

## ONLINE APPENDIX

Paper: Optimal Sovereign Default

Authors: Klaus Adam and Michael Grill

American Economic Journal: Macroeconomics

## C1. First Order Equilibrium Conditions

This appendix derives the necessary and sufficient first order conditions for problem (7). Using equation (8) to express beginning-of-period wealth, the problem is given by

$$\max_{\{b_t, a_t \geq 0, \tilde{k}_t \geq 0, \tilde{c}_t \geq 0\}} E_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t)$$

s.t. :

$$\begin{aligned} \tilde{c}_t &= z_t \tilde{k}_{t-1}^\alpha + (1-d)k_{t-1} + b_{t-1} + (1-\lambda)a_{t-1}(z_t) - \bar{c} - \tilde{k}_t - \frac{1}{1+r}b_t - p'_t a_t \\ z_{t+1} \tilde{k}_t^\alpha + (1-d)k_t + b_t + (1-\lambda)a_t(z_{t+1}) &\geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \\ \tilde{w}_0 &= w_0, z_0 \text{ given,} \end{aligned}$$

We now formulate the Lagrangian  $\Lambda$ , letting  $\eta_t$  denote the multiplier on the budget constraint in period  $t$ ,  $v_t(z^j)$  the multiplier for the short-selling constraint on the Arrow security that pays off in state  $z^j$  in  $t+1$ , and  $\omega_t \in R^N$  the vector of multipliers associated with the natural borrowing limits for each possible realization of productivity in  $t+1$ , where  $w_t(z^j)$  denotes the entry of the vector pertaining to productivity state  $z^j$ :

$$\begin{aligned} \Lambda &= E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) \right. \\ &\quad + \beta^t \eta_t \left( -\tilde{c}_t + z_t \tilde{k}_{t-1}^\alpha + (1-d)k_{t-1} + b_{t-1} + (1-\lambda)a_{t-1}(z_t) - \bar{c} - \tilde{k}_t - \frac{1}{1+r}b_t - p'_t a_t \right) \\ &\quad + \beta^t \sum_{j=1}^N v_t(z^j) a_t(z^j) \\ &\quad \left. + \beta^t \sum_{j=1}^N \omega_t(z^j) \left( z^j \tilde{k}_t^\alpha + (1-d)k_t + b_t + (1-\lambda)a_t(z^j) - NBL(z^j) \right) \right] \end{aligned}$$

We drop the inequality constraints for  $\tilde{k}_t$  and  $\tilde{c}_t$ , as the Inada conditions guarantee an interior solution for these variables whenever  $\tilde{w}_0 > NBL(z^0)$ . Differentiating the La-

grangian with respect to the choice variables, one obtains

$$\begin{aligned} \tilde{c}_t &: u'(\tilde{c}_t) - \eta_t = 0 \\ b_t &: -\eta_t \frac{1}{1+r} + \beta E_t \eta_{t+1} + \sum_{n=1}^N \omega_t(z^n) = 0 \\ a_t(z^j) &: -\eta_t p_t(z^j) + \beta \pi(z^j|z^t) \eta_{t+1}(z^j)(1-\lambda) \\ &\quad + v_t(z^j) + \omega_t(z^j)(1-\lambda) = 0 \quad \text{for } j = 1, \dots, N \\ \tilde{k}_t &: -\eta_t + \beta E_t [(a_{z_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - d) \eta_{t+1}] + (\alpha \tilde{k}_t^{\alpha-1} + 1 - d) \sum_{j=1}^N \omega_t(z^j) z^j = 0 \end{aligned}$$

Using the FOC for consumption to replace  $\eta_t$  in the last three FOCs, delivers three Euler equations:

(C2a)

$$Bond : -u'(\tilde{c}_t) \frac{1}{1+r} + \beta E_t u'(\tilde{c}_{t+1}) + \sum_{j=1}^N \omega_t(z^j) = 0$$

$$Arrow : -u'(\tilde{c}_t) p_t(z^j) + \beta \pi(z^j|z_t) u'(\tilde{c}_{t+1}(z^j))(1-\lambda)$$

(C2b)

$$+ v_t(z^j) + \omega_t(z^j)(1-\lambda) = 0 \quad \text{for } j = 1, \dots, N$$

(C2c)

$$Capital : -u'(\tilde{c}_t) + \beta E_t [(a_{z_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - d) u'(\tilde{c}_{t+1})] + (\alpha \tilde{k}_t^{\alpha-1} + 1 - d) \sum_{j=1}^N \omega_t(z^j) z^j = 0$$

In addition, we have the following complementarity conditions for  $j = 1, \dots, N$ :

(C2d)

$$0 \leq a_t(z^j), v_t(z^j) \geq 0, \text{ one holds strictly}$$

(C2e)

$$0 \leq z^j \tilde{k}_t^\alpha + (1-d)k_t + b_t + (1-\lambda)a_t(z^j) - NBL(z^j), \omega_{t+1}(z^j) \geq 0, \text{ one holds strictly}$$

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the necessary and sufficient optimality conditions for problem (7).

