

Aggregate Recruiting Intensity - Online Appendix

Alessandro Gavazza, Simon Mongey and Giovanni L. Violante *

March 29, 2018

This Appendix is organized as follows. Section **A** contains the derivations of the hiring cost function that we introduced in Section 2. Section **B** provides additional figures referenced in the main text. Section **C** details the algorithms for the computation of the stationary equilibrium, transitional dynamics, and estimation of the model's parameters.

A The hiring cost function

In this section we show that, once we postulate the hiring cost function

$$\mathcal{C}(n, e, v) = \left[\frac{\kappa_1}{\gamma_1} e^{\gamma_1} + \frac{\kappa_2}{\gamma_2 + 1} \left(\frac{v}{n} \right)^{\gamma_2} \right] v, \quad (\text{A1})$$

then, through the firm's optimization, we obtain a log-linear cross-sectional relationship between the job-filling rate and the employment growth rate that is consistent with the empirical findings in **DFH**. Next, by substituting the firm's first-order conditions into **(A1)**, we derive a formulation of the cost only in terms of (n, n') that we use in the intertemporal problem (10) in the main text.

As we explained in Section 2.1, the firm solves a static cost minimization problem: given a choice of n' , it determines the lowest cost combination of (e, v) that can deliver n' . The hiring

*Gavazza: London School of Economics and CEPR, a.gavazza@lse.ac.uk. Mongey: Federal Reserve Bank of Minneapolis, simonmongey@gmail.com. Violante: Princeton University, CEPR, IFS, IZA, and NBER, violante@princeton.edu. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. The authors declare that they have no relevant or material financial interests that relate to the research described in this paper.

firm's cost minimization problem is

$$\begin{aligned} \mathcal{C}(n, n') &= \min_{e, v} \left[\frac{\kappa_1}{\gamma_1} e^{\gamma_1} + \frac{\kappa_2}{\gamma_2 + 1} \left(\frac{v}{n} \right)^{\gamma_2} \right] v \\ \text{s.t.} \quad & n' - n \leq q(\theta^*) e v \\ & v \geq 0 \end{aligned} \quad (\text{A2})$$

Convexity of the cost function (A1) in (e, v) requires $\gamma_1 \geq 1$ and $\gamma_2 \geq 0$. After setting up the Lagrangian, one can easily derive the two first-order conditions with respect to e and v that, combined, yield a relationship between the optimal choice of e and the optimal choice of the vacancy rate v/n :

$$e = \left[\frac{\kappa_2}{\kappa_1} \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \right]^{\frac{1}{\gamma_1}} \left(\frac{v}{n} \right)^{\frac{\gamma_2}{\gamma_1}}. \quad (\text{A3})$$

Note that if $\gamma_2 = 0$, as in [Pissarides \(2000\)](#), recruiting intensity is equal to a constant for all firms and it is independent of aggregate labor market conditions—both counterfactual implications. The following changes in parameters (*ceteris paribus*) result in a substitution away from vacancies and towards effort: $\uparrow \kappa_2, \downarrow \kappa_1, \uparrow \gamma_2$, and $\downarrow \gamma_1$. The effect of the cost shifter is obvious. A higher curvature on the vacancy rate in the cost function ($\uparrow \gamma_2$) makes the marginal cost of creating vacancies rise faster than the marginal cost of recruiting effort. Since the gain in terms of additional hires from a marginal unit of effort or vacancies is unaffected by γ_2 , it is optimal for the firm to use relatively more effort.

Now, substituting the law of motion for employment at the firm level into (A3), we obtain the optimal recruitment effort choice, expressed only as a function of the firm-level variables (n, n') :

$$e(n, n') = \left[\frac{\kappa_2}{\kappa_1} \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \right]^{\frac{1}{\gamma_1 + \gamma_2}} q(\theta^*)^{-\frac{\gamma_2}{\gamma_1 + \gamma_2}} \left(\frac{n' - n}{n} \right)^{\frac{\gamma_2}{\gamma_1 + \gamma_2}} \quad (\text{A4})$$

which, in turn implies, for the job-filling rate,

$$f(n, n') = q(\theta^*) e(n, n') = \left[\frac{\kappa_2}{\kappa_1} \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \right]^{\frac{1}{\gamma_1 + \gamma_2}} q(\theta^*)^{\frac{\gamma_1}{\gamma_1 + \gamma_2}} \left(\frac{n' - n}{n} \right)^{\frac{\gamma_2}{\gamma_1 + \gamma_2}}. \quad (\text{A5})$$

This equation demonstrates that the model implies a log-linear relation between the job-filling rate and employment growth at the firm level, with elasticity $\gamma_2 / (\gamma_1 + \gamma_2) < 1$, as in the data.

