Online Appendix to "Make and Buy: Outsourcing, Vertical Integration, and Cost Reduction"

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Abstract

This is the Online Appendix to: Loertscher and Riordan (2018): "Make and Buy: Outsourcing, Vertical Integration, and Cost Reduction", *American Economic Journal: Microeconomics*.

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Online Appendix

A Extensions

In this section, we detail the various extensions discussed in Sections II.c and III.

A.1 Acquisition Game

This subsection considers an acquisition game when the initial market structure is non-integration. The game proceeds as follows.

In Stage 1, the customer makes sequential take-it-or-leave-it offers t_i to the independent suppliers $i = 1, .., n$. The sequence in which offers are made is pre-determined but since suppliers are symmetric *ex ante* this is arbitrary. Without loss of generality, we assume that supplier i receives the i-th offer. If i accepts, the acquisition stage (i.e. Stage 1) ends and the Stage 2 subgame with vertical integration analyzed above ensues. If firm $i < n$ rejects, the customer makes the offer t_{i+1} to firm $i+1$. If supplier n receives an offer but rejects it, the Stage 2 subgame with non-integration analyzed above ensues.

The equilibrium behavior in Stage 1 is readily determined. Suppose first that $\Phi(n, \mu) < 0$. That is, vertical integration is jointly profitable. Then the subgame perfect equilibrium offers are $t_i = \Pi_I^*$ for $i < n$ and $t_n = \Pi_N^*$. On and off the equilibrium path, these offers are accepted. Notice that in order for supplier n to accept the offer he receives, he must be offered $t_n \geq \Pi_N^*$ because the alternative to his accepting is that the game with the non-integrated market structure ensues, in which case he nets Π_N^*. Anticipating that the last supplier would accept the offer if and only if he is offered Π_N^*, the alternative for any supplier $i < n$ when rejecting is that the ensuing market structure will be non-integration if $\Phi < 0$ and integration, with i as an independent supplier netting Π_I^* otherwise. Therefore, it suffices to offer $t_i = \Pi_I^*$ to i with $i = 1, .., n-1$, provided $t_n = \Pi_N^*$. But as the latter is only a credible threat if $\Phi(n, \mu) \leq 0$, it follows that vertical integration is more profitable than the necessary (and sufficient) condition for it to be an equilibrium outcome suggests: $\Phi(n, \mu) \leq 0$ must be the case for integration to occur on the equilibrium path, but if $\Phi(n, \mu) \leq 0$, the profit of integration to the customer is actually strictly larger than $-\Phi(n, \mu)$ because she has to pay less than Π_N^* on the equilibrium path.

Lastly, if $\Phi(n, \mu) > 0$, vertical integration is not jointly profitable and the customer will only make offers that will be rejected (e.g. $t_i \leq 0$ for all i would be a sequence of such offers).

A.2 Alternative Cost Distributions

Uniform Model We first consider the model with uniformly distributed costs, that is, for investment x the costs are distributed according to $G(c + x) = c - (\beta - x)$ for $c \in [\beta - x, 1 + \beta - x]$, and assume $\Psi(x) = ax^2/2$. For $n > 2$, this requires solving numerically for the equilibrium bidding under integration as mentioned in the main text.

Figure 2 plots the benefits from non-integration minus the payoff from vertical integration, $\Phi(n)$, as a function of n for $a = 1.75$.

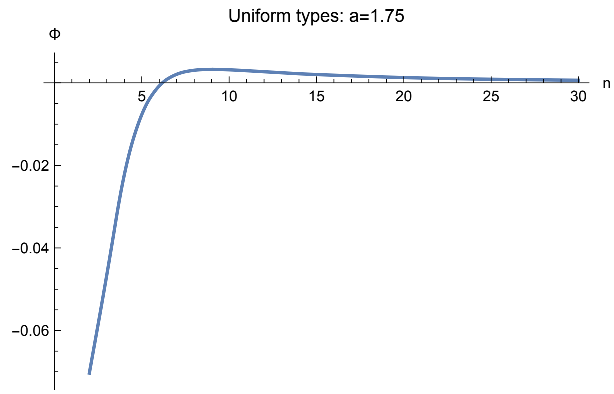

Figure 2: $\Phi(n)$ for Uniformly Distributed Costs.

An intuitive conjecture is that vertical integration has the advantage of squeezing (rather than just avoiding) markups. Analysis of the exponential case has already shown this intuition is not correct in general.[22] For the uniform case, equilibrium bid markups indeed decrease with vertical integration seemingly in line with the intuition. However, closer analysis reveals that the reason for this is the effect of vertical structure on equilibrium investments because, keeping investments fixed, vertical integration does not affect equilibrium bidding.[23] Figure 3 depicts the equilibrium bids given equilibrium investments.

Fixed-Support Exponential Model In the fixed-support exponential model, the distribution of the costs c given investment x is $G(c; x) = 1 - e^{-\mu x(c-\beta)}$. We assume quadratic costs of effort and set $a = 1 = \mu$. This is without loss of generality by

[22]For the case of a fixed cost distribution with a convex decreasing inverse hazard rate, Burguet and Perry (2009) argue that a right of first refusal granted to a preferred supplier is profitable in part because it causes independent suppliers to bid more aggressively. The exponential cost distribution is a limiting case, in which the hazard rate is constant and the bid distribution does not change with vertical integration, consistent with a more basic markup avoidance motive for granting a right of first refusal.

[23]To see this, notice that in a standard first-price procurement auction with n bidders and costs independently drawn from the uniform distribution with support $[\underline{c}, \overline{c}]$ the equilibrium bidding function is $\beta(c) = \overline{c}/n + (n-1)c/n$. With one integrated supplier whose bid is equal to his realized cost c_1 and $n-2$ competing independent suppliers who all bid according to $\beta_I(c) = \alpha_0 + \alpha_1 c$, satisfying the boundary condition $\beta_I(\overline{c}) = \overline{c}$ (which implies $\alpha_0 = \overline{c}(1-\alpha_1)$), the optimal bid of a representative independent bidder i, b_i, solves the problem of maximizing $(1/\alpha_1)^{n-2}\,(1/(\overline{c}-\underline{c}))^{n-1}\,(\overline{c}-b_i)^{n-1}(b_i - c_i)$, yielding $b_i = \overline{c}/n + (n-1)c_i/n$. The second-order condition is readily seen to be satisfied. This invariance is due to the linearity of equilibrium bidding strategies with uniformly distributed costs on identical supports. It reflects the fact that the equilibrium bidding strategy $\beta(c)$ is the best response to any collection of linear bidding strategies of the form $b_i(c) = \alpha_{0,i} + \alpha_{1,i}c$ that satisfy the boundary condition, i.e. $\alpha_{i,0} = \overline{c}(1-\alpha_{i,1})$. The integrated supplier has a particularly simple linear bidding strategy with $\alpha_{1,0} = 0$ and $\alpha_{1,1} = 1$.

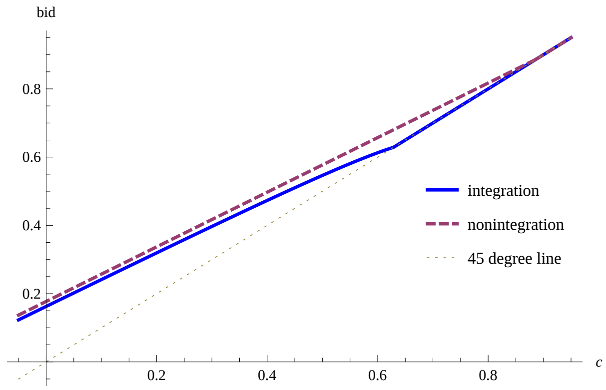

Figure 3: Equilibrium bidding with uniformly distributed costs.

appropriately choosing units of measurement for c and x. We also set $\beta = 0$ to simplify derivations.

Equilibrium bids by independent suppliers are again a constant markup on cost. The difference from the baseline model is that the markups depend endogenously on investments. In the case of non-integration the bid function is

$$b_N(c; n) = c + \frac{1}{(n-1)x_N},$$

where x_N is the symmetric investment of n independent suppliers. In the case of vertical integration, the bid function is

$$b_I(c; n) = c + \frac{1}{x_1 + (n-2)x_2},$$

where x_1 is the investment of the integrated supplier and x_2 the symmetric investment of the $n-1$ independent suppliers.

Equilibrium investments are derived from first-order conditions as before. In a symmetric equilibrium of the non-integrated environment, each of the suppliers invests an amount equal to 1 over the cube root on n^2, that is, $x_N = \frac{1}{\sqrt[3]{n^2}}$. For the integrated environment, let $z = \frac{x_2}{x_1}$. The symmetric best response investments can be written as functions of z, $x_1 = x_1(z)$ and $x_2 = x_2(z)$, respectively. Equilibrium investments are then given by $x_1 = x_1(z(n))$ and $x_2 = x_2(z(n))$, where $z(n)$ is the unique fixed point to the equation

$$z = \frac{x_2(z)}{x_1(z)}.$$

For a given $z > 0$, the integrated supplier optimally invests

$$x_1(z) = \sqrt[3]{1 - \frac{(n-1)z[3 + z(2z-6) + 2n(4 + (n-3)z)]e^{-\frac{1}{1+(n-2)z}}}{[1+(n-2)z][1+(n-1)z]^2}}$$

and the independent suppliers symmetrically invest

$$x_2(z) = \sqrt[3]{\frac{z^2 e^{-\frac{1}{1+(n-2)z}}}{[1+(n-1)z]^2}}.$$

Dividing $x_2(z)$ by $x_1(z)$ and simplifying yields the fixed point

$$z = \sqrt[3]{\frac{z^2}{[1+(n-1)z]^2 e^{\frac{1}{1+(n-2)z}} - \frac{(n-1)\{3z+z^2[(2z-6)+4n+n(n-3)z]\}}{1+(n-2)z}}}.$$

A simple graphical analysis shows that $z(n)$ is increasing in n.