## C2. Proof of Proposition 1

Consider some state-contingent beginning-of-period wealth profile  $w_t$  arising from some combination of bond holdings, default decisions and capital investment  $(F_{t-1}, D_{t-1}, \Delta_{t-1}, k_{t-1})$  in problem (5). We show below that one can generate the same state contingent beginning-

of-period wealth profile  $\tilde{w}_t = w_t$  in problem (7) by choosing  $\tilde{k}_{t-1} = k_{t-1}$  and by choosing an appropriate investment profile  $(a_{t-1}, b_{t-1})$ . Moreover, the same amount of funds are required to purchase  $(a_{t-1}, b_{t-1})$  in  $t - 1$  than to purchase  $(F_{t-1}, D_{t-1})$  when the default profile is  $\Delta_{t-1}$ . With the costs of financial investments generating a particular future payout profile being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (5b) and (7b) that a consumption path which is feasible in (5) is also feasible in (7).

To simplify notation we establish the previous claim for the case with  $N = 2$  productivity states only. The extension to more states is straightforward. Consider the following state-contingent initial wealth profile

$$\begin{pmatrix} w_t(z^1) \\ w_t(z^2) \end{pmatrix} = \begin{pmatrix} z^1 k_{t-1}^\alpha + (1-d)k_{t-1} + F_{t-1} - D_{t-1}(1 - (1-\lambda)\delta_{t-1}(z^1)) \\ z^2 k_{t-1}^\alpha + (1-d)k_{t-1} + F_{t-1} - D_{t-1}(1 - (1-\lambda)\delta_{t-1}(z^2)) \end{pmatrix}.$$

As is easily verified, this beginning-of-period wealth profile in problem (7) can be replicated by choosing  $\tilde{k}_{t-1} = k_{t-1}$  and by choosing the portfolio

$$(C3) \quad b_{t-1} = F_{t-1} - D_{t-1},$$

$$(C4) \quad a_{t-1} = \begin{pmatrix} D_{t-1}\delta_{t-1}(z^1) \\ D_{t-1}\delta_{t-1}(z^2) \end{pmatrix}$$

The funds  $f_{t-1}$  required to purchase and issue  $(F_{t-1}, D_{t-1})$  under the default profile  $\Delta_{t-1} = (\delta_{t-1}(z^1), \delta_{t-1}(z^2))$  are given by

$$f_{t-1} = \frac{1}{1+r} F_{t-1} - \frac{1}{1+R(z_{t-1}, \Delta_{t-1})} D_{t-1}$$

where the interest rate satisfies

$$\frac{1}{1+R(z_{t-1}, \Delta_{t-1})} = \frac{1}{1+r} \left( (1 - \delta_{t-1}(z^1))\pi(z^1|z_{t-1}) + (1 - \delta_{t-1}(z^2))\pi(z^2|z_{t-1}) \right).$$

The funds  $\tilde{f}_{t-1}$  required to purchase  $(b_{t-1}, a_{t-1})$  are given by

$$\tilde{f}_{t-1} = \frac{1}{1+r} (F_{t-1} - D_{t-1}) + \frac{1}{1+r} \left( \delta_{t-1}(z^1)\pi(z^1|z_{t-1}) + \delta_{t-1}(z^2)\pi(z^2|z_{t-1}) \right) G_{t-1}^S,$$

where we used the price of the Arrow security in (6). As can be easily seen  $\tilde{f}_{t-1} = f_{t-1}$ , as claimed. This completes the proof that a consumption path which is feasible in (5) is also feasible in (7). Since  $a \geq 0$ , the reverse is also true, because equations (C3) and (C4) can then be solved for values  $(D_{t-1}, F_{t-1}, \Delta_{t-1})$  satisfying  $D_{t-1} \geq 0$ ,  $F_{t-1} \geq 0$ , and  $\Delta_{t-1} \in [0, 1]^N$ , so that it is possible in problem (5) to obtain a portfolio with the

same contingent payouts.<sup>60</sup> Again, this portfolio has the same costs and thus admits the same consumption choices. This completes the equivalence proof.

### C3. Proof of Proposition 2

We look for fixed point solutions satisfying (13). In a first step, we derive the unique solution to (11) under the assumption that the NBLs in the constraints of (11) satisfy (13). We then show in a second step, that the NBLs implied by the objective function in (11) also satisfy (13), so that problem (11) defines a mapping from the set of NBLs satisfying (13) to the set of NBLs satisfying (13). As a last step, we show that this mapping generically has a unique solution.