Finally, substituting (A5) into the firm-level law of motion for employment yields an expression for the vacancy rate

$$\frac{v}{n} = \left[\frac{\kappa_2}{\kappa_1} \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \right]^{-\frac{1}{\gamma_1 + \gamma_2}} q(\theta^*)^{-\frac{\gamma_1}{\gamma_1 + \gamma_2}} \left(\frac{n' - n}{n} \right)^{\frac{\gamma_1}{\gamma_1 + \gamma_2}}. \quad (\text{A6})$$

Now, note that by substituting the optimal choice for recruitment effort (A3) into (A1), we obtain the following formulation for the cost function:

$$\mathcal{C}(n, v) = \left[\kappa_2 \frac{\gamma_1 + \gamma_2}{(\gamma_1 - 1)(\gamma_2 + 1)} \left(\frac{v}{n} \right)^{\gamma_2} \right] v, \quad (\text{A7})$$

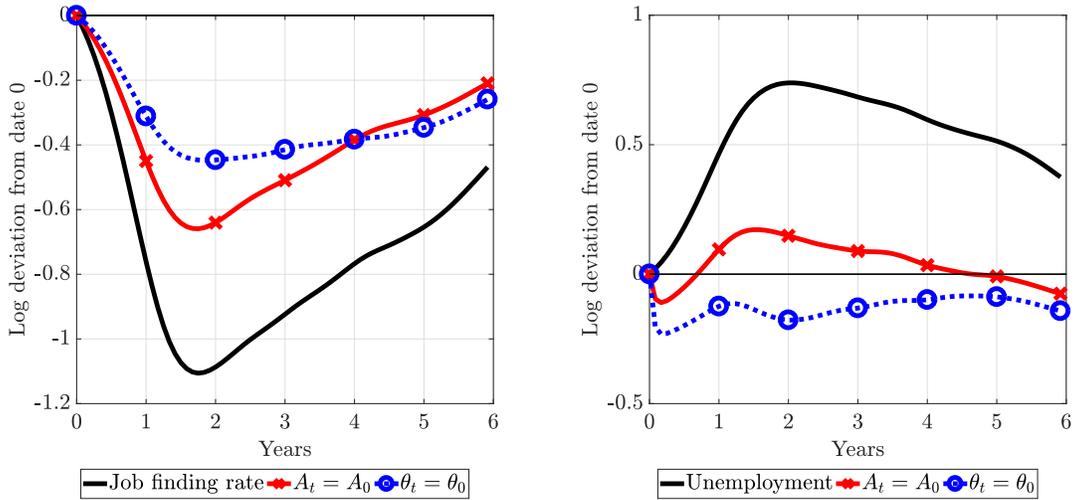
which is one of the specifications invoked by [Kaas and Kircher \(2015\)](#).

Finally, if we use (A6) in (A7), we obtain a version of the cost function only as a function of (n, n') that we can use directly in the dynamic problem (10):

$$\mathcal{C}^*(n, n') = \kappa_2 \left[\frac{\gamma_1 + \gamma_2}{(\gamma_1 - 1)(\gamma_2 + 1)} \right] \left\{ \left[\frac{\kappa_2}{\kappa_1} \left(\frac{\gamma_1}{\gamma_1 - 1} \right) \right]^{\frac{1}{\gamma_1 + \gamma_2}} q(\theta^*)^{-\frac{\gamma_1}{\gamma_1 + \gamma_2}} \left(\frac{n' - n}{n} \right)^{\frac{\gamma_1}{\gamma_1 + \gamma_2}} \right\}^{1 + \gamma_2} n.$$

