-q_n x_{n+1}h_{n+1}}\varphi_{n,n} \\ &+ e^{-q_{n-1}x_n h_n}e^{-q_{n-1}(1-x_{n-1})}\varphi_{n,n-1} + e^{-q_n x_n h_n}\left[1 - e^{-q_{n-1}x_n h_n}\right](\varphi_{n,n-1} - \varphi_{n-1,n-1}) = 0 \end{aligned}$$

These equations characterize the equilibrium allocation.

Uniqueness

As in the $N = 2$ case, uniqueness comes from a monotonicity argument. Condition (C_3^p) implies that there will be some high-skilled workers descending even when all mid-skilled workers are applying in island 2. Condition (C_2^p) implies that mid-skilled workers descend even when none of the high-skilled workers are applying in their island. As in the case $N = 2$, some high skilled workers always apply in island 3 in island 3 because $\varphi_{3,3} > \varphi_{3,2}$.

Under this set of conditions, we now show that there exists a unique equilibrium. With two types of actors, the relative gain depends on the others' behaviors: there is

[10] As noted earlier, despite the presence of competing applicants, a single type n applicant can extract some of the surplus thanks to due to his productivity advantage over the other applicants.

a complementarity between their choices. To see it, let us write down the conditions under which wages are equal for workers 3 in islands 2 and 3, and workers 2 in islands 1 and 2.

$$\begin{cases} \varphi_{3,3}e^{-q_3(1-x_3)} = \left[\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right] e^{-q_2 x_3 h_3} & (A_3) \\ \varphi_{2,2}e^{-q_2(1-x_2)-q_2 x_3 h_3} = \left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right] e^{-q_1 x_2 h_2} & (A_2) \end{cases}$$

The two curves (A_3) and (A_2) both describe a positive relationship between x_3 and x_2, respectively the pull $x_3 = f_3(x_2)$ and push $x_3 = f_2(x_2)$ effects. Any interior equilibrium should be at the intersection of those two curves. It can be shown that:

$$\begin{cases} f_3'(x_2) = \frac{q_1 h_2 + q_2}{q_2 h_3} \\ f_2'(x_2) = \frac{q_2}{q_3 + q_2 h_3} \frac{\varphi_{2,2}e^{-q_2(1-x_2)}}{\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}} \end{cases}$$

It can be easily verified that $\frac{q_2}{q_3+q_2h_3}\frac{\varphi_{2,2}}{\varphi_{3,2}} < \frac{q_1h_2+q_2}{q_2h_3}$. As a consequence, (A_2) is always steeper than (A_3), e.g. the push effect is always stronger than the pull effect and uniqueness derives from this observation (see figure A4). $\square$

As illustrated in figure A4, the two arbitrage equations (A_3) and (A_2) implicitly define a unique equilibrium allocation (x_2, x_3). Both curves are increasing, but the (A_2) curve is always steeper than the (A_3) curve.

To get some intuition, recall that the (A_2) curve captures the decision of type-2 workers to search in island 2 or 1. The (A_2) curve is increasing, because an increase in x_3, the number of type-3 workers in island 2, raises congestion in island 2, which "pushes" type-2 workers down to island 1 and increases x_2. The (A_3) curve captures the decision of type-3 workers to search in island 3 or 2. The (A_3) curve is increasing, because an increase in x_2, the number of type-2 workers in island 1, lowers congestion in island 2, which attracts, i.e., "pulls", type-3 workers down to island 2 and increases x_3. The fact that the (A_2) curve is always steeper than the (A_3) curve means that the "pushing effect" is always stronger than the "pulling effect".

Mechanisms Compared to the $N = 2$ case, under-employment is determined by the interactions of two forces: (i) a force that "pushes" workers down the ladder, captured by the (A_2) curve: as higher type workers invade the island below, they push the lower types further down the ladder, as in the $N = 2$ case, and (ii) a force that "pulls" workers down the ladder, captured by the (A_3) curve: as the lower types move down the ladder, they free up space in their islands, which pulls the higher types even further into their island.

In order to understand how these two forces interact, consider the thought experiment in which island $n = 1$ was closed for agents of type $n = 2, 3$. This initial point corresponds to the point E_0 in Figure A4 and is identical to the $N = 2$ case previously discussed: type-2 agents are stuck in island 2 and $x_2 = 0$. Imagine that island 1 suddenly opens, allowing anyone to look for a job in island 1.