Under the assumption that the NBLs in the constraints of (11) satisfy (13), we can show that the solution to (11) is given as follows. For any productivity state  $z^n$  ( $n = 1, \dots, N$ ), define the critical future productivity state  $n^*$

$$(C5) \quad n^* = \arg \max_{i \in \{1, \dots, N\}} i$$

$$s.t. \sum_{j=i}^N \pi(z^j | z^n) \geq 1 - \lambda.$$

As we establish below, the optimal choices in state state  $z^n$  are

$$\tilde{k}_t = \left( \frac{\alpha\beta}{1 + (1-d)\beta} \left( \frac{\left( \sum_{j=1}^{n^*} \pi(z^j | z^n) \right) - \lambda}{1 - \lambda} z^{n^*} + \frac{\sum_{j=n^*+1}^N \pi(z^j | z^n) z^j}{1 - \lambda} \right) \right)^{\frac{1}{1-\alpha}}$$

(C6)

$$\tilde{k}^n = \left( \frac{\alpha\beta}{1 + (1-d)\beta} \left( \frac{\left( \sum_{j=1}^{n^*} \pi(z^j | z^n) \right) - \lambda}{1 - \lambda} z^{n^*} + \frac{\sum_{j=n^*+1}^N \pi(z^j | z^n) z^j}{1 - \lambda} \right) \right)^{\frac{1}{1-\alpha}}$$

(C7)

$$b^n = NBL(z^{n^*}) - z^{n^*} (\tilde{k}^n)^\alpha - (1-d)\tilde{k}^n$$

(C8)

$$a^n(z^j) = 0 \text{ for } j \leq n^*$$

(C9)

$$a^n(z^j) = \frac{NBL(z^j) - z^j (\tilde{k}^n)^\alpha - (1-d)\tilde{k}^n - b^n}{(1-\lambda)} > 0 \text{ for } j > n^*$$

<sup>60</sup>Note that for  $a < 0$ , no such choices would exist, which shows that  $a \geq 0$  is required to obtain equivalence.

The NBLs in the objective function of (11) implied by the previous solution also satisfies (13). To see this, consider two productivity states  $z^n$  and  $z^m$  with  $n < m$  and the associated optimal choices. The optimal choices  $a^m, b^m, \tilde{k}^m$  for state  $z^m$  also satisfy the constraints of problem (11) for state  $z^n$ , i.e., are feasible choices in state  $z^n$ . Moreover, since  $a^m(z^j)$  is increasing in  $j$ , it follows from (12) that

$$\sum_{j=1}^N a^m(z^j) p^n(z^j) \leq \sum_{j=1}^N a^m(z^j) p^m(z^j),$$

i.e., the purchase of the risky assets  $a^m$  is cheaper in state  $z^n$  than in state  $z^m$ . Since the cost of investments in capital and bonds do not depend on the productivity state, this implies that the NBL in state  $z^n$  must be weakly laxer than the one in state  $z^m$ , as claimed.

We now show that (C6)-(C9) satisfy the necessary and sufficient first order conditions of problem (11). Letting  $\omega^n(z^j)$  denote the Lagrange multipliers for the first set of constraints in (11) and  $v^n(z^j)$  the multipliers for the second set of constraints, the first order necessary conditions are given by

$$(C10) \quad \tilde{k}^n : 1 + a(\tilde{k}^n)^{\alpha-1} \sum_{j=1}^N \omega^n(z^j) z^j + (1-d) \sum_{j=1}^N \omega^n(z^j) = 0$$

$$(C11) \quad b^n : \frac{1}{1+r} + \sum_{j=1}^N \omega^n(z^j) = 0$$

$$(C12) \quad a^n(z^j) : p^n(z^j) + \omega^n(z^j)(1-\lambda) + v^n(z^j) = 0.$$

We also have the constraints

$$(C13) \quad z^j (\tilde{k}^n)^\alpha + (1-d)\tilde{k}^n + b^n + (1-\lambda)a^n(z^j) - NBL(z^j) \geq 0, \omega^n(z^j) \leq 0, \text{ one holding strictly}$$

$$(C14) \quad a^n(z^j) \geq 0, v^n(z^j) \leq 0, \text{ one holding strictly}$$

Conditions (C12) and (C14) can equivalently be summarized as

$$(C15) \quad p^n(z^j) + \omega^n(z^j)(1-\lambda) \geq 0, a^n(z^j) \geq 0, \text{ one holding strictly}$$

so that the first order conditions are given by (C10),(C11), (C13) and (C15). Since the objective is linear and the constraint set convex, the first order conditions are necessary and sufficient. We now show that the postulated solution satisfies these first order conditions.

Since  $a^n(z^j) > 0$  for  $j > n^*$  for the conjectured solution, condition (C15) implies

$$\begin{aligned}\omega^n(z^j) &= -\frac{p^n(z^j)}{1-\lambda} \\ &= -\frac{1}{1+r} \frac{\pi(z^j|z^n)}{1-\lambda} < 0 \quad \text{for all } j > n^*\end{aligned}$$

We now conjecture (and verify later) that

$$(C16) \quad \omega^n(z^j) = 0 \quad \text{for all } j < n^*$$

$$(C17) \quad \omega^n(z^{n^*}) = -\frac{1}{1+r} - \sum_{j=n^*+1}^N \omega^n(z^j)$$

