Two-sided investment and matching with multi-dimensional cost types and attributes: Online Appendix

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CMP's "constrained efficiency" property

CMP noted an indirect but interesting constrained efficiency property of ex-post contracting equilibria. Attributes that are part of a pair of attributes that some buyer and some seller could use for blocking the equilibrium outcome in a world of ex-ante contracting (net surplus exceeds the sum of net equilibrium payoffs) cannot exist in the endogenous market. I always use equilibrium conditions directly in this paper (i.e., I do not invoke constrained efficiency), but it seems worthwhile to state CMP's result in the present notation.[1]

Lemma OA.1 (Lemma 2 of CMP). *Let* $((\beta,\sigma,\pi_0),(\pi_1^*,\psi_X^*))$ *be an ex-post contracting equilibrium. Suppose that there are* $b \in \text{Supp}(\mu_B)$*,* $s \in \text{Supp}(\mu_S)$ *and* $(x,y) \in X \times Y$ *such that* $h(x,y|b,s) > r_B(b) + r_S(s)$*. Then,* $x \notin \text{Supp}(\mu_X)$ *and* $y \notin \text{Supp}(\mu_Y)$*.*

Proof of Lemma OA.1. Assume to the contrary that $x \in \text{Supp}(\mu_X)$. Then,

$$r_S(s) + \psi_X^*(x) - c_B(x,b) \geq v(x,y) - \psi_X^*(x) - c_S(y,s) + \psi_X^*(x) - c_B(x,b) > r_B(b) + r_S(s).$$

The first inequality follows from the definition of r_S, and the second holds by assumption. It follows that $\psi_X^*(x) - c_B(x,b) > r_B(b)$, a contradiction. The proof for $y \notin \text{Supp}(\mu_Y)$ is analogous. □

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[1]Nöldeke and Samuelson (2015) have recently clarified the relationship between constrained efficiency, appropriately defined to allow for non-separable and ITU environments, and ex-post contracting equilibrium. Their findings imply in particular that the two concepts are equivalent under the separability assumptions of CMP and the present paper.

Some basic facts about the 1-d supermodular framework

As is well known, strict supermodularity of v forces optimal matchings to be positively assortative for any attribute assignment game. The Kantorovich duality result can be used for a very short proof.

Lemma OA.2. *Let Condition 1 hold. Then, for any* (μ_X, μ_Y, v)*, the unique optimal matching is the positively assortative one.*

Proof of Lemma OA.2. By Kantorovich duality, the support of any optimal matching π_1^* is a v-cyclically monotone set. In particular, for any $(x,y),(x',y') \in \text{Supp}(\pi_1^*)$ with $x > x'$, $v(x,y) + v(x',y') \geq v(x,y') + v(x',y)$ and hence $v(x,y) - v(x',y) \geq v(x,y') - v(x',y')$. As v has strictly increasing differences, it follows that $y \geq y'$. □

Lemma OA.3. *Let Condition 1 hold. Then, in any ex-post contracting equilibrium, attribute choices are non-decreasing with respect to agents' own type.*

Proof of Lemma OA.3. From Definition 5, $\beta(b,s) \in \text{argmax}_{x \in X}(r_X(x) - c_B(x,b))$. The objective is strictly supermodular in (x,b). By Theorem 2.8.4 from Topkis (1998), all selections from the solution correspondence are non-decreasing in b. The argument for sellers is analogous. □

Corollary OA.1. *Let Condition 1 hold. Then every ex-post contracting equilibrium is equivalent to an equilibrium for which the equilibrium type-matching is positively assortative, in the sense that each type makes the same investment and gets the same (gross and net) payoff in each of the two equilibria.*

The positively assortative matching may assign buyers of the same type to different seller types, and vice versa, whenever μ_B or μ_S have atoms, but this does not affect the result.

