Online Appendix

Nominal Exchange Rate Determinacy Under the Threat of Currency Counterfeiting

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This online appendix contains all the omitted proofs and two generalizations of the model from the paper.[14]

Proof of Proposition 1

Consider a generalization of the model in the paper where DM agents can also trade internationally. The key takeaway from this section is the following: Even if we generalize the environment in the main paper to allow agents to shop internationally in the DM—i.e., a Home buyer can also buy from a Foreign seller and vice-versa—we still get the same characterization of the private information game in Proposition 1, where we have the equilibrium liquidity constraints. As a consequence, the main insight regarding the determinacy of the nominal exchange (having binding liquidity constraints) is robust to a more general setting where DM buyers can also buy from Foreign DM sellers.

Now in each DM, a Home DM-buyer has a probability $\xi \in (0,1]$ to be in Home DM. With probability $1-\xi$ a Home DM-buyer is to relocated to the Foreign DM. Ex ante, at the end of the CM (and prior to the DM), the buyer must take this possibility into account. The buyer at the end of CM will make a plan for what terms of trade to offer to a Home DM-seller, which we denote by $\omega := (q_{+1}, d_{+1}, d^f_{+1})$, and, what terms of trade to offer a Foreign DM-seller, denoted by $\omega^\star := (q^\star_{+1}, d_{+1}, d^f_{+1})$. The outcome of the plan would be contingent on the ex post realization of whether the buyer will shop in the Home DM or in the Foreign DM. For simplicity, assume that the probability of a match between a buyer and a seller in both Home and Foreign DM is identical and given by σ.

Given a commitment to the plan $(\omega, \omega^\star)$, and before the DM opens, the buyer then decides on his counterfeiting mixed strategy, $(\eta, \eta^f) := (\eta(\omega,\omega^\star), \eta^f(\omega,\omega^\star))$. As in the main paper, the buyer conditions his strategy on his beliefs about a Home (or Foreign) and the seller's probability of accepting his payment, $\hat{\pi}_{+1} := \hat{\pi}_{+1}(\omega,\omega^\star)$ (or $\hat{\pi}_{\star,+1} := \hat{\pi}_{\star,+1}(\omega,\omega^\star)$). From a seller's point of view, the Home (or Foreign) seller will ex-post play a mixed strategy $\pi_{+1} := \pi_{+1}(\omega)$ (or $\pi_{\star,+1} := \pi_{\star,+1}(\omega^\star)$).

A1. DM-buyers and DM-sellers

Since the model does not have aggregate uncertainty and agents have perfect foresight, let us re-write the game and work backwards from a DM(t) and then

[14] This document is available from: https://github.com/phantomachine/gkwcurse

to a preceding date's CM($t-1$). Given a fixed strategy $(\omega, \omega^{\star}, \eta, \eta^f)$, which is determined at the end of the preceding CM, the induced beginning-of-DM value to a buyer with portfolio (a, a^f) is given by

$$\begin{aligned} V^b\left(a, a^f\right) &= \xi\sigma\hat{\pi}\left\{u(q) + W^b\left[\eta(a-d), \eta^f(a^f - d^f)\right]\right\} \\ &+ (1-\xi)\sigma\hat{\pi}_{\star}\left\{u(q^{\star}) + W^b\left[\eta(a-d, \eta^f(a^f - d^f)\right]\right\} \\ &+ \xi\left(1-\sigma\hat{\pi}\right)W^b\left(\eta a, \eta^f a^f\right) \\ &+ (1-\xi)\left(1-\sigma\hat{\pi}_{\star}\right)W^b\left(\eta a, \eta^f a^f\right). \end{aligned} \tag{A1}$$

Since DM sellers are do not face a probability of being relocated, the seller's problem is the same as that in the main paper. The difference now is that in each period a Home (Foreign) DM-seller may end up meeting with either a Home or a Foreign DM-buyer. Nevertheless, the seller's optimal strategy will still be the same given any buyer's offer.

After some algebra, and as a consequence of quasilinearity of all agent's per-period payoff functions, in the last stage of the counterfeiting-bargaining game (in the Home DM(t)), the seller maximizes expected profit by playing a mixed strategy π such that

$$\pi \in \left\{\arg\max_{\pi_{\star}\in[0,1]} \pi_{\star}\left[-c(q) + \phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right)\right]\right\}, \tag{A2}$$

taking as given a buyer's offer ω and the seller's beliefs about the buyer's counterfeiting probabilities, $(\hat{\eta}, \hat{\eta}^f)$. One can also write down a corresponding problem for a Foreign DM seller where the mixed strategy is denoted by $\pi_{\star}$:

$$\pi_{\star} \in \left\{\arg\max_{\pi\in[0,1]} \pi\left[-c(q^{\star}) + \phi^f\left(\hat{\eta}d/e + \hat{\eta}^f d^f\right)\right]\right\}. \tag{A3}$$

Note that since the law of one price holds in terms of the CM good, the Foreign seller's problem equivalently yields,

$$\pi_{\star} \in \left\{\arg\max_{\pi\in[0,1]} \pi\left[-c(q^{\star}) + \phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right)\right]\right\}. \tag{A3'}$$

In the penultimate stage, in CM($t-1$), the buyer chooses the counterfeiting lottery (η, η^f) to solve the following cost-minimization problem:

$$\max\Big\{-\kappa(1-\eta) - \kappa^f(1-\eta^f) - \xi\beta\sigma\hat{\pi}\left(\eta\phi d + \eta^f\phi e d^f\right) \\ -(1-\xi)\beta\sigma\hat{\pi}_\star\left(\eta\phi_{+1}d + \eta^f\phi e d^f\right) \\ -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f\Big\}, \tag{A4}$$

given his earlier commitment ω and his beliefs about a Home (or Foreign) seller's acceptance probability $\hat{\pi}$ (or $\hat{\pi}_\star$).

The buyer chooses a TIOLI offer at the beginning of the game to maximize his payoff given his belief functions $(\hat{\eta}, \hat{\eta}^f, \hat{\pi}, \hat{\pi}_\star)$ about the continuation play. The buyer commits to an optimal plan of contingent offers, ω and $\omega^\star$, to maximize

$$-\kappa(1-\hat{\eta}) - \kappa^f\left(1-\hat{\eta}^f\right) + \xi\beta\sigma\hat{\pi}\left[u(\hat{q}) - \phi\left(\hat{\eta}\hat{d} + \hat{\eta}^f e\hat{d}^f\right)\right] \\ -(1-\xi)\beta\sigma\hat{\pi}_\star\left[u(q^\star) - \phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right)\right] \\ -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f. \tag{A5}$$

Given these descriptions, we may proceed directly to a generalization of the private-information bargaining game's description (as in Proposition 1).

PROPOSITION 5: *An equilibrium of the counterfeiting-bargaining game is such that*

1) *Each Home seller accepts with probability* $\hat{\pi} = \pi = 1$ *and each Foreign seller accepts with probability* $\hat{\pi}_\star = \pi_\star = 1$*;*

2) *Each buyer does not counterfeit:*

$$(\hat{\eta}, \hat{\eta}^f, \hat{\eta}, \hat{\eta}^f) = (\eta, \eta^f, \eta, \eta^f) = (1, 1, 1, 1);$$

and

3) *Each buyer's travel-contingent TIOLI offer* $\omega := (q, d, d^f)$ *and* $\omega^\star := (q^\star, d, d^f)$

attains

(A6)

$$\max\left\{\left(\frac{\phi_{-1}}{\phi}-\beta\right)\phi m-\left(\frac{\phi_{-1}e_{-1}}{\phi e}-\beta\right)\phi e m^f+\xi\beta\sigma\left[u(q)-\phi\left(d+ed^f\right)\right]\right.$$
$$+(1-\xi)\beta\sigma\left[u(q^\star)-\phi\left(d+ed^f\right)\right]$$

s.t.

$$(\zeta):\quad \phi\left(d+ed^f\right)-c(q)\geq 0,$$
$$(\zeta^f):\quad \phi\left(d+ed^f\right)-c(q^\star)\geq 0,$$
$$(\nu):\quad 0\leq d,$$
$$(\mu):\quad d\leq m,$$
$$(\nu^f):\quad 0\leq d^f$$
$$(\mu^f):\quad d^f\leq m^f,$$
$$(\lambda):\quad \phi m\leq\frac{\kappa}{\phi_{-1}/\phi-\beta(1-\sigma)}$$
$$\left.(\lambda^f):\quad \phi e m^f\leq\frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e-\beta(1-\sigma)}\right\}$$

and the equilibrium is unique.

One can also write a symmetric characterization for the corresponding Foreign country. The intuitive explanation of this resulting characterization, and in particular, its endogenous liquidity constraints are similar to that in the main paper. Below we provide the detailed proof of this result. Note that by setting $\xi = 1$, we also have the proof of Proposition 1. A similar proof for this special case discussed in the main paper can also be found in our working paper Gomis-Porqueras, Kam and Waller (2015).

PROOF:

Denote the maximum value of the program in (A6), when $\hat{\pi} = \pi = 1$, $\hat{\pi}_\star = \pi_\star = 1$, and $(\hat{\eta}, \hat{\eta}^f) = (\eta, \eta^f) = (1, 1)$, as $(U^b)^*$. The aim is to show that an equilibrium strategy yields the same value as $(U^b)^*$, and it satisfies the characterization in Proposition 5 (Case 1); and that any other candidate strategy under beliefs $\hat{\pi} \neq 1$, $\hat{\pi}_\star \neq 1$, and/or $(\hat{\eta}, \hat{\eta}^f) \neq (1, 1)$ will induce a buyer's valuation that is strictly less than $(U^b)^*$, and therefore cannot constitute an equilibrium (Cases 2-5).

Consider the subgame following an offer plan $(\omega, \omega_\star)$. Let $\rho(\chi, \chi^f)$ denote the joint probability measure on events $\{(\chi, \chi^f)\}$, where the pure actions over counterfeiting are $(\chi, \chi^f) \in \{0, 1\}^2$. Denote $P := 2^{\{0,1\}^2}$ as the power set of $\{0, 1\}^2$. By the definition of probability measures, it must be that $\sum_{\{z\}\in P} \rho(z) = 1$.

Consider the subgame where we have reached the seller's problem. At this stage, the event of a particular buyer going to the Foreign DM, or staying in the Home DM, is already realized. Therefore, without loss, let us focus on a DM-seller's problem in the Home DM. The seller's problem in (9) is equivalent to:

$$\text{(A7)}\qquad \pi \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \Bigg[\phi \Bigg([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \Bigg) - c(q) \Bigg] \right\}.$$

This is a linear programming problem in π, given the seller's rational belief system $\hat{\rho}$ and buyer's offer ω. Thus the seller's best response satisfies:

$$\text{(A8)}\qquad \left(\phi \left([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \right) - c(q) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \right) \Rightarrow \left(\pi(\omega) \begin{cases} = 1 \\ = 0 \\ \in [0,1] \end{cases} \right).$$

We can also write down a similar characterization for an ex-post Foreign DM seller who meets a Home buyer. The conclusion would be the same.

Now consider the preceding stage, where a buyer has already committed to some travel-contingent plan of offer $(\omega, \omega_\star)$, i.e., before the buyer knows which DM (Home or Foreign) he has to travel to. At this stage, a buyer is deciding on counterfeiting choices and has to compare alternative payoffs from mixing over pure counterfeiting strategies. We will need to define some convenient notation here: Let $U^b_{\{z\}} \equiv U^b[\omega, \{z\}, \hat{\pi} | \mathbf{s}_{-1}, \phi, e]$ denote the buyer's expected payoff from *realizing* pure actions (χ^h, χ^f), given offer ω and rational belief system $\hat{\pi}, \hat{\pi}_\star \in [0,1]$, where $\{z\} \in P$. From the pure counterfeiting strategy induced payoffs, we can construct those arising from non-degenerating mixed strategies below.

We have the following possible payoffs following each non-empty (pure-strategy) counterfeiting event $\{z\} \in P$:

(A9)

$$U^b_{\{(0,0)\}} = -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f$$
$$+ \xi\beta\sigma\hat{\pi}\left[u(q) - \phi\left(d + ed^f\right)\right] + (1-\xi)\beta\sigma\hat{\pi}_\star\left[u(q^\star) - \phi\left(d + ed^f\right)\right];$$

(A10)

$$U^b_{\{(0,1)\}} = -\kappa^f - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d$$
$$+ \xi\beta\sigma\hat{\pi}\left[u(q) - \phi d\right] + (1-\xi)\beta\sigma\hat{\pi}_\star\left[u(q^\star) - \phi d\right];$$

(A11)

$$U^b_{\{(1,0)\}} = -\kappa - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f$$
$$+ \xi\beta\sigma\hat{\pi}\left[u(q) - \phi e d^f\right] + (1-\xi)\beta\sigma\hat{\pi}_\star\left[u(q^\star) - \phi e d^f\right];$$

(A12)

$$U^b_{\{(1,1)\}} = -\kappa^f - \kappa + \xi\beta\sigma\hat{\pi}u(q) + (1-\xi)\beta\sigma\hat{\pi}_\star u(q^\star)$$

Observe that

$$U^b_{\{(0,1)\}} + U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} + U^b_{\{(1,1)\}}. \tag{A13}$$

There are five cases to consider.

Case 1. Suppose there is a set of candidate equilibria such that $\rho(0,0) = 1$ and $\rho(z) = 0$, for all $\{z\} \in P$ and $z \neq (0,0)$. Then, we have $U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$. Since $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and $U^b_{\{(0,0)\}} > U^b_{\{(0,1)\}}$, then, from (A9)-(A12) we can derive that

$$\phi m < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}, \tag{A14}$$

and,

$$\phi e m^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}, \tag{A15}$$

where $m = d$ and $m^f = d^f$. (Since inflation is costly, and since utility is quasilinear, the portfolio (m, m^f) is such that the balance carried into the DM is exactly equivalent to the value of payments offered by the buyer's plan.)

The interpretation from (A14) and (A15) is that the liquidity constraints on either currencies are slack. Therefore the buyer's expected payoff in this case can be evaluated from (A9). If $\hat{\pi} < 1$ or $\hat{\pi}_\star < 1$, then from the Home or Foreign seller's decision rule (A8), or its Foreign equivalent, we can deduce $\omega \equiv (q, d, d^f)$ and $\omega_\star \equiv (q^\star, d, d^f)$ must be such that the seller's participation/incentive constraint binds. That is, if it is the Home seller, then

$$c(q) = \phi(d + ed^f). \tag{A16}$$

If it is the Foreign seller who will be the Home buyer in the DM, then

$$c(q^\star) = \phi(d + ed^f). \tag{A17}$$

Since (A16) or (A17) must hold ex post, all we need to do is verify the buyer's payoff. Since, the buyer's liquidity constraints (A14) and (A15) do not bind at $\hat{\pi} < 1$ and/or $\hat{\pi}_\star < 1$, a small increment in either payment offered, d or d^f if the buyer shops at Home, and $d^\star$ or $d^{f,\star}$ if the buyer shops abroad, relaxes (A16) and (A17). This serve to raise $\hat{\pi}$ and $\hat{\pi}_\star$—i.e., the buyer's rational belief that sellers in either contingency will be more likely to accept his offer—and thus the buyer's payoff (A9). The maximal payoff to the buyer, keeping the seller in participation, is when the sellers' best responses are consistent with the buyer's belief system: $\pi = \hat{\pi} = 1$ and $\pi_\star = \hat{\pi}_\star = 1$, and the offer plan $(\overline{\omega}, \overline{\omega}_\star)$ is such that

$$\begin{aligned}\overline{U}^b \equiv & U^b_{\{(0,0)\}}[\overline{\omega}, \overline{\omega}_\star | \pi = \hat{\pi} = \pi_\star = \hat{\pi}_\star = 1] \\ = & \sup_{\omega} \Bigg\{ U^b_{\{(0,0)\}}[\overline{\omega}, \overline{\omega}_\star | \pi = \hat{\pi} = \pi_\star = \hat{\pi}_\star = 1] : \phi m \leq \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1-\sigma)}, \\ & \phi e m^f \leq \frac{\kappa^f}{\frac{\phi_{-1} e_{-1}}{\phi e} - \beta(1-\sigma)}, c(q) \leq \phi(d + ed^f), c(q^\star) \leq \phi(d + ed^f) \Bigg\}.\end{aligned}$$

Then it is easily verified that this maximal value coincides with the maximum value of the program given in (A6) in Proposition 5, i.e. $\overline{U}^b = (U^b)^*$, since the payoff function is continuous, and the constraints also define a nonempty, compact subset of the feasible set. Since a seller has no incentive to deviate from $\pi = 1$ or $\pi_\star = 1$, then a behavior strategy $(\omega, \omega_\star, (\eta, \eta^f), (\pi, \pi_\star)) = \langle \overline{\omega}, \overline{\omega}_\star, (1,1), (1,1) \rangle$ inducing the TIOLI payoff $\overline{U}^b$ is a PBE.

Case 2. Note that in any equilibrium, a seller will never accept an offer if $\rho(1,1) = 1$, and, a buyer will never counterfeit both assets with probability 1—counterfeiting for sure costs $\kappa + \kappa^f$ and the buyer gains nothing. Therefore, $\rho(1,1) < 1$ is a necessary condition for an equilibrium in the subgame following ω. Likewise, all unions of disjoint events with this event of counterfeiting all assets—i.e. $\{(\chi, \chi^f)\} \in \{(0,1)\} \cup \{(1,1)\}$ or $\{(\chi, \chi^f)\} \in \{(1,0)\} \cup \{(1,1)\}$—such that $\rho(0,1) + \rho(1,1) = 1$ or $\rho(1,0) + \rho(1,1) = 1$, respectively, cannot be on any equilibrium path.

Case 3. Suppose instead we have equilibria in which $\rho(0,0) + \rho(1,0) = 1$, $\rho(1,0) \neq 0$, and $\rho(1,1)+\rho(0,1) = 0$, so $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} > \max\{U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$.

Given this case, and from (A13), we have $U^b_{\{(0,1)\}} = U^b_{\{(1,1)\}}$. From $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}}$, and (A9) and (A11), respectively, we have:

$$\tilde{\pi} \equiv \xi\hat{\pi} + (1-\xi)\hat{\pi}_\star = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}, \tag{A18}$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1-\sigma\tilde{\pi})}. \tag{A19}$$

If $\hat{\pi} < 1$, then from the Home seller's decision rule (A8) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$\begin{aligned} c(q) &= \phi[(1-\rho(1,0)-\rho(1,1))d + (1-\rho(0,1)-\rho(1,1))ed^*] \\ &= \phi[(1-\rho(1,0))d + ed^*]. \end{aligned} \tag{A20}$$

Likewise, if $\hat{\pi}_\star < 1$, then a Foreign seller's participation constraint will be binding.

The buyer's payoff can be evaluated from (A11). If $\hat{\pi} < 1$ and/or $\hat{\pi}_\star < 1$, then reducing d infinitesimally will increase $\tilde{\pi}$ in (A18), and this uniformly increases the buyer's payoff in (A11). The buyer would like to attain $\hat{\pi} = \hat{\pi}_\star = 1$ since the sellers' participation constraint will still be respected:

$$c(q) \leq \phi[(1-\rho(1,0))d + ed^f], \qquad c(q^\star) \leq \phi[(1-\rho(1,0))d + ed^f]. \tag{A21}$$

Let the maximum of the buyer's TIOLI value (A11) such that the constraints (A18), (A19) and (A21) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1,0) \neq 0$, it is easily verified that

$$\begin{aligned} (U^b)^\dagger &< U^b_{\{(0,0)\}}[\overline{\omega}|\pi = \hat{\pi} = \pi_\star = \hat{\pi}_\star = 1; \rho(1,0) = 0] \\ &= \sup_{\omega, \rho(1,0)} \{U^b_{\{(1,0)\}}|(A18), (A19), (A21)\} = (U^b)^*, \end{aligned}$$

in which the last equality is attained when $\rho(1,0) = 0$. This contradicts the claim that $\rho(0,0) + \rho(1,0) = 1$ and $\rho(1,0) \neq 0$ is a component of a PBE.

Case 4. Suppose there are equilibria consisting of $\rho(0,0) + \rho(0,1) = 1$ with $\rho(0,1) \neq 0$, and $\rho(1,0) = \rho(1,1) = 0$. The buyer's payoff is such that $U^b_{\{(0,1)\}} =$

$U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(1,1)\}}\}$. Given this assumption, we have from (A13) that $U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. From (A9) and (A10), we can derive

$$\text{(A22)} \qquad \tilde{\pi} \equiv \xi\hat{\pi} + (1-\xi)\hat{\pi}_\star = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi e d^f}{\beta\sigma\phi e d^f}.$$

From the case that $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and (A9)-(A11), we have:

$$\text{(A23)} \qquad \phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1-\sigma\tilde{\pi})}.$$

The buyer's payoff can be evaluated from (A10). If $\hat{\pi} < 1$ or $\hat{\pi}_\star < 1$, from (A8), we can deduce that the Home seller's participation constraint is binding. Again, the same goes for the Foreign seller. If $\tilde{\pi} < 1$, then reducing d^f infinitesimally will increase these acceptance probabilities in (A22) and thus $\tilde{\pi}$; and this serves to increase the buyer's payoff in (A10). The buyer would like to attain $\hat{\pi} = \hat{\pi}_\star = 1$ since the sellers' participation constraint will still be respected at that point:

$$\text{(A24)} \qquad c(q) \leq \phi[d + (1-\rho(0,1))ed^f], \qquad c(q^\star) \leq [\phi d + e(1-\rho(1,0))d^f].$$

Let the maximum of the buyer's TIOLI value (A10) such that the constraints (A22), (A23) and (A24) are respected, in this case be $(U^b)^{\dagger\dagger}$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^{\dagger\dagger} < U^b_{\{(0,0)\}}[\overline{\omega}|\pi = \hat{\pi} = \pi_\star = \hat{\pi}_\star = 1; \rho(0,1) = 0] = \sup_\omega\{U^b_{\{(0,1)\}}|(A22),(A23),(A24)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,1) = 0$. This contradicts the claim that $\rho(0,0) + \rho(0,1) = 1$ and $\rho(0,1) \neq 0$ is a component of a PBE.

Case 5. Suppose a candidate equilibrium is such that $\sum_{\{z\}\in P}\rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, and that $U^b_{\{(0,1)\}} = U^b_{\{(0,0)\}} = U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. Then from (A10) and (A12), and from (A11) and (A12), respectively, we can derive

$$\text{(A25)} \quad \tilde{\pi} \equiv \xi\hat{\pi} + (1-\xi)\hat{\pi}_\star = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi e d^f}{\beta\sigma\phi e d^f} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}.$$

If the payment offered (d, d^f) are such that $\hat{\pi} < 1$ or $\hat{\pi}_\star < 1$, then from the seller's decision rule (A8) we can deduce $\omega \equiv (q, d, d^f)$ and $\omega \equiv (q^\star, d, d^f)$ must be such that the Home seller's participation/incentive constraint binds:

$$\text{(A26)} \qquad c(q) = \phi[(1-\rho(1,0)-\rho(1,1))d + (1-\rho(0,1)-\rho(1,1))ed^f],$$

Likewise, for a Foreign seller. However, the buyer can increase his expected payoff in (A12) by reducing both (d, d^*), thus raising $\hat{\pi}$ and $\hat{\pi}_\star$ in (A25) while still ensuring that the sellers participate, until $\hat{\pi} = \hat{\pi}_\star = 1$, where

$$\text{(A27)} \qquad c(q) \leq \phi[(1-\rho(1,0)-\rho(1,1))d + (1-\rho(0,1)-\rho(1,1))ed^f].$$

Let the maximum of the buyer's TIOLI value (A12) such that the constraints (A25) and (A27) are respected, in this case be $(U^b)^\ddagger$.

However, since $\rho(1,0), \rho(0,1), \rho(1,1) \neq 0$, it is easily verified that $(U^b)^\ddagger < U^b_{\{(1,1)\}}[\overline{\omega}|\pi = \hat{\pi} = \pi_\star = \hat{\pi}_\star = 1; \rho(0,0) = 1] = \sup_\omega\{U^b_{\{(1,1)\}}|(A25), (A24)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,0) = 1$. This contradicts the claim that $\sum_{\{z\}\in P} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, is a component of a PBE.

Summary. From Cases 1 to 5, we have shown that the only mixed-strategy Nash equilibrium in the subgame following an offer ω must be one such that $\langle \rho(0,0), \pi \rangle = \langle 1, 1 \rangle$, and that the offer ω satisfies the program in (A6) in Proposition 5.

Finally, since $u(.)$ and $-c(.)$ are strictly concave functions and the inequality constraints in program (A6) define a convex feasible set, the program (A6) has a unique solution.

Proof of Proposition 2

For $\lambda = \lambda^f > 0$ the solution to the buyer's problem is given by

$$\text{(B1)} \qquad \phi m = \phi e m^f = \frac{\kappa}{\Pi - \beta(1-\sigma)},$$

$$\text{(B2)} \qquad c(\hat{q}) = \frac{2\kappa}{\Pi - \beta(1-\sigma)},$$

$$\text{(B3)} \qquad \zeta = \beta\sigma \frac{u'(\hat{q})}{c'(\hat{q})},$$

$$\text{(B4)} \qquad \mu = \Pi - \beta,$$

$$\text{(B5)} \qquad \lambda = \beta\sigma \left[\frac{u'(\hat{q})}{c'(\hat{q})} - 1\right] + \beta - \Pi.$$

Equation (B1) shows the real balances are the same and equal to the liquidity bounds. Given these solutions, (B2) yields the solution for $\hat{q}$. The last three are the solutions for the multipliers. Finally, to solve for m and m^f we use the market clearing conditions

$$\begin{aligned} M &= m + \tilde{m} \\ M^f &= m^f + \tilde{m}^f. \end{aligned}$$

In an equilibrium where the two countries are identical in every respect, buyers in each country face the same liquidity constraints. Thus we have $\phi m = \phi \tilde{m} =$

$\kappa / \left(\Pi - \beta(1-\sigma)\right)$. It then follows that $m = M/2, m^f = M^f/2$. Substituting these expressions into (B1) yields

$$\phi = \frac{2\kappa}{\left[\Pi - \beta(1-\sigma)\right] M}, \quad \text{and,} \quad e = \frac{M}{M^f}. \tag{B6}$$

The only thing that is left to do is choose parameter values such that the solutions are valid. From (B5) we need $\hat{q} < q^*$. As a result, the solution from (B2) for $\hat{q}$ must satisfy this restriction. Since q^* is independent of κ and Π, it is clear that this condition is satisfied for a sufficiently small cost of counterfeiting and/or a sufficiently high inflation rate.

Proof of Proposition 3

For this equilibrium the solution to the buyer's problem yields

$$\begin{aligned} \Pi - \beta &= \beta\sigma \left[\frac{u'(\hat{q})}{c'(\hat{q})} - 1\right], \qquad c(\hat{q}) = \phi m + \phi e m^f, \\ \zeta &= \beta\sigma \frac{u'(\hat{q})}{c'(\hat{q})}, \qquad \text{and,} \qquad \mu = \Pi - \beta. \end{aligned}$$

In this case, $\hat{q}$ is pinned down by the first equation. The second equation then gives us the total real value of the buyer's currency portfolio. Using the second equation for both countries, in conjunction with the currency market clearing conditions, we obtain

$$\phi\left(M + eM^f\right) = 2c(\hat{q}),$$

which gives us one equation in two unknowns, ϕ and e. Thus, for any value of ϕ there is a nominal exchange rate that solves this expression. As a result, the nominal exchange rate is indeterminate.

Proportional Bargaining Trading Protocol

Let us consider an alternative trading protocol: The proportional bargaining solution of Kalai and Smorodinsky (1975). The buyer now proposes $\omega := (q_{+1}, d_{+1}, d^f_{+1})$ and commit to ω before making any (C, N) decisions in CM(t). Note that the underlying private information problem faced by the seller is still present under this new trading protocol. This is the case as the seller still can not differentiate between genuine and counterfeited currencies.

The buyers' payoff under this new trading protocol is given by

(D1)
$$W^b\left(m, m^f\right) = \max_{\omega, m_{+1}, m^f_{+1}, \eta(\omega), \eta^f(\omega)} \Big\{ \mathcal{U}(C^*) - \phi(m + em^f + \tau) + \beta W^b(0,0) - \left(\frac{\phi}{\phi_{+1}} - \beta\right)\phi_{+1}m_{+1} - \left(\frac{\phi e}{\phi_{+1}e_{+1}} - \beta\right)\phi_{+1}e_{+1}m^f_{+1} - \kappa\left[1 - \eta(\omega)\right] - \kappa^f\left[1 - \eta^f(\omega)\right] + \beta\sigma\hat{\pi}_{+1}(\omega)\left[u(q_{+1}) - \eta(\omega)\phi_{+1}d_{+1} - \eta^f(\omega)\phi_{+1}e_{+1}d^f_{+1}\right]\Big\},$$

while the seller's payoff is given by

$$W^s\left(m_s, m^f_s\right) = \mathcal{U}(C^*) - \phi(m_s + em^f_s) + \beta W^s(0,0) + \max_{\pi_{+1}(\omega)}\left[\beta\sigma\pi_{+1}(\omega)\left(-c(q_{+1}) + \hat{\eta}(\omega)\phi_{+1}d_{+1} + \hat{\eta}^f(\omega)\phi_{+1}e_{+1}d^f_{+1}\right)\right], \tag{D2}$$

where the only difference with respect to the buyer's TIOLI is the payment (d, d^f) and DM quantitate traded (q).

The seller plays a mixed strategy π to maximize her expected payoff, which is given by

$$\pi(\omega) \in \left\{\arg\max_{\pi\in[0,1]} \pi(\omega)\left[-c(q) + \phi\left(\hat{\eta}(\omega)d + \hat{\eta}^f(\omega)ed^f\right)\right]\right\}, \tag{D3}$$

taking the posted offer and the buyer's belief about the seller's best response $\hat{\pi}(\omega)$ as given. To determine the exact terms of trade, the buyer solves the following problem:

$$\max_{\eta(\omega), \eta^f(\omega)\in[0,1]} \Big\{ -\kappa\left[1 - \eta(\omega)\right] - \kappa^f\left[1 - \eta^f(\omega)\right] + \beta\sigma\hat{\pi}_{+1}[\omega)\left(\eta(\omega)\phi_{+1}d_{+1} + \eta^f(\omega)\phi_{+1}e_{+1}d^f_{+1}\right] - \left(\frac{\phi}{\phi_{+1}} - \beta\right)\phi_{+1}m_{+1} - \left(\frac{\phi e}{\phi_{+1}e_{+1}} - \beta\right)\phi_{+1}e_{+1}m^f_{+1}\Big\}. \tag{D4}$$

Taking the counterfeiting probabilities as given, the buyer then chooses the proportional bargaining offer at the beginning of the game to maximize his payoff given the conjecture $(\hat{\eta}(\omega), \hat{\eta}^f(\omega), \hat{\pi}(\omega))$ of the continuation play. Consequently, the buyer commits to an optimal offer $\omega \equiv (\hat{q}, \hat{d}, \hat{d}^f)$ solving

(D5)
$$\max \Big\{ -\kappa\left(1-\hat{\eta}\left(\omega\right)\right) - \kappa^f \left(1-\hat{\eta}^f\left(\omega\right)\right) + \beta\sigma\hat{\pi}\left[u(\hat{q}) - \phi\left(\hat{\eta}\left(\omega\right)\hat{d} + \hat{\eta}^f\left(\omega\right)e\hat{d}^f\right)\right] - \left(\frac{\phi}{\phi_{+1}} - \beta\right)\phi_{+1}m_{+1} - \left(\frac{\phi e}{\phi_{+1}e_{+1}} - \beta\right)\phi_{+1}e_{+1}m^f_{+1} \Big\},$$

where under proportional bargaining we have that the buyer and seller surpluses satisfy[15]

$$B_s \equiv u(\hat{q}) - \phi\left(\hat{\eta}\left(\omega\right)\hat{d} + \hat{\eta}^f\left(\omega\right)e\hat{d}^f\right) = \frac{\theta}{1-\theta}\left[\phi\left(\hat{\eta}\hat{d} + \hat{\eta}^f e\hat{d}^f\right) - c(\hat{q})\right] \equiv S_s > 0.$$

This contrasts with TIOLI, where the seller's surplus is $\phi\left(\hat{\eta}\left(\omega\right)\hat{d} + \hat{\eta}^f\left(\omega\right)e\hat{d}^f\right) - c(\hat{q}) = 0$.

PROPOSITION 6: *An equilibrium of the counterfeiting-bargaining game is such that*

1) *Each seller accepts with probability* $\hat{\pi}\left(\omega\right) = \pi(\omega) = 1$*;*

2) *Each buyer does not counterfeit:* $\left(\hat{\eta}\left(\omega\right), \hat{\eta}^f\left(\omega\right)\right) = (\eta(\omega), \eta^f(\omega)) = (1, 1)$*; and*

[15]Upon a successful DM match between a buyer and a seller, the maximum total surplus is $T_s = u(q) - c(q)$. Under the proportional bargaining trading protocol, the buyer's (B_s) and seller's (S_s) maximal surpluses are a fraction of the total surplus and are given by:

$$B_s = \theta[u(q) - c(q)] \quad S_s = (1-\theta)[u(q) - c(q)]$$

where $\theta \in [0, 1]$ represents the bargaining power of the buyer.

3) Each buyer's TIOLI offer ω is such that

(D6)

$$\omega \in \left\{ arg\max_{\omega} \left[-\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \right.\right.$$
$$\left. + \beta\sigma\left[u(q) - \phi\left(d + ed^f\right)\right]\right] \qquad s.t.$$
$$(\zeta): \quad u(q) - \phi(d + ed^f) = \frac{\theta}{1-\theta}\left(\phi(d + ed^f) - c(q)\right) > 0,$$
$$(\nu): \quad 0 \le d,$$
$$(\mu): \quad d \le m,$$
$$(\nu^f): \quad 0 \le d^f,$$
$$(\mu^f): \quad d^f \le m^f,$$
$$(\ \lambda): \quad \phi d \le \frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)}$$
$$\left.(\lambda^f): \quad \phi e d^f \le \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)}\right\}$$

and the equilibrium is unique.

Since the buyer does not counterfeit, we have that $z = z^f = 0$ and $d = m$, $d^f = m^f$ for $\phi_{-1}/\phi > \beta$ and $\phi_{-1}e_{-1}/\phi e > \beta$. Furthermore, there is no reason to incur the costs of acquiring genuine currencies and then make an offer that the seller will reject. So $\hat{\pi}(\omega) = 1$. If the buyer counterfeits then the marginal cost of producing a counterfeit is zero while the marginal cost of acquiring genuine currency is positive. In this case $m, m^f = 0$. Similarly, there is no reason to incur the cost of counterfeiting if the offer is rejected by the seller. So, again set $\hat{\pi} = 1$. Again, slightly abusing notaiton, let $W^b\left[\eta, \eta^f\right]$ denote the value function for the pure strategies $\eta(\omega), \eta^f(\omega) \in \{0, 1\}$ where $\eta(\omega) = \eta^f(\omega) = 1$ represents "no counterfeiting of currencies." For no counterfeiting to be an optimal strategy, it must be the case $W^b[1,1] \ge W^b[0,1], W^b[1,0], W^b[0,0]$. Using (D1), $W^b[1,1] \ge W^b[0,1]$ reduces to

$$-\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\left[u(q) - \phi\left(d + ed^f\right)\right]$$
$$\ge -\kappa - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\left[u(q) - \phi e d^f\right],$$

which collapses to the second to the last constraint in (D6). A similar exercise for $W^b[1,1] \ge W^b[1,0]$ yields the last constraint in (D6). Finally, it is straightforward to show that if the previous two conditions are satisfied, then $W^b[1,1] \ge$

$W^b[0,0]$ holds as well.

Relative to the TIOLI case, the only difference in the monetary equilibrium is the ζ constraint as the seller always obtains some strictly positive surplus. It is worth emphasising, however, that the same underlying mechanism to determine the nominal exchange rate follow the same logic as in the TIOLI presented in the main text of the paper. More precisely, the nominal exchange rate can not be determined unless the liquidity constraints bind.

Let us now focus on the stationary monetary equilibrium associated with an economy with $\Pi = \Pi^f > \beta$ and $\kappa = \kappa^f$. This implies that $d = m$ and $d^f = m^f$. The DM-buyer's problem can be written as follows:[16]

$$\text{(D7)} \quad \mathcal{U}(C^*) - \phi(m_{-1} + e_{-1}m^f_{-1} + \tau_{-1}) + \max_q \Bigg\{ -(\Pi - \beta)\left[(1-\theta)u(q) + \theta c(q)\right] + \beta\sigma\theta\left[u(q) - c(q)\right] : \phi m \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)}, \text{ and, } \phi e m^f \leq \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)} \Bigg\},$$

where the buyer's total payment is such that

$$\text{(D8)} \qquad \phi(d + ed^f) = \phi(m + em^f) = (1-\theta)u(q) + \theta c(q).$$

The corresponding first order condition is then given by

$$-(\Pi - \beta)\left[(1-\theta)u'(q) + \theta c'(q)\right] + \beta\sigma\theta\left[u'(q) - c'(q)\right] - (\lambda + \lambda^f)\left[(1-\theta)u'(q) + \theta c'(q)\right] = 0.$$

We can now establish a result analogous to Proposition 2 presented in the main text under TIOLI, but now for the case of proportional bargaining. In particular we have that

PROPOSITION 7: *If* $\lambda = \lambda^f > 0$, *then* $e = M/M^f$.

PROOF:

For $\lambda = \lambda^f > 0$ we have that the various payments are such that

$$\phi m = \phi e m^f = \frac{\kappa}{\Pi - \beta(1-\sigma)}.$$

Using the total payment equation (8), the quantity traded in DM, q, satisfies the

[16] Here we have substituted the various constraints so we can write the total payment in terms on DM output and imposed that buyers spend all their currency as holding it is costly.

following implicit equation

$$(1-\theta)u(q)+\theta c(q)=\frac{2\kappa}{\Pi-\beta(1-\sigma)}.$$

Now using the first order condition of the DM-buyer's problem, we have that the Lagrange multiplier λ is given by

$$\lambda=\frac{\beta\sigma\theta}{2}\frac{u'(q)-c'(q)}{(1-\theta)u'(q)+\theta c'(q)}-\frac{(\Pi-\beta)}{2}.$$

Finally, to solve for m and m^f we use the market clearing conditions

$$\begin{aligned} M &= m+\tilde{m} \\ M^f &= m^f+\tilde{m}^f \end{aligned}$$

where $\tilde{m}$ denotes the foreign demand of the domestic currency.

In an economy where both countries are identical, buyers in each country face the same liquidity constraints. Thus we have $\phi m=\phi\tilde{m}=\kappa/\left(\Pi-\beta(1-\sigma)\right)$. It then follows that $m=M/2, m^f=M^f/2$. Substituting these expressions we have that

$$\phi=\frac{2\kappa}{\left[\Pi-\beta(1-\sigma)\right]M} \quad \text{and} \quad e=\frac{M}{M^f}. \tag{D9}$$