For the previous conjecture, equation (C11) holds by construction. Also, the second inequality of (C13) holds for all  $j \in \{1, \dots, N\}$  because we have

$$\sum_{j=n^*+1}^N \omega^n(z^j) = \frac{1}{1+r} \frac{\sum_{j=n^*+1}^N \pi(z^j|z^n)}{1-\lambda} < \frac{1}{1+r}$$

from the definition of  $n^*$ , so that  $\omega^n(z^{n^*}) < 0$ . Equations (C8) and (C9) then imply that (C15) hold. Furthermore, (C6) implies that (C10) holds. It thus only remains to show that the first inequality for (C13) also holds. For  $j \geq n^*$  this follows from (C9). For  $j < n^*$  this is also true because  $-NBL(z^j)$  is increasing as  $j$  falls under the assumed ordering for the NBLs in the constraints of (11),  $z^j (\tilde{k}^n)^\alpha + (1-d)\tilde{k}^n$  is equally increasing as  $j$  falls due to the assumed ordering of the productivity levels, and the first inequality of (C13) holds with equality for  $j = n^*$  due to (C7). As a result, the first inequality in (C13) holds strictly for  $j < n^*$ , justifying our conjecture (C16). This proves that all first order conditions hold for the conjectured solution (C6)-(C9).

Since the solution (C6)-(C9) is linear in the NBLs showing up in the constraints of (11), the minimized objective is also a linear function of these NBLs. The fixed point problem defined by (11) is thus characterized by a system of equations that is linear in the NBLs, which generically admits a unique solution. This completes the proof.

#### C4. Proof of Proposition 3

Suppose that in some period  $t$  and for some productivity state  $z^n$  ( $n \in \{1, \dots, N\}$ ), beginning-of-period wealth falls short of the limits implied by the marginally binding NBL, i.e.

$$(C18) \quad \tilde{w}_t(z^n) = NBL(z^n) - \varepsilon,$$

for some  $\varepsilon > 0$ . We then prove below that for at least one contingency  $z^j$  in  $t + 1$ , which can be reached from  $z^n$  in  $t$  with positive probability, it must hold that

$$(C19) \quad \tilde{w}_{t+1}(z^j) \leq NBL(z^j) - \varepsilon(1 + r),$$

such that along this contingency the distance to the marginally binding NBL is increasing at the rate  $1 + r > 1$  per period. Since the same reasoning also applies for future periods, and since the marginally binding NBLs assume finite values, this implies the existence of a path of productivity realizations along which wealth far in the future becomes unboundedly negative, such that any finite borrowing limit will be violated with positive probability.

It remains to prove that if (C18) holds in period  $t$  and contingency  $z^n$ , this implies that (C19) holds for some contingency  $z^j$  in  $t + 1$  ( $j \in \{1, \dots, N\}$ ) and that  $z^j$  can be reached from  $z^n$  with positive probability. Suppose for contradiction that

$$(C20) \quad \tilde{w}_{t+1}(z^h) > NBL(z^h) - \varepsilon(1 + r)$$

for all  $h \in \{1, \dots, N\}$  that can be reached from  $z^n$ , i.e. for which  $\pi(z^h|z^n) > 0$ . The cost-minimizing way to satisfy the constraints (C20) for all  $h$ , when replacing the strict inequality by a weak one, is to choose the solution (C6), (C8)- (C9) and  $b_t = NBL(z^{n_t}) - z^{n_t} k_t^\alpha - (1 - d)k_t - \varepsilon$ . It follows from the proof of proposition 2 that this holds true whenever the NBLs in the constraint satisfy the ordering (13). Achieving this requires  $NBL(z^n) - \varepsilon$  units of funds in  $t$ , which in turn implies that satisfying constraints (C20) with strict inequality requires strictly more funds than are available in  $t$ , whenever the state can be reached with positive probability. As a result, (C19) must hold for at least one  $j$  that can be reached from  $z^n$  with positive probability.

#### C5. Proof of Proposition 4

We first show that the proposed consumption solution (14) and investment policy (15) satisfy the budget constraint, that the inequality constraints  $a \geq 0$  are not binding, and that the NBLs are not binding either. Thereafter, we show that the remaining first order conditions of problem (7), as derived in appendix C.C1, hold.

We start by showing that the portfolio implementing (14) in period  $t = 1$  is consistent with the flow budget constraint and  $a \geq 0$ . The result for subsequent periods follows by induction. In period  $t = 1$  with productivity state  $z^n$ , beginning-of-period wealth under the investment policy (15) is given by

$$(C21) \quad \tilde{w}_1^n \equiv z^n (k^*(z_0))^\alpha + (1 - d)k^*(z_0) + b_0 + a_0(z^n)$$