B Additional figures

Figure B1: Counterfactual job finding rate and unemployment



Notes: The counterfactual series are constructed as follows. First take the aggregate matching function $H_t = A_t U_t^\alpha V_t^{1-\alpha}$. This implies a job finding rate $f_t = H_t/U_t = A_t \theta_t^{1-\alpha}$, where $\theta_t = V_t/U_t$. Consider also the empirical law of motion $U_t = (1 - f_{t-1}) U_{t-1} + S_{t-1}$, which defines a series for entry into unemployment. In the data this comes from either separations of employed workers or transitions of out of the labor force to unemployment, the exact source is not of consequence to this exercise. With data on H_t , U_t , V_t , we can use the above to construct $\{\theta_t, S_t, A_t\}$. This can also be inverted. Given a counterfactual $\{\theta_t^d, S_t^d, A_t^d\}$, we can compute a counterfactual f_t and U_t . The counterfactual series in the figure involve setting $X_t = X_0$ one by one for $\{\theta_t^d, S_t^d, A_t^d\}$, and computing the counterfactual series for f_t and U_t .

Figure B2: Dynamics of output under the two shocks

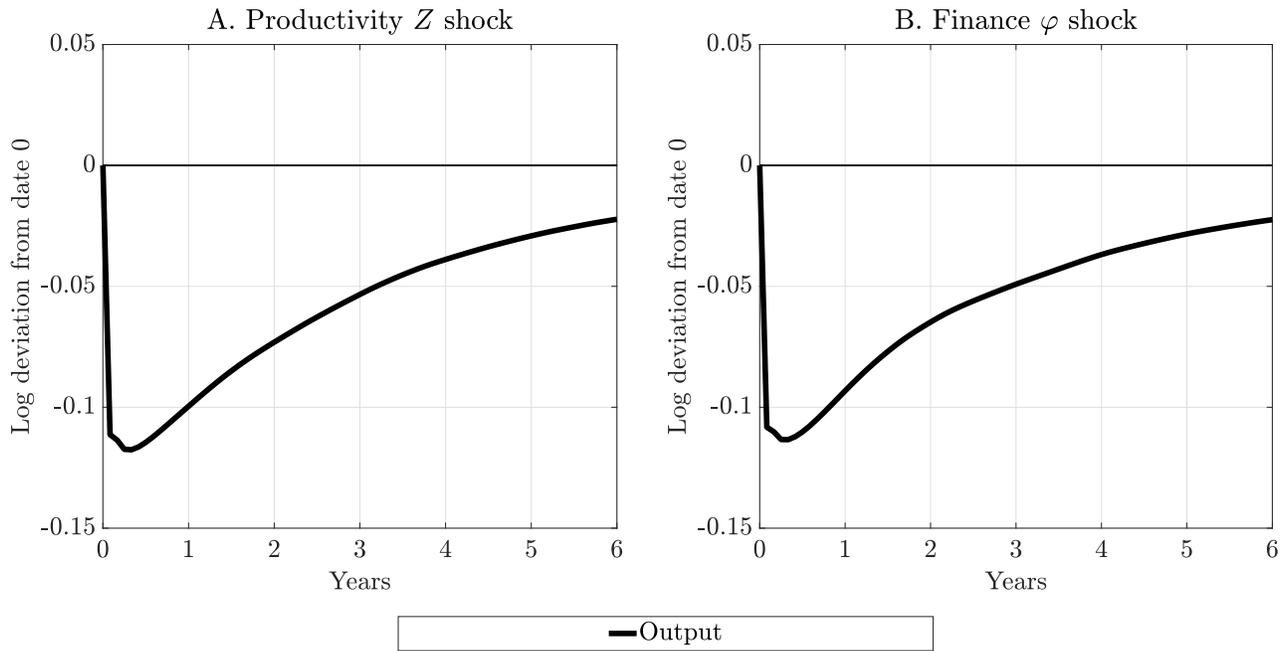
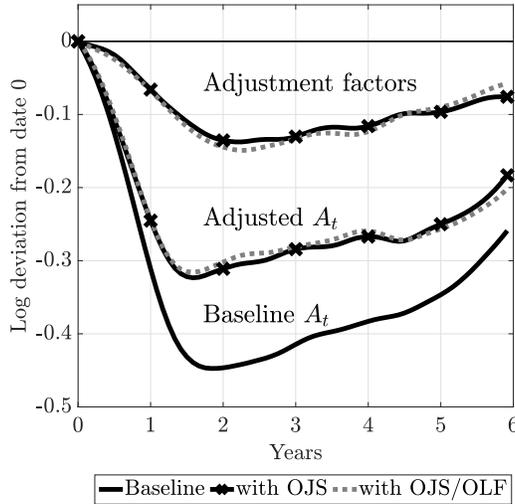


