Online Appendix

Decentralized Exchange

Semyon Malamud[35] and Marzena Rostek[36]

C Additional Existence and Uniqueness Results

Proof of Lemmas A.2 and A.1. $\{\Lambda_i\}_i$ is an equilibrium if and only if $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$. By direct calculation, defining $X_i = (\Lambda_i + \alpha_i \mathbf{V}_{N(i)})^{-1}$, we get $\{\Lambda_i\}_i = F(\{\Lambda_i\}_i)$ if, and only if, $\{X_i\}_i = G(\{X_i\}_i)$. That F and G are monotone increasing follows because matrix inversion $Y \to Y^{-1}$ is monotone decreasing (e.g., Horn and Johnson (2013)), and projection onto $N(i)$ $Y \to Y_{N(i)}$ is monotone increasing in the positive semidefinite order. ■

Proof of Proposition 6. Pick an arbitrary starting tuple $\{X_i^0\}_i$ such that $\{X_i^0\}_i \leq G(\{X_i^0\}_i)$. By direct calculation, the corresponding price impacts $\Lambda_i^0 = (X_i^0)^{-1} - \alpha_i \mathbf{V}_{N(i)}$ satisfy $\{\Lambda_i^0\}_i \geq F(\{\Lambda_i^0\}_i)$. Since map F is continuous and monotone with respect to the defined partial order, recursively applying F to inequality $\{\Lambda_i^0\}_i \geq F(\{\Lambda_i^0\}_i)$, we can see that $F^n(\{\Lambda_i^0\}_i)$ is monotone decreasing and hence converges to a fixed point of F. For the price impact tuple satisfying $\{\Lambda_i^0\}_i \leq F(\{\Lambda_i^0\}_i)$, the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing. Therefore, to prove convergence to a fixed point, we need to show that it is bounded from above. To this end, pick $\alpha > 0$ sufficiently large so that $\{\tilde{\Lambda}_i\}_i$ defined by

$$\tilde{\Lambda}_i = \alpha \text{diag}((I(n) - 2)^{-1})_{n \in N(i)} \tag{42}$$

satisfies $\{\Lambda_i^0\}_i \leq \{\tilde{\Lambda}_i\}_i$, where $I(n)$ is the number of agents in exchange n. An analogous argument implies $F(\{\tilde{\Lambda}_i\}_i) \leq \{\tilde{\Lambda}_i\}_i$. Let $\Omega = \{\{\Lambda_j\}_j \in \mathcal{S}^M : \Lambda_j \leq \tilde{\Lambda}_j, \ \forall j\}$. Then, for any $\{\Lambda_j\}_j \in \Omega$,

$$F(\{\Lambda_j\}_j) \leq F(\{\tilde{\Lambda}_j\}_j) \leq \{\tilde{\Lambda}_i\}_i \tag{43}$$

and hence F maps Ω into itself. Therefore, the sequence $F^n(\{\Lambda_i^0\}_i)$ is monotone increasing, bounded from above by $\{\tilde{\Lambda}_i\}_i$, and hence converges to a fixed point of F. ■

Equilibrium uniqueness is equivalent to the uniqueness of the fixed point of map F. It therefore suffices to show that F is a contraction on a suitably defined normed space. We can identify the strategy of an agent in the game (i.e., his demand schedule) with its slope $(\alpha_i \mathbf{V}_{N(i)} + \Lambda_i)^{-1}$ and find conditions on the demand slopes for this to be the case.

Lemma C.1 For any i, suppose that $0 \leq \{B_i\}_i \leq \{A_i\}_i$ are such that any equilibrium tuple $\{\Lambda_i\}_i$ satisfies $\{B_i\}_i \leq \{(\alpha_i \bar{\mathbf{V}}_{N(i)} + \bar{\Lambda}_i)^{-1}\}_i \leq \{A_i\}_i$. Suppose that for any $\{X_i\}_i$, $\{B_i\}_i \leq \{X_i\}_i \leq \{A_i\}_i$,

$$(M_j - 1)X_j^2 + \sum_{i \neq j} M_i X_i^2 < \left((M_j - 1)X_j + \sum_{i \neq j} M_i X_i \right)^2, \ j = 1, \cdots, I\,. \tag{44}$$

[35] EPF Lausanne, Swiss Finance Institute and CEPR, semyon.malamud@epfl.ch.

[36] University of Wisconsin-Madison, Department of Economics, mrostek@ssc.wisc.edu.

Then, map F is a contraction on the set $\{B_i\}_i \leq \{X_i\}_i \leq \{A_i\}_i$, and hence there exists a unique equilibrium.

Note that when X_i are positive numbers (or commuting matrices, in which case they can be simultaneously diagonalized), a direct calculation implies that condition (44) holds. However, absent commutativity, this is generally not true. The usefulness of Lemma C.1 depends on a good choice of bounds $\{B_i\}_i$ and $\{A_i\}_i$. The next result provides a simple and easily verifiable condition that guarantees the applicability of Lemma C.1, based on the choice $\{B_i\}_i = \{(\bar{\Lambda}^0_{i,\max} + \alpha_i \bar{\mathbf{V}}_{N(i)})^{-1}\}_i$ and $\{A_i\}_i = \{(\bar{\Lambda}^0_{i,\min} + \alpha_i \bar{\mathbf{V}}_{N(i)})^{-1}\}_i$.[37]

Corollary 3 Suppose that

$$\min_n \frac{I(n)-2}{\lambda^{**}(n)} \geq \max_n \frac{I(n)-2}{(I(n)-1)\lambda^{**}(n)}. \tag{45}$$

Then, equilibrium is unique.

Intuitively, the left-hand side of (45) measures how competitive an exchange is, whereas the right-hand side reflects the dispersion of payoff riskiness across exchanges. If this dispersion is high, there is a lot of 'room' for non-commutativity and uniqueness can only be guaranteed when strategic effects are small; that is, when $I(n)$ is sufficiently large.

To proceed further, we first establish auxiliary results.

Lemma C.2 If there is only one asset and one exchange, then equilibrium is unique.

Proof. The proof follows by Lemma C.1. Indeed, in this case, the conditions of Lemma C.1 hold, and therefore map F is a contraction and has a unique fixed point. ■

Lemma C.3 Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Consider a map $F_A : \mathcal{S}^I \to \mathcal{S}^I$ defined via

$$F_{A,i}(\{\Lambda_j\}_j) = \left(\left(\sum_{j \neq i} (\alpha_j \bar{A}_j + \bar{\Lambda}_j)^{-1} \right)^{-1} \right)_{N(i)}.$$

This map has a unique fixed point in the class of diagonal matrices.

Proof. Since F is a contraction on the set of diagonal matrices, Lemma C.1 gives the proof. ■

Lemma C.4 Let $\{A_i\}_i \in \mathcal{S}^I$ be a tuple of diagonal matrices. Then, map F_A from Lemma C.3 has a unique fixed point.

[37] As an example, consider the case when all pairs of assets are equally correlated with correlation ρ, all agents have the same risk aversion α, and $\max_i |N(i)| \leq \hat{N}$. Then, $\max(\text{eig}(C(\mathbf{V}_{N(i)}))) \leq \max\{1+\rho(\hat{N}-1), 1-\rho\}$, $\min(\text{eig}(C(\mathbf{V}_{N(i)}))) \geq \min\{1+\rho(\hat{N}-1), 1-\rho\}$, and (45) imposes upper and lower bounds on the correlation ρ. For example, in the symmetric case when $I(n) = \hat{I}$ is independent of n and $\rho > 0$, we obtain the simple condition $\rho < \frac{\hat{I}-2}{\hat{I}+\hat{N}-2}$. For Corollary 3, one could also pick $\{B_i\}_i = \{(\bar{\Lambda}^k_{i,\max} + \alpha_i \bar{\mathbf{V}}_{N(i)})^{-1}\}_i$ and $\{A_i\}_i = \{(\bar{\Lambda}^k_{i,\min} + \alpha_i \bar{\mathbf{V}}_{N(i)})^{-1}\}_i$ for any $k \geq 1$.

Proof. Let $\{\Lambda_i^A\}_i$ be an arbitrary fixed point of F_A, and let $\{\Lambda_i^{**}\}_i$ be a diagonal fixed point, which is unique by Lemma C.3. Pick $\beta_1 \in \mathbb{R}_+$ so that β_1 satisfies $\beta_1 \mathrm{Id}_{N(i)} \leq \Lambda_i^A$ for all i and $\beta_1 \leq \min_i \min(\mathrm{eig}(A_i))$. Similarly, pick $\beta_2 \in \mathbb{R}_+$ so that β_2 satisfies $\beta_2 \mathrm{Id}_{N(i)} \geq \Lambda_i^A$ for all i and $\beta_2 \geq \max_i \max(\mathrm{eig}(A_i))$. Define $\{B_k\}_i \equiv \{\beta_k \mathrm{Id}_{N(i)}\}_i$, $k = 1, 2$, and let F_{B_k}, $k = 1, 2$ be the corresponding maps. Then define $\{\Lambda_i^{B_k}\}_i = \beta_k \{\mathrm{diag}((I(n) - 2)^{-1})_{N(i)}\}_i$. We have

$$\{\Lambda_i^{B_k}\}_i = F_{B_k}(\{\Lambda_i^{B_k}\}_i).$$

Iterating the inequality

$$\{\Lambda_i^{B_1}\}_i = F_{B_1}(\{\Lambda_i^{B_1}\}_i) \leq F_A(\{\Lambda_i^{B_1}\}_i),$$

we obtain that $F_A^n(\{\Lambda_i^{B_1}\}_i)$ converges to a diagonal fixed point of F_A, and hence by Lemma C.3 converges to $\{\Lambda_i^{**}\}_i$. A similar argument implies that $F_A^n(\{\Lambda_i^{B_2}\}_i)$ also converges to $\{\Lambda_i^{**}\}_i$.