Lemma OA.4. *Let Condition 1 hold, and assume that for all* $b \in \text{Supp}(\mu_B)$ *and* $s \in \text{Supp}(\mu_S)$*, the FA game between b and s has a unique NE . Then every ex-post contracting equilibrium is ex-ante efficient.*

Proof of Lemma OA.4. By Corollary OA.1, every equilibrium is equivalent to an equilibrium with positively assortative equilibrium type-matching. In particular, this is true for the ex-ante efficient equilibrium of Proposition 2. By Proposition 3, inefficiency of joint investments is impossible. This proves the claim. □

The case $a_H < a_2$ in Example 1

I show here that for $a_H < a_2$, a non-trivial, mismatch inefficient equilibrium exists if and only if

$$\frac{4-2\alpha}{4-\alpha}\frac{b_2}{b_1} \geq \frac{\left(\frac{s_H}{s_L}\right)^{\frac{4-2\alpha}{4-\alpha}} - 1}{\frac{s_H}{s_L} - 1}.$$

Inefficiency requires that there are both $((0,b_2),(s_H,s_H))$- and $((b_1,0),(s_L,s_L))$-matches. Some $((0,b_2),(s_L,s_L))$-matches must form as well, given that $a_H < a_2$. The additional existence of $((b_1,0),(s_H,s_H))$-pairs would lead to an immediate contradiction. So, the only possibility is that all (s_H,s_H)-sellers invest for and match with buyers from sector 2. It follows that $r_S(s_L,s_L) = 0$, $r_B(0,b_2) = \kappa b_2 s_L$, $r_S(s_H,s_H) = \kappa b_2(s_H - s_L)$ and $r_B(b_1,0) = \kappa b_1 s_L$. Buyers and (s_L,s_L)-sellers have no profitable deviations. The remaining equilibrium condition for (s_H,s_H) is $\kappa b_2\,(s_H - s_L) \geq \frac{4-\alpha}{4-2\alpha}\kappa b_1 s_H^{\frac{4-2\alpha}{4-\alpha}} s_L^{\frac{\alpha}{4-\alpha}} - \frac{4-\alpha}{4-2\alpha}\kappa b_1\, s_L$, which may be rewritten as $\frac{4-2\alpha}{4-\alpha}\frac{b_2}{b_1} \geq \frac{\left(\frac{s_H}{s_L}\right)^{\frac{4-2\alpha}{4-\alpha}}-1}{\frac{s_H}{s_L}-1}$. This condition is most stringent if $\frac{s_H}{s_L}$ is close to 1, in which case the investments made by the more productive sector of buyers are very suitable also for (s_H,s_H)-sellers.

An example of mismatch under technological multiplicity

The following example illustrates that mismatch may become a common feature of inefficient equilibria in environments with technological multiplicity that do not fit into the 1-d supermodular framework. The example combines Example 2 with an under-investment example à la CMP.

Example OA.1. $\mathrm{Supp}(\mu_S) = \{(s_1,s_1)|s_L \leq s_1 \leq s_H\}$, *where* $s_L < s_H$. $\mathrm{Supp}(\mu_B)$ *is the union of* $\{b_\varnothing\}$ *and two compact intervals* $\{(0,b_2)|b_{2,L} \leq b_2 \leq b_{2,H}\}$ *(*$b_{2,L} < b_{2,H}$*), and* $\{(b_1,0)|b_{1,L} \leq b_1 \leq b_{1,H}\}$ *(*$b_{1,L} < b_{1,H}$*).* μ_S *and* $\mu_B(\cdot|b \neq b_\varnothing)$ *have bounded densities, uniformly bounded away from zero, with respect to Lebesgue measure on their supports. Let* $v(x,y) = x_1 y_1 + \max(f_{1,1}, f_{\frac{1}{2},\frac{3}{2}})(x_2 y_2)$, $c_B(x,b) = \frac{x_1^4}{b_1^2} + \frac{x_2^4}{b_2^2}$ *and* $c_S(y,s) = \frac{y_1^4}{s_1^2} + \frac{y_2^4}{s_2^2}$.