1. Given x_3^0, the initial fraction of type-3 workers in island 2, workers in island 2 have an incentive to look for a job in island 1, because E_0 is above the (A_2) curve, so that $(A_2(0, x_3)) > 0$ and $E\omega_{2,1} > E\omega_{2,2}$. As a result a fraction x_2^1 of type-2 workers moves down to island 1, up until the point where $(A_2(x_2^1, x_3)) = 0$ (point E_1 in figure A4). In effect, type-2 workers are "pushed down" the ladder by type-3 workers, and this "pushing" effect is captured by the curve (A_2).

2. Following the downward movement of type-2 workers, island 2 is less congested than when type-3 agents initially made their island choice, and E_1 is below the (A_3) curve, so that $E\omega_{3,3} < E\omega_{3,2}$. As a result, more type-3 workers will descend to island 2 up until the point where $(A_3(x_2^1, x_3^2)) = 0$ with x_3^2 the new number of type-3 workers in island 2 (point E_2 in figure A4). In effect, type-3 workers are "pulled down" the ladder by type-2 workers leaving their island, and this "pulling" effect is captured by the curve (A_3).

3. Again, type-2 workers respond to the increased number of type-3 workers by further descending to island 1, which triggers a response from type-3 workers and so on. This cascade ends at the equilibrium point E.

Job polarization We now discuss one comparative statics exercise to illustrate how the interactions between agents' decisions across islands (when $N > 2$) play out in equilibrium, and how a local shock can end up affecting all workers.

Consider an adverse labor demand shock hitting the middle productivity island, i.e., an increase in the queue length q_2. This thought experiment can be seen as studying the effect of job polarization and the disappearance of jobs in middle-skill occupations (the "hollowing out" of the skill distribution, Autor (2010)) on the allocation of workers.

Job polarization has two effects (see Figure A5). On the one hand, the (A_2) curve shifts down, because of fewer job opportunities for type-2 workers in island 2, which would increase x_2, i.e., under-employment. On the other hand, the (A_3) curve also shifts down, because of fewer job opportunities for type-3 workers in island 2. This shift decreases x_2, because there are fewer type-3 workers pushing type-2 workers down the ladder. Overall, the effect of job polarization on the under-employment rate of middle-skill workers is thus ambiguous. However, under-employment amongst high-skill workers will unambiguously decrease.

In terms of expected income, it is easy to show that job polarization leads the expected income of type-3 (high-skill) workers to decrease and the expected income of type-2 (middle-skill) workers to decrease. However, the expected income of type-1 (low-skill) workers can either increase or decrease, depending on the effect of job polarization on the under-employment rate of middle-skill workers.

3.1 General equilibrium with endogenous labor demand

The equilibrium with three types of workers and firms is characterized by the following Proposition:

Proposition A2. *With $N = 3$, there is a unique equilibrium allocation satisfying*

- *The arbitrage conditions characterizing the allocation of workers*
  - *Type 3 workers are indifferent between islands 2 and 3, and x_3, the share of type 3 workers searching in island 2, is given by the arbitrage condition*

$$A(x_3, x_2, q_2) = -E\omega_{3,3} + E\omega_{3,2}(x_3, x_2, q_2) = 0$$

  - *Type 2 workers are indifferent between islands 1 and 2, and x_2, the share of type 2 workers searching in island 1, is given by the arbitrage condition*

$$A(x_3, x_2, q_2, q_1) = -E\omega_{2,2}(x_3, x_2, q_2) + E\omega_{2,1}(x_2, q_1) = 0$$

  - *Type 1 workers only look for jobs in island 1: $x_1 = 0$.*
- *Firms' free entry conditions (market clearing) in islands 1, 2 and 3*

$$\begin{cases} \pi_1(x_2, q_1) = c_1 \\ \pi_2(x_3, x_2, q_2) = c_2 \\ \pi_3(x_3, q_3) = c_3 \end{cases}$$

Proof. The proof for the $N = 3$ case is very similar to the $N = 2$ case. The equilibrium is characterized by the allocation of workers of type 3 and 2, and the free-entry conditions in islands 1, 2 and 3. First, the free entry condition imposes that $q_3(1-x_3)$ is constant, and thus the expected wage in island 3, $E\omega_{3,3}$, is constant.