Now by the definition of β_k, $k = 1, 2$, we also have

$$F_{B_1}(\{\Lambda_i\}_i) \leq F_A(\{\Lambda_i\}_i) \leq F_{B_2}(\{\Lambda_i\}_i)$$

for any $\{\Lambda_i\}_i \in \mathcal{S}^I$. Therefore, by the monotonicity of map F_A,

$$F_A^n(\{\Lambda_i^{B_1}\}_i) \leq F_A^n(\{\Lambda_i^A\}_i) \leq F_A^n(\{\Lambda_i^{B_2}\}_i)\,.$$

Taking $n \to \infty$ and using the fact that $F_A^n(\{\Lambda_i^A\}_i) = \{\Lambda_i^A\}_i$, we get $\{\Lambda_i^{**}\}_i \leq \{\Lambda_i^A\}_i \leq \{\Lambda_i^{**}\}_i$, and the claim follows. ■

In the sequel, we use the following convenient notation:
Notation. For any $x, y \in \mathbb{R}^N$, we write $y^T x = \langle x, y \rangle$. Given a triplet X, A, B of symmetric matrices of the same dimension, we use notation $X \in [B, A]$ when $B \leq X \leq A$.

Proof of Corollary 3. For simplicity, we work directly with the correlation matrix and assume that $\mathbf{V} = C(\mathbf{V})$. By the proof of Theorem 1 (iv), any equilibrium $\{\Lambda_i\}_i$ satisfies

$$\Lambda_{i,\min}^0 \leq \Lambda_{i,\min} \leq \Lambda_i \leq \Lambda_{i,\max} \leq \Lambda_{i,\max}^0,\ i \in I.$$

Therefore,

$$\mathrm{diag}\left(\frac{I(n) - 2}{(I(n) - 1)\lambda^{**}(n)}\right)_{N(i)} \leq (\Lambda_i + \alpha_i \mathbf{V}_{N(i)})^{-1} \leq \mathrm{diag}\left(\frac{I(n) - 2}{(I(n) - 1)\lambda_{**}(n)}\right)_{N(i)},\ i \in I\,,$$

and the claim follows from Lemma E.5. ■

D Additional Comparative Statics and Welfare

Proof of Lemma B.1. Suppose that there are N traders such that

$$\tilde{\Lambda}_i^{-1} + (\alpha_i \tilde{\mathbf{V}}_i + \tilde{\Lambda}_i)^{-1} = \tilde{B}^{-1},\ i = 1, \cdots, N.$$

Pick an arbitrary $\hat{\Lambda}_1 \leq \Lambda_1$ and define $\hat{B}$ via $\hat{\Lambda}_1^{-1} + (\alpha_1 \tilde{\mathbf{V}}_1 + \hat{\Lambda}_1)^{-1} = \hat{B}^{-1}$. Then, define $\hat{\Lambda}_i$ via

$$\hat{\Lambda}_i^{-1} + (\alpha_i \tilde{\mathbf{V}}_i + \hat{\Lambda}_i)^{-1} = \hat{B}^{-1}, \ i = 2, \cdots, N.$$

This defines $\hat{\Lambda}_i$ uniquely: $\hat{\Lambda}_i = \tilde{\mathbf{V}}_i^{1/2} f(\tilde{\mathbf{V}}_i^{1/2} \hat{B} \tilde{\mathbf{V}}_i^{1/2}) \tilde{\mathbf{V}}_i^{1/2}$ and $(\hat{\Lambda}_i + \tilde{\mathbf{V}}_i)^{-1} = \tilde{\mathbf{V}}_i^{-1/2} g(\tilde{\mathbf{V}}_i^{1/2} \hat{B} \tilde{\mathbf{V}}_i^{1/2}) \tilde{\mathbf{V}}_i^{-1/2}$. Then, define $A \equiv \sum_i (\hat{\Lambda}_i + \tilde{\mathbf{V}}_i)^{-1}$. It therefore suffices to show that there exist $M \geq 1$ and $\tilde{\mathbf{V}}_{N+1}$ and the corresponding price impact $\hat{\Lambda}_{N+1}$ such that

$$\hat{\Lambda}_{N+1}^{-1} + (\alpha_{N+1} \tilde{\mathbf{V}}_{N+1} + \hat{\Lambda}_{N+1})^{-1} = \hat{B}^{-1} \text{ and } \ A + (M_3 - 1)(\alpha_{N+1} \tilde{\mathbf{V}}_{N+1} + \hat{\Lambda}_{N+1})^{-1} = \hat{\Lambda}_{N+1}^{-1}.$$

This gives the required matrix $\tilde{\mathbf{V}}_3 = (M_3^{-1}(\hat{B} - A))^{-1} - (M_3^{-1}(A + (M_3 - 1)\hat{B}^{-1}))^{-1}$. ■

The following result gives the analytic characterization of equilibrium in Proposition 1.

Lemma D.1 (Functional Calculus for Symmetric Matrices) For any continuous function $f(x)$ and any symmetric matrix A, we can define $f(A)$ as follows. By the eigen-decomposition theorem, there exists an orthogonal matrix U and a diagonal matrix D such that $A = U^T D U$, where $D = \text{diag}(d_i)$ and d_i are the eigenvalues of A. Then,

$$f(A) = U^T \text{diag}(f(d_i)) U.$$

In general, matrix U is not unique. The uniqueness holds only if the eigenvalues of A are all distinct. However, even if U is not unique, $f(A)$ is uniquely determined, and so it is well-defined. The following lemma explicitly links price impact Λ_i with the aggregate liquidity measure $B = \sum_j (\alpha_j \bar{\mathbf{V}}_{N(j)} + \bar{\Lambda}_j)^{-1}$. Let $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$ and $f(a) = f_1(a)/a$.

Lemma D.2 Let $Y_i = (B^{-1})_{N(i)}$. Then

$$\Lambda = Y_i^{1/2} f_1(Y_i^{-1/2} \alpha_i \mathbf{V}_{N(i)} Y_i^{-1/2}) Y_i^{1/2}.$$

If $\mathbf{V}_{N(i)}$ is invertible, then

$$\Lambda_i = \alpha_i \mathbf{V}_{N(i)}^{1/2} f(\alpha_i \mathbf{V}_{N(i)}^{1/2} Y_i^{-1} \mathbf{V}_{N(i)}^{1/2}) \mathbf{V}_{N(i)}^{1/2}. \tag{46}$$

Proof of Lemma D.2. The assertion is a direct consequence of Lemma E.6. ■

Proof of Corollary 2. The expressions follow by direct calculation from market clearing, $\sum_{j=1}^{I} \left(\alpha_j \bar{\mathbf{V}}_{N(j)} + \bar{\Lambda}_j(B)\right)^{-1} (d - p - \alpha_j \bar{V}_{N(j)} q_j^0) = 0$. ■

Next, we consider how the extent to which a trader is connected with the market and his more or less central position in the market, measured by participation in different exchanges, influence his equilibrium price impact relative to other traders in a given exchange. The equilibrium price impacts of different market participants are linked through the aggregate liquidity measure B. Namely, let

$$\Phi(\Lambda_i, \alpha_i \mathbf{V}_{N(i)}) \equiv (\Lambda_i^{-1} + (\alpha_i \mathbf{V}_{N(i)} + \Lambda_i)^{-1})^{-1} \tag{47}$$

be the harmonic mean of two matrices Λ_i and $\Lambda_i + \alpha_i \mathbf{V}_{N(i)}$. Then, by Theorem 1, $\Phi(\Lambda_i, \alpha_i \mathbf{V}_{N(i)}) = (B^{-1})_{N(i)}$, for any class i. In particular, the price impacts of two classes i and j that are connected (i.e., $N(i) \cap N(j) \neq \emptyset$) are related as follows

$$\big(\Phi(\Lambda_i, \alpha_i \mathbf{V}_{N(i)})\big)_{N(i)\cap N(j)} = \big(\Phi(\Lambda_j, \alpha_j \mathbf{V}_{N(j)})\big)_{N(i)\cap N(j)} = (B^{-1})_{N(i)\cap N(j)}. \tag{48}$$

Suppose that $N(i) \supset N(j)$; for instance, class i is better connected than class j. A concavity property of the harmonic mean (47) implies the following relationship among the price impacts in the exchanges in which both classes i and j participate, $(\Lambda_i)_{N(j)}$ and Λ_j.