Under non-integration, the equilibrium (expected) procurement cost of the buyer as a function of symmetric supplier investments x_N is

$$PC_N = \int_0^\infty b_N(c;n)dG(c;nx_N) = \frac{2n-1}{n(n-1)x_N}$$

and the (expected) profit of a supplier is

$$\Pi_N = \int_0^\infty [b_N(c;n)-c][1-G(c;(n-1)x_N)]dG(c;x_N) - \frac{1}{2}x_N^2 = \frac{1}{n(n-1)x_N} - \frac{1}{2}x_N^2.$$

Substituting $x_N(n)$ into these expressions yields equilibrium values of procurement cost and profits as functions of the number of suppliers

$$PC_N(n) = \frac{2n-1}{(n-1)\sqrt[3]{n}} \quad \text{and} \quad \Pi_N(n) = \frac{n+1}{2n(n-1)\sqrt[3]{n}}.$$

Procurement cost under vertical integration can be expressed as a function of x_1 and z:

$$\begin{aligned} PC_I &= \int_0^{x_1} c dG(c;x_1) + \frac{1}{2}x_1^2 \\ &- \int_{\frac{1}{x_1+(n-2)x_2}}^\infty \int_0^{c_1-\frac{1}{x_1+(n-2)x_2}} [c-b_I(c;n)]dG(c;(n-1)x_1 z)dG(c_1;x_1) \\ &= \frac{1}{x_1} + \frac{1}{2}x_1^2 - \frac{(n-1)ze^{-\frac{1}{1+(n-2)z}}}{1+(n-1)z}. \end{aligned}$$

Substituting $x_1 = x_1(z(n))$ and $z = z(n)$ yields procurement cost $PC_I(n)$ as a function of n. Since $z(n)$ lacks a closed form solution, so does $PC_I(n)$.

Divestiture is more profitable than vertical integration if

$$\Phi(n) \equiv PC_I(n) + \Pi_N(n) - PC_N(n)$$

is positive. Figure 4 shows that $\Phi(n) < 0$ if and only if $n < 10$. Thus, as in the baseline model, non-integration and a complete reliance on outsourcing is more profitable than vertical integration if the upstream market is sufficiently competitive.

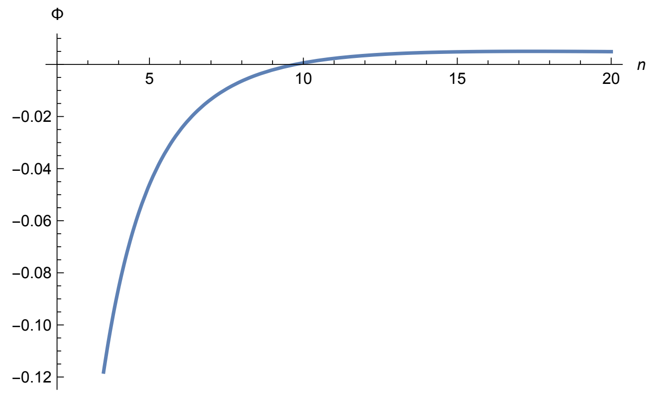

Figure 4: The benefit from divestiture, $\Phi(n)$ for the fixed-support exponential model.

It is also interesting to compare the independent bid functions under integration and non-integration. The difference in markups is

$$\Delta b(n) = \frac{1}{x_1(z(n)) + (n-2)x_2(z(n))} - \frac{1}{(n-1)x_N(n)}.$$

Figure 5 shows that $\Delta b(n) < 0$ if and only if $n < 6$. That is, the equilibrium markup is lower under vertical integration if and only if upstream competition is limited. Surprisingly, vertical integration fails to reduce markups for more competitive upstream market structures. The reason is an additional negative consequence of the investment discouragement effect: reduced investment by independent suppliers increases cost heterogeneity, causing the independent firms to bid more aggressively.

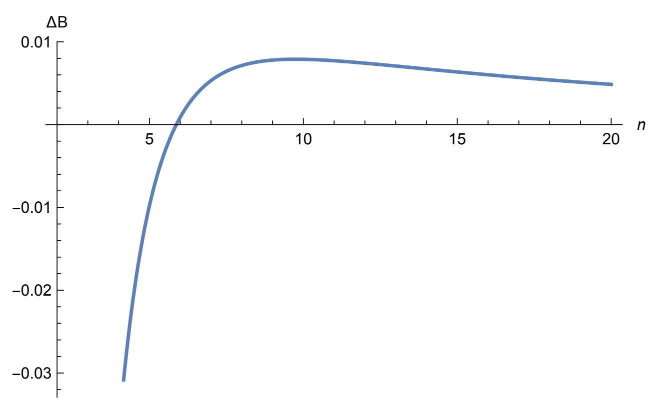

Figure 5: The function $\Delta b(n)$.

Furthermore, it can be shown that in this case vertical integration always decreases total investment, i.e. $x_1(z(n)) + (n-1)x_2(z(n)) < nx_N(n)$.

A.3 Elastic Demand

Setup To model elastic demand, we assume that the customer (or buyer, indicated with subscript B) has value v for the input, drawn from an exponential probability distribution $G_B(v) = 1 - e^{-\lambda(v-\alpha)}$ with support $[\alpha, \infty)$. The mean of the exponential distribution is $\alpha + \frac{1}{\lambda}$ and can be interpreted to indicate the expected profitability of the downstream market. The variance, which is $\frac{1}{\lambda^2}$, can be interpreted to indicate uncertainty about product differentiation. This model converges to the inelastic case as $\lambda \to 0$. The customer learns the realization of v before making the purchase (or production) decision.

Under vertical integration, the investment x_1 in cost reduction is made before the customer learns the realized v. Independent suppliers know G_B but not v. All other assumptions regarding timing are as in Section 2. The cost of exerting effort x is $\frac{a}{2}x^2$ and given investment x_i supplier i's cost is drawn from the exponential distribution $1 - e^{-\mu(c+x_i-\beta)}$ with support $[\beta - x_i, \infty)$ for all $i = 1, .., n$ and with $\mu \leq a$. To simplify the equilibrium analysis, we impose the parameter restriction

$$\beta - \alpha \geq \frac{\mu}{a(\lambda + n\mu)} - \frac{1}{\lambda + (n-1)\mu}, \tag{17}$$

which makes sure that under non-integration (and therefore also under integration) the lowest equilibrium bid is always larger than the lowest possible draw of v. Observe that the right-hand side in (17) is negative, so that $\beta \geq \alpha$ is sufficient for the condition.[24]

Bidding As in the inelastic demand case, the bidding function is the same with or without vertical integration. The bidding function with elastic demand is denoted by $b_E(c; n)$ and given by

$$b_E(c; n) = c + \frac{1}{\lambda + \mu(n-1)} \tag{18}$$

for all $c \geq \beta - \frac{\mu}{a(\lambda+n\mu)}$ as shown next.

We begin with *non-integration*. Given symmetric investments x, a symmetric equilibrium bidding strategy $b(c)$ is such that

$$c = \arg\max_z \left\{ [b(z) - c] \, [1 - G_B(b(z))] \, [1 - G(c + x)]^{n-1} \right\}.$$

For G_B and G exponential, a representative supplier's problem becomes

$$\max_z (b(z) - c) e^{-\lambda(b(z)-\alpha) - \mu(n-1)(z+x-\beta)}.$$

[24] To see where (17) comes from, notice that supplier i's expected profit when investing $x_i \leq x$ while all rivals invest x and when all suppliers bid according to (18) is $\int_{\beta - x_i}^{\infty} (1/(\lambda + (n-1)\mu))\mu e^{-\lambda(c+1/(\lambda+(n-1)\mu)-\alpha)-\mu(n-1)(c+x-\beta)-\mu(c+x_i-\beta)} dc - a x_i^2/2$. The first-order condition at $x_i = x$ is

$$\frac{\mu}{\lambda + \mu n} e^{\lambda\left(x + \alpha - \beta - \frac{1}{\lambda + (n-1)\mu}\right)} = ax.$$

If $x + \alpha - \beta - 1/(\lambda + (n-1)\mu) < 0$, the first-order condition implies $x < \frac{\mu}{a(\lambda+n\mu)}$. Plugging this back into the preceding inequality gives (17).

Taking the derivative with respect to z and imposing the boundary condition $\lim_{c\to\infty}(b(c)-c)/c = 0$ yields the unique solution

$$b(c) = c + \frac{1}{\lambda + (n-1)\mu},$$

as claimed.