Figure B3: Match efficiency and the composition of job seekers



Notes: The adjusted measures of match efficiency are constructed as follows. Consider the case of allowing for on-the-job search (OJS). The matching function is $H_t = A_t S_t^\alpha V_t^{1-\alpha}$, where the measure of effective units of search is $S_t = U_t + s_t^E E_t$, in which U_t is the number of unemployed workers, and E_t the number of employed workers. Rearranging the matching function obtains $H_t = A_t [1 + s_t^E (E_t/U_t)] U_t^\alpha V_t^{1-\alpha}$. The first term A_t is the adjusted path of match efficiency, and the term in square brackets is the adjustment factor. To measure s_t^E we exploit constant returns to scale in the matching function which implies that hires out of employment are $H_t^E = s_t f_t E_t$, where f_t is the job-finding rate of unemployed workers: $H_t^U = f_t U_t$. Relative search intensity of the employed s_t^E can therefore be measured by $s_t^E = EE_t/EU_t$ where EE_t and EU_t are the BLS employment–employment and unemployment–employment transition rates. A corresponding exercise where $S_t = U_t + s_t^N N_t + s_t^E E_t$ includes N_t workers out of the labor force (OLF), is responsible for the grey lines.

Figure B4: Employment-weighted growth rate distribution in the model

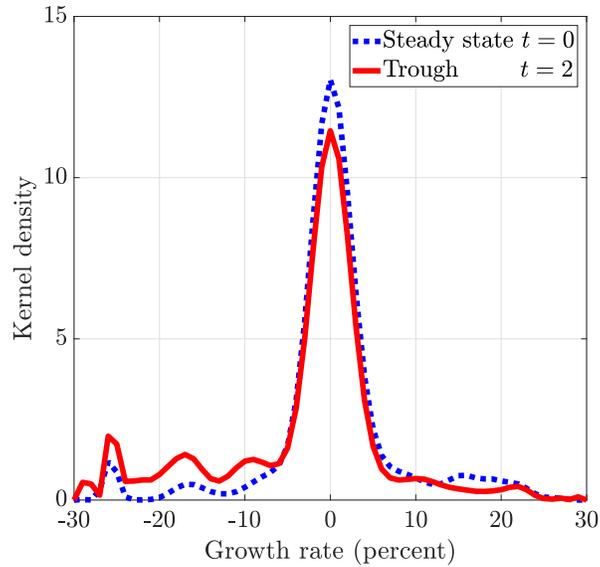
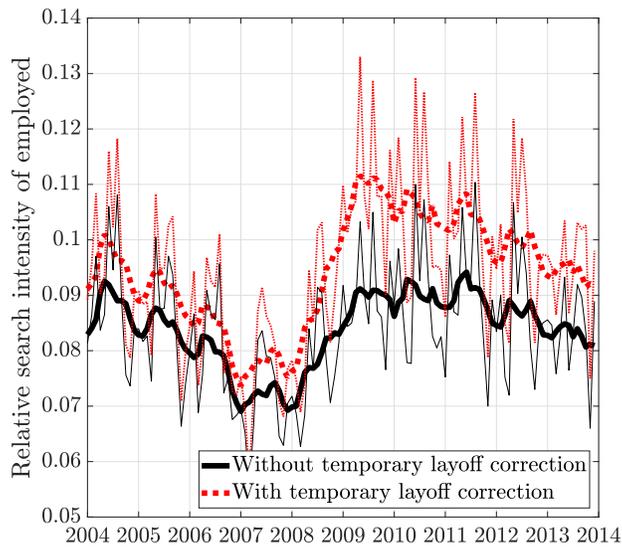


Figure B5: Relative search intensity of employed workers



Note: The two series are constructed as the ratio of the EE flow rate to the UE flow rate. In one case, the UE rate is taken directly from the BLS, and in the other case, we adjust the UE rate by subtracting unemployed workers on temporary layoff from the denominator and rehires from temporary layoffs from the numerator.