Lemma D.3 Suppose that class i has greater market participation than class j, $N(i) \supset N(j)$. Then

$$\Phi\big((\Lambda_i)_{N(j)}, \alpha_i \mathbf{V}_{N(j)}\big) \geq \Phi\big(\Lambda_j\,,\, \alpha_j \mathbf{V}_{N(j)}\big)\,. \tag{49}$$

Proof of Lemma D.3. By (48), $\big(\Phi\big(\Lambda_i, \alpha_i \mathbf{V}_{N(i)}\big)\big)_{N(j)} = \Phi\big(\Lambda_j, \alpha_j \mathbf{V}_{N(j)}\big)$. By Theorem 5 in Anderson (1971),

$$\big(\Phi\big(\Lambda_i, \alpha_i \mathbf{V}_{N(i)}\big)\big)_{N(j)} \leq \Phi\big((\Lambda_i)_{N(j)}, \alpha_i \mathbf{V}_{N(j)}\big)\,,$$

and the claim follows. ■

Function $\Phi(\Lambda, \alpha \mathbf{V})$ is monotone increasing in Λ, and therefore, for the case of scalar Λ_j, inequality (49) immediately yields the last item of Theorem 2.

Nevertheless, with many assets, one cannot extrapolate this result by using (49) to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$. The non-commutativity is, again, the key. Let $A_1 \equiv \Phi\big((\Lambda_i)_{N(j)}\,,\, \alpha_i \mathbf{V}_{N(j)}\big)$ and $A_2 \equiv \Phi\big(\Lambda_j\,,\, \alpha_j \mathbf{V}_{N(j)}\big)$. Then (using Lemma D.2),

$$(\Lambda_i)_{N(j)} = \alpha_i \mathbf{V}^{1/2}_{N(j)} f(\alpha_i \mathbf{V}^{1/2}_{N(j)} A_1^{-1} \mathbf{V}^{1/2}_{N(j)}) \mathbf{V}^{1/2}_{N(j)}\ ,\ \Lambda_j = \alpha_j \mathbf{V}^{1/2}_{N(j)} f(\alpha_j \mathbf{V}^{1/2}_{N(j)} A_2^{-1} \mathbf{V}^{1/2}_{N(j)}) \mathbf{V}^{1/2}_{N(j)}\,,$$

where

$$f(a) = \frac{2 - a + \sqrt{a^2+4}}{2a} \tag{50}$$

is monotone decreasing in a. Inequality $A_1 \geq A_2$ (Proposition D.3) implies $X_1 \equiv \mathbf{V}^{1/2}_{N(j)} A_1^{-1} \mathbf{V}^{1/2}_{N(j)} \leq \mathbf{V}^{1/2}_{N(j)} A_2^{-1} \mathbf{V}^{1/2}_{N(j)} \equiv X_2$. However, given two non-commuting symmetric matrices X_1 and X_2 and a monotone decreasing function $f(x)$, inequality $X_1 \leq X_2$ does *not* generally imply $f(X_1) \geq f(X_2)$. A function f that satisfies $f(X_1) \geq f(X_2)$ for any $X_1 \leq X_2$ is called *matrix monotone.* In particular, to conclude that $(\Lambda_i)_{N(j)} \geq \Lambda_j$, function f in (50) must be matrix-monotone, which is not the case.[38]

[38] In fact, f is not matrix monotone on any interval. This noteworthy property does not have any scalar analogues. This implies that with sufficiently many assets, for any $A \geq 0$ there exists B, $B \leq A$, such that B is sufficiently close to A and the monotonicity fails (by the Löwner's Theorem). A function $f(z)$ is matrix monotone on some (even an arbitrarily small) interval if and only if it can be approximated by convex combinations of simple hyperbolic functions $\frac{\alpha}{z+\beta}$, $\alpha \in R_+$, $\beta \in R$. For the general theory of monotone matrix functions, see Löwner (1934) and Donoghue (1974).

Note that non-commutativity is essential here. If A and B commute, they can be diagonalized in the same basis and, clearly, the implication $A \geq B \Rightarrow f(A) \leq f(B)$ holds for diagonal matrices.

One can still compare price impacts through an eigenvalue order instead of the (weaker) positive semidefinite order, using that with positive semidefinite matrices, there is a min-max interpretation of eigenvalues. For the eigenvalues of a symmetric $m \times m$ matrix A ordered to be decreasing, $\text{eig}(A) = \{\mu_1(A) \geq \cdots \geq \mu_m(A)\}$, we write $\text{eig}(A) \geq \text{eig}(B)$ if $\mu_i(A) \geq \mu_i(B)$ for all $i = 1, \cdots, m$.

Lemma D.4 (Relative Price Impact: Many Assets) Suppose that class i has greater market participation than class j, $N(i) \supset N(j)$. Then, if $\alpha_i \leq \alpha_j$, equilibrium price impact of class i in exchanges $N(j)$ is larger than that of class j in the following sense:

$$\text{eig}\big(\alpha_i^{-1}\mathbf{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathbf{V}_{N(j)}^{-1/2}\big) \geq \text{eig}\big(\alpha_j^{-1}\mathbf{V}_{N(j)}^{-1/2}\Lambda_j\mathbf{V}_{N(j)}^{-1/2}\big).$$

If the matrices $\mathbf{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathbf{V}_{N(j)}^{-1/2}$ and $\mathbf{V}_{N(j)}^{-1/2}\Lambda_j\mathbf{V}_{N(j)}^{-1/2}$ commute, then the stronger inequality (2) holds.

Proof of Lemma D.4. Let $W_1 = \alpha_i^{-1}\mathbf{V}_{N(j)}^{-1/2}(\Lambda_i)_{N(j)}\mathbf{V}_{N(j)}^{-1/2}$ and $W_2 = \alpha_j^{-1}\mathbf{V}_{N(j)}^{-1/2}\Lambda_j\mathbf{V}_{N(j)}^{-1/2}$. Then, $W_k = f(\alpha_i\mathbf{V}_{N(j)}^{1/2}A_k^{-1}\mathbf{V}_{N(j)}^{1/2})$, $k = 1, 2$. Since eigenvalues are increasing in the positive semidefinite order, Proposition D.3 implies that

$$\text{eig}(\alpha_i\mathbf{V}_{N(j)}^{1/2}A_1^{-1}\mathbf{V}_{N(j)}^{1/2}) \leq \text{eig}(\alpha_j\mathbf{V}_{N(j)}^{1/2}A_2^{-1}\mathbf{V}_{N(j)}^{1/2})\,.$$

Therefore, $\text{eig}(W_1) = f(\text{eig}(\alpha_i\mathbf{V}_{N(j)}^{1/2}A_1^{-1}\mathbf{V}_{N(j)}^{1/2})) \geq f(\text{eig}(\alpha_j\mathbf{V}_{N(j)}^{1/2}A_2^{-1}\mathbf{V}_{N(j)}^{1/2})) = \text{eig}(W_2)$. If W_1 and W_2 commute, diagonalizing them in the same basis implies that eigenvalue order and the positive semidefinite order are equivalent. ■

To prove part (2) of Proposition 5, we first characterize a class of markets. For each i, write

$$\mathbf{V}_{N(i)} = \begin{pmatrix} \mathbf{V}_{i,i} & \mathbf{V}_{i,-i} \\ \mathbf{V}_{-i,-i} & \mathbf{V}_{-i,-i} \end{pmatrix},$$

the block decomposition of $\mathbf{V}$ in $\mathbb{R}^{N(i)} = \mathbb{R}^{N(i)\setminus N(j)} \oplus \mathbb{R}^{N(i)\cap N(j)}$. For any $i \neq j$, $\mathbf{V}_{i\setminus j} \equiv \mathbf{V}_{i,i} - \mathbf{V}_{i,-i}(\mathbf{V}_{-i,-i})^{-1}\mathbf{V}_{i,-i} \in \mathbb{R}^{(N(i)\setminus N(j))\times(N(i)\setminus N(j))}$ is the conditional covariance for the residual risks in $N(i) \setminus N(j)$, which cannot be hedged in the liquid exchange $N(j)$.

Proposition 7 (Price Impact and *Residual* Riskiness) Let I_j be a set of agents with risk aversion α_j and let $N(j)$ be the set of exchanges in which they participate. Assume $I_j \geq 2$. Then, in the limit as $\alpha_j/I_j \to 0$, equilibrium price impacts in exchanges $N(j)$ become zero, $\Lambda_j \to 0$, whereas equilibrium price impacts in exchanges $N(i) \setminus N(j)$, $\Lambda_{i\setminus j} \equiv \Lambda_{i,N(i)\setminus N(j)}$, solve the system

$$\Lambda_{i\setminus j} = \left(\left(\sum_{k\notin I_j}(\alpha_k\bar{\mathbf{V}}_{k\setminus j} + \bar{\Lambda}_{k\setminus j})^{-1}\right)^{-1}\right)_{N(i)\setminus N(j)}, \; i \in I.$$

Furthermore, the demand slope of agent i in exchanges $N(i)$ coincides with $(\alpha_i\bar{\mathbf{V}}_{i\setminus j} + \bar{\Lambda}_{i\setminus j})^{-1}$ and $(\mathbf{VQ})_{N(j)} = 0$.