With *integration*, G_B and G exponential and $x_1 \geq x_2$, a representative non-integrated supplier's problem is

$$\max_z (b(z) - c)e^{-\lambda(b(z)-\alpha)-\mu(b(z)+x_1-\beta)-\mu(n-2)(z+x_2-\beta)}.$$

Taking the derivative with respect to z yields the first-order condition

$$b'(c) - [(\lambda+\mu)b'(c) + (n-2)\mu][b(c) - c] = 0.$$

Imposing the boundary condition $\lim_{c\to\infty}(b(c) - c)/c = 0$ then gives the unique solution

$$b(c) = c + \frac{1}{\lambda + (n-1)\mu},$$

which is the same as b_E defined in (18).

Profits Consider first non-integration when the symmetric investments of the independent suppliers are x. The profit $\Pi^B_{EN}(x)$ accruing to the buyer is

$$\Pi^B_{EN}(x) = n\int_{b_E(\beta-x;n)}^{\infty}\int_{\beta-x}^{y(v)}[v - b_E(c;n)][1-G(c+x)]^{n-1}dG(c+x)dG_B(v),$$

where $y(v) = v - \frac{1}{\lambda+\mu(n-1)}$ denotes the inverse of the bidding function $b_E(c;n)$ with respect to c.

The expected profit $\Pi_{EN}(x_i, x)$ of an independent supplier under non-integration who invests x_i while each of the other suppliers is expected to invest x with $x_i \leq x$ is[25]

$$\Pi_{EN}(x_i, x) = \int_{b_E(\beta-x_i;n)}^{\infty}\int_{\beta-x_i}^{y(v)}[b_E(c;n) - c][1-G(c+x)]^{n-1}dG(c+x_i)dG_B(v) - \frac{a}{2}x_i^2.$$

With integration, the buyer's profit is $\Pi^B_{EI}(x_1, x_2) =$

$$\begin{aligned}
& \int_{\alpha}^{\infty}\int_{\beta-x_1}^{\max\{v,\beta-x_1\}}[v - c_1]dG(c_1+x_1)dG_B(v) \\
+ & \int_{\beta-x_1}^{\infty}(1-G_B(c_1))\int_{\beta-x_2}^{\max\{y(c_1),\beta-x_2\}}[c_1 - b_E(c_2;n)]dL(c_2+x_2;n-1)dG(c_1+x_1) \\
+ & \int_{\alpha}^{\infty}(1-G(v+x_1))\int_{\beta-x_2}^{\max\{y(v),\beta-x_2\}}[v - b_E(c_2;n)]dL(c_2+x_2;n-1)dG_B(v) - \frac{a}{2}x_1^2.
\end{aligned}$$

[25] For $x_i = x + \varepsilon$ with $\varepsilon > 0$ small, the expected profit function has a different functional form. However, the profit function $\Pi_{EN}(x_i, x)$ is continuously differentiable at $x_i = x$.

This profit is computed by deriving the expected profit from internal sourcing, which is done in the first line in the above expression, then adding the cost savings from sourcing from the independent supplier with the lowest bid, which is captured in the second line, and finally adding in the third line the expansion effect of external sourcing that arises whenever $c_1 > v$ and $b_E(\min\{\mathbf{c}_{-1}\}) < v$, where $\mathbf{c}_{-1} = (c_2, .., c_n)$.

Given its own investment x_i, investments $x_2 \geq x_i$ by all other non-integrated suppliers and x_1 by the integrated supplier, the expected profit $\Pi_{EI}(x_i, x_1, x_2)$ of an independent supplier under vertical integration is $\Pi_{EI}(x_i, x_1, x_2) =$

$$\int_{\beta - x_i}^{\infty} [b_E(c;n) - c][1 - G_B(b_E(c;n))][1 - G(b_E(c;n) + x_1)][1 - G(c + x_2)]^{n-2} dG(c + x_i) - \frac{a}{2}x_i^2.$$

Equilibrium Investments Under non-integration, the necessary first-order conditions for the symmetric equilibrium investment x is

$$x = \frac{1}{a}\frac{\mu}{\lambda + n\mu} e^{-\lambda\left[\frac{1}{\lambda + (n-1)\mu} + \beta - \alpha - x\right]}. \tag{19}$$

With vertical integration, the vertically integrated supplier invests x_1 and all $n - 1$ independent suppliers invest x_2 satisfying

$$x_1 = x_2 + \frac{1}{a}\frac{\mu}{\lambda + \mu} e^{-\mu(x_1 - x_2)} \left[e^{\mu(\beta - \alpha - x_2)} - e^{-\lambda(\beta - \alpha - x_2) - \frac{\lambda + \mu}{\lambda + (n-1)\mu}} \right] \tag{20}$$

and

$$x_2 = \frac{1}{a}\frac{\mu}{\lambda + n\mu} e^{-\lambda(\beta - \alpha - x_2) - \mu(x_1 - x_2) - \frac{\lambda + \mu}{\lambda + (n-1)\mu}} \tag{21}$$

according to the necessary first-order conditions for equilibrium. We proceed by presuming that these conditions are also sufficient. (For the parameters underlying Figure 6 this can be verified numerically.)

Profitability of Non-Integration Evaluating (19), (20) and (21) numerically we can determine the buyer's and the independent suppliers' equilibrium profits under non-integration and vertical integration. Denoting these equilibrium payoffs with an asterisk, the analogue for the case of elastic demand to the function $\Phi(n, \mu)$ defined in (16) is

$$\Phi_E(n, \mu, \alpha, \lambda, \beta) := \Pi_{EN}^{B*} + \Pi_{EN}^{*} - \Pi_{EI}^{B*}.$$

Figure 6 contains contour sets of $\Phi_E(n, \mu, \alpha, \lambda) = 0$ for different values of n in (α, λ)-space with $\mu = 1$ and $\beta = 0$. Non-integration is profitable for a given n for values of α and λ below the corresponding curve.

Social Welfare Effects In the model with inelastic demand, non-integration is always socially optimal because it minimizes the sum of expected production and investment costs although it is not always an equilibrium outcome. In contrast, with elastic demand vertical integration has an additional, socially beneficial effect because it increases the

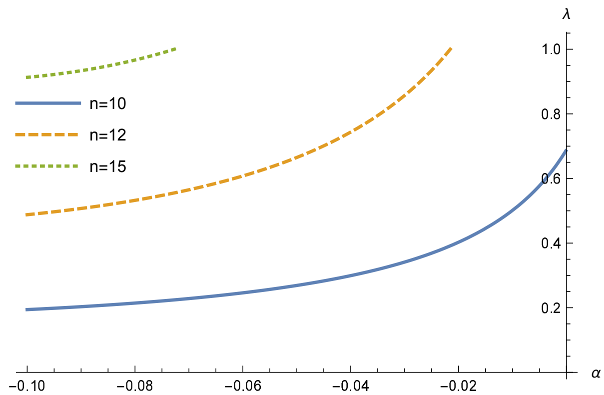

Figure 6: $\Phi_E(n, \mu, \alpha, \lambda) = 0$ for selected parameters.

market demand by inducing production for realizations of costs and values for which there is no production under non-integration, (and because it decreases the lowest cost of production by increasing investment by the integrated supplier).

The numerical analysis for the shifting support exponential model with elastic demand, displayed in Figure 7, reveals that vertical integration is better than non-integration when n is small. As before Φ is the private benefit from divestiture while ΔW is the difference between social welfare under divestiture and under vertical integration. The figure plots Φ and ΔW for $\beta = 0$ and $a = 1$. The figure illustrates a substantial range of upstream market structures for which vertical integration is privately optimal but socially inefficient.

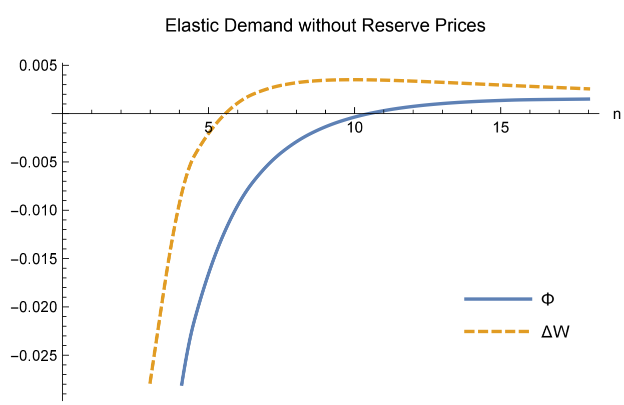

Figure 7: Φ and ΔW as functions of n (without any reserve prices).

A.4 Reserve Prices

We perform the analysis of the effect of reserve prices within the exponential-quadratic model with inelastic demand, setting $a = 1$. Suppose that the vertically integrated customer commits to a reserve price r after learning the cost of internal supply c_1. Given the symmetric equilibrium investment of independent firms x_2, the optimal reserve price satisfies

$$c_1 = r + \frac{G(r+x_2)}{g(r+x_2)} \equiv \Gamma_{x_2}(r)$$

while the symmetric bidding function $b(c,r)$ depends on the reserve price r according to[26]

$$b(c,r) = c + \frac{1}{\mu(n-1)}\left[1 - e^{-\mu(n-1)(r-c)}\right]$$

where we drop its dependence on n for notational ease.