Note that the technology for sector 1 is as in Example 2, but match surplus in sector 2 has an additional regime of increased complementarity for high investments. By Lemma 3, the surplus for sector 2 is strictly supermodular. If the surplus for sector 2 were globally defined by $f_{1,1}$, then $(x,y) = \left(\left(0, \frac{1}{2} b_2^{\frac{3}{4}} s_1^{\frac{1}{4}}\right), \left(0, \frac{1}{2} b_2^{\frac{1}{4}} s_1^{\frac{3}{4}}\right)\right)$ would be the unique non-trivial NE of the FA game between $(0,b_2)$ and (s_1,s_1), yielding net surplus $\frac{1}{8} b_2 s_1$. The expressions for $f_{\frac{1}{2},\frac{3}{2}}$ are $(x,y) = \left(\left(0, \frac{3}{16} b_2^{\frac{5}{4}} s_1^{\frac{3}{4}}\right), \left(0, \frac{3}{16} b_2^{\frac{3}{4}} s_1^{\frac{5}{4}}\right)\right)$ and $\kappa\left(\frac{3}{2},\frac{1}{2}\right)(b_2 s_1)^3 = \frac{3^3}{2^{15}}(b_2 s_1)^3$. Hence, pairs with $b_2 s_1 < \frac{2^6}{3^{\frac{3}{2}}} =: \tau$ are better off with the $f_{1,1}$-technology, and pairs with $b_2 s_1 > \tau$ are better off with the $f_{\frac{1}{2},\frac{3}{2}}$-technology. The true technology is defined via $f_{1,1}$ for $x_2 y_2 < z_{12} = 4$ and via $f_{\frac{1}{2},\frac{3}{2}}$ for $x_2 y_2 > 4$. Still, the identified attributes are the jointly optimal choices for all b_2 and s_1, as $x_2 y_2 = \frac{1}{4} b_2 s_1$ and $x_2 y_2 = \frac{3^2}{2^8}(b_2 s_1)^2$ evaluated at the indifference pairs $b_2 s_1 = \tau$ are equal to $\frac{2^4}{3^{\frac{3}{2}}} < 4$ and $\frac{2^4}{3} > 4$ respectively. However, making "low regime" investments still is a NE of the FA game for some range of b_2 and s_1 with $b_2 s_1 > \tau$.

Consider a situation in which ex-ante efficiency requires that high cost investments are made in sector 2. This is the case if and only if $(0, b_{2,H})$ is matched to a type (s_1^*, s_1^*) satisfying $b_{2,H} s_1^* > \tau$ in the ex-ante efficient equilibrium.[2]

If all sector 2 pairs invest according to the low cost regime - which is inefficient by assumption - then Claim 2 implies that $(0, b_{2,H})$ is matched to the seller type $(s_{1,q}, s_{1,q})$ who satisfies $\mu_S(\{(s_1, s_1) | s_1 \geq s_{1,q}\}) = q$, for $q = \mu_B(\{b | b_1 + b_2 \geq b_{2,H}\})$. In contrast to Example 2, this means a mismatch in the present case! This inefficient situation (in which efficient investment opportunities in sector 2 are missed, and some high seller types invest for sector 1 while they should invest for sector 2) is ruled out if and only if the low regime investments are in fact not a NE of the FA game between $(0, b_{2,H})$ and $(s_{1,q}, s_{1,q})$. Whether this is true depends crucially on q, and hence on sector 1 of the buyer population. In particular, whether the coordination failure is precluded or not depends on the full ex-ante populations, not just on supports (as in CMP). Finally, note that if the inefficient equilibrium exists, it exhibits inefficiency of joint investments if $b_{2,H} s_{1,q} > \tau$, whereas all agents make jointly optimal investments if $b_{2,H} s_{1,q} \leq \tau$.

[2]In contrast to Example 2, w is not globally supermodular with regard to 1-d sufficient statistics, so that the problem of finding the ex-ante optimal matching is non-local and difficult. However, for the present purposes, it is not necessary to solve the ex-ante assignment problem explicitly.