Proof of Proposition 7. Suppose that $M_i \geq 2$. Since, by assumption, there are at least three agents participating in each exchange, an $\varepsilon > 0$ exists such that

$$\begin{aligned}\Lambda_i &= \left(\left(\bar{\Lambda}_i)^{-1} + \sum_{j\neq i}(\alpha_j \bar{\mathbf{V}}_{N(j)} + \bar{\Lambda}_j)^{-1}\right)^{-1}\right)_{N(i)} \\ &\leq \left(\left(\varepsilon \mathrm{Id} + (M_i - 1)(\alpha_i \|\mathbf{V}\| \mathrm{Id} + \bar{\Lambda}_i)^{-1}\right)^{-1}\right)_{N(i)} = \left(\varepsilon \mathrm{Id} + (M_i - 1)(\alpha_i \|\mathbf{V}\| \mathrm{Id} + \Lambda_i)^{-1}\right)^{-1}.\end{aligned}$$

Let $\ell \geq 0$ be the largest eigenvalue of Λ_i. Then, we get $\ell \leq \left(\varepsilon \mathrm{Id} + (M_i - 1)(\alpha_j \|\mathbf{V}\| \mathrm{Id} + \ell)^{-1}\right)^{-1}$. By direct calculation, this inequality implies that $\ell \to 0$ as $\alpha_j \to 0$ or $M_j \to \infty$.

Pick any trader $i \neq j$. Then,

$$(\Lambda_i)_{N(j)\cap N(i)} \leq \left(\left(M_j(\alpha_j \|\mathcal{V}\| + \bar{\Lambda}_j)^{-1} + \varepsilon \mathrm{Id}\right)^{-1}\right)_{N(j)\cap N(i)} = \left(\left(M_j(\alpha_j \|\mathbf{V}\| + \Lambda_j)^{-1} + \varepsilon \mathrm{Id}\right)^{-1}\right)_{N(j)\cap N(i)}.$$

Since $(\alpha_j \|\mathbf{V}\| + \Lambda_i)^{-1}$ diverges to ∞, we get the required result.

Finally, the last claim follows because $\lim_{\alpha_j \to 0} \Lambda_i = (\bar{\Lambda}_{i\setminus j})_{N(i)}$, and hence, using the Frobenius formula (Lemma E.1), $\left((V_{N(i)} + \Lambda_i)^{-1}\right)_{N(j)\cap N(i)} \to (\Lambda_{i\setminus j} + S(V_{N(i)}, N(i) \setminus N(j)))^{-1}$.

To prove the result about the limit allocation, we need to study the asymptotic behavior in greater detail. This is done in the following proposition.

Proposition 8 Let $M_j > 2$. Then, for sufficiently small $\alpha = \alpha_j$, an equilibrium price impact tuple $\{\Lambda_i(\alpha)\}_i$ exists that satisfies $\Lambda_j(\alpha) \approx \frac{\alpha}{M_j - 2}\mathcal{V}_{N(j)}$, and for all $i \neq j$, to the first order in α,

$$\Lambda_i(\alpha) \approx \begin{pmatrix} \alpha \mathbf{V}_{N(j)} \frac{M_j - 1}{(M_j - 2)M_j} & -\alpha \frac{M_j - 1}{(M_j - 2)M_j} \mathbf{V}_{N(j)} W_{12}(i) W_{22}(i)^{-1} \\ -\alpha \frac{M_j - 1}{(M_j - 2)M_j} W_{22}(i)^{-1} W_{12}(i)^T \mathcal{V}_{N(j)} & \Lambda_{i\setminus j} + \alpha \Lambda^{(1)}_{i\setminus j} \end{pmatrix}_{N(i)},$$

where

$$W(i) = \begin{pmatrix} W_{11}(i) & W_{12}(i) \\ W_{12}^T(i) & W_{22}(i) \end{pmatrix} \equiv \sum_{k\neq i,j} (\alpha_k \bar{\mathbf{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

The first order equilibrium response $\{\Lambda^{(1)}_{i\setminus j}\}_{j\neq i}$ is the unique solution to the system

$$\Lambda^{(1)}_{i\setminus j} = \left(W_{22}(i)^{-1}\left(\sum_{k\neq i,j} Z_k \Lambda^{(1)}_{k\setminus j} Z_k + W_{12}(i)^T \mathbf{V}_{11} \frac{M_j - 1}{(M_j - 2)M_j} W_{12}(i)\right) W_{22}(i)^{-1}\right)_{N(i)\setminus N(i)},$$

where $Z_i \equiv (\alpha_i \bar{\mathbf{V}}_{N(i)} + \bar{\Lambda}_i(0))^{-1}$, $i \neq j$.

Proof. The fixed point equation is

$$\Lambda_i(\alpha) = \left(((\alpha \bar{\mathbf{V}}_{N(j)} + \bar{\Lambda}_j)^{-1} + W(i, \alpha))^{-1}\right)_{N(i)}$$

and the claim follows by direct calculation from the Frobenius formula (Lemma E.1). Furthermore,

$$B^{-1} \approx \begin{pmatrix} \alpha_j \mathbf{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} & -\alpha_j \frac{M_j-1}{(M_j-2)M_j} \mathbf{V}_{N(j)} W_{12} W_{22}^{-1} \\ -\alpha_j \frac{M_j-1}{(M_j-2)M_j} W_{22}^{-1} W_{12}^T \mathbf{V}_{N(j)} & W_{2\setminus 1} + \alpha_j W_{2\setminus 1}^{(1)} \end{pmatrix},$$

where

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{pmatrix} \equiv \sum_{k \neq j} M_k (\alpha_k \bar{\mathbf{V}}_{N(k)} + \bar{\Lambda}_k(0))^{-1}.$$

Thus, the trade of an agent j is approximately given by

$$(\alpha_j \mathbf{V}_{N(j)} + \Lambda_j)^{-1} \mathbf{Q}_{N(j)} - \frac{M_j - 2}{M_j - 1} q_j^0 = \alpha_j^{-1} \frac{M_j - 2}{M_j - 1} \mathbf{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} - \frac{M_j - 2}{M_j - 1} q_j^0.$$

We have[39]

$$\alpha_j^{-1} \frac{M_j - 2}{M_j - 1} \mathbf{V}_{N(j)}^{-1} \mathbf{Q}_{N(j)} \approx M_j^{-1} \left(X_{N(j)}^{(0)} - W_{12} W_{22}^{-1} X_{N \setminus N(j)}^{(0)} \right),$$

where

$$X^{(0)} = \sum_{j \neq i} \left(\alpha_j \bar{\mathbf{V}}_{N(j)} + \bar{\Lambda}_j(0) \right)^{-1} \alpha_j \bar{\mathbf{V}}_{N(j)} Q_j^0 + \frac{M_i - 2}{M_i - 1} Q_i^0.$$

In contrast, agents $i \neq j$ trade $(\alpha_i \mathbf{V}_{N(i)} + \bar{\Lambda}_i(0))^{-1} \left(\mathbf{Q}_{N(i)\setminus N(j)} - \alpha_i \mathbf{V}_{N(i)} q_i^0 \right)$, since $\mathbf{Q}_{N(j)} = 0$. ■ ■

We will now restrict our analysis to markets from Proposition 7. For simplicity, we will assume that there is a single illiquid exchange in which all agents participate, so that $n = N(i) \setminus N(j)$ is the same for all agents. By Proposition 7, the problem reduces to studying the price impact $\tilde{\Lambda}_i = \Lambda_{i\setminus j}$ of agent i in the (illiquid) exchange n. Let $\Pi_{K(n)}$ be the orthogonal projection onto the subspace of assets traded in exchange n and let $\tilde{\mathbf{Q}} \equiv (\mathbf{V}\mathbf{Q})_{K(n)}$ and $\tilde{q}_i^0 \equiv (q_i^0)_{K(n)}$. With $\tilde{\mathbf{V}}_i = \mathbf{V}_{i\setminus j}$ defined as in Proposition 7, $\tilde{\Upsilon}_i \equiv (\alpha_i \tilde{\mathbf{V}}_i)^{-1} \tilde{\Upsilon}_i(\tilde{\mathbf{V}}_i, \tilde{\Lambda}_i)(\alpha_i \tilde{\mathbf{V}}_i)^{-1}$, $\tilde{\Delta}_i \equiv (\alpha_i \tilde{\mathbf{V}}_i)^{-1} \tilde{\Delta}_i(\tilde{\mathcal{V}}_i, \tilde{\Lambda}_i)(\alpha_i \tilde{\mathbf{V}}_i)^{-1}$, and $\tilde{\Lambda}_i^{-1} + (\tilde{\Lambda}_i + \alpha_i \tilde{V}_i)^{-1} = \tilde{B}$ for all i, which implies a global upper bound on the price impact of all agents,

$$\tilde{\Lambda}_i < 2\tilde{B}^{-1}. \tag{51}$$

Lemma D.5 In the markets from Proposition 7, $\tilde{\Upsilon}_j = \frac{1}{2}\left(\tilde{B}\tilde{\Lambda}_j \tilde{B} - \tilde{B} \right)$ and $2\tilde{\Delta}_j = 3\tilde{\Lambda}_j \tilde{B} \tilde{\Lambda}_j - \tilde{\Lambda}_j \tilde{B} \tilde{\Lambda}_j \tilde{B} \tilde{\Lambda}_j$.