In equilibrium, the vertically integrated firm chooses its own investment x_1 to minimize expected procurement cost given x_2, and each independent supplier invests to maximize expected profit given x_1 and x_2. The optimal reserve given $c_1 \geq \beta - x_2$ then satisfies

$$r(c_1) := \Gamma_{x_2}^{-1}(c_1). \tag{22}$$

Total equilibrium procurement cost (net of investment cost) is equal to the expected cost of internal supply, denoted $E_{x_1}[c_1] = \beta - x_1 + \frac{1}{\mu}$, minus the expected cost savings from sourcing externally:

$$E_{x_1}[c_1] - \int_{\beta-x_2}^{\infty}\int_{\beta-x_2}^{r(c_1)} [c_1 - b(c, r(c_1)] dL(c+x_2; n-1) dG(c_1+x_1). \tag{23}$$

Assuming $x_1 > x_2$, the expected profit of a representative independent firm choosing x in the neighborhood of x_2 is equal to the expected value of the markup times the probability of winning the auction:

$$\int_{\beta-x_2}^{\infty}\int_{\beta-x}^{r(c_1)} [b(c, r(c_1)) - c][1 - L(c+x_2; n-2)] dG(c+x) dG(c_1+x_1)$$

In equilibrium each independent supplier chooses $x = x_2$. We compute the equilibrium investments levels (x_1, x_2) solving the necessary first-order conditions, presuming the appropriate second-order conditions are satisfied.

The condition for non-integration to be preferred to vertical integration is similar to before. Figure 8 graphs Φ as a function of n for $\mu = 1$ and compares it to the case without reserves, depicted also in Figure 1. The curve is shifted to the right compared to the base model in which there is no reserve price. Although an optimal reserve price does lower procurement costs under vertical integration, non-integration nevertheless is preferred for n sufficiently large.

Elastic Demand with Reserve The analysis with elastic demand can also be extended to account for optimal reserves. Under non-integration, the optimal reserve is $r(v)$, where the function $r(.)$ is defined in (22). With vertical integration, the optimal

[26] In the exponential case, the virtual cost function $\Gamma_{x_2}(r)$ is strictly increasing in r for given x_2, and therefore invertible. We denote its inverse by $\Gamma_{x_2}^{-1}(c_1)$. The bid function $b(c,r)$ solves the usual necessary differential equation for optimal bidding with the boundary condition $b(r,r) = r$.

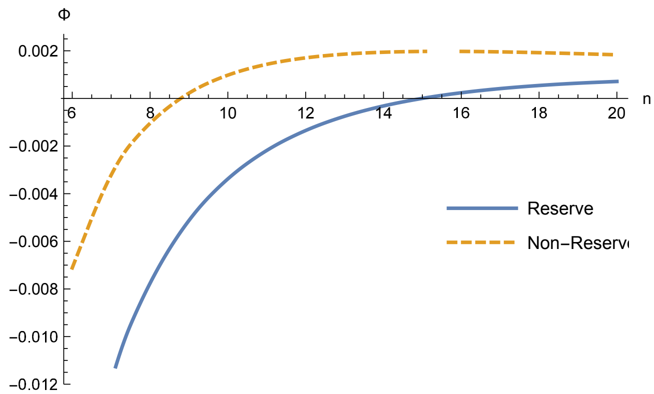

Figure 8: The function Φ with and without reserve prices for $\mu = 1$.

reserve will be given by the same function $r(.)$, evaluated at $\hat{v} := \min\{c_1, v\}$. Because of continuity, it is intuitive that, with elastic demand and optimal reserves, non-integration will be profitable in the neighborhood of the parameter region for which it is profitable with perfectly inelastic demand and a reserve, that is, for values of λ close to zero. This intuition is corroborated by numerical analysis. Figure 9 plots the buyer's gain from non-integration with reserves, denoted Φ_{ER}, and her gain from non-integration without reserves, Φ_E, as a function of λ for $n = 16$ and $\alpha = \beta = 0$.

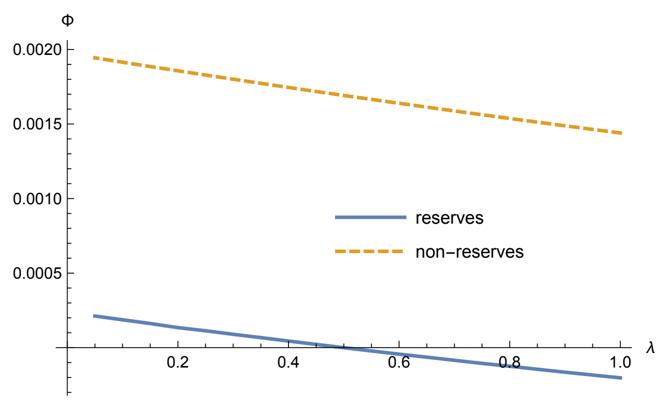

Figure 9: Φ_{ER} and Φ_E as functions of λ.

Figure 10 plots the social welfare effects of and the private incentives for divestiture for elastic demand when the customer can set a reserve price. Comparing Figure 7 to Figure 10 reveals that the ability to set a reserve hardly matters for the social welfare effects but increases the private benefits from vertical integration, thereby increasing the range in which vertical integration is an equilibrium outcome but not socially desirable.

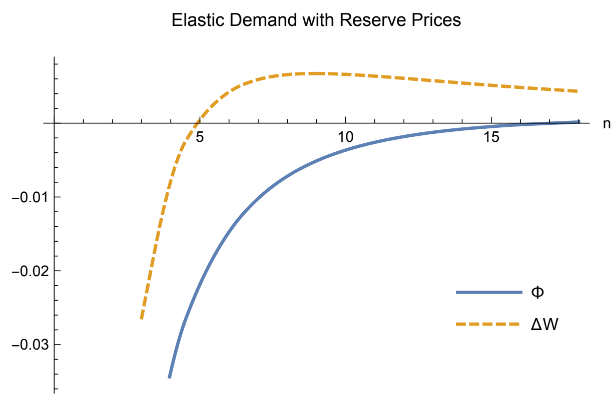

Figure 10: Φ and ΔW as functions of n with reserves.

B Proofs

Proof of Corollary 1: The necessary conditions have been derived in the main text. We are now going to show that the profit function is $\Pi_N(x_i, x)$ is quasiconcave in x_i (and continuously differentiable).

For $x_i \geq x$, the first derivative of $\Pi_N(x_i, x)$ with respect to x_i is

$$\frac{\partial \Pi_N(x_i, x)}{\partial x_i} = 1 - \frac{n-1}{n} e^{-\mu(x - x_i)} - a x_i.$$

Observe that this partial is decreasing in x_i. It is thus largest at $x_i = x$, at which point it is $\frac{1}{n} - a x_i$.

For $x_i < x$, the first partial of $\Pi_N(x_i, x)$ with respect to x_i is

$$\frac{\partial \Pi_N(x_i, x)}{\partial x_i} = \frac{1}{n} e^{-\mu(n-1)(x - x_i)} - a x_i. \tag{24}$$

Because $\frac{1}{n} e^{-\mu(n-1)(x-x_i)}$ is increasing and convex in x_i while $a x_i$ is increasing and linear in x_i, the functions $\frac{1}{n} e^{-\mu(n-1)(x-x_i)}$ and $a x_i$ have either (i) no point of intersection on $[0, \infty)$ or, generically, (ii) two points of intersection. If (i) is the case, we have $\frac{\partial \Pi_N(x_i,x)}{\partial x_i} > 0$ for all $x_i \leq x$. In case (ii), for $\mu < \frac{n}{n-1} a$ and $x = \frac{1}{an}$, the smallest point of intersection is $x_i = x$. To see this, evaluate the second left-hand partial at $x_i = x$ to obtain

$$\frac{\partial^2 \Pi_N(x_i, x)}{\partial x_i^2}|_{x_i = x} = \mu \frac{n-1}{n} - a,$$

which is negative if and only if $\mu < \frac{n}{n-1} a$. This means that at $x_i = x$, $\frac{1}{n} e^{-\mu(n-1)(x-x_i)}$ is less steep than $a x_i$. Evaluated at $x_i = x$, the first-order condition $\frac{\partial \Pi_N(x_i,x)}{\partial x_i} = 0$ implies $x = \frac{1}{an}$. Thus, $x_i = x = \frac{1}{an}$ is the smallest point of intersection under the conditions stated. (The non-generic case occurs if $\mu = \frac{n}{n-1} a$, in which case $x_i = x = \frac{1}{an}$ is the unique point of intersection.) Consequently, the profit function is quasiconcave and the second-order condition is satisfied if and only if $\frac{\mu}{a} < \frac{n}{n-1}$.