C Computational details

C.1 Value and policy functions

We use collocation methods to solve the firm's value function problem (4)-(7). Let $s = (n, a, z)$ be the firm's idiosyncratic state, abstracting from heterogeneity in σ which is permanent. We solve for an approximant of the *expected* value function $\mathbb{V}^e(n', a', z)$ which gives the firm's expected value conditional on current decisions for net worth and employment:

$$\mathbb{V}^e(n', a', z) = \int_{\mathcal{Z}} \mathbb{V}(n', a', z') d\Gamma(z, z'),$$

where the integrand is the value given in (6).

We set up a grid of collocation nodes $S = N \times A \times Z$ where $N = \{n_1, \dots, n_{N_n}\}$, with $N_n = N_a = N_z = 10$. We construct Z by first creating equi-spaced nodes from 0.001 to 0.999, which we then invert through the cumulative distribution function of the stationary distribution implied by the AR(1) process for z to obtain Z . This ensures better coverage in the higher probability regions for z . We choose A and N to have a higher density at lower values. The upper bound for employment, \bar{n} , is chosen so that the optimal size of the highest productivity firm $n^*(\bar{z})$ is less than \bar{n} . We choose the upper bound for net worth, \bar{a} , so that the maximum optimal capital $k^*(\bar{z})$ can be financed, that is, $k^*(\bar{z}) < \varphi\bar{a}$. Note that N , A , and Z are parameter dependent, and therefore recomputed for each new vector of parameters considered in estimation.

We approximate $\mathbb{V}^e(s)$ on S using a linear spline with $N_s = N_n \times N_a \times N_z$ coefficients. Given a guess for the spline's coefficients, we iterate towards a vector of coefficients that solve the system of N_s Bellman equations, which are linear in the N_s unknown coefficients. Each iteration proceeds as follows. Given the spline coefficients, we use golden search to compute the optimal policies for all states $s \in S$ and the value function $\mathbb{V}(s)$. We then fit another spline to $\mathbb{V}(s)$, which facilitates integration of productivity shocks $\varepsilon \sim \mathcal{N}(0, \vartheta_z)$. To compute $\mathbb{V}^e(s)$ on S , we approximate the integral by

$$\mathbb{V}^e(n, a, z) = \sum_{i=1}^{N_\varepsilon} w_i \mathbb{V}(n, a, \exp(\rho_z \log(z) + \varepsilon_i)).$$

Here, $N_\varepsilon = 80$, and the values of ε_i are constructed by creating a grid of equi-spaced nodes

between 0.001 to 0.999, then using the inverse cumulative distribution function of the shocks (normal) to create a grid in ε . The weights w_i are given by the probability mass of the normal distribution centered on each ε_i . Note that this differs from quadrature schemes in which one is trying to minimize the number of evaluations of the integrand, usually with N_ε around four. Since $\mathbb{V}(s)$ is already given by an approximant at this step, and the integral is only computed once each iteration, this is not a concern and we compute the integral very precisely. We then fit an updated vector of coefficients to $\mathbb{V}^e(s)$ and continue.¹

C.2 Stationary distribution

To construct the stationary distribution, we use the method of nonstochastic simulation from [Young \(2010\)](#), modified to accommodate a continuously distributed stochastic state. We create a new, fine grid of points S^f on which we approximate the stationary distribution using a histogram, setting $N_n^f = N_n^f = N_z^f = 100$. Given our approximation of the expected continuation value, we solve for the policy functions $n'(s^f)$ and $a'(s^f)$ on the new grid and use these to create two transition matrices \mathbf{Q}_n and \mathbf{Q}_a , which determine how mass shifts from points $s^f \in S^f$ to points in N^f and A^f , respectively. We construct \mathbf{Q}_x as follows for $x \in \{a, n\}$:

$$\mathbf{Q}_x[i, j] = \left[\mathbf{1}_{x'(s_i^f) \in [X_{j-1}^f, X_j^f]} \frac{x'(s_i^f) - X_j^f}{X_j^f - X_{j-1}^f} + \mathbf{1}_{x'(s_i^f) \in [X_j^f, X_{j+1}^f]} \frac{X_{j+1}^f - x'(s_i^f)}{X_{j+1}^f - X_j^f} \right],$$

for $i = 1, \dots, N_s^f$ and $j = 1, \dots, N_x^f$.² This approach ensures that aggregates computed from the stationary distribution will be unbiased. For example if $x'(s) \in (X_j, X_{j+1})$, then masses w_j and w_{j+1} are allocated to X_j and X_{j+1} such that $w_j X_j + w_{j+1} X_{j+1} = x'(s)$. The transition matrix for the process for z is computed by $\mathbf{Q}_z = \sum_{i=1}^{N_\varepsilon} w_i \mathbf{Q}_z^i$, where \mathbf{Q}_z^i is computed as above under $z'(s^f) = \exp(\rho_z \log z + \varepsilon_i)$. The overall incumbent transition matrix \mathbf{Q} is simply the tensor product $\mathbf{Q} = \mathbf{Q}_z \otimes \mathbf{Q}_a \otimes \mathbf{Q}_n$.

To compute the stationary distribution, we still need the distribution of entrants. To allow

¹In practice, instead of this simple iterative approach to solve for the coefficients, we follow a Newton algorithm as in [Miranda and Fackler \(2002\)](#), which is two orders of magnitude faster. The Newton algorithm requires computing the Jacobian of the system of Bellman equations with respect to the coefficient vector. The insight of [Miranda and Fackler \(2002\)](#) is that this is simple to compute given the linearity of the system in the coefficients.

²If exit is optimal on grid point s_i^f , then we set row i of \mathbf{Q}_x to zero.

for entry cutoffs to move smoothly, we compute entrant policies on a dense grid of $N_z^0 = 500$ productivities. This is clearly important for us since it ensures that entry does not jump in the transition dynamics or across parameters in calibration. The grid Z^0 is constructed by taking an equally spaced grid in $[0.01, 0.99]$ and inverting it through the cumulative distribution function of potential entrant productivities (exponential). Let the corresponding vector of weights be given by \mathbf{P}_0 . Given the approximation of the continuation value $\mathbb{V}^e(s)$, we can solve the potential entrant's policies $n'_0(s_0)$ and $a'_0(s_0)$, conditional on entry. We can then solve the firm's discrete entry decision. Finally, we compute an equivalent transition matrix \mathbf{Q}_0 using these policies, where nonentry results in a row of zeros in \mathbf{Q}_0 .

The discretized stationary distribution \mathbf{L} on S^f is then found by the following approximation to the law of motion (15)

$$\mathbf{L} = (1 - \zeta)\mathbf{Q}'\mathbf{L} + \lambda_0\mathbf{Q}'_0\mathbf{P}_0,$$

which is a contraction on \mathbf{L} , solved by iterating on a guess for \mathbf{L} . The final stationary distribution is found by choosing λ_0 such that $\sum_{i=1}^{N_s^f} L_i = 1$.

C.3 Computation of moments

We compute an aggregate moment X by integrating λ over firm policies $x(s)$. Using the above approximation, this is simply $X = \mathbf{L}'x(s)$.

For age-based statistics, our moments in the data refer to firm ages in years. We therefore generate an "age zero" measure of firms by allowing for 12 months of entry. We then iterate this distribution forward to compute age statistics such as average debt to output for age 1 firms or the distribution of vacancies by age.

For statistics such as the average annual growth rate conditional on survival, we need to simulate the model. In this case, we draw 100,000 firms on S^f in proportion to \mathbf{L} and simulate these forward solving (rather than interpolating) firm policies each period and evolving productivity with draws from the continuous distribution of innovations ε . To remove the effect of the starting grid, we simulate for 36 months and compute our statistics, comparing firms across months 24 and 36.

C.4 Estimation

The model has a large number of unknown parameters and a criterion function that is potentially nonsmooth. Furthermore, the model does not have an equilibrium for large regions of the parameter space.³ For these reasons, using a sequential optimizer that takes the information from successive draws from the parameter space and updates its guess is prohibitive. For example, a Nelder-Mead optimizer both needs to be returned values for the objective function at each evaluation and needs to make many evaluations of the function when taking each “step”.