We can thus restrict our analysis to the arbitrage conditions for workers of type 2 and 3 coupled with the free entry conditions in island 1 and 2. □

The general equilibrium allocation with $N = 3$ is the vector $(x_3, x_2, q_3, q_2, q_1)$ determined by firms' free entry conditions in islands 1, 2 and 3, and the arbitrage equations for type-2 and type-3 workers.

3.2 Constrained efficient allocation

The following proposition states that the decentralized allocation is generally inefficient when $N = 3$. In the constrained allocation, there is less under-employment of type-2 workers (lower x_2) and less under-employment of type-3 workers (lower x_3) than in the decentralized allocation.

Proposition A3. *When $N = 3$, the constrained optimal allocation $(x_2^*, x_3^*, q_1^*, q_2^*, q_3^*)$ does not coincide with the decentralized allocation. It is characterized by the same free entry conditions in islands 1, 2 and 3 but the difference in expected income between two islands for type-3 and type-2 workers is now respectively*

$$\begin{aligned}
A_3(x_2^*, x_3^*, q_1^*, q_2^*, q_3^*) &= -E\omega_{3,3} + E\omega_{3,2} \\
&= \frac{(1-\beta)\, h_3 h_2 (1-x_2^*)^2 q_2^* \varphi_{2,2} e^{-2q_2^* x_3^* h_3 - q_2^*(1-x_2^*)} \left(\varphi_{3,2} - \varphi_{2,2}\right)}{\frac{\partial \pi_2(x_3^*, x_2^*, q_2^*)}{\partial q_2}} \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
A_2(x_2^*, x_3^*, q_1^*, q_2^*) &= -E\omega_{2,2} + E\omega_{2,1} \\
&= \frac{(1-\beta)\, h_2 q_1^* \varphi_{1,1} e^{-2q_1^* x_2^* h_2 - q_1^*} \left(\varphi_{2,1} - \varphi_{1,1}\right)}{\frac{\partial \pi_1(x_2^*, q_1^*)}{\partial q_1}} \\
&+ \frac{(1-\beta)(1-x_2^*)(\varphi_{3,2} - \varphi_{2,2})\varphi_{2,2} h_2 q_2^* x_3^* h_3 e^{-2q_2^* x_3^* h_3 - q_2^*(1-x_2^*)}}{\frac{\partial \pi_2(x_3^*, x_2^*, q_2^*)}{\partial q_2}} \\
&\geq 0
\end{aligned}$$

with the expression for $\frac{\partial \pi_2(x_3, x_2, q_2)}{\partial q_2} > 0$ and $\frac{\partial \pi_1(x_2, q_1)}{\partial q_1} > 0$ given in the proof.

Proof. We proceed here exactly as we did for Proposition 4. The maximization program of the central planner can be written as follows (denote Y the aggregate output of the economy):

$$\max_{x_2, x_3, q_1, q_2, q_3} \{Y\}$$

subject to

$$\begin{cases} \pi_3(x_3, q_3) = c_3 \\ \pi_2(x_3, x_2, q_2) = c_2 \\ \pi_1(x_2, q_1) = c_1 \end{cases}$$

As before, two remarks help us simplify the program. First, with free entry, the aggregate profit of firms (net of investment costs) is zero: the central planner maximizes the wage bill of workers. Second, free entry in island 3 imposes that q_3 is set such as to make $(1-x_3)q_3$ constant.

$$(1-x_3)q_3 = f^{-1}\left(\frac{c_3}{\varphi_{3,3}}\right)$$

The program then sums up to (where each line represents wages earn by agents of different types):

$$\max_{x_2,x_3,q_1,q_2} \left\{ \begin{array}{l} h_2h_3(1-x_3)E\omega_{3,3} + h_2h_3x_3e^{-q_2h_3x_3}\left[\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right] \\ +h_2(1-x_2)\varphi_{2,2}e^{-q_2x_3h_3-q_2(1-x_2)} + h_2x_2\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right)e^{-q_1x_2h_2} \\ +\varphi_{1,1}e^{-q_1x_2h_2-q_1} \end{array} \right\}$$

subject to

$$\begin{cases} \varphi_{3,2} - e^{-x_3h_3q_2}\left[\left(\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right)(1+(2-\beta)x_3h_3q_2) + (2-\beta)\varphi_{2,2}(1-x_2)e^{-q_2(1-x_2)}\right] = c_2 \\ \varphi_{2,1} - e^{-x_2h_2q_1}\left[(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1})(1+(2-\beta)x_2h_2q_1) + (2-\beta)\varphi_{1,1}e^{-q_1}\right] = c_1 \end{cases}$$