To see that PC_N^* decreases in n, observe that

$$\frac{\partial PC_N^*}{\partial n} = \frac{(\mu - a)(n-1)^2 - an^2}{\mu a n^2 (n-1)^2},$$

which is negative if and only if $\frac{\mu}{a} < 1 + \frac{n^2}{(n-1)^2}$. The derivative of Π_N^* with respect to n is

$$\frac{\partial \Pi_N^*}{\partial n} = \frac{\mu(n-1)^2 - an(2n-1)}{\mu a n^3 (n-1)^2},$$

which has the same sign as $\mu(n-1) - an\left(1 + \frac{n}{n-1}\right)$. This is negative if and only if $\frac{\mu}{a} < \frac{n}{n-1}\left(1 + \frac{n}{n-1}\right)$. Both inequalities are satisfied if $\mu < a\frac{n}{n-1}$. ■

Proof of Proposition 2: Part (a): Equations (9) and (10) are the necessary first-order conditions as shown in the text.

Part (b): Denote by $x_2^1(x_1)$ and $x_2^2(x_1)$, respectively, the solutions to (9) and (10) in x_2. Invoking the implicit function theorem, we have

$$\frac{dx_2^2(x_1)}{dx_1} = -\frac{s_1(x_1, x_2^2(x_1))}{s_2(x_1, x_2^2(x_1)) - \psi'(x_2^2(x_1))} < 0 \tag{25}$$

and

$$\frac{dx_2^1(x_1)}{dx_1} = -\frac{s_1(x_1, x_2^1(x_1)) + \frac{1}{n-1}\psi'(x_1)}{s_2(x_1, x_2^1(x_1))} < 0, \tag{26}$$

where the inequalities hold by second-order conditions and assumption (ii).

Assume that there is a point of intersection of $x_2^1(x_1)$ and $x_2^2(x_1)$, that is, there is at least one value of x_1, denoted x_1', such that $x_2^1(x_1') = x_2^2(x_1')$. Under assumptions (i) and (ii), we have

$$\frac{dx_2^1(x_1)}{dx_1}|_{x_1 = x_1'} < \frac{dx_2^2(x_1)}{dx_1}|_{x_1 = x_1'} < 0, \tag{27}$$

which proves uniqueness of such a point of intersection. Next, we establish that such a point exists, is an equilibrium, and satisfies $x_1' = x_1^* > x_2^* = x_2^2(x_1')$.

Let $\overline{x}$ be the smallest number such that $1 - (n-1)s(\overline{x}, \overline{x}) = \psi(\overline{x})$ and let $\hat{x}$ be the smallest number such that $s(\hat{x}, \hat{x}) = \psi(\hat{x})$. Because $s(x, x) < 1/n$ as noted in Section II.B, it follows that $\overline{x} > \psi^{-1}(1/n) > \hat{x}$. This implies that $x_2^2(\hat{x}) = \hat{x} < \overline{x} = x_2^1(\overline{x})$.

Next, let $\overline{x}_1$ be such that $1 - (n-1)s(\overline{x}_1, 0) = \psi(\overline{x}_1)$. Notice that $\overline{x}_1 > \overline{x}$. Because of assumptions (ii) and (iii), we know that $\overline{x}_1 < \psi^{-1}(1)$. Therefore, $s(\overline{x}_1, 0) > 0$. Consequently, $x_2^2(\overline{x}_1) > 0$. Lastly, let $\tilde{x}_1$ be such that $s(\tilde{x}_1, 0) = 0$. Notice that $\tilde{x}_1$ may be infinity. Because $s_1 < 0$, $\tilde{x}_1 > \overline{x}_1$ follows.

Taken together we have thus shown that $x_2^2(x_1)$ is a continuously decreasing function in x_1 on $[\hat{x}, \tilde{x}_1]$ satisfying $x_2^2(\overline{x}_1) > 0$ and $x_2^2(\overline{x}) < x_2^2(\tilde{x}) < x_2^1(\overline{x})$. Moreover, on $[\overline{x}, \overline{x}_1]$,

$x_2^1(x_1)$ is a continuous function satisfying $x_2^1(\overline{x}) > x_2^2(\overline{x})$ and $x_2^1(\overline{x}_1) = 0 < x_2^2(\overline{x}_1)$. Thus, the functions $x_2^1(x_1)$ and $x_2^2(x_1)$ have a point of intersection on $[\overline{x}, \overline{x}_1]$.

Quasi-concavity and quasi-convexity imply that this point of intersection is an equilibrium. For all $x_1 \in (\hat{x}, \overline{x}_1]$, we have $x_2^2(x_1) < x_1$, which proves that $x_1^* := x_1' > x_2^* := x_2^2(x_1')$. Finally, $x_1^* > x^*$ and $x^* > x_2^*$ then follows from the first-order condition under non-integration, $1/n = \psi(x^*)$, and $s(x_1^*, x_2^*) < 1/n$, which holds because $s(x, x) < 1/n$, $x_1^* > x_2^*$ as just shown, and $s_1 < 0 < s_2$ by assumption (ii). ■

Proof of Corollary 2: Under non-integration, equilibrium effort is given by $\psi(x^*) = \frac{1}{n}$. On the other hand, rewriting the consolidated equilibrium condition with vertical integration, (11), as $\frac{n-1}{n}\psi(x_2) + \frac{1}{n}\psi(x_1) = \frac{1}{n}$, it follows from Jensen's inequality that $(n-1)x_2 + x_1 = nx^*$ if $\psi'' = 0$ and $(n-1)x_2 + x_1 > nx^*$ if $\psi'' < 0$ and $(n-1)x_2 + x_1 < nx^*$ if $\psi'' > 0$. ■

Proof of Corollary 3: The arguments in the main text imply that PC_I^* and Π_I^* are the equilibrium payoffs of the integrated firm and the independent suppliers with x_1 and x_2 given by (14).

Having already argued in the main text why assumptions (ii) and (iii) are satisfied, we are thus left to show that assumption (i) is satisfied.

We begin by establishing *quasi-concavity* of $\Pi_I(x_i, x_1, x_2)$.

Case 1: $x_i < x_2$. We first look at a downward deviation $x_i < x_2$ by a non-integrated supplier. Applying the exponential-quadratic model to the definition of $\Pi_I(x_i, x_1, x_2)$ given in Section II.B, we have that the first and second partials of $\Pi_I(x_i, x_1, x_2)$ with respect to x_i are

$$\frac{\partial \Pi_I(x_i, x_1, x_2)}{\partial x_i} = \frac{1}{n} e^{-\mu\Delta - \frac{1}{n-1} + \mu(n-1)(x_i - x_2)} - a x_i$$

and

$$\frac{\partial^2 \Pi_I(x_i, x_1, x_2)}{\partial x_i^2} = \frac{\mu(n-1)}{n} e^{-\mu\Delta - \frac{1}{n-1} + \mu(n-1)(x_i - x_2)} - a,$$

where $\Delta \equiv x_1 - x_2$. The profit function is thus concave on $[0, x_2]$ if and only if $\frac{\mu(n-1)}{n} e^{-\mu\Delta - \frac{1}{n-1} + \mu(n-1)(x_i - x_2)} - a \leq 0$. As the term $\frac{\mu(n-1)}{n} e^{-\mu\Delta - \frac{1}{n-1} + \mu(n-1)(x_i - x_2)}$ increases in x_i, this second-order condition is thus satisfied for all $x_i \in [0, x_2]$ if and only if

$$\frac{\mu}{a} \leq \frac{n}{(n-1)(1 - a\Delta)},$$

where $1 - a\Delta = e^{-\mu\Delta - \frac{1}{n-1}}$ has been used. Since $a\Delta < 1$, this second-order condition is always satisfied if the necessary condition for a symmetric equilibrium under non-integration holds.

Let $\hat{\hat{x}} = x_2 + \frac{n-2}{\mu(n-1)}$.

Case 2: $x_i \in [x_2, \hat{\hat{x}}]$. Next we consider deviations by i such that $c_i \in \left[\beta - x_2 - \frac{1}{\mu}\frac{n-2}{n-1}, \beta - x_2\right]$ occur with positive probability, and no lower c_i can occur. From Lemma 2 we know that for cost realizations in this interval, the optimal bid by i will be the constant bid $\beta - x_2 + \frac{1}{\mu(n-1)}$.

For $x_i \in [x_2, \hat{\hat{x}}]$ the profit function for the deviating supplier i is

$$\begin{aligned}
\Pi_I(x_i, x_1, x_2) &= \frac{1}{n-1}\int_{\beta-x_2}^{\infty} e^{-\mu[n(c_i-\beta)+x_1+(n-2)x_2+x_i+\frac{1}{\mu(n-1)}]}dc_i \\
&+ \int_{\beta-x_i}^{\beta-x_2} \mu\left(\beta - x_2 + \frac{1}{\mu(n-1)} - c_i\right) e^{-\mu\Delta-\frac{1}{n-1}-\mu(c_i+x_i-\beta)}dc_i - \frac{a}{2}x_i^2 \\
&= e^{-\mu\Delta-\frac{1}{n-1}}\left[x_i - x_2 - \frac{n-2}{\mu(n-1)} + e^{-\mu(x_i-x_2)}\frac{n-1}{\mu n}\right] - \frac{a}{2}x_i^2.
\end{aligned}$$

The first and second partial derivatives are

$$\begin{aligned}
\frac{\partial \Pi_I(x_i, x_1, x_2)}{\partial x_i} &= e^{-\mu\Delta-\frac{1}{n-1}}\left[1 - \frac{n-1}{n}e^{-\mu(x_i-x_2)}\right] - ax_i \\
\frac{\partial^2 \Pi_I(x_i, x_1, x_2)}{\partial x_i^2} &= e^{-\mu\Delta-\frac{1}{n-1}}\left[\mu\frac{n-1}{n}e^{-\mu(x_i-x_2)}\right] - a.
\end{aligned}$$

Therefore, on $[x_2, \hat{\hat{x}}]$, the deviator's profit function is concave in x_i, and maximized at $x_i = x_2$ if and only if

$$\frac{\mu}{a} < \frac{n}{(n-1)(1-a\Delta)},$$

which is the same condition derived in Case 1.