Proof of Lemma D.5. For brevity, let $\Upsilon = \Upsilon_{j\setminus i}$, $\mathbf{V} = \alpha_j \tilde{\mathbf{V}}_j$, $\Lambda = \tilde{\Lambda}_j$, $B = \tilde{B}$. Then,

$$(\alpha_j \mathbf{V}_{N(j)})^{-1} \Upsilon (\alpha_j \mathbf{V}_{N(j)})^{-1} = \frac{1}{2}(\mathbf{V}+\Lambda)^{-1} + \frac{1}{2}(\mathbf{V}+\Lambda)^{-1}\Lambda(\mathbf{V}+\Lambda)^{-1} = \frac{1}{2}(B - \Lambda^{-1}) + \frac{1}{2}(B-\Lambda^{-1})\Lambda(B-\Lambda^{-1})$$

and the claim follows. For Δ, we have

$$\begin{aligned} (\alpha_j \mathbf{V}_{N(j)})^{-1} \Delta (\alpha_j \mathbf{V}_{N(j)})^{-1} &= \Lambda(\Lambda + \mathbf{V})^{-1} \mathbf{V} (\Lambda + \mathbf{V})^{-1} \Lambda = \\ \Lambda(\Lambda + \mathbf{V})^{-1}\Lambda - \Lambda(\Lambda + \mathbf{V})^{-1}\Lambda(\Lambda + \mathbf{V})^{-1}\Lambda &= \Lambda(B - \Lambda^{-1})\Lambda - \Lambda(B - \Lambda^{-1})\Lambda(B - \Lambda^{-1})\Lambda \end{aligned}$$

[39] $\alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathbf{V}_{N(j)}^{-1} \alpha_j \mathbf{V}_{N(j)} \frac{M_j-1}{(M_j-2)M_j} X_{N(j)}^{(0)} - \alpha_j^{-1} \frac{M_j-2}{M_j-1} \mathbf{V}_{N(j)}^{-1} \alpha_j \frac{M_j-1}{(M_j-2)M_j} \mathbf{V}_{N(j)} W_{12} W_{22}^{-1} X_{N\setminus N(j)}^{(0)}$ approximates the left hand side and equals the right hand side.

and the claim follows by direct calculation. ■

The following lemma shows that within the general framework of Proposition 7 we can directly study welfare with the "reduced" matrices of price impact and surplus from trade.

Lemma D.6 Let

$$\tilde{\Upsilon}_i \equiv (\alpha_i\tilde{\mathbf{V}}_j+\tilde{\Lambda}_j)^{-1}\left(\frac{1}{2}\alpha_i\tilde{\mathbf{V}}_j+\tilde{\Lambda}_j\right)(\alpha_i\tilde{\mathbf{V}}_j+\tilde{\Lambda}_j)^{-1} \text{ and } \tilde{\Delta}_j \equiv \frac{1}{2}\tilde{\Lambda}(\alpha_i\tilde{\mathbf{V}}_j+\tilde{\Lambda}_j)^{-1}\alpha_i\tilde{\mathbf{V}}_j(\alpha_i\tilde{\mathbf{V}}_j+\tilde{\Lambda}_j)^{-1}\tilde{\Lambda}_j.$$

Then, the utility U_j of agent from class j with initial holdings q_k^0 with $\left(q_k^0\right)_{N\setminus K(n)} = 0$ (i.e., no initial holdings in exchanges $N \setminus K(n)$) is given by

$$U_j(\Lambda_j; q_k^0) = \langle\tilde{\Upsilon}_j\tilde{\mathbf{Q}},\tilde{\mathbf{Q}}\rangle - 2\langle\alpha_j\tilde{\mathbf{V}}_j\tilde{\Upsilon}_j\tilde{\mathbf{Q}},\tilde{q}_k^0\rangle - \langle\tilde{\Delta}_j\tilde{q}_k^0,\tilde{q}_k^0\rangle. \tag{52}$$

Proof. Let $\mathbf{V}_j \equiv \mathbf{V}_{N(j)}$. Then,

$$\Upsilon_j = \frac{1}{2}(\alpha_i\mathbf{V}_j+\Lambda_j)^{-1} + \frac{1}{2}(\alpha_j\mathbf{V}_j+\Lambda_j)^{-1}\Lambda_j(\alpha_j\mathbf{V}_j+\Lambda_j)^{-1},$$

and hence $\langle\Upsilon_i\mathbf{Q}_{N(i)},\mathbf{Q}_{N(i)}\rangle = \langle\tilde{\Upsilon}_j\tilde{\mathbf{Q}},\tilde{\mathbf{Q}}\rangle$, because $\mathbf{Q}_{\kappa(n)} = 0$ and price impact in exchanges other than $\kappa(n)$ also vanishes. Furthermore, a direct calculation implies that

$$\alpha_i\mathbf{V}_i\Upsilon_i = \frac{1}{2}(\mathrm{Id}-\Lambda_i(\alpha\mathbf{V}_i+\Lambda_i)^{-1}) + \frac{1}{2}(\mathrm{Id}-\Lambda_i(\alpha\mathbf{V}_i+\Lambda_i)^{-1})\Lambda_i(\alpha\mathbf{V}_i+\Lambda_i)^{-1},$$

and hence $\langle\alpha_i\mathbf{V}_i\Upsilon_i\mathbf{Q}_{N(i)},q_k^0\rangle = \langle\alpha_j\tilde{\mathbf{V}}_j\tilde{\Upsilon}_j\tilde{\mathbf{Q}},\tilde{q}_k^0\rangle$. ■

Proposition 9 (Commutativity, Connectedness and Price impact) If $\tilde{B}^{1/2}\tilde{\mathbf{V}}_{j_1}\tilde{B}^{1/2}$ and $\tilde{B}^{1/2}\tilde{\mathbf{V}}_{j_2}\tilde{B}^{1/2}$ commute and $\tilde{\mathbf{V}}_{j_1} \le \tilde{\mathbf{V}}_{j_2}$, then $\tilde{\Lambda}_{j_1} \ge \tilde{\Lambda}_{j_2}$. However, for any $\tilde{B}$ and $\tilde{\mathbf{V}}_{j_1}$ that do not commute and satisfy $\tilde{B} > 2\tilde{\mathbf{V}}_{j_1}^{-1}$, there exists $\mathbf{V}_{j_2} \ge \mathbf{V}_{j_1}$ such that $\tilde{\Lambda}_{j_1} \not\ge \tilde{\Lambda}_{j_2}$.

Proof of Proposition 9. For simplicity, we normalize all risk aversions to 1. Let $j_1 = 1,\ j_2 = 2$. We first show that for any $\tilde{\mathbf{V}}_1,\ \tilde{\mathbf{V}}_2$ and $\tilde{B}$ there exists a market in which they are realized. To prove this, consider a market with three classes and let us show that we can pick $\tilde{\mathbf{V}}_3$ accordingly. First, equation $\tilde{\Lambda}_i^{-1} + (\tilde{\mathbf{V}}_i+\tilde{\Lambda}_i)^{-1} = \tilde{B}$ implies (by Lemma D.2) that

$$\tilde{\Lambda}_i \ = \ \tilde{\mathbf{V}}_i^{1/2} f(\tilde{\mathbf{V}}_i^{1/2}\tilde{B}\tilde{\mathbf{V}}_i^{1/2})\tilde{\mathbf{V}}_i^{1/2}$$

and

$$(\tilde{\Lambda}_i+\tilde{\mathbf{V}}_i)^{-1} = \tilde{\mathbf{V}}_i^{-1/2} g(\tilde{\mathbf{V}}_i^{1/2}\tilde{B}\tilde{\mathbf{V}}_i^{1/2})\tilde{\mathbf{V}}_i^{-1/2}.$$

Denote $A \ \equiv (\tilde{\mathbf{V}}_1+\tilde{\Lambda}_1)^{-1} + (\tilde{\mathbf{V}}_2+\tilde{\Lambda}_2)^{-1}$. Then, $\tilde{\Lambda}_3$ satisfies

$$\tilde{\Lambda}_3 = (A + (M_3-1)(\tilde{\mathbf{V}}_3+\tilde{\Lambda}_3)^{-1})^{-1} = (\tilde{B} - (\tilde{\mathbf{V}}_3+\tilde{\Lambda}_3)^{-1})^{-1}$$

and therefore to complete the proof it suffices to show that there exist positive definite matrices $\tilde{\Lambda}_3,\ \tilde{\mathbf{V}}_3$ satisfying

$$\tilde{\Lambda}_3^{-1} - (M_3 - 1)(\tilde{\mathbf{V}}_3 + \tilde{\Lambda}_3)^{-1} = A,\ \tilde{\Lambda}_3^{-1} + (\tilde{\mathbf{V}}_3 + \tilde{\Lambda}_3)^{-1} = \tilde{B}.$$

Solving this system, we get

$$(\tilde{\mathbf{V}}_3 + \tilde{\Lambda}_3)^{-1} = M_3^{-1}(\tilde{B} - A),\ \tilde{\Lambda}_3^{-1} = M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1})\,.$$

Since $A + (M_3 - 1)\tilde{B}^{-1} > \tilde{B}^{-1} - A$ whenever $M_3 > 1$, the matrix

$$\tilde{\mathbf{V}}_3 = (M_3^{-1}(\tilde{B} - A))^{-1} - (M_3^{-1}(A + (M_3 - 1)\tilde{B}^{-1}))^{-1}$$

is positive definite. Finally, $\tilde{B} > 2\tilde{\mathbf{V}}_1^{-1} > \tilde{\mathbf{V}}_1^{-1} + \tilde{\mathbf{V}}_2^{-1} > A$, completing the proof of existence.

Lemma D.7 None of the functions $f,\ f_1,\ g$ is matrix monotone.

Proof. By the Löwner Theorem (Donoghue (1974)), it suffices to show that none of these functions can be analytically continued to the whole upper half-plane. This follows directly from the fact that $2i$ is a branching point for all these functions. ■

By Lemma, $f_1 = \frac{2-a+\sqrt{a^2+4}}{2}$ is not matrix monotone on any interval, and consequently, for any positive definite matrix X_1 of sufficiently high dimension, there exists a matrix $X_2 \geq X_1$ such that $f_1(X_2) \not\leq f_1(X_1)$. Let $X_1 = \tilde{B}^{-1/2}\tilde{\mathbf{V}}_1\tilde{B}^{-1/2}$. Then, define $\tilde{\mathbf{V}}_2 \equiv \tilde{B}^{1/2}X_2\tilde{B}^{1/2}$.

Now, by Lemma *D.2*,

$$\Lambda_1 = \tilde{B}^{1/2}f_1(X_1)\tilde{B}^{1/2} \not\geq \tilde{B}^{1/2}f_1(X_2)\tilde{B}^{1/2} = \Lambda_2,$$

and the proof of Proposition 9 is complete. ■

The proof of Proposition 3 shows that the eigenvalues of the map $\mathcal{E}^{\mathbb{M}}$ play a crucial role in determining the welfare properties of decentralized markets. We complete this Appendix with a discussion of the structure of these eigenvalues and their link to price impact. For simplicity, we assume that there is only one asset traded on all exchanges.[40]

Corollary 4 (One Asset: Equilibrium Allocations) Consider a decentralized exchange for a single asset (Example 1 (ii)). Suppose that Λ_i is nonsingular.[41] Letting $(1 - \gamma_i^{\mathbb{M}}) \equiv \frac{1}{1+\alpha_i\sigma^2\mathrm{tr}(\Lambda_i^{-1})}$, the allocation of trader i is given by

$$q_i + q_i^0 = (1 - \gamma_i^{\mathbb{M}})\,(\mathbf{1}^T\bar{\Lambda}_i^{-1}\mathbf{Q}^* + q_i^0)\,. \tag{53}$$

[40] The case of multiple assets is similar, but the notation is more cumbersome.

[41] This is without loss of generality and is always the case if the market structure is a tree. See Malamud and Rostek (2016).

The scalar $\gamma_i^{\mathbb{M}}$ is the decentralized-market counterpart of the degree of diversification by trader i. The overall liquidity $\mathrm{tr}(\Lambda_i^{-1})$ of a trader who participates in multiple exchanges for a homogenous asset determines the fraction of the initial endowment he retains. In one-asset markets, equilibrium price impacts of all traders are diagonal without loss (see Malamud and Rostek (2016)), allowing us to use the trace. □

In a noncompetitive market (either centralized or decentralized), the map $\mathcal{E}^{DM}$ still keeps the efficient allocation unchanged, $\mathcal{E}^{DM}(\alpha_i^{-1})_i = (\alpha_i^{-1})_i$ (Corollary 2). However, in addition to this eigenvalue of one, it has other eigenvalues that are non-zero. The magnitude of these eigenvalues shows precisely how efficient the map $\mathcal{E}^{DM}$ is in terms of eliminating the inefficient parts of the initial allocation.[42] Recall that $\gamma_i \equiv 1 - \frac{1}{1+\alpha_i\sigma^2\mathrm{tr}(\Lambda_i^{-1})}$ (see Corollary 4) is the degree of diversification by trader i in the decentralized market. In order to characterize the eigenvectors e of $\mathcal{E}^{\mathbb{M}}$, we substitute the eigenvalue condition $\mathcal{E}^{\mathbb{M}}e = \nu e$ into (53) and arrive at the following result.

Proposition 10 Suppose that there is one asset and that $(1-\gamma_i) \neq \delta_j$ for all $i \neq j$, and the market hypergraph is connected.[43] Then, the map $\mathcal{E}^{\mathbb{M}}$ has I different eigenvalues $\nu_i,\ i = 1, \cdots, I$ satisfying

$$0 < (1-\gamma_1) \ < \ \nu_1 \ < \ (1-\gamma_2) < \cdots < \ \nu_{I-1} < (1-\gamma_I) < \nu_I = 1\,. \tag{54}$$

The eigenvector for the eigenvalue $\nu_I = 1$ is the efficient allocation $e_I = (\alpha_i^{-1})_i$, while for an eigenvalue $\nu = \nu_i,\ i \leq I-1$ the corresponding eigenvector is given by $e(\nu) = (q_j(\nu))_j$, where

$$q_j(\nu) \ = \ -\frac{\delta_j}{\delta_j - \nu}\bar{\mathbf{1}}^T\bar{\Lambda}_j^{-1}\mathbf{Q}^*(\nu)\,, \tag{55}$$

$\mathbf{Q}^*(\nu)$ is the corresponding aggregate risk, and the eigenvalues $\nu_i,\ i = 1, \cdots, I-1$ are determined by the zero aggregate endowment condition $\mathbf{1}^T e(\nu) = 0$ for all $i \leq I-1$.

The zero aggregate endowment condition is straightforward: By market clearing, we always have $\mathbf{1}^T\mathcal{E}^{\mathbb{M}}e = \mathbf{1}^T e$ and hence, if $e(\nu)$ is an eigenvector with $\nu \neq 0$, we must have $\nu\mathbf{1}^T e = \mathbf{1}^T e$ implying that either $\nu = 1$ (and then e is the efficient allocation) or $\mathbf{1}^T e = 0$. By the zero aggregate endowment condition, the efficient (competitive) trade would have $\mathcal{E}^{\mathrm{CM}}e = 0$. But price impacts do not allow the agents to diversify away their endowment risk. An eigenvector $e(\nu)$ corresponds to initial endowments for which exactly the fraction $1-\nu$ of initial endowment is be diversified away.

The eigenvectors and eigenvalues of the map $\mathcal{E}$ can be used to derive the decomposition of the action of $\mathcal{E}^{\mathbb{M}}$. Indeed, let V be the matrix of eigenvectors of $\mathcal{E}$ so that $\mathcal{E}^{\mathbb{M}} = V\mathcal{D}V^{-1}$. For any vector $\mathcal{Q}^0 = (q_i^0)$ define $\mathcal{Q}_j^0 \equiv (V^{-1}\mathcal{Q}^0)_j$ to be the coordinates of $\mathcal{Q}^0$ in the basis of eigenvectors of $\mathcal{E}^{\mathbb{M}}$. Then, $\mathcal{Q}^0 = \sum_i \mathcal{Q}_i^0 e_i$ and

$$\mathcal{E}^{\mathbb{M}}\mathcal{Q}^0 \ = \ \sum_i \nu_i \mathcal{Q}_i^0\,. \tag{56}$$

[42] Sannikov and Skrzypacz (2015) use a similar decomposition in a slightly different setting and show that these eigenvalues determine how quickly the initial allocation converges to the efficient one over time.

[43] That is, for any two exchanges, there is a trading path connecting them. The general case is analogous, but the notation is more complicated.

If the eigenvalues $\nu_i,\ i < I$, are sufficiently small, the map $\mathcal{E}^{\mathbb{M}}$ essentially eliminates the inefficiency of the allocation and pushes it all the way to the efficient one. Otherwise, it only "contracts" the inefficiencies along the directions of the corresponding eigenvectors, and the degree of contraction is given by $\nu_i \in (0,1)$. The smaller ν_i is, the more efficient the market structure is in diversifying away the initial endowment risk. In particular, if the initial endowments are given by the eigenvector e_1 corresponding to the minimal eigenvalue ν_1, then the given market structure is highly efficient in diversifying initial endowment risk. In general, if a vector of initial endowments is a linear combination of $e_1, \cdots, e_L$, then, on average, a fraction $\mu \in [1-\nu_L, 1-\nu_1]$ of initial risk can be diversified away.

By Proposition 10, the degree of diversification $1-\nu_i$ is locked between the individual traders' degrees of diversification γ_i, γ_{i+1}. By Theorem 2, $(1-\gamma_i)$ are always monotone increasing as the market becomes more decentralized. Hence, if a given vector e of endowments is an eigenvector of both $\mathcal{E}^{\mathbb{M}}, \mathcal{E}^{\mathbb{M}'}$ and $\mathbb{M}'$ is more decentralized than $\mathbb{M}$, we expect that the diversification gains lower in the more decentralized market. However, a key property of decentralized markets is that the eigenvectors are different across different market structures: An endowment vector with a high degree of diversification in market $\mathbb{M}$ may have a low degree of diversification in $\mathbb{M}'$. For example, $e_1^{\mathbb{M}}$ may be close to $e_{I-1}^{\mathbb{M}'}$ in which case $e_1^{\mathbb{M}}$ will be almost diversifiable in $\mathbb{M}$ and almost non-diversifiable in $\mathbb{M}'$.

Welfare Calculation in Example 2 For $\alpha_3 \to \infty$, class 3 agents do not trade, and hence $x_1 = x_1^{Split},\ x_2 = x_2^{Split}$. When $\alpha_3 < \infty$, then $x_3 > 0$, and hence $x_1 + x_2 = -x_3 < 0 = x_1^{Split} + x_2^{Split}$. That is, when α_3 is sufficiently large, then $x_1 + x_2 < 0$, implying that x_1 decreases faster than x_2 increases as α_3 decreases. It follows that $0 > x_1^{Split} > x_1$ for large α_3; class 1 agents buy more despite the larger price impact.

Given $x_3^{Split} = 0$, the split market dominates the centralized market in total welfare (3) if and only if the agents' utility losses from risk exposure satisfy

$$\alpha_1 (x_1^{Split})^2 + \alpha_2 (x_2^{Split})^2 < \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 \,.$$

Suppose, for simplicity, that $\alpha_2 = \alpha_3$. Since $|x_1^{Split}| < |x_1|$, it suffices to verify that the total welfare loss of classes 2 and 3 is lower in the split market, that is, that $(x_2^{Split})^2 + 0^2 < x_2^2 + x_3^2$. We have $x_3 + x_2 = -x_1 > -x_1^{Split} = x_2^{Split} = x_3^{Split} + x_2^{Split}$. Using $x_3^2 + x_2^2 = 0.5(x_2 + x_3)^2 + 0.5(x_3 - x_2)^2$, the inequality $x_2 + x_3 > x_2^{Split}$ combined with sufficiently large $|x_3 - x_2|$ gives the desired inequality.

Example 6 (Intermediated Market Can (Further) Increase Welfare) Consider the market from Example 4, in which agent $1d$ participates in both exchanges and equilibrium allocation is given by (20). Example 2 demonstrated that total welfare in the split market with allocation $(x_1^{Split}, x_2^{Split}, x_3^{Split})$ is higher than in the centralized market. We wish to understand whether the intermediated market can increase welfare even more, and if so, why this may be the case; that is, under what conditions we may have

$$\alpha_2 M(x_2^2 + x_3^2) + \alpha_1((M-1)x_1^2 + x_{1d}^2) < \alpha_2 M((x_3^{Split})^2 + (x_2^{Split})^2) + \alpha_1 M (x_1^{Split})^2 \,, \tag{57}$$

where we assume $\alpha_1 < \alpha_2 = \alpha_3$ (two classes) for simplicity, as in Example 2.

The common participating trader lowers price impact in exchange 1 (Theorem 2), while also changing the aggregate risk in the exchanges. In particular, intermediation does not fully eliminate the inefficient trade of class 3. However, as we show next, the efficiency improvements over the centralized market due to heterogeneity in aggregate risk across exchanges are affected only slightly by (the single endowment of) the intermediary. When class 1 is large enough, the welfare benefits due to lower price impact dominate and total welfare increases.

Equilibrium aggregate risk satisfies $\mathbf{Q}_1^* > q^{Split,1}$, given the lower price impact in exchange 1 and recalling that $q^{Split,1}$ is the aggregate risk portfolio in exchange 1 of the split market.[44] In turn, $\mathbf{Q}_2^* < 0$, since the endowment of $1d$ is negative and the endowment of class 3 agents is zero.

The price impact of intermediary $1d$ is higher than that of other agents of class 1 *in exchange* 1 (this is, in fact, a general result for intermediaries; Theorem 2 (3)). If, as in Example 2, α_3 is sufficiently high, class 3 agents trade little with $1d$ and his trade in exchange 1 is insufficient to maintain the amount of trade he does in the split market: $0 > x_1 > x_{1d}$. Since price impact (Theorem 2) and price ($\mathbf{Q}_1^* > q_1^{Split}$) are lower in exchange 1, class 1 agents buy more of the asset: $0 > x_1 > x_1^{Split} > x_{1d}$. If M is not too small, the welfare gain of non-intermediating agents dominates the utility loss of the intermediary,

$$\alpha_1((M-1)x_1^2 + x_{1d}^2) < \alpha_1 M (x_1^{Split})^2 .$$

The utility change of class 2 is ambiguous. Their aggregate risk in exchange 1, $\mathbf{Q}_1^*$, is higher, and the lower price discourages selling. At the same time, their price impact is lower than in the split market (by Theorem 2), since the intermediated market is less decentralized than the split market. When α_2 is sufficiently high, the contribution of the aggregate risk, $(\Lambda_2 + \alpha_2)^{-1}\mathbf{Q}_1^*$, is small and the effect of lower price impact dominates, efficiently lowering (the natural sellers') allocation $x_2 = (\Lambda_2 + \alpha_2)^{-1}\mathbf{Q}_1^* + (1 - \Gamma_2)q$ relative to the split market. Since risk aversion $\alpha_3 = \alpha_2$ is sufficiently large, the inefficient trade and allocation of class 3 is small, and altogether we have $\alpha_2 M(x_2^2 + x_3^2) < \alpha_2 M((x_2^{Split})^2 + (x_3^{Split})^2)$. □

In Example 6, welfare improves even with a common participating trader who takes the *same* (buying) position in both exchanges, while the intermediary's utility decreases.

E Useful Linear-Algebraic Results

Lemma E.1 (Frobenius Formula) By direct calculation,

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}B^T)^{-1} & -A^{-1}B(D - B^TA^{-1}B)^{-1} \\ -(D - B^TA^{-1}B)^{-1}B^TA^{-1} & (D - B^TA^{-1}B)^{-1} \end{pmatrix}.$$

Lemma E.2 For a positive definite matrix A, $A^{-1} \geq \bar{A}_{N(i)}^{-1}$.

[44] Heuristically, $1d$ trades only a fraction μ the endowment in exchange 1, and the total endowment in exchange 1 is not $-Mq + Mq = 0$ but $-\mu q - (M-1)q + Mq = (1-\mu)q > 0$.

Proof of Lemma E.2. Let $B = A^{-1}$, $x \in \mathbb{R}^{N(i)}$ and $y \in Y^{N\setminus N(i)}$. Then

$$\min_{y\in\mathbb{R}^{N\setminus N(i)}} \langle B(x,y),(x,y)\rangle = \langle (B_{11} - B_{12}B_{22}^{-1}B_{21})x, x\rangle. \tag{58}$$

By the Frobenius formula (Lemma E.1), $B_{11} - B_{12}B_{22}^{-1}B_{21} = ((B^{-1})_{11})^{-1} = A_{11}^{-1}$. Therefore,

$$\langle A^{-1}(x,y),(x,y)\rangle = \langle B(x,y),(x,y)\rangle \geq \langle A_{11}^{-1}x, x\rangle = \langle \bar{A}_{11}^{-1}(x,y),(x,y)\rangle$$

for any $(x,y) \in \mathbb{R}^N$, and the claim follows since $A_{11} = A_{N(i)}$. ■

Proof of Lemma C.1. Let us calculate the derivative of map F. That is, consider an infinitesimal change $\{\Lambda_i\}_i \to \{\Lambda_i + \varepsilon Y_i\}_i$. Then, a direct calculation based on the identity, used twice,

$$(U+\varepsilon V)^{-1} \approx U^{-1} - \varepsilon U^{-1}VU^{-1}$$

implies that the Frechet derivative of F, $\frac{\partial F_j}{\partial(\{\Lambda_i\}_i)}(\{Y_i\}_i)$, is given by

$$\left(\left(\sum_{i\neq j} X_i\right)^{-1}\left(\sum_{i\neq j} X_iY_iX_i\right)\left(\sum_{i\neq j} X_i\right)^{-1}\right)_{N(j)}.$$

Introduce a norm of the set of I-tuples of positive semidefinite matrices via $|||\{Y_i\}_i||| = \max_i \|Y_i\|_{N(i)}$, where $\|\cdot\|_{N(i)}$ is the standard norm on matrices in $\mathbb{R}^{N(i)}$ defined by

$$\|Y\| = \max_{x\in\mathbb{R}^N, x\neq 0} \frac{\|Yx\|}{\|x\|}.$$

For simplicity, in the sequel we omit the index $N(i)$ for the norms. For a symmetric matrix, $\|Y\| = \max|\mathrm{eig}(Y)|$, and therefore, $Y_i \in [-\|Y_i\|\mathrm{Id}_{N(i)}, \|Y_i\|\mathrm{Id}_{N(i)}]$. Suppose now that condition (44) holds. Then,

$$\left\|\left(\sum_{i\neq j} X_i\right)^{-1}\left(\sum_{i\neq j} X_i^2\right)\left(\sum_{i\neq j} X_i\right)^{-1}\right\| \leq 1,$$

and hence,

$$\frac{\partial F_j}{\partial\{\Lambda_i\}_i}(\{Y_i\}_i) \leq \left(\left(\sum_{i\neq j} X_i\right)^{-1}\left(\sum_{i\neq j} X_i|||\{Y_i\}_i|||\mathrm{Id}_{N(i)}X_i\right)\right)_{N(j)} \leq |||\{Y_i\}_i|||\mathrm{Id}_{N(j)}.$$

The same argument implies

$$\frac{\partial F_j}{\partial\{\Lambda_i\}_i}(\{Y_i\}_i) > -|||\{Y_i\}_i|||\mathrm{Id}_{N(j)};$$

that is,

$$\left\||\frac{\partial F_j}{\partial\{\Lambda\}_i}(\{Y_i\}_i)\right\| < |||\{Y_i\}_i|||.$$

Hence, map F is a contraction on this set and cannot have more that one fixed point. ■

Lemma E.3 If $X \in [B, A]$ and $Xq = z$ then $\langle q, z\rangle \geq \max\{\langle Bq, q\rangle, \langle A^{-1}z, z\rangle\}$.

Proof. Since $X \geq B$, we have $\langle Bq, q\rangle \leq \langle Xq, q\rangle = \langle q, z\rangle$ and the first claim follows. To prove the second claim, pick an $\varepsilon > 0$. Then, $X \leq A$ implies $(X + \varepsilon \mathrm{Id})^{-1} \geq (A + \varepsilon \mathrm{Id})^{-1}$, and therefore,

$$XA^{-1}X \leq X(X + \varepsilon \mathrm{Id})^{-1}X.$$

Since $x^2(x + \varepsilon)^{-1}x \leq x$ for any $x \geq 0$, functional calculus implies $X(X + \varepsilon \mathrm{Id})^{-1}X \leq X$. Taking the limit as $\varepsilon \downarrow 0$, we get $XA^{-1}X \leq X$ and the following completes the proof

$$\langle A^{-1}z, z\rangle = \langle A^{-1}Xq, Xq\rangle = \langle XA^{-1}Xq, q\rangle \leq \langle Xq, q\rangle = \langle z, q\rangle.$$

■

Lemma E.4 Consider function $\Psi_j(z_1, \cdots, z_I) \equiv \sum_i \|z_i\|^2 - \|\sum_{i\neq j} z_i\|^2$ and let

$$\mu(q) \;=\; \max\{\Psi_j(z_1, \cdots, z_I) \;:\; z_i \in \mathbb{R}^{N(i)}, \langle q, z_i\rangle \;\geq\; \max\{\langle \bar{B}_i q, q\rangle, \langle A_i^{-1}z_i, z_i\rangle\},\; i \in I\}.$$

If $\max_{q\in\mathbb{R}^N} \mu(q) < 0$, then the conditions of Lemma C.1 are satisfied.

Proof. The claim follows directly from Lemma E.3 if we define $\bar{X}_i q = z_i$. ■

Lemma E.5 Let $a_i = \|A_i\|$ and $a = \max_{i\in I} a_i$. Suppose that $a\mathrm{Id} \leq \sum_i \bar{B}_i$. Then, the hypothesis of Lemma E.4 is satisfied.

Proof. Pick a tuple $z_i \in \mathbb{R}^{N(i)}, i \in I$, satisfying $\langle q, z_i\rangle \;\geq\; \max\{\langle \bar{B}_i q, q\rangle,\; \langle A_i^{-1}z_i, z_i\rangle\},\; i \in I$. Then,

$$a_i^{-1}\|z_i\|^2 \leq \langle A_i^{-1}z_i, z_i\rangle \leq \langle q, z_i\rangle,\; i \in I.$$

Normalize q so that $\|q\| = 1$. Then, we can decompose $z_i \;=\; \langle q, z_i\rangle q + z_i^{\perp}$ with $z_i^{\perp} \in \mathbb{R}^N$, $\langle z_i^{\perp}, q\rangle = 0$. Let $\beta_i \equiv \langle q, z_i\rangle$. Then,

$$\|\sum_{i\neq j} z_i\|^2 = \left(\sum_i \beta_i\right)^2 + \|\sum_i z_i^{\perp}\|^2 \geq \left(\sum_i \beta_i\right)^2,$$

and therefore,

$$\begin{aligned}\Psi_j(z_1, \cdots, z_I) \;&\equiv\; \sum_i \|z_i\|^2 - \|\sum_{i\neq j} z_i\|^2 \leq \sum_i a_i\beta_i - \|\sum_{i\neq j} z_i\|^2 \leq \sum_i a_i\beta_i - \left(\sum_i \beta_i\right)^2 \\ &\leq\; \left(\sum_i \beta_i\right)\left(a - \sum_i \beta_i\right)\end{aligned}$$

and the claim follows because by assumption, $\sum_i \beta_i \geq \sum_i \langle q, z_i\rangle \geq \sum_i \langle q, \bar{B}_i q\rangle \geq a$. ■

Lemma E.6 Let Y, Z be nonnegative definite, with Y positive definite. The unique positive definite symmetric matrix Λ solving

$$\Lambda = (Y^{-1} - (Z + \Lambda)^{-1})^{-1}$$

is given by

$$\Lambda = Y^{1/2} f_1(Y^{-1/2} Z Y^{-1/2}) Y^{1/2},$$

where $f_1(a) = (2 - a + \sqrt{a^2 + 4})/2$. If Z is invertible, then we can also write

$$\Lambda = Z^{1/2} f(Z^{1/2} Y^{-1} Z^{1/2}) Z^{1/2}$$

with $f(a) = f_1(a)/a$. Furthermore,

$$(Z + \Lambda)^{-1} = Z^{-1/2} g(Z^{1/2} Y^{-1} Z^{1/2}) Z^{-1/2}$$

with $g(a) = (f(a) + 1)^{-1} = 2a/(2 + a + \sqrt{a^2 + 4})$.

Proof. Multiplying by $(Y^{-1} - (Z + \Lambda)^{-1})$ from the right gives

$$\Lambda(Y^{-1} - (Z + \Lambda)^{-1}) = \mathrm{Id}.$$

Multiplying by Λ^{-1} from the left gives

$$Y^{-1} = \Lambda^{-1} + (Z + \Lambda)^{-1}. \tag{59}$$

Multiplying from the left and right by $Y^{1/2}$ (we do this to preserve symmetry), we have

$$\mathrm{Id} = L^{-1} + (Y^{-1/2} Z Y^{-1/2} + L)^{-1},$$

where $L = (Y^{-1/2} \Lambda Y^{-1/2})$. Let $A = Y^{-1/2} Z Y^{-1/2}$. Let us first show that A and L commute. Indeed, multiplying $(A + L)$ from the left and right gives

$$(A + L)L^{-1} + \mathrm{Id} = (A + L) = L^{-1}(A + L) + \mathrm{Id}.$$

Subtracting Id from both sides and multiplying by L from the left and right gives

$$LA + L^2 = L(A + L) = (A + L)L = AL + L^2$$

and the claim follows. Thus, A and L commute, and therefore, there exists an orthonormal basis in which both A and L are diagonal in this basis. For an orthogonal matrix U, both UAU^T and ULU^T are diagonal and

$$\mathrm{Id} = U\mathrm{Id}U^T = UL^{-1}U^T + U(A + L)^{-1}U^T = (ULU^T)^{-1} + (UAU^T + ULU^T)^{-1}.$$

Since all matrices on both sides are diagonal, each diagonal element has the same form with the

unique positive solution $f(a)$ of

$$1 = \frac{1}{x} + \frac{1}{a+x}.$$

Therefore, we obtain

$$L = U^T f(UAU^T)U = f(A) = f(Y^{-1/2}ZY^{-1/2}).$$

Similarly, assume that Z is positive definite. Then, there exists a positive-definite invertible matrix $Z^{1/2}$. Multiplying (59) by $Z^{1/2}$ from the left and right, we get

$$K = B^{-1} + (\mathrm{Id} + B)^{-1},$$

where $K = Z^{1/2}Y^{-1}Z^{1/2}$ and $B = Z^{-1/2}\Lambda Z^{-1/2}$. Multiplying $(\mathrm{Id} + B)$ from the left and right,

$$K + BK = (\mathrm{Id} + B)K = B^{-1} + 2\mathrm{Id} = K(\mathrm{Id} + B) = K + KB,$$

which implies that K and B commute. By an argument analogous to the above, with the unique positive solution $f_1(a)$ to

$$a = \frac{1}{x} + \frac{1}{1+x},$$

we get that $B = f_1(K)$. ■

Lemma E.7 Let $H \subset \mathbb{R}^N$ be a subspace, let B a symmetric positive definite matrix on H and let A be a positive definite matrix on $\mathbb{R}^N$. Then, $A \geq \bar{B}$ if and only if $(A^{-1})_H \leq B^{-1}$.

Proof. We have

$$A - \bar{B} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} - B \end{pmatrix},$$

and hence by (58), $A - \bar{B} \geq 0$ if and only if $A_{22} - A_{12}^T A_{11}^{-1} A_{12} - B \geq 0$. By Lemma E.1, this is equivalent to $(A^{-1})_{22} \leq B^{-1}$. ■

Lemma E.8 There exists a matrix $B \leq A$ such that $Bq = z$ if, and only if, $\langle A^{-1}z, z\rangle \ \leq \langle q, z\rangle$.

Proof. We normalize z so that $\|z\| = 1$. Suppose first that $B \leq A$ satisfies $Bq = z$. Then, $\langle q, z\rangle = \langle B^{-1}z, z\rangle \geq \langle A^{-1}z, z\rangle$. Suppose that $\langle A^{-1}z, z\rangle \ \leq \langle q, z\rangle$ and define $B = (\langle q, z\rangle)^{-1}\langle \cdot, z\rangle z$. Let H be the span of vector z. By Lemma E.7, it suffices to check that $(A^{-1})_H \leq B^{-1}$. Since $(A^{-1})_H = \langle A^{-1}z, z\rangle$ and B^{-1} acts as $(\langle q, z\rangle)$ on this subspace, the claim follows. ■

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