We need now to write the four first-order conditions:

$$\begin{cases} A_3(x_3,x_2,q_2) - B_{x_3}(x_3,x_2,q_2) - \lambda_2C_{x_3}(x_3,x_2,q_2) = 0 & [x_3] \\ -B_{q_2}(x_3,x_2,q_2) + \lambda_2C_{q_2}(x_3,x_2,q_2) = 0 & [q_2] \\ A_2(x_3,x_2,q_2,q_1) - B_{x_2}(x_3,x_2,q_2,q_1) - \lambda_2C_{x_2}(x_3,x_2,q_2) - \lambda_1D_{x_2}(x_3,x_2,q_1) = 0 & [x_2] \\ -B_{q_1}(x_2,q_1) - \lambda_2D_{q_1}(x_2,q_1) = 0 & [q_1] \end{cases}$$

Let us detail the notations, A_3 (resp. A_2) denotes the difference between wages earned in level 2 and 3 (resp. 1 and 2) for workers of type 3 (resp. 2). B_{x_3} and B_{q_2} represents the additional terms deriving from differentiating W with respect to x_3 and q_2. We report their exact expression below.

$$\begin{cases} B_{x_3}(x_3,x_2,q_2) = & h_3q_2e^{-x_3q_2h_3}\left[\left(\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right)(1+x_3h_3q_2) + \varphi_{2,2}q_2(1-x_2)e^{-q_2(1-x_2)}\right] \\ B_{q_2}(x_3,x_2,q_2) = & \frac{x_3}{q_2}B_{x_3}(x_3,x_2,q_2) + (x_3h_3 + 1 - x_2)\,h_2\,(1-x_2)\,\varphi_{2,2}e^{-q_2h_3x_3-q_2(1-x_2)} \end{cases}$$

B_{x_2} and B_{q_2} represents the additional terms deriving from differentiating W with respect to x_3 and x_2. We report their exact expression below.

$$\begin{cases} B_{x_2}(x_3,x_2,q_2,q_1) = & -q_2h_2\varphi_{2,2}e^{-q_2h_3x_3-q_2(1-x_2)}\left(x_3h_3+1-x_2\right) \\ & +q_1h_2e^{-q_1h_2x_2}\left[h_2x_2(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1})+\varphi_{1,1}e^{-q_1}\right] \\ B_{q_1}(x_2,q_1) = & x_2h_2e^{-q_1h_2x_2}\left[h_2x_2(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1})+\varphi_{1,1}e^{-q_1}\right] \\ & +(1+x_2h_2)\varphi_{1,1}e^{-q_1h_2x_2-q_1} \end{cases}$$

C_l represents the additional terms deriving from differentiating the profits in island 2 with respect to l. D_l represents the additional terms deriving from differentiating the profits in island 1 with respect to l. We report their exact expression below.

$$\begin{cases} C_{x_3}(x_3,x_2,q_2) = & h_3q_2e^{-x_3h_3q_2}\big[\left(\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}\right)\left((2-\beta)x_3h_3q_2-(1-\beta)\right) \\ & +(2-\beta)\varphi_{2,2}q_2(1-x_2)e^{-q_2(1-x_2)}\big] \\ C_{q_2}(x_3,x_2,q_2) = & \frac{x_3}{q_2}C_{x_3}(x_3,x_2,q_2)+\varphi_{2,2}(1-x_2)e^{-x_3h_3q_2-q_2(1-x_2)}\left[\left(x_3h_3+1-x_2\right)(2-\beta)q_2-(1-\beta)\right] \\ D_{x_2}(x_3,x_2,q_1) = & h_2q_1e^{-x_2h_2q_1}\big[\left(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}\right)\left((2-\beta)x_2h_2q_1-(1-\beta)\right) \\ & +(2-\beta)\varphi_{1,1}q_1e^{-q_1}\big] \\ D_{q_1}(x_2,q_1) = & \frac{x_2}{q_1}D_{x_2}(x_3,x_2,q_1)+\varphi_{1,1}e^{-x_2h_2q_1-q_1}\left[\left(x_2h_2+1\right)(2-\beta)q_1-(1-\beta)\right] \\ C_{x_2}(x_3,x_2,q_2) = & -q_2\varphi_{2,2}\left[\left(x_3h_3+1-x_2\right)(2-\beta)q_2-(1-\beta)\right]e^{-x_3h_3q_2-q_2(1-x_2)} \end{cases}$$

The main difference with the $N=2$ case comes from an additional interaction term between workers of type-2 and 3. Workers of type-2 influences the profits that firms can make in island 2. C_{x_2} represents this gain in profits. We eliminate the shadow prices in the first-order conditions:

$$\begin{cases} A_3(x_3,x_2,q_2) = B_{x_3}(x_3,x_2,q_2) - \frac{C_{x_3}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}B_{q_2}(x_3,x_2,q_2) \\ A_2(x_3,x_2,q_2,q_1) = B_{x_2}(x_3,x_2,q_2,q_1) - \frac{C_{x_2}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}B_{q_2}(x_3,x_2,q_2) - \frac{D_{x_2}(x_3,x_2,q_1)}{D_{q_1}(x_2,q_1)}B_{q_1}(x_2,q_1) \end{cases}$$

Let us focus on the first equation:

$$A_3(x_3,x_2,q_2) = \frac{B_{x_3}(x_3,x_2,q_2)C_{q_2}(x_3,x_2,q_2) - C_{x_3}(x_3,x_2,q_2)B_{q_2}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}$$

As in proposition 4,

$$A_3(x_3,x_2,q_2) = \frac{(1-\beta)\,h_3h_2(1-x_2)^2q_2\varphi_{2,2}e^{-2q_2x_3h_3-q_2(1-x_2)}\left(\varphi_{3,2}-\varphi_{2,2}\right)}{C_{q_2}(x_3,x_2,q_2)}$$

It is easy to see that $A_3(x_3,x_2,q_2)>0$. The centralized allocation gives a higher wage to agents 3 in island 2 than what they would receive in island 3. x_3 is lower than in the decentralized allocation, q_3 is higher.

We now turn to the second equation.

$$A_2(x_3, x_2, q_2, q_1) = \frac{B_{x_2} D_{q_1} - D_{x_2} B_{q_1}}{D_{q_1}} - \frac{C_{x_2}}{C_{q_2}} B_{q_2}$$

$$\begin{aligned} A(x_2, q_1) = & \tfrac{(1-\beta) h_2 q_1 \varphi_{1,1} e^{-2q_1 x_2 h_2 - q_1} (\varphi_{2,1} - \varphi_{1,1})}{D_{q_1}(x_2, q_1)} \\ & + \tfrac{(1-\beta)(1-x_2)(\varphi_{3,2} - \varphi_{2,2}) \varphi_{2,2} h_2 q_2 x_3 h_3 e^{-2q_2 x_3 h_3 - q_2(1-x_2)}}{C_{q_2}} \end{aligned}$$

□

Compared to the $N = 2$ case, interactions across agents' decision introduces an additional effect. Comparing with the $N = 2$ case, we can notice an additional (positive) term in A_2 in the $N = 3$ case, which brings the constrained allocation further away from the decentralized one. This additional term captures the fact that, when deciding to search in island 1, type-2 workers affect not only the ratio of type-1 to type-2 workers in island 1, which affects the job creation decision of firms in island 1 (as is the $N = 2$ case), but also the ratio of type-2 to type-3 workers in island 2, which affects the job creation decision of firms in island 2.

Additional figures

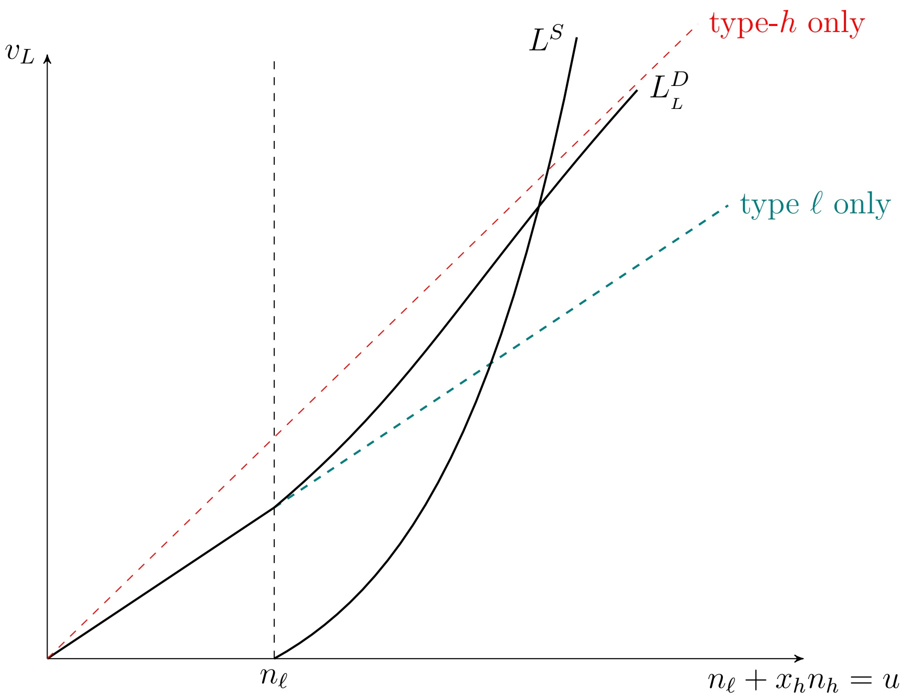

Figure A1. Labor market equilibrium—N=2.

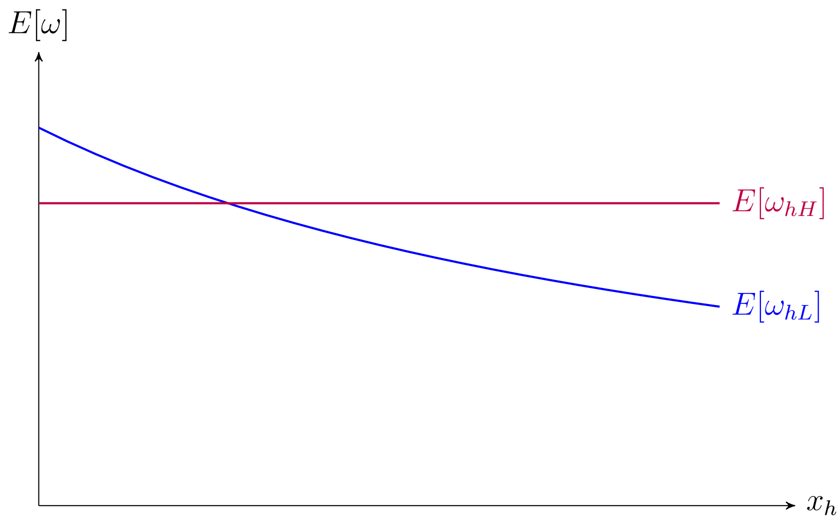

Figure A2. General Equilibrium—wages for type-h workers.

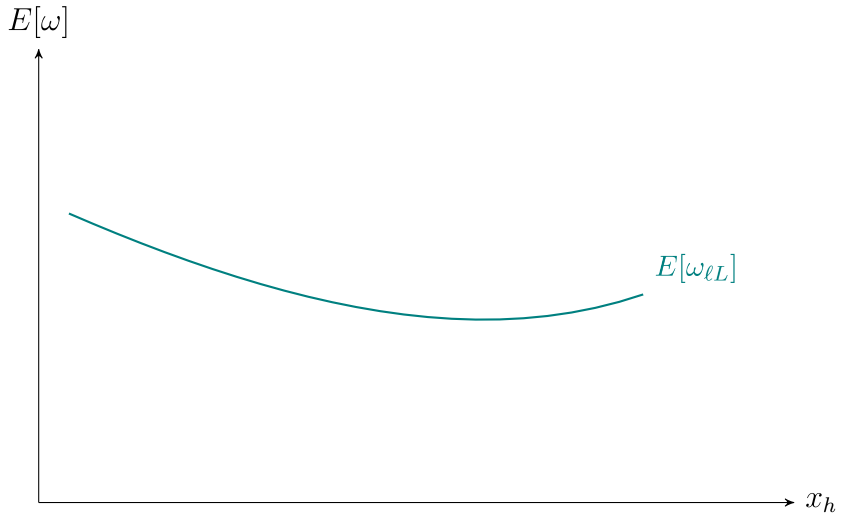

Figure A3. General Equilibrium—wages for type-ℓ workers.

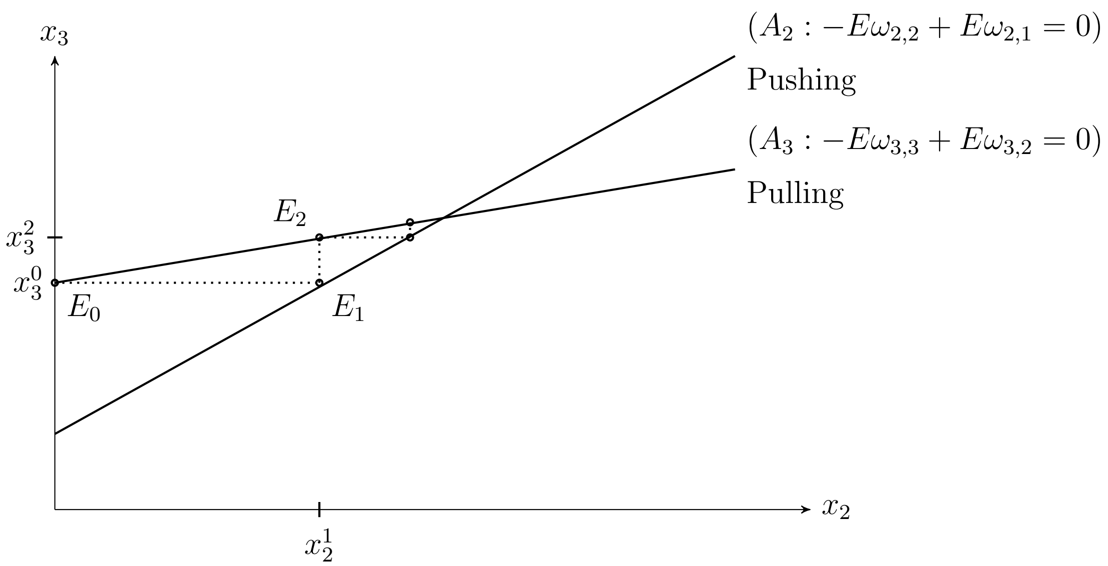

Figure A4. Partial Equilibrium – N=3.

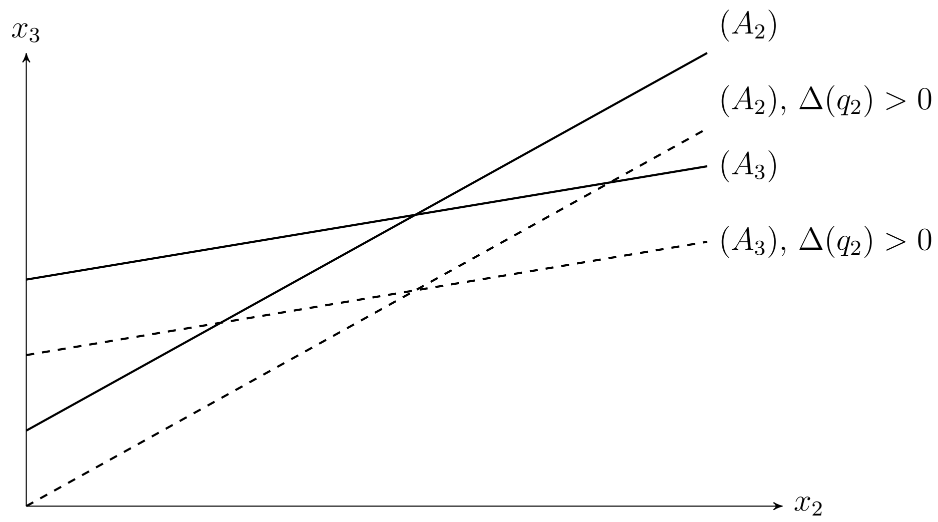

Figure A5. Partial Equilibrium – Effect of job polarization – N=3.

References

Autor, David. 2010. "The Polarization of Job Opportunities in the U.S. Labor Market." *The Hamilton Project and The Center for American Progress*, 1 – 40.

Julien, Benoit, John Kennes, and Ian King. 2000. "Bidding for Labor." *Review of Economic Dynamics*, 3(4): 619–649.

Pissarides, Christopher. 1985. "Short-run Equilibrium Dynamics of Unemployment Vacancies, and Real Wages." *American Economic Review*, 75(4): 676–90.

Pissarides, Christopher A. 2000. *Equilibrium Unemployment Theory, 2nd Edition.* Vol. 1 of *MIT Press Books*, The MIT Press.

Shimer, Robert. 1999. "Job Auctions."