Numerical Investigation : Model and Algorithm

1 Model: money investment model

1.1 steady state without shock

- correction: $F(K,H) = K^\alpha H^{1-\alpha}$: fortran program $B = 1$

$$
\begin{aligned}
x_h &= \left[\frac{(\frac{1}{\beta} - 1 + (1-\phi)\delta_h)\delta_h^{\frac{\phi}{1-\phi}}}{\phi A^{\frac{1}{1-\phi}} w}\right]^{\frac{1-\phi}{2\phi-1}} \\
h &= \left[\frac{A x_h^\phi}{\delta_h}\right]^{\frac{1}{1-\phi}} \\
r &= \frac{1}{\beta} - 1 \\
\frac{k}{h} &= \left(\frac{r+\delta_k}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
w &= (1-\alpha)\left(\frac{k}{h}\right)^\alpha
\end{aligned}
$$

1.2 Agent's problem

$$
v(k,h,A) \;=\; max_{k',s} u((1+r)k + wh - k' - x'_h) + \beta E_{\eta'} v(k', \eta'\{(1-\delta_h)h + A x_h^\phi h^\phi\}, A) + \lambda k' + \theta x_h
$$

$$
\begin{aligned}
&(FOC1) \quad u'((1+r)k + wh - k' - x_h) = \beta E_{\eta'} v_1(k', \eta'\{(1-\delta_h)h + A x_h^\phi h^\phi\}, A) + \lambda \\
&(FOC2) \quad u'((1+r)k + wh - k' - x_h) = g_{x_h}(A,h,x_h)\beta E_{\eta'} \eta' v_2(k', \eta'\{(1-\delta_h)h + A x_h^\phi h^\phi\}, A) + \theta
\end{aligned}
$$

$$
\Rightarrow \theta = 0 : \quad (\because \lim_{x_h \to 0} g_{x_h} = \infty)
$$

Envelope conditions are following.

$$
\begin{aligned}
(Env1) \quad v_1(k,h,A) &= (1+r)u'((1+r)k + wh - k' - x_h) \\
(Env2) \quad v_2(k,h,A) &= wu'((1+r)k + wh - k' - x_h) \\
&\quad + g_h(A,h,x_h)\beta E_{\eta'} \eta' v_2(k', \eta'\{(1-\delta_h)h + A x_h^\phi h^\phi\}, A) \\
&= [w + \frac{g_h(A,h,x_h)}{g_{x_h}(A,h,x_h)} h] u'(((1+r)k + wh - k' - x_h)
\end{aligned}
$$

Then, using the (FOC)'s and (Env)'s we can get following Euler equations

$$\begin{aligned}
(EE1):&\quad ((1+r)k + wh - k' - x_h)^{-\sigma} = (1+r)\beta E_{\eta'}((1+r)k' + wh' - k'' - x_h')^{-\sigma} \\
(EE2):&\quad ((1+r)k + wh - k' - x_h)^{-\sigma} \\
&\quad = A\phi x_h^{\phi-1} h^{\phi-1} \beta E_{\eta'}\left[\eta'\{\frac{1-\delta_h + A\phi x_h'^{\phi} h'^{\phi-1}}{A\phi x_h'^{\phi-1} h'^{\phi}}\}((1+r)k' + wh' - k'' - x_h')^{-\sigma}\right]
\end{aligned}$$

1.3 Agent's problem Algorithm (EGM)

1. Guess initial derivatives of value functions

$$\begin{aligned}
cbdd &= \max\{0.0001, kr + hw\} \\
dval1^{(0)}(k,h,A) &= (1+r)(cgdd)^{-\sigma} \\
dval1^{(0)}(k,h,A) &= \left\{ w + \frac{(1-\delta_h) + A\phi h^{\phi-1} x_{h,ss}^{\phi}}{A\phi h^{\phi} x_{h,ss}^{\phi-1}} \right\}(cgdd)^{-\sigma} \\
where &\qquad x_{h,ss} = \left[\frac{\delta_h h^{1-\phi}}{A}\right]^{\frac{1}{\phi}}
\end{aligned}$$

2. Fix A_m, h_j. For each $k_l' \in kgrid$ ($k_l' \geq 0$), solve for $x_h(k_l', h_j, A_m)$.
   $x_h^{1-\phi} E_{\eta'} v_1(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A) \leq A\phi h^{\phi} \beta E_{\eta'} \eta' v_2(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A)$
   $\Rightarrow Eulers(s: k_l', h_j, A_m) = x_h^{1-\phi} E_{\eta'} v_1(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A)$
   $\qquad - A\phi h^{\phi} \beta E_{\eta'} \eta' v_2(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A)$
   $\Rightarrow$ solve for x_h and get $x_h(k_l', h_j, A_m)$

3. $c(k_l', h_j, A_m) = \max\{0.01, [\beta E v_1(k_l', \eta' g(s^*, h_j, A_m), A_m)]^{-\frac{1}{\sigma}}\}$

4. $k(k_l', h_j, A_m) = \frac{c(k_l', h_j, A_m) - wh_j + k_l' + x_h^*}{1+r}$

5. $kgridend(l, h_j, A_m) = k(k_l', h_j, A_m)$
   $polkend(k_l', h_j, A_m) = k_l'$
   $polxhend(k_l', h_j, A_m) = x_h(k_l', h_j, A_m)$

6. interpolation on kgrid
   -for $j1 = 1 : ngrid \quad (A_m)$
   - for $j3 = 1 : dimgridh \quad (h_j)$
   - for $j2 = 1 : dimgridk \quad (k_i)$
   - $k = kgrid(j2)$
   - if $k > kgridend(1, j3, j1)$
   - $polk(k_i, h_j, A_m) = interp(kgridend, polkend, k)$
   - $polxh(k_i, h_j, A_m) = interp(kgridend, polsend, k)$
   - else
   - $polk(k_i, h_j, A_m) = 0$
   - $polxh(k_i, h_j, A_m) =$ solution of

- $0 = Eulerkink(x_h)$
- $= u'((1+r)k + wh - x_h) - A\phi h^{\phi} x_h^{\phi-1} \beta E \eta' v_2(k'=0, \eta' g(A,h,x_h), A)$
- end

7. Evaluate new derivatives
$dval1new = (1+r)(k(1+r) + wh - polknew - polxhnew)^{-\sigma}$
$dval2new = (w + ((1-\delta_h) + A\phi h^{\phi-1} x_h^{\phi})/(A\phi x_h^{\phi-1} h^{\phi})) * ((k(1+r) + wh - polknew - polxhnew)^{-\sigma}$

1.4 Main algorithm for competitive equilibrium

1. random number generation for $\eta shock$
2. Guess for $\frac{K}{H} \Rightarrow$ get price r, w
3. Solve for HH's problem for given r and w
4. compute K^{new} and H^{new} using simulated data and compute new r^{new}
5. Check if $|r - r^{new}| < tol$. If not, update $\frac{K}{H}$ and go back to 2.

1.5 Planner's problem

$$
\begin{aligned}
\Omega(\Phi) &= \max_{\substack{k'(k,h,A) \\ s(k,h,A)}} \int_{S=K\times H\times A} u((1+r)k + wh - k' - x_h) d\Phi + \beta\Omega(\Phi') \\
&= \int_{S=K\times H\times A} V(k,h,A) d\Phi
\end{aligned}
$$

$$
\begin{aligned}
(FOC1) \quad & u'((1+r)k + wh - k' - x_h) = \beta E_{\eta'} v_1(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A) + \Delta_k + \lambda(k,h,A) \\
where \quad & \Delta_k = \beta \int_S u'((1+r)k' + wh' - k'' - x_h')[k' F_{KK} + h' F_{LK}] d\Phi' \\
(FOC2) \quad & u'((1+r)k + wh - k' - x_h) \\
& = g_{x_h}(A,h,x_h) \beta E_{\eta'} \eta' v_2(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi} h^{\phi}\}, A) + g_{x_h}(A,h,x_h)\Delta_h \\
where \quad & \Delta_h = \beta \int_{S\times\eta'} \eta' u'((1+r)k' + wh' - k'' - x_h')[k' F_{KL} + h' F_{LL}] d\Phi'^{+\eta'}
\end{aligned}
$$

Envelope conditions are following.

$$
\begin{aligned}
(Env1) \quad v_1(k,h,A) &= (1+r)u'((1+r)k + wh - k' - x_h) \\
(Env2) \quad v_2(k,h,A) &= [w + \frac{g_h(A,h,x_h)}{g_{x_h}(A,h,x_h)}] u'(((1+r)k + wh - k' - x_h)
\end{aligned}
$$

1.6 Planner's problem Algorithm - HH problem (EGM)

- For fixed $\Delta_k, \Delta_h, \frac{K}{H}, r, w$

1. Fix A_m, h_j. For each $k'_l \in kgrid$ ($k'_l \geq 0$), solve for $x_h(k'_l, h_j, A_m)$.
$$Eulers(s : k'_l, h_j, A_m) = x_h^{1-\phi}\left\{\beta E_{\eta'} v_1(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi}h^{\phi}\}, A) + \Delta_k\right\}$$
$$- A\phi h^{\phi}\left\{\beta E_{\eta'}\eta' v_2(k', \eta'\{(1-\delta_h)h + Ax_h^{\phi}h^{\phi}\}, A) + \Delta_h\right\}$$

2. $c(k'_l, h_j, A_m) = \max\{0.001, [\beta E v_1(k'_l, \eta' g(A_m, h_j, x_h^*), A_m) + \Delta_k]^{-\frac{1}{\sigma}}\}$

3. $k(k'_l, h_j, A_m) = \frac{c(k'_l,h_j,A_m) - wh_j + k'_l + x_h^*}{1+r}$

4. $kgridend(l, h_j, A_m) = k(k'_l, h_j, A_m)$
$polkend(k'_l, h_j, A_m) = k'_l$
$polxhend(k'_l, h_j, A_m) = x_h(k'_l, h_j, A_m)$

5. interpolation on kgrid
-for $j1 = 1 : ngrid \quad (A_m)$
- for $j3 = 1 : dimgridh \quad (h_j)$
- for $j2 = 1 : dimgridk \quad (k_i)$
- $k = kgrid(j2)$
- if $k > kgridend(1, j3, j1)$
- $polk(k_i, h_j, A_m) = interp(kgridend, polkend, k)$
- $polxh(k_i, h_j, A_m) = interp(kgridend, polxhend, k)$
- else
- $polk(k_i, h_j, A_m) = 0$
- $polxh(k_i, h_j, A_m) =$ solution of
- $0 = Eulerkink(x_h)$
- $= u'((1+r)k + wh - x_h) - A\phi h^{\phi} x_h^{\phi-1}\{\beta E\eta' v_2(k' = 0, \eta' g(A, h, x_h), A) + \Delta_h\}$
- end

6. Evaluate new derivatives
$dval1new = (1+r)(k(1+r) + wh - polknew - polxhnew)^{-\sigma}$
$dval2new = (w + ((1-\delta_h) + A\phi h^{\phi-1} polxhnew^{\phi})/(A\phi polxhnew^{\phi-1}h^{\phi})) * ((k(1+r) + wh - polknew - polxhnew)^{-\sigma}$

(Main algorithm)
- Given Δ_k, solve for $\frac{K}{H}$.
- Search over Δ_k (fixed point).