Case 3: $x_i \in [\hat{\hat{x}}, x_1 + \frac{1}{\mu}]$. We next consider investments $x_i \in [\hat{\hat{x}}, x_1 + \frac{1}{\mu}]$. This implies that there is a range of cost realizations such that i is effectively a monopoly supplier facing downward sloping demand. The expected profit of the deviating supplier is

$$\begin{aligned}
\Pi_I(x_i, x_1, x_2) &= \frac{1}{n-1}\int_{\beta-x_2}^{\infty} e^{-\mu[n(c_i-\beta)+x_1+(n-2)x_2+x_i+\frac{1}{\mu(n-1)}]}dc_i \\
&+ \int_{\beta-x_2-\frac{n-2}{\mu(n-1)}}^{\beta-x_2} \mu\left(\beta - x_2 + \frac{1}{\mu(n-1)} - c_i\right) e^{-\mu\Delta-\frac{1}{n-1}-\mu(c_i+x_i-\beta)}dc_i \\
&+ \int_{\beta-x_i}^{\beta-x_2-\frac{n-2}{\mu(n-1)}} e^{-\mu[2(c-\beta)+x_1+x_i]-1}dc_i - \frac{a}{2}x_i^2 \\
&= e^{-\mu\Delta-\frac{1}{n-1}-\mu(x_i-x_2)}\frac{1}{\mu}\frac{n-1}{n} - \frac{1}{2\mu}e^{-\mu\Delta-\frac{1}{n-1}}\left[e^{-\mu(x_i-x_2)+\frac{n-2}{n-1}} - e^{\mu(x_i-x_2)-\frac{n-2}{n-1}}\right] - \frac{a}{2}x_i^2
\end{aligned}$$

as the profit function for a deviating independent supplier choosing investment $x_i \in [\hat{\hat{x}}, x_1 + \frac{1}{\mu}]$. Using the facts that $1 - a\Delta = e^{-\mu\Delta-\frac{1}{n-1}}$ and $x_2 = \frac{1}{an}(1-a\Delta)$ and defining $y := \mu(x_i - x_2) - \frac{n-2}{n-1}$, we can express the deviator's profit equivalently as

$$\hat{\Pi}_I(y, x_1, x_2) = \frac{1-a\Delta}{\mu}\left[e^{-y-\frac{n-2}{n-1}}\frac{n-1}{n} + \frac{1}{2}\left[e^y - e^{-y}\right]\right] - \frac{a}{2}\left(\frac{1}{\mu}\left[y + \frac{n-2}{n-1}\right] + \frac{1}{an}(1-a\Delta)\right)^2,$$

for $y \in [0, \mu\Delta + \frac{1}{n-1}]$.

We are now going to show that $\hat{\Pi}_I(y, x_1, x_2)$ is decreasing in y for all $y \in [0, \mu\Delta + \frac{1}{n-1}]$. We do so by first establishing that $\frac{\partial \hat{\Pi}_I(y,x_1,x_2)}{\partial y}|_{y=0} < 0$. Second, we show that the third derivative with respect to y is positive. This implies that the second derivative is largest over this interval at $y = \mu\Delta + \frac{1}{n-1}$. The final step in the argument is then to show that $\frac{\partial^2 \hat{\Pi}_I(y,x_1,x_2)}{\partial y^2}|_{y=\mu\Delta+\frac{1}{n-1}} < 0$, which then implies that $\hat{\Pi}_I(y, x_1, x_2)$ is decreasing over the interval in question.

Step 1:

$$\frac{\partial \hat{\Pi}_I(y, x_1, x_2)}{\partial y} = \frac{1 - a\Delta}{\mu} \left[-e^{-y-\frac{n-2}{n-1}} \frac{n-1}{n} + \frac{1}{2} \left(e^y + e^{-y} \right) \right] - \frac{a}{\mu} \left[\frac{1}{\mu}(y + \frac{n-2}{n-1}) + \frac{1}{an}(1 - a\Delta) \right].$$

At $y = 0$, we get

$$\frac{\partial \hat{\Pi}_I(y, x_1, x_2)}{\partial y}|_{y=0} = \frac{1}{\mu} \left\{ \frac{n-1}{n}(1 - a\Delta) \left[1 - e^{\frac{n-2}{n-1}} \right] - \frac{a}{\mu} \frac{n-2}{n-1} \right\}.$$

Since $(1 - a\Delta) < 1$ and $\frac{a}{\mu} \geq \frac{n-1}{n}$ under the necessary and sufficient condition for the existence of a symmetric equilibrium under non-integration, we have

$$\frac{\partial \hat{\Pi}_I(y, x_1, x_2)}{\partial y}|_{y=0} < \frac{1}{\mu} \frac{n-1}{n} \left\{ 1 - e^{\frac{n-2}{n-1}} - \frac{n-2}{n-1} \right\}.$$

The term in brackets is decreasing in n and equal to 0 at $n = 2$. Thus, $\frac{\partial \hat{\Pi}_I(y,x_1,x_2)}{\partial y}|_{y=0} < 0$ holds for all n.

Step 2: Differentiating further we get

$$\begin{aligned} \frac{\partial^2 \hat{\Pi}_I(y, x_1, x_2)}{\partial y^2} &= \frac{1 - a\Delta}{\mu} \left[e^{-y-\frac{n-2}{n-1}} \frac{n-1}{n} + \frac{1}{2} \left(e^y - e^{-y} \right) \right] - \frac{a}{\mu^2} \\ &= \frac{1 - a\Delta}{\mu} \left[\frac{1}{2} e^y + \left(\frac{n-1}{n} e^{-\frac{n-2}{n-1}} - \frac{1}{2} \right) e^{-y} \right] - \frac{a}{\mu^2}, \end{aligned}$$

where $\frac{n-1}{n} e^{-\frac{n-2}{n-1}} - \frac{1}{2} \leq 0$ for all $n \geq 2$ with strict inequality for $n > 2$ (at $n = 2$, it is equal to 0; differentiating with respect to n yields $-\frac{e^{-\frac{n-2}{n-1}}}{n^2(n-1)}$, which is negative), and

$$\frac{\partial^3 \hat{\Pi}_I(y, x_1, x_2)}{\partial y^3} = \frac{1 - a\Delta}{\mu} \left[\frac{1}{2} e^y - \left(\frac{n-1}{n} e^{-\frac{n-2}{n-1}} - \frac{1}{2} \right) e^{-y} \right] > 0.$$

Thus, $\frac{\partial^2 \hat{\Pi}_I(y,x_1,x_2)}{\partial y^2}$ is an increasing function of y and hence largest at $y = \mu\Delta + \frac{1}{n-1}$.

Step 3: Evaluating $\frac{\partial^2 \hat{\Pi}_I(y,x_1,x_2)}{\partial y^2}$ at $y = \mu\Delta + \frac{1}{n-1}$ one gets

$$\frac{\partial^2 \hat{\Pi}_I(y, x_1, x_2)}{\partial y^2}|_{y=\mu\Delta+\frac{1}{n-1}} = \frac{1 - a\Delta}{\mu} \left[\frac{1}{2} e^{\mu\Delta + \frac{1}{n-1}} + \left(\frac{n-1}{n} e^{-\frac{n-2}{n-1}} - \frac{1}{2} \right) e^{-\mu\Delta - \frac{1}{n-1}} \right] - \frac{a}{\mu^2}.$$

Replacing $e^{-\mu\Delta-\frac{1}{n-1}}$ by $1-a\Delta$ and $e^{\mu\Delta+\frac{1}{n-1}}$ by $\frac{1}{1-a\Delta}$ and collecting terms yields

$$\frac{\partial^2\hat{\Pi}_I(y,x_1,x_2)}{\partial y^2}|_{y=\mu\Delta+\frac{1}{n-1}} = \frac{\mu-2a}{2\mu^2}+\left(\frac{n-1}{n}e^{-\frac{n-2}{n-1}}-\frac{1}{2}\right)\frac{(1-a\Delta)^2}{\mu}.$$

As just noticed, the last term on the right-hand side is not positive. Therefore,

$$\frac{\partial^2\hat{\Pi}_I(y,x_1,x_2)}{\partial y^2}|_{y=\mu\Delta+\frac{1}{n-1}} < 0$$

if $\frac{\mu}{a} < 2$, which is certainly the case if $\frac{\mu}{a} < \frac{n}{n-1}$, which is the necessary and sufficient condition for the existence of a symmetric equilibrium under non-integration.