Our solution is to use an algorithm that we can very easily parallelize, that efficiently explores the parameter space, and for which we can ignore cases with no equilibrium. We set up a hyper-cube in the parameter space and then initialize a Sobol sequence to explore it. A Sobol sequence is a quasi-random low-discrepancy sequence that maintains a maximum dispersion in each dimension and far outperforms standard random number generators. We then partition the sequence and submit each partition to a separate CPU on a high performance computer (HPC). From each evaluation of the parameter hyper-cube, we save the vector of model moments and regularly splice these together, choosing one that minimizes the criterion function. Starting with wide bounds on the parameters, we run this procedure a number of times, shrinking the hyper-cube each time.

This procedure has a number of benefits. First, we trade in the optimization steps associated with a traditional solver for scale. Instead of using a 10 CPU machine to run a Nelder-Mead algorithm, we can simultaneously solve the model on 300+ CPUs. Second, the output of the exercise gives a strong intuition for the identification of the model. From an optimizer one may retrieve the moments of the model along the path of the parameter vector chosen by the algorithm. In our case, we retrieve thousands of evaluations, knowing that the low-discrepancy property of the Sobol sequence implies that for an interval of any one parameter, the remaining parameters are drawn uniformly. Plotting moments against parameters therefore shows the effect of a parameter on a certain moment, conditional on local draws of all other parameters. Plotting a histogram of the moments returned gives a strong indication as to which moments may be difficult to match for the current bounds of the parameter space.

³For example, if the value of home production is very low then unemployment derived from the labor demand condition may be *negative*. Wages are so low that labor demand eclipses the fixed supply \bar{L} .

C.5 Transition dynamics

We solve for transition dynamics as follows. Consider the case of a shock to aggregate productivity Z . We specify a path for $\{Z_t\}_{t=0}^T$ with $Z_0 = Z_T = \bar{Z}$. Given a conjectured path for equilibrium market tightness $\{\tilde{\theta}_t^*\}_{t=0}^T$ and the assumption that the date T continuation values of the firm are the same as they are in steady state, one can solve *backward* for expected value functions \mathbf{V}_t^e at all dates $T-1, T-2, \dots, 1$. Setting the aggregate states $U_0 = \bar{U}$ and $\lambda_0 = \bar{\lambda}$, and using the conjectured path θ_t^* , the shocks, and continuation values, one can then solve *forward* for a new market clearing $\theta_t^{*'} that equates unemployment from labor demand U_t^{demand} and worker flows U_t^{flows} in every period using the labor demand and evolution of unemployment equations,$

$$\begin{aligned} U_{t+1}^{flows} &= U_t - H(\theta_t^*) + F(\theta_t^*) - \lambda_{e,t} n_0 \\ U_{t+1}^{demand} &= \bar{L} - \int n'(s, \theta_t^*, A_t, \mathbf{V}_t^e) d\lambda_t \end{aligned}$$

Once we reach $t = T$, we set $\tilde{\theta}_t^* = \theta_t^*$ and iterate until the proposed path and equilibrium path for market tightness converge.

References

- Davis, Steven J., R. Jason Faberman, and John C. Haltiwanger.** 2013. "The Establishment-Level Behavior of Vacancies and Hiring." *Quarterly Journal of Economics*, 128(2): 581–622.
- Kaas, Leo, and Philipp Kircher.** 2015. "Efficient Firm Dynamics in a Frictional Labor Market." *American Economic Review*, 105(10): 3030–60.
- Miranda, Mario J., and Paul L. Fackler.** 2002. *Applied Computational Economics and Finance*. Cambridge, MA: MIT Press.
- Pissarides, Christopher A.** 2000. *Equilibrium Unemployment Theory*. Cambridge, MA: MIT Press.
- Young, Eric R.** 2010. "Solving the Incomplete Markets Model with Aggregate Uncertainty Using the Krusell-Smith Algorithm and Non-stochastic Simulations." *Journal of Economic Dynamics and Control*, 34(1): 36–41.