To insure that consumption can stay constant from  $t = 1$  onwards we again need

$$(C22) \quad \tilde{c} = (1 - \beta)(\Pi(z^n) + \tilde{w}_1^n)$$

for all possible productivity realizations  $n = 1, \dots, N$ . This provides  $N$  conditions that can be used to determine the  $N + 1$  variables  $b_0$  and  $a_0(z^n)$  for  $n = 1, \dots, N$ . We also have the condition  $a_0(z^n) \geq 0$  for all  $n$  and by choosing  $\min_n a_0(z^n) = 0$ , we obtain one more condition that makes it possible to pin down a unique portfolio  $(b_0, a_0)$ . Note that the inequality constraints on  $a$  do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers  $v_1(z^n)$  in Appendix C.C1 are zero for all  $n = 1, \dots, N$ . It remains to be shown that the portfolio achieving (C22) is feasible given the initial wealth  $\tilde{w}_0$ . Using (C21) to substitute  $\tilde{w}_1^n$  in equation (C22) we obtain

$$\tilde{c} = (1 - \beta)(\Pi(z^n) + z^n (k^*(z_0))^\alpha + (1 - d)k^*(z_0) + b_0 + a_0(z^n)) \quad \forall n = 1, \dots, N.$$

Combining with (14) we obtain

$$\Pi(z^n) + z^n (k^*(z_0))^\alpha + (1 - d)k^*(z_0) + b_0 + a_0(z^n) = \Pi(z_0) + \tilde{w}_0$$

Multiplying the previous equation with  $\pi(z^n|z_0)$  and summing over all  $n$  one obtains

$$E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + b_0 + \sum_{n=1}^N \pi(z^n|z_0) a_0(z^n) = \Pi(z_0) + \tilde{w}_0.$$

Using  $\Pi(z_0) = -k^*(z_0) + \beta E_0 [z_1 (k^*(z_0))^\alpha] + \beta(1 - d)k^*(z_0) - \bar{c} + \beta E_0 [\Pi(z_1)]$  and (6) the previous equation delivers

$$(1 - \beta)E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha + (1 - d)k^*(z_0)] + b_0 + (1 + r)p'_0 a_0 = -k^*(z_0) + \tilde{w}_0 - \bar{c}$$

Using  $\beta = 1/(1 + r)$  this can be written as

$$(1 - \beta) \left( E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha + (1 - d)k^*(z_0)] + \frac{1}{\beta} p_0 a_0 + b_0 \right) + \frac{1}{1 + r} b_0 + p'_0 a_0 = -k^*(z_0) + \tilde{w}_0 - \bar{c} \quad (\text{C23})$$

From substituting (C21) into (C22), multiplying the result with  $\pi(z^n|z_0)$  and summing over all  $n$ , it follows that the terms in the first line of the previous equation are equal to

$$(1 - \beta) \left( E_0 [\Pi(z_1) + z_1 (k^*(z_0))^\alpha] + \frac{1}{\beta} p'_0 a_0 + b_0 \right) = \tilde{c}$$

where we also used (6) and  $1 + r = 1/\beta$ . Using this result to substitute the first line in (C23) shows that (C23) is just the flow budget equation (7b) for period zero. This proves that the portfolio giving rise to (C22) in  $t = 1$  for all  $n = 1, \dots, N$  satisfies the budget constraint of period  $t = 0$ . The results for  $t \geq 1$  follow by induction. The result (16) follows from substituting (C21) into (C22) and noting that  $b_0$  and  $(1 - d)k^*(z_0)$  are not



state contingent.

From equation (C22) and the fact that  $\Pi(z_t)$  is bounded, it follows that  $\tilde{w}_t$  is bounded so that the process for beginning-of-period wealth automatically satisfies the marginally binding NBLs. The multipliers  $\omega_{t+1}$  in appendix C.C1 are thus equal to zero for all  $t$  and all contingencies. Using  $v_t(z^n) \equiv 0$ ,  $\omega_{t+1} \equiv 0$ , the fact that capital investment is given by (15) and that the Arrow security price is (6), the Euler conditions (C2a) - (C2c) all hold when consumption is given by (14). This completes the proof.

### C6. Proof of proposition 5

Since we start with a beginning-of-period wealth level at the marginally binding NBL, we necessarily have  $\tilde{c}_t = 0$ . Otherwise one could afford an even lower initial wealth level and satisfy all constraints, which would be inconsistent with the definition of the marginally binding NBLs given in (11). Indeed, the available beginning of period wealth  $\tilde{w}_t$  is just enough to insure that  $\tilde{w}_{t+1} \geq NBL(z_{t+1})$  for all possible future productivity states  $z_{t+1}$ . The optimal choices  $a_t \in R^N$ ,  $b_t$ ,  $k_t$  are thus given by the cost-minimizing choices satisfying  $a_t \geq 0$  plus the marginally binding NBLs in  $t + 1$  for all possible future productivity states. Formally,

$$\min_{a_t, b_t, \tilde{k}_t} \bar{c} + \tilde{k}_t + \frac{1}{1+r} b_t + \sum_{j=1}^N a_t(z^j) p_t(z^j)$$

s.t.

$$z^j \tilde{k}_t^\alpha + (1-d)\tilde{k}_t + b_t + (1-\lambda)a_t(z^j) \geq NBL(z^j) \text{ for } j = 1, \dots, N$$

$$a_t(z^j) \geq 0 \text{ for } j = 1, \dots, N$$

The optimal choices are thus equivalent to those solving problem (11). From appendix C.C3 follows that under the stated assumptions, the optimal choices are given by (C6)-(C9).