Case 4: $x_i > x_1 + \frac{1}{\mu}$. Finally, consider investments $x_i > x_1 + \frac{1}{\mu}$. This implies that, in addition to the range of cost realizations covered by Case 3, there is now also a range of low cost realizations such that the optimal bid is $\beta - x_1$ for all these costs. This bid is always accepted. For such investments, the expected profit of a deviating non-integrated supplier is

$$\begin{aligned}
\Pi_I(x_i,x_1,x_2) &= \frac{1}{n-1}\int_{\beta-x_2}^{\infty} e^{-\mu[n(c_i-\beta)+x_1+(n-2)x_2+x_i+\frac{1}{\mu(n-1)}]}dc_i \\
&+ \int_{\beta-x_2-\frac{n-2}{\mu(n-1)}}^{\beta-x_2} \mu\left(\beta-x_2+\frac{1}{\mu(n-1)}-c_i\right)e^{-\mu\Delta-\frac{1}{n-1}-\mu(c_i+x_i-\beta)}dc_i \\
&+ \int_{\beta-x_1-\frac{1}{\mu}}^{\beta-x_2-\frac{n-2}{\mu(n-1)}} e^{-\mu[2(c-\beta)+x_1+x_i]-1}dc_i \\
&+ \int_{\beta-x_i}^{\beta-x_1-\frac{1}{\mu}} (\beta-x_1-c_i)\mu e^{-\mu(c_i+x_i-\beta}dc_i - \frac{a}{2}x_i^2 \\
&= \frac{1}{\mu}\frac{n-1}{n}e^{-\mu\Delta-\frac{1}{n-1}-\mu(x_i-x_2)} \\
&+ \frac{1}{2\mu}e^{-\mu(x_i-x_i)+1}\left[1-e^{-2(\mu\Delta+\frac{1}{n-1})}\right] \\
&+ x_i-x_1-\frac{1}{\mu}-\frac{a}{2}x_i^2.
\end{aligned}$$

The key observation is that the terms in the third to last and second to last lines decrease in x_i. The derivative of the last line with respect to x_i is $1-ax_i$. Since $x_i \geq x_1+\frac{1}{\mu} \geq \frac{1}{an}+\frac{1}{\mu}$, we have

$$1-ax_i \leq \frac{n-1}{n}-\frac{a}{\mu} \leq 0,$$

where the inequality follows because it is equivalent to $\frac{\mu}{a} \leq \frac{n}{n-1}$.

We now turn to establishing *quasi-convexity* of $PC_I(x_1,x_2)$ over the relevant range.

For the exponential model with $x_1 \geq x_2 - \frac{1}{\mu(n-1)}$, $PC_I(x_1, x_2)$ is as defined in (8) because $b_I^*(\beta - x_2) = \beta - x_2 + \frac{1}{\mu(n-1)} \geq \beta - x_1$ under this condition. With quadratic costs of effort, we have for any $x_1 \geq x_2 - \frac{1}{\mu(n-1)}$,

$$\frac{\partial PC_I(x_1, x_2)}{\partial x_1} = -(1 - (n-1)s(x_1, x_2) - ax_1).$$

Noticing that $s_1(x_1, x_2) = -\mu s(x_1, x_2)$, the second-order condition is

$$\frac{\partial^2 PC_I(x_1, x_2)}{\partial x_1^2} = -(n-1)\mu s(x_1, x_2) + a \geq 0.$$

Observe that for all $x_1 \geq x_2 - \frac{1}{\mu(n-1)}$, $s(x_1, x_2) \leq \frac{1}{n}$, with equality only if $x_1 = x_2 - \frac{1}{\mu(n-1)}$. Thus, $-(n-1)\mu s(x_1, x_2) + a \geq a - (n-1)\mu/n \geq 0$, where the last inequality holds because of the assumption $\frac{\mu}{a} \leq \frac{n}{n-1}$. Thus, on $[x_2 - \frac{1}{\mu(n-1)}, \infty)$, $PC_I(x_1, x_2)$ is convex in x_1.

For $x_1 \leq x_2 - \frac{1}{\mu(n-1)}$, the procurement cost of the integrated supplier can be written as

$$\begin{aligned} PC_I(x_1, x_2) &= \Psi(x_1) + \int_{\beta - x_2}^{\infty} b_I^*(c; \mathbf{x}, n) dL(c + x_2; n-1) \\ &+ \int_{\beta - x_1}^{\infty} \int_{b_I^{-1}(c)}^{\infty} (c - b_I^*(y; \mathbf{x}, n)) dL(y + x_2; n-1) dG(c + x_1), \end{aligned} \tag{28}$$

where $b_I^{-1}(c)$ is the inverse of $b_I^*(y; \mathbf{x}, n)$, i.e. $b_I^*(b_I^{-1}(c); \mathbf{x}, n) = c$ (for example, for the exponential $b_I^{-1}(c) = c - \frac{1}{\mu(n-1)}$). Here the first line captures cost of effort plus the cost of always procuring the good from the independent suppliers. The second line represents the cost savings from avoiding the markup by producing internally. Observe that the integral in the first line does not depend on x_1 if x_1 is a deviation from equilibrium (only the equilibrium level of x_1 affects $b_I^*(c; \mathbf{x}, n)$ with $\mathbf{x} = (x_1, x_2)$). Therefore, for the purpose of cost minimization, it can be treated as a constant, denoted K.

Making use of the exponential-quadratic assumptions, we obtain

$$PC_I(x_1, x_2) = ax_1^2/2 + K - \frac{1}{\mu(n-1)} \frac{e^{\mu(n-1)(x_1 - x_2) + 1}}{n},$$

whose derivatives are

$$\frac{\partial PC_I(x_1, x_2)}{\partial x_1} = -\frac{e^{\mu(n-1)(x_1 - x_2) + 1}}{n} + ax_1$$

and

$$\frac{\partial^2 PC_I(x_1, x_2)}{\partial x_1^2} = -\mu(n-1)\frac{e^{\mu(n-1)(x_1 - x_2) + 1}}{n} + a.$$

Thus, the function is (twice) continuously differentiable.

Notice also that because $x_2 \leq \frac{1}{a(n-1)}$, $x_1 \leq x_2 - \frac{1}{\mu(n-1)}$ is not possible if $\mu \leq a$ as $\mu \leq a$ implies $\frac{1}{a(n-1)} - \frac{1}{\mu(n-1)} \leq 0$, which would thus require $x_1 \leq 0$. In the following analysis, we can thus assume $1 \leq \frac{\mu}{a} \leq \frac{n}{n-1}$.

For a fixed $x_2 > 0$, let $\underline{x}_1(x_2)$ denote the smallest non-negative value of x_1 such that $\frac{\partial PC_I(x_1,x_2)}{\partial x_1} = 0$, that is, $\underline{x}_1(x_2)$ is such that:

$$\frac{e^{\mu(\underline{x}_1(x_2)-x_2)+1}}{n} = a\underline{x}_1(x_2).$$

(If no such value exists, we set $\underline{x}_1(x_2) = \infty$). Because $h(x_1) := \frac{e^{\mu(x_1-x_2)+1}}{n}$ is increasing and convex in x_1, satisfying $\frac{e^{\mu(-x_2)+1}}{n} > 0$, while ax_1 is increasing linearly in x_1 and equal to 0 at $x_1 = 0$, it follows that $\underline{x}_1(x_2) > 0$ and $\partial PC_I(x_1, x_2)/\partial x_1 < 0$ for all $x_1 < \underline{x}_1(x_2)$.

We are now going to show that $\underline{x}_1(x_2) > x_2 - \frac{1}{\mu(n-1)}$. This then completes the proof of quasiconvexity. Observe that $h\left(x_2 - \frac{1}{\mu(n-1)}\right) = \frac{1}{n}$. This is larger than $a\left(x_2 - \frac{1}{\mu(n-1)}\right)$ if and only if

$$\frac{1}{n} > ax_2 - \frac{a}{\mu(n-1)}.$$

Because $x_2 \leq \frac{1}{a(n-1)}$, the right-hand side is not more than $\frac{1}{n-1}\left(1 - \frac{a}{\mu}\right)$ and because $\frac{\mu}{a} \leq \frac{n}{n-1}$, we have in turn

$$\frac{1}{n-1}\left(1 - \frac{a}{\mu}\right) \leq \frac{1}{n-1}\frac{1}{n}.$$

But this is less than $\frac{1}{n}$, which thereby completes the proof. ■

Proof of Corollary 4: Inserting the expressions obtained in Corollaries 1 and 3, one obtains

$$\beta + \frac{a-\mu}{\mu}x_1 + \frac{a}{2}x_1^2 + \frac{1}{n}\left[\frac{1}{\mu(n-1)} - \frac{1}{2an}\right]$$

for $PC_I^* + \Pi_N^*$. As $PC_N^* = \beta - \frac{1}{an} + \frac{1}{\mu}\frac{2n-1}{n(n-1)}$, vertical divestiture is thus jointly profitable if and only if

$$\beta + \frac{a-\mu}{\mu}x_1 + \frac{a}{2}x_1^2 + \frac{1}{n}\left[\frac{1}{\mu(n-1)} - \frac{1}{2an}\right] > \beta - \frac{1}{an} + \frac{1}{\mu}\frac{2n-1}{n(n-1)},$$

which is equivalent to the inequality in the corollary. ■

Proof of Proposition 4: The proof uses symmetry and quasiconvexity of the function $TC(\mathbf{x})$.

The function $TC(\mathbf{x})$ is *symmetric* in the sense that, for $x_i = x$ and $x_j = x'$ with $i \neq j$, we have

$$TC(x, x', \mathbf{x}_{-i-j}) = TC(x', x, \mathbf{x}_{-i-j}),$$

where $\mathbf{x}_{-i-j} = (x_k)_{k\neq i,j}$.

The rest of the proof is by contradiction. That is, suppose to the contrary that $\min_{\mathbf{x}} TC(\mathbf{x}) = TC(\hat{\mathbf{x}})$, where $\hat{\mathbf{x}}$ is not symmetric, i.e. $\hat{x}_i \neq \hat{x}_j$ for some $i \neq j$, and that there is no symmetric investment, denoted $\mathbf{x}^S$ such that $TC(\mathbf{x}^S) = \min_{\mathbf{x}} TC(\mathbf{x})$.

Without loss of generality, let $i = 1$ and $j = 2$. Let $\hat{\hat{\mathbf{x}}} = (\hat{x}_2, \hat{x}_1, \hat{x}_3, .., \hat{x}_n)$. That is, $\hat{\hat{\mathbf{x}}}$ is a permutation of $\hat{\mathbf{x}}$. By symmetry, we have

$$TC(\hat{\hat{\mathbf{x}}}) = TC(\hat{\mathbf{x}}).$$

But by quasiconvexity, we have, for any $t \in (0, 1)$,

$$TC(t\hat{\hat{\mathbf{x}}} + (1 - t)\hat{\mathbf{x}}) \leq TC(\hat{\mathbf{x}}),$$

which is a contradiction to the hypothesis that TC is minimized at $\hat{\mathbf{x}}$ and not at a symmetric investment $\mathbf{x}^S$.

The last part of the statement follows by noting that at symmetry, i.e. $x_i = x$ for all i, total cost, denoted $TC_S(x)$ is

$$TC_S(x) = \int_{\beta - x}^{\infty} cl(c + x; n)dc + n\Psi(x),$$

Noting $\partial l(c + x; n)/\partial x = \partial l(c + x; n)/\partial c$, we can write the derivative $TC'_S(x)$ using integration by parts as

$$TC'_S(x) = -\underbrace{\int_{\beta - x}^{\infty} l(c + x; n)dc}_{=1} + n\psi(x).$$

Setting $TC'_S(x) = 0$, we thus get $x = \psi^{-1}(1/n)$. Moreover, $TC''_S(x) = n\psi'(x) > 0$, so this is indeed a minimum. ■

Proof of Corollary 5: We first show that $TC(\mathbf{x})$ is quasiconvex if $\mu \leq a$ by showing that there is a unique solution to the system of first-order conditions. Second, we show that for $\mu > a$, the symmetric solution to the first-order conditions is not socially optimal. Although this is not required to prove the corollary, we state it here because we referred to this result in the text.

Substituting the expressions for the exponential case gives us the following expression for the expected production cost:

$$EC(\mathbf{x}) = \mu \sum_{j=1}^{n} je^{-\mu X_j} \int_{\beta - x_j}^{\beta - x_{j+1}} ce^{-j\mu(c - \beta)}dc,$$

where $X_j := \sum_{i=1}^{j} x_i$, $x_{n+1} := -\infty$, and $TC(\mathbf{x}) = EC(\mathbf{x}) + \sum_i \Psi(x_i)$. Letting

$$S_j := e^{-\mu(X_j - jx_j)} \left[\beta - x_j + \frac{1}{j\mu} - \left(\beta - x_{j+1} + \frac{1}{j\mu} \right) e^{-j\mu(x_j - x_{j+1})} \right],$$

it then follows that

$$\frac{\partial EC(\mathbf{x})}{\partial x_j} = \mu e^{-\mu(X_j - jx_j)}(\beta - x_j) - \mu \sum_{i=j}^{n} S_i$$

for all $j = 1, .., n$ and

$$\frac{\partial EC(\mathbf{x})}{\partial x_j} - \frac{\partial EC(\mathbf{x})}{\partial x_{j+1}} = -\frac{1}{j} e^{-\mu(X_j - jx_j)} (-1 + e^{-\mu(x_j - x_{j+1})})$$

for all $j < n$.

Finally,

$$\frac{\partial S_n}{\partial x_n} = \mu(n-1) e^{-\mu(X_n - nx_n)} \left(\beta - x_n + \frac{1}{n\mu} \right)$$

and the derivative of $EC(\mathbf{x})$ with respect to x_n is

$$\frac{\partial EC(\mathbf{x})}{\partial x_n} = \frac{\partial S_n}{\partial x_n} + \frac{\partial S_{n-1}}{\partial x_n} = -\frac{1}{n} e^{-\mu(X_n - nx_n)}.$$

Observe that

$$\partial TC(\mathbf{x})/\partial x_i = \partial EC(\mathbf{x})/\partial x_i + \psi(x_i).$$

Using the first-order condition $\partial TC(\mathbf{x})/\partial x_n = 0$, we obtain the boundary condition

$$\frac{1}{n} e^{-\mu(X_n - nx_n)} = ax_n. \tag{29}$$

Subtracting $\frac{\partial TC(\mathbf{x})}{\partial x_i}$ from $\frac{\partial TC(\mathbf{x})}{\partial x_{i+1}}$ and simplifying yields for $i = 1, .., n-2$ with $n > 2$ a system of first-order difference equations

$$\frac{1}{i} e^{-\mu X_i} \left[e^{i\mu x_{i+1}} - e^{i\mu x_i} \right] = a(x_{i+1} - x_i) \tag{30}$$

with the boundary condition (29) and the constraints $x_i \geq x_{i+1}$. Notice that the symmetric solution $x_i = \frac{1}{an}$ for all $i = 1, .., n$ is always a solution of this system. We are now going to show that for $a \geq \mu$ it is the unique solution.

Notice first that the right-hand side of (30) is linear in x_{i+1} with slope a. The left-hand side of (30) is increasing and convex in x_{i+1} with slope μ at symmetry. Fix then an arbitrary x_1. Provided $\mu \leq a$, $x_2 = x_1$ is the unique solution to (30). Iterating the argument, we get that $x_i = x_1$ is the unique solution to (30) for all $i = 1, .., n-1$. Notice then that the left-hand side of (29) is convex and increasing in x_n with slope $\mu\frac{n-1}{n}$ at symmetry. Since $\mu \leq a$ implies $\mu\frac{n-1}{n} < a$, where a is the slope of the right-hand side of (29), it follows that $x_n = x_1$ is the unique solution to (29). But at symmetry, (29) implies $x_n = \frac{1}{an}$. Thus, for $\mu \leq a$, $x_i = \frac{1}{an}$ for all $i = 1, .., n$ is the unique solution. This completes the proof of the claim in to corollary.

The remainder of the proof shows that the symmetric solution is not a minimizer of $TC(\mathbf{x})$ if $\mu > a$ by showing that $\mathbf{x} = (x_1, x_2, ..., x_2)$ with $x_1 > x_2$ optimally chosen does strictly better. The second own and cross partial of $TC(x_1, x_2)$ are

$$\begin{aligned} \frac{\partial^2 TC(x_1, x_2)}{\partial x_1^2} &= a - \mu\frac{n-1}{n} e^{-\mu(x_1 - x_2)} = \frac{\partial^2 TC(x_1, x_2)}{\partial x_2^2} \\ \frac{\partial^2 TC(x_1, x_2)}{\partial x_1 \partial x_2} &= \mu\frac{n-1}{n} e^{-\mu(x_1 - x_2)}. \end{aligned}$$

At $x_1 = x_2$, the Hessian matrix $\mathbf{H}$ can be shown to positive semi-definite if and only if $\mu \leq a$. This can be shown by noting that the product $\mathbf{z} \cdot \mathbf{H} \cdot \mathbf{z}$ with $\mathbf{z} = (z_1, z_2) \neq \mathbf{0}$ is quasiconvex (quasiconcave) in z_2 if $\mu \leq an$ ($\mu > an$) and minimized (maximized) at $z_2 = -z_1/(n-1)$. Evaluated at $z_2 = -z_1/(n-1)$, we have

$$\mathbf{z} \cdot \mathbf{H} \cdot \mathbf{z} = \frac{anz_1^2(a-\mu)}{an-\mu}.$$

For $a < \mu < an$, $\mathbf{z} \cdot \mathbf{H} \cdot \mathbf{z} > 0$, which proves that $\mathbf{H}$ is positive semi-definite. Because of our restriction $\mu \leq \frac{n}{n-1}a$, we know that $\mu < an$. (For completeness, if $\mu > an$, we know that $\mathbf{z} \cdot \mathbf{H} \cdot \mathbf{z}$ is maximized at $z_2 = -z_1/(n-1)$ and negative at this point. Consequently, $\mathbf{z} \cdot \mathbf{H} \cdot \mathbf{z} < 0$ for all $\mathbf{z} \neq \mathbf{0}$ if $\mu > an$.) ■