Public Debt and Low Interest Rates Appendices

Olivier Blanchard

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1. Appendix A. Construction of growth and interest rates.

1.1 Data sources

- Real GDP:

  The proximate source is Measuringworth.com. Original sources are: From 1800 to 1929: The series are linked to data from 1790 to 1929 that were constructed by Louis Johnston and Samuel H.Williamson. (More information here: https://www.measuringworth.com/datasets/usgdp/sourcegdp.php). From 1929 to the present: U.S. Bureau of Economic Analysis. The real GDP data in the .dta file used is different from the one that can be currently downloaded from Measuringworth.com as the deflator has changed. The differences are minimal (2009=100 for the series used in the paper and 2012=100 for the series they currently have on their web site).

- 1-year T-bill rate:

  Federal Reserve Board: 1-year Treasury bill secondary market rate, quoted on discount basis. Source: From 1950 to 1952: 9-12 months rate, Economic Reports of the President. From 1952 to the present: 1-year Treasury bill secondary market rate, quoted on discount basis. FRED data base.

- 10-year bond rate.

  Federal Reserve Board: Market yield on U.S. Treasury securities at 10-year constant maturity, quoted on investment basis. Source: Measuringworth.com [The series on the FRED site starts in 1962. The two series are not identical, but very close. Mean 1962-2017: Measuringworth.com: 6.13. FRED: 6.25]

1.2 Construction of the adjusted rate

- Maturity of debt held by private investors.

  The series for the average length and maturity distribution of marketable interest-bearing public debt held by private investors is only available from 1974 on. (Sources: 1974 to 2012: Economic Report of the President, 2013, (Table B88, excel file).www.gpo.gov/fdsys/pkg/ERP-2013. 2012 to present: Treasury, (table FD5, p.26) www.fiscal.treasury.gov/fsreports/rpt/treasBulletin/b2017_2.pdf

  Before that, the series is the average length and maturity distribution of marketable interest-bearing public debt. (Source: 1976 ERP, p255 Table B.72 https://fraser.stlouisfed.org/files/docs/publications/ERP/1976/ERP_1976.pdf

  For the years for which both series are available (1974 and 1975), the average maturities for the two series are very close.

- Tax rate paid by marginal holders.

  The tax rate of the marginal holder is computed as the difference between the yield on municipal bonds and the yield on treasuries. The series from Kuenz (2014) covers 1977 to 1993, for both 1-year and 10-year maturities. The series from Bloomberg (3M AAA muni bond; BVMB3M Index, 10y AAA muni bond BVMB10T Index)covers 1994-2017. As no series available before 1977, I assume the value from 1950 to 1976 to be equal to the 1977 value.

- Proportion of investors who pay taxes.

The series is constructed by excluding foreign holders, private and public, Federal retirement programs, and Fed holdings from holders of public debt. Source: Flow of funds L210 (Z1 publication Federal Reserve Board https://www.federalreserve.gov/releases/z1/20160609/data.htm)

2. Appendix B. Derivation of results in Section 2

2.1 Certainty

Under certainty, utility is given by:

$$U = (1-\beta)U(C_1) + \beta U(C_2)$$

First and second period budget constraints are given by:

$$C_1 = W - K - D\ ; C_2 = R\ K + D$$

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Production is given by a constant returns to scale production function:

$$Y = F(K, N)$$

Labor is normalized to 1, so $N = 1$.

The first order condition for utility maximisation is given by:

$$(1-\beta)\ U'(C_1) = \beta R\ U'(C_2)$$

The effect of a small increase in the transfer D on utility is given by:

$$dU = [-(1-\beta)U'(C_1) + \beta U'(C_2)]\ dD + [(1-\beta)U'(C_1)\ dW + \beta K U'(C_2)\ dR]$$

The first term, call it dU_a, represents the direct effect of the transfer, the second term, call it dU_b, the effect of the transfer through the change in wages and rates of return.

Consider the **first term**, the effect of debt on utility given labor and capital prices. Using the first-order condition gives:

$$dU_a = [\beta(-R\, U'(C_2) + U'(C_2))]\ dD = \beta(1-R)U'(C_2)\ dD \tag{1}$$

Consider the **second term**, the effect of debt on utility through the changes in W and R.

Start with $W = F(K,1) - F_K(K,1)K$ and $R = F_K(K,1)$. This implies:

$$dW = (F_K(K,1) - F_K(K,1) - F_{KK}(K,1)K)\ dK = -F_{KK}(K,1)K\ dK$$

$$dR = F_{KK}(K,1)\ dK \text{ so } dR = -(1/K)dW$$

Replace dW in the second term to get:

$$dU_b = -[(1-\beta)U'(C_1) - \beta U'(C_2)]K\ dR$$

Using the first order condition for utility maximization:

$$dU_b = -[\beta(R-1)U'(C_2)]K\ dR$$

Or, using the relation between dR and dK:

$$dU_b = [-\beta(R-1)U'(C_2)]KF_{KK}(K,1)\; dK \tag{2}$$

Using the definition of the elasticity of substitution in production, $\eta \equiv (F_K F_N)/(F_{KN}K)$, the share of labor in production, $(\alpha \equiv F_N/F)$, and the relation between second derivatives, $(F_{KN} = -F_{KK}K)$, this equation can be rewritten as:

$$dU_b = [\beta(R-1)U'(C_2)](1/\eta)\alpha R\; dK \tag{3}$$

2.2 Uncertainty

Under uncertainty, expected utility is given by:

$$U = (1-\beta)U(C_{1_t}) + \beta EU(C_{2t+1})$$

First and second period budget constraints are given by

$$C_{1t} = W_t - K_t - D\;; C_{2t+1} = R_{t+1}\; K_t + D$$

Production is given by:

$$Y_t = A_t F(K_{t-1}, N)$$

where $N = 1$, with wage W_t and rate of return on capital R_t, and A_t is stochastic.

At time t, the first order condition for utility maximization is given by:

$$(1-\beta)\; U'(C_{1t}) = \beta E[R_{t+1}\; U'(C_{2t+1})]$$

We can define a shadow safe rate R^f_{t+1} which must satisfy:

$$R^f_{t+1} EU'(C_{2t+1}) = E[R_{t+1}\, U'(C_{2t+1})]$$

The effect of a small increase in D on utility at time t is given by:

$$dU_t = dU_{at} + dU_{bt}$$

where

$$dU_{at} = [-(1-\beta)U'(C_{1t}) + \beta EU'(C_{2t+1})]\; dD$$

$$dU_{bt} = (1-\beta)U'(C_{1t})\; dW_t + \beta K_t E[U'(C_{2t+1})\; dR_{t+1}]$$

As before, the first term, dU_{at}, reflects the direct effect of the transfer, the second term, dU_{bt}, reflects the effect through the change in wages and rates of return.

Take the **first term**, the effect of debt on utility given prices. Using the first order condition gives:

$$dU_{at} = -\beta E[R_{t+1}\, U'(C_{2t+1})] + \beta E[U'(C_{2t+1})]\; dD$$

So, using the definition of the safe rate:

$$dU_{at} = \beta(1 - R^f_{t+1}) EU'(C_{2t+1})\; dD \tag{4}$$

Take the **second term**, the effect of debt on utility through prices:

$$dU_{bt} = (1-\beta)U'(C_{1t})\; dW_t + \beta K_t E[U'(C_{2t+1})\; dR_{t+1}]$$

Use the first order condition to get:

$$dU_{bt} = \beta E[R_{t+1}U'(C_{2t+1})]\ dW_t + \beta K_t E[U'(C_{2t+1})\ dR_{t+1}]$$

Note that:

$$W_t = A_t F(K_{t-1}, 1) - A_t F_K(K_{t-1}, 1)K_{t-1}$$

$$R_{t+1} = A_{t+1}F_K(K_t, 1)$$

$$dW_t = -K_{t-1}dR_t$$

Replace dW_t in the expression for dU_{bt}

$$dU_{bt} = -\beta K_{t-1}E[R_{t+1}U'(C_{2t+1})]dR_t + \beta K_t E[U'(C_{2t+1})dR_{t+1}]$$

Use the relation between dR_t and R_t, and between dR_{t+1} and R_{t+1}:

$$dR_t = (F_{KK}(K_{t-1}, 1)/F_K(K_{t-1}, 1))R_t dK_{t-1}$$

$$dR_{t+1} = (F_{KK}(K_t, 1)/F_K(K_t, 1))R_{t+1}dK_t$$

Replace, to get:

$$\begin{aligned} dU_{bt} &= -\beta \frac{K_{t-1}F_{KK}(K_{t-1}, 1)}{F_K(K_{t-1}, 1)}\ E[R_{t+1}U'(C_{2t+1})]R_t dK_{t-1} \\ &\quad +\beta \frac{K_t F_{KK}(K_t, 1)}{F_K(K_t, 1)}\ E[R_{t+1}U'(C_{2t+1})]\ dK_t \end{aligned} \tag{5}$$

Evaluate it at $K_t = K_{t-1} = K$ and $dK_t = dK_{t+1} = dK$. Then:

$$dU_{bt} = [-\beta \frac{KF_{KK}(K)}{F_K(K)} \ E[U'(C_{2t+1})R_{t+1}] \ (R_t - 1) \ dK$$

Use the fact that $KF_{KK}(K)/F_K(K) = -(1/\eta)\alpha$, where α is the share of labor, and η is the elasticity of substitution between capital and labor.

Use the relation between the marginal product and the safe rate, $E[U'(C_{2t+1})R_{t+1}] = R^f_{t+1} EU'(C_{2t+1})$,

Rewrite the equation as:

$$dU_{bt} = [\beta \ \frac{1}{\eta} \ \alpha \ E[U'(C_{2t+1})R_{t+1}]] \ (R_t - 1) \ dK$$

Thus the relevant rate in assessing the sign of the welfare effect of the transfer through this second term is the risky rate, the marginal product of capital.

The best way to give the intuition is to rewrite it as:

$$dU_{bt} = [\beta \ \frac{1}{\eta} \ \alpha \ R_t] \ E[(R_{t+1} - \frac{R_{t+1}}{R_t})U'(C_{2t+1})]]dK$$

Capital delivers R_{t+1}. The implicit transfer delivers R_{t+1}/R_t, which is less if $R_t > 1$.

To get a sense of relative magnitudes of the two effects, evaluate the two terms at the average values of the safe and the risky rates:

$$dU/dD = [(1 - ER^f) - \frac{1}{\eta}\alpha ER^f (ER - 1)(-dK/dD)]\beta E[U'(C_2)]$$

so that:

$$\text{sign dU} \equiv \text{sign } (1 - ER^f) - [(1/\eta)\alpha ER^f(-dK/dD)](ER - 1) \qquad (6)$$

An approximation to dK/dD can be derived from the accumulation equation. Ignore uncertainty, and ignore the (small) wealth effect of the transfer on saving, so that steady state capital satisfies:

$$K = \beta W - D = \beta F_N(K,1) - D$$

This implies, for a small transfer dD:

$$dK = \beta F_{KN}(K,1) - dD$$

Using the definition of the elasticity of substitution, and the labor share, this can be rewritten as:

$$dK/dD = \beta\alpha(1/\eta)R\ dK/dD - 1$$

Or:

$$dK/dD = -\frac{1}{1-\beta\alpha(1/\eta)\ ER}$$

In the Cobb Douglas case ($\eta = 1$), using the approximation $ER \approx (1-\alpha)/(\alpha\beta)$ (this again ignores uncertainty and Jensen's inequality, which is small), the equation reduces to the simpler formula:

$$\text{sign dU} \equiv \text{sign}\ [(1-ER^f) - ER^f(ER-1)] \tag{7}$$

Or

$$\text{sign dU} \equiv \text{sign}\ [1-ER^f\ ER] \tag{8}$$

3. Appendix C. Calibration and simulations

3.1 Functional forms

Expected utility is given by the Epstein-Zin-Weil specification:

$$(1-\beta)\ln C_{1t} + \beta \frac{1}{1-\gamma} \ln E(C_{2t+1}^{1-\gamma})$$

The elasticity of substitution is equal to 1, and γ is the coefficient of relative risk aversion.

The production function is given by:

$$Y_t = A_t \left(bK_{t-1}^{\rho} + (1-b)N^{\rho}\right)^{1/\rho} = A_t(bK_{t-1}^{\rho} + (1-b))^{1/\rho}$$

where A_t is white noise, log normal, $\ln A_t \sim \mathcal{N}(\mu;\sigma^2)$ and $\rho = (\eta-1)/\eta$, where η is the elasticity of substitution.

When $\eta = \infty, \ \rho = 1$ and we get the linear case. Cobb Douglas is the limit when η tends to 1, and thus ρ tends to zero.

The wage and the rate of return to capital are given by:

$$W_t = A_t(1-b)(\frac{Y_t}{A_t N_t})^{1-\rho} = A_t(1-b)(\frac{Y_t}{A_t})^{1-\rho} \equiv W(K_{t-1}, A_t)$$

$$R_t = A_t b(\frac{Y_t}{A_t K_{t-1}})^{1-\rho} \equiv R(K_{t-1}, A_t)$$

$$R_{t+1} = A_{t+1} b(\frac{Y_{t+1}}{A_{t+1} K_t})^{1-\rho} \equiv R(K_t, A_{t+1})$$

3.2 Choice of parameters

The parameters are chosen so as to fit a set of pairs of values for the average safe rate and the average risky rate:

I consider net annual average risky rates (marginal products of capital) minus the growth rate (here equal to zero) between 0% and 4%. These imply values of the average 25-year gross risky rate, ER, between 1.00 and 2.66.

I consider net annual average safe rates minus the growth rate between -2% and 1%; these imply values of the average 25-year gross safe rate, ER^f, between 0.60 and 1.28.

The following coefficients are chosen a priori:

- b (which is equal to the capital share in the Cobb-Douglas case)is equal to 1/3.
- σ is equal to 0.20.
- η takes one of two values: $\eta = \infty$ which corresponds to the linear production function case, and $\eta = 1$, the Cobb-Douglas case.

The parameters used to fit each pair ER, ER^f are, on the one hand, β and μ, and on the other, γ.

- The parameters β and μ determine (together with σ, which plays a minor role) the average level of capital accumulation and the average marginal product of capital—the average risky rate. In general, both parameters matter.

  In the linear production case however, the marginal product of capital is independent of the level of capital, and thus depends only on μ; thus, I choose μ to fit the relevant average value of the marginal product of capital.

  In the Cobb-Douglas case, the marginal product of capital is instead independent of μ and depends only on β; thus I choose β to fit the relevant average value of the marginal product of capital.

- The parameter γ determines, together with σ the spread between the risky rate and the safe rate. In the absence of transfers, the following relation holds between the two rates.

$$\ln R^f_{t+1} - \ln ER_{t+1} = -\gamma\sigma^2$$

Thus, a given spread between the two average rates determines γ.

For reasons explained in the text, I also assume that people, when young, receive a non stochastic endowment X. For each pair ER, ER^f, I choose X to be equal to the average steady state wage corresponding to that pair, absent transfers or debt.

3.3 Equations of motion, absent intergenerational transfers or debt

For $D = 0$, the dynamics are given by:

$$K_t = \beta\ (X + W(K_{t-1}, A_t))$$

$$C_{1t} = (1-\beta)(X + W(K_{t-1}, A_t))$$

$$C_{2t+1} = \beta(X + W(K_{t-1}, A_t))\ R(K_t, A_{t+1})$$

with the functions $W(.)$ and $R(.)$ given from above.

The relation between the safe and the risky rate is given by:

$$\ln R^f_{t+1} = \ln ER_{t+1} - \gamma\sigma^2$$

These equations are used to solve for the underlying parameters β, μ, γ

which generate each pair ER and ER^f. Once the parameters have been chosen, a simulation based on 30,000 realizations of A is used to determine the average level of capital, which is then used as the starting value of capital when transfers or debt are introduced.

3.4 Equations of motion, with intergenerational transfers D

The dynamics of capital accumulation are given implicitly by the first order conditions for utility maximization:

$$(1-\beta)\ (1/C_{1t}) = \beta E[C_{2t+1}^{-\gamma} R_{t+1}]/E[C_{2t+1}^{1-\gamma}]$$

Or, equivalently:

$$\frac{1-\beta}{(X+W_t-K_t-D)} = \beta\ \frac{E[(R_{t+1}(R_{t+1}K_t+D)]^{-\gamma})}{E[(R_{t+1}K_t+D)^{1-\gamma}]} \tag{9}$$

which leads to a function (which, for $D>0$, must be found numerically) of the form:

$$K_t = K(W_t, D)$$

This function is approximated using a third-order polynomial, and thus the dynamics of capital are given by:

$$K_t = K(W(K_{t-1}, A_t), D)$$

Consumption is given in turn by:

$$C_{1t} = X + W(K_{t-1}, A_t) - K_t - D$$

$$C_{2t+1} = K_t\ R(K_t, A_{t+1}) + D$$

These equations are used to solve for the change in steady state welfare in the absence or the presence of intergenerational transfers, for each pair (ER, ER^f). I do this by generating a sequence of 2,000 realizations of A, and computing the relevant averages. The results are reported in Figures 7 to 10.

3.5 Equations of motion, with a debt rollover

The equation of motion for capital is given by:

$$K_t = \beta\,(X + W(K_{t-1}, A_t))$$

The associated values of consumption are given by:

$$C_{1t} = W_t + X - (K_t + D_t)$$

$$C_{2t+1} = R_{t+1}K_t + D_t R^f_{t+1}$$

And debt follows:

$$D_t = R^f_t\ D_{t-1}$$

While saving is divided between capital and debt:

$$S_t = K_t + D_t$$

The safe rate is defined by the first order condition:

$$R^f_{t+1} E[C_{2t+1}{}^{-\gamma}] = E[R_{t+1}(C_{2t+1})^{-\gamma}]$$

Or equivalently:

$$R^f_{t+1} = \frac{\mathbb{E}_t\left[R_{t+1} \cdot C^{-\gamma}_{2,t+1}\right]}{\mathbb{E}_t\left[C^{-\gamma}_{2,t+1}\right]} \tag{10}$$

$$R^f_{t+1} = \frac{\mathbb{E}_t\left[R_{t+1}\left(R_{t+1}(S_t - D_t) + R^f_{t+1}D_t\right)^{-\gamma}\right]}{\mathbb{E}_t\left[\left(R_{t+1}(S_t - D_t) + R^f_{t+1}D_t\right)^{-\gamma}\right]} \tag{11}$$

Define $Q_t = \frac{D_t}{S_t}$ and rewrite the previous equation as:

$$R^f_{t+1} = \frac{\mathbb{E}_t\left[R_{t+1}\left(R_{t+1} + (R^f_{t+1} - R_{t+1})Q_t\right)^{-\gamma}\right]}{\mathbb{E}_t\left[\left(R_{t+1} + (R^f_{t+1} - R_{t+1})Q_t\right)^{-\gamma}\right]} \tag{12}$$

This implicitly defines R^f_{t+1} as a function of K_t and Q_t. I approximate this function by a third-order polynomial.

I then use these equations to derive the dynamics of the debt to saving ratio and of utility, by generating 1000 simulations, each one based on 5 draws (150 years) of A. In each simulation, I start from the average steady state value of capital in the absence of debt, and introduce debt at time 0.

To decide whether a particular debt rollover fails, I use as an (arbitrary) threshold: the condition that the safe rate exceed 1% a year. The motivation is that, if this is the case, then, unless the economy experiences a series of unusually good shocks which return the safe rate to a negative number (a very low probability event), debt dynamics are likely to lead to a steady increase in the debt to saving ratio from then on. If the threshold is exceeded, I assume (again, arbitrarily) that debt is fully paid off through a tax on the young.

These simulations are reported in Figures 11 to 14.

4. Appendix D. Profit rates and marginal products

4.1 Data sources

- Profit.

  When computing the ratio of earnings to replacement cost: Table 1.14, line 24, since 1929, BEA data site (sum of corporate profits, plus net interest, plus transfers) Gross Value Added of Domestic Corporate Business in Current Dollars and Gross Value Added of Nonfinancial Domestic Corporate Business in Current and Chained Dollars [Billions of dollars] Seasonally adjusted at annual rates

  When computing the ratio of earnings to market value: Flow of funds. https://www.federalreserve.gov/releases/z1/20091210/data.htm Net operating surplus minus taxes. Table F102, line 1 minus line 2

- Capital at replacement cost.

  Table 6-1, line 2, since 1925, BEA data site. Current-Cost Net Stock of Private Fixed Assets by Industry Group and Legal Form of Organization. [Billions of dollars; year end estimates]

- Market value of capital

  Flow of funds https://www.federalreserve.gov/releases/z1/20091210/data.htm Table B102. Market value of non financial capital, defined as market value of equities outstanding (line 35) plus liabilities (line 21) minus financial assets (line 6)