### C7. Proof of proposition 6

Using the assumed policies,  $\frac{1}{1+r} = \beta$ ,  $p_t(z^j) = \frac{\pi(z^j|z_t)}{1+r}$ , and the fact that the NBLs are not binding for sufficiently high wealth levels, the Euler equations (C2a)-(C2c) for  $i = 0$  imply

$$(C24a) \quad u'(\tilde{c}_t) = E_t u'(\tilde{c}_{t+1})$$

$$(C24b) \quad v_t(z^j) = \beta \pi(z^j|z_t) (u'(\tilde{c}_t) - u'(\tilde{c}_{t+1}(z^j))(1-\lambda)) \text{ for } j = 1, \dots, N$$

$$(C24c) \quad 0 = -u'(\tilde{c}_t) + \alpha \tilde{k}_t^{\alpha-1} \beta E_t u'(\tilde{c}_{t+1}) z_{t+1} + \beta(1-d)$$

We show below that the Euler equation errors for  $i = 0$  converge to zero and that  $v_t(z^j) \geq 0$  as the wealth  $\tilde{w}_t$  increases without bound. Under the assumed policies, wealth evolves

according to

$$\begin{aligned}
\tilde{w}_{t+1} &= z_{t+1}k^*(z_t)^\alpha + b_t \\
&= z_{t+1}k^*(z_t)^\alpha + \frac{1}{\beta} (\tilde{w}_t - k^*(z_t) - (1 - \beta)(\Pi(z_t) + \tilde{w}_t) - \bar{c}) \\
&= \tilde{w}_t + z_{t+1}k^*(z_t)^\alpha - \frac{1}{\beta}k^*(z_t) - \frac{(1 - \beta)}{\beta}\Pi(z_t) - \frac{1}{\beta}\bar{c} \\
&= \tilde{w}_t + \Pi(z_t) + z_{t+1}k^*(z_t)^\alpha - \frac{1}{\beta}k^*(z_t) - \frac{1}{\beta}(\Pi(z_t)) - \frac{1}{\beta}\bar{c} \\
&= \tilde{w}_t + \Pi(z_t) - E_t[\Pi(z_{t+1})] + (z_{t+1} - E_t z_{t+1})k^*(z_t)^\alpha
\end{aligned}$$

Since the fluctuations in  $z_t$ ,  $k^*(z_t)$  and  $\Pi(z_t)$  are all bounded, fluctuations in wealth are also bounded over any finite number of periods. Moreover, the fluctuations in wealth are independent of the initial wealth level. As a result, fluctuations in consumption are also bounded and of a size that is not dependent on the wealth level under the proposed consumption policy.

We now show that for  $i = 0$  and a sufficiently high wealth level the Euler equation residuals remains below  $\epsilon$ . Using the assumed consumption policy and the result from the previous equation, we have

$$\begin{aligned}
E_t[\tilde{c}_{t+1}] &= (1 - \beta)E_t[\Pi(z_{t+1}) + \tilde{w}_{t+1}] \\
&= (1 - \beta)(\tilde{w}_t + \Pi(z_t)) \\
&= c_t
\end{aligned}$$

i.e. consumption follows a random walk. Now consider equation (C24a), which requires

$$\begin{aligned}
u'(\tilde{c}_t) &= E_t u'(\tilde{c}_{t+1}) \\
\text{(C25)} \quad &= \sum_{j=1}^N \pi(z^j | z_t) u'(\tilde{c}_{t+1}(z^j))
\end{aligned}$$

From Taylor's theorem we have

$$u'(\tilde{c}_{t+1}(z^j)) = u'(\tilde{c}_t) + u''(c^j)(\tilde{c}_{t+1}(z^j) - \tilde{c}_t)$$

where  $c^j$  can be chosen from the bounded interval

$$[\min\{\tilde{c}_t, \min_j \tilde{c}_{t+1}(z^j)\}, \max\{\tilde{c}_t, \max_j \tilde{c}_{t+1}(z^j)\}]$$

whose width is independent of the wealth level  $\tilde{w}_t$  (as fluctuations in consumption do not depend on wealth as shown above). Moreover, under the assumed consumption policy, the lower bound of this interval - and thus also  $c^j$  increases without bound, as  $\tilde{w}_t$

increases without bound. Using the earlier result, (C25) can be rewritten as

$$0 = \sum_{j=1}^N \pi(z^j) (u''(c^j) (\tilde{c}_{t+1}(z^j) - \tilde{c}_t))$$

where the sum on the right-hand side of the equation denotes the Euler equation residual whenever it is not equal to zero. For the considered consumption policies, we have that  $\tilde{c}_{t+1}(z^j) - \tilde{c}_t$  is bounded and invariant to wealth. Moreover, for sufficiently large wealth,  $c^j$  increases without bound, therefore  $u''(c^j) \rightarrow 0$  under the maintained assumption about preferences. This implies that for any given  $\epsilon > 0$  we can find a wealth level  $w^*$  so that the Euler equation residual falls below  $\epsilon$ . Since the fluctuations in wealth are bounded over the finite horizon  $T$  and do not depend on the initial wealth level, we can find an initial wealth level  $\bar{w}$  high enough such that over the next  $T$  periods wealth stays above  $w^*$ . The Euler equation errors then remain below  $\epsilon$  over the next  $T - 1$  periods, as claimed.

Similar arguments can be made to show that (C24c) holds and that (C24b) implies  $v_t(z^j) \geq 0$  for a sufficiently large initial wealth level. We omit the proof here for the sake of brevity.

#### C8. Default Costs Borne by Lender

This appendix shows that if a consumption allocation is feasible in a setting in which default costs are borne by the borrower, then it is also feasible in a setting in which some or all of these costs are borne by the lender instead. For simplicity, we only consider the extreme alternative where all costs are born by the lender. Intermediate cases can be covered at the cost of some more cumbersome notation.

Consider a feasible choice  $\{F_t \geq 0, D_t \geq 0, \Delta_t \in [0, 1]^N, k_t \geq 0, c_t \geq 0\}_{t=0}^{\infty}$ , i.e. a choice that satisfies the constraints of the government's problem (5), which assumes  $\lambda^l = 0$  and  $\lambda^b = \lambda$ . Let variables with a hat denote the corresponding choices in a setting in which the lender bears all default costs, i.e., where  $\lambda^l = \lambda$  and  $\lambda^b = 0$ . We show below that it is then feasible to choose the same real allocation, i.e., to choose  $\hat{k}_t = k_t$  and  $\hat{c}_t = c_t$ , provided one selects appropriate values for  $\hat{F}_t, \hat{D}_t$  and  $\hat{\Delta}_t$ .

First, note that in a setting where foreign investors bear all settlement costs, the interest rate  $\hat{R}(z_t, \hat{\Delta}_t)$  on domestic bonds satisfies

$$(C26) \quad 1 + r = \frac{1 - (1 + \lambda) \sum_{n=1}^n \hat{\delta}_t^n \Pi(z^n | z_t)}{\frac{1}{1 + \hat{R}(z_t, \hat{\Delta}_t)}}$$

where the denominator on the right-hand side denotes the issuance price of the bond and the numerator the expected repayment net of the lender's settlement cost. The previous equation thus equates the expected returns of the domestic bonds with the expected return

on the foreign bond.

Next, consider the following financial policies:<sup>61</sup>

$$(C27a) \quad \widehat{\Delta}_t = (1 - \lambda)\Delta_t \frac{D_t}{\widehat{D}_t}$$

$$(C27b) \quad \widehat{D}_t = \frac{1 + \widetilde{R}(z_t, \widehat{\Delta}_t)}{\widetilde{R}(z_t, \widehat{\Delta}_t)} \frac{R(z_t, \Delta_t)}{1 + R(z_t, \Delta_t)} D_t$$

$$(C27c) \quad \widehat{F}_t = F_t + \widehat{D}_t - D_t$$

As we show below, in a setting in which settlement costs are borne by the lender, the financial policies  $\{\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t\}_{t=0}^{\infty}$  give rise to the same state-contingent financial payoffs as generated by the policies  $\{F_t, D_t, \Delta_t\}_{t=0}^{\infty}$  in a setting in which default cost are borne by the borrower. Therefore, as claimed, the former policies allow the implementation of the same real allocations.

Consider the financial flows generated by the policy component  $(\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t)$ . In period  $t$ , the financial inflows are given by

$$\frac{\widehat{D}_t}{1 + \widehat{R}(z_t, \widehat{\Delta}_t)} - \widehat{F}_t$$

Using the definitions (C27), it is straightforward to show that these inflows are equal to

$$\frac{1}{1 + R(z_t, \Delta_t)} D_t - F_t$$

which are the inflows under the policy  $(F_t, D_t, \Delta_t)$  in a setting where default costs are borne by the lender.

We show next that the financial flows in  $t + 1$  are also identical under the two policies. The financial inflows generated by the policy choices  $(\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t)$  in some future contingency  $n \in \{1, \dots, N\}$  in period  $t + 1$  are given by

$$-\widehat{D}_t(1 - \widehat{\delta}_t^n) + \widehat{F}_t$$

From the first and last equation in (C27), we determine that these flows are equal to

$$-(1 - (1 - \lambda)\delta_t^n)D_t + F_t$$

which are the inflows generated by the policy  $(F_t, D_t, \Delta_t)$  in a setting where default costs are borne by the lender.

<sup>61</sup>Lengthy but straightforward calculations, which are available upon request, show that these policies satisfy  $\widehat{\Delta}_t \in [0, 1]^N$ , although they may imply  $\widehat{F}_t < 0$ , which requires the government also to issue safe bonds, i.e. bonds that promise full repayment.

Finally, since the policies  $\{F_t, D_t, \Delta_t\}_{t=0}^{\infty}$  satisfy the marginally binding natural borrowing limits in the government's problem (5), it must generate bounded financial flows, with the same therefore also applying for the policies  $\{\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t\}_{t=0}^{\infty}$ . These policies thus also satisfy the marginally binding natural borrowing limits, which completes the proof.

### C9. Estimation of Lender's Default Costs

Consider a non-contingent one period bond that in explicit legal terms promises to repay one unit and that has an associated implicit default profile  $\Delta = (\delta^1, \dots, \delta^n) \in [0, 1]^n$ . A risk-neutral foreign lender, who bears proportional default costs  $\lambda^b$  in the event of default and can earn the gross return  $1 + r$  on alternative safe investments, will price this bond according to equation (27). As explained below, the asset pricing equation (27) can be used to obtain an estimate for  $\lambda^l$ .

We start by defining the *ex post* return  $epr_t$  on a government bond

$$1 + epr_t = \frac{1 - \sum_{n=1}^N \delta^n \Pi(z^n | z_t)}{\frac{1}{1 + R(z_t, \Delta)}},$$

which is the bond return that accounts for losses due to non-repayment but not for potential default costs. *Ex post* returns can be measured from financial market data. Using the previous equation to substitute  $\sum_{n=1}^N (1 - \delta^n) \cdot \pi(z^n | z_t)$  on the r.h.s. of equation (27) and applying the unconditional expectations operator<sup>62</sup>, one obtains

$$(C28) \quad \lambda^l = \frac{E[epr_t - r]}{E\left[(1 + R(z_t, \Delta)) \sum_{n=1}^N \delta^n \Pi(z^n | z_t)\right]}.$$

Information about the average excess return, which shows up in the numerator of the previous equation, can be obtained from Klingen, Weder, and Zettelmeyer 2004, who consider 21 emerging market economies over the period 1970-2000. Using data from table 3 in Klingen, Weder, and Zettelmeyer 2004, the average excess return varies between -0.2% and +0.5% for publicly guaranteed debt, depending on the estimation method used.<sup>63,64</sup> We use the average of the estimated values and set  $E[epr_t - r] = 0.15\%$ .

<sup>62</sup>The expectations operator integrates over the set of possible histories  $z^t = \{z_t, z_{t-1}, \dots\}$ .

<sup>63</sup>As suggested in Klingen, Weder, and Zettelmeyer 2004, we use the return on a three-year US government debt instrument as the safe asset, since it approximately has the same maturity as the considered emerging market debt.

<sup>64</sup>The fact that *ex post* excess returns on risky sovereign debt are relatively small or sometimes even negative is confirmed by data provided in Eichengreen and Portes 1986 who compute *ex post* excess returns using interwar data. The

We now turn to the denominator on the r.h.s. of equation (C28). Using a first order approximation we obtain

$$(C29) \quad E \left[ (1 + R(z_t, \Delta)) \sum_{n=1}^N \delta^n \Pi(z^n | z_t) \right] \approx E [1 + R(z_t, \Delta)] E \left[ \sum_{n=1}^N \delta^n \Pi(z^n | z_t) \right],$$

where the last term equals (again to a first order approximation)

$$E \left[ \sum_{n=1}^N \delta^n \Pi(z^n | z_t) \right] \approx \Pr(\delta > 0) E[\delta | \delta > 0].$$

Using data compiled by Cruces and Trebesch 2011, who kindly provided us with the required information, we observe for the 21 countries considered in Klingen et al. 2004 and for the period 1970-2000 a total of 58 default events, thus the average yearly default probability equals 8.9%. The average haircut conditional on a default was 25%; these figures therefore imply

$$E \left[ \sum_{n=1}^N \delta^n \Pi(z^n | z_t) \right] \approx 2.22\%.$$

The average *ex ante* interest rate  $R(z_t, \Delta)$  appearing in equation (C29) can be computed by adding to the average *ex post* return of 8.8% reported in table 3 in Klingen, Weder, and Zettelmeyer 2004 for publicly guaranteed debt, the average loss due to default, which equals 2.22% to first order, such that  $R(z_t, \Delta) \approx 11.02\%$ . Combining these results to evaluate  $\lambda^l$  in equation (C28) delivers our estimate for the default costs accruing to lenders reported in the main text.

#### C10. Numerical Solution Approach

We solve the recursive version of the Ramsey problem from section I.B using global solution methods, so as to account for the non-linear nature of the optimal policies. The state space  $S$  of the problem is given by

$$S = \{z^1 \times [NBL(z^1), w_{max}], \dots, z^N \times [NBL(z^N), w_{max}]\}$$

where  $NBL(z^n)$  denotes the marginally binding natural borrowing limits and  $w_{max}$  is a suitably chosen and sufficiently large upper bound for the country's wealth level.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

$$\tilde{f} : (z_t, w_t) \rightarrow (c_t, k_t, b_t, a_t),$$

negative *ex post* excess returns likely arise due to the presence of sampling uncertainty: the high volatility of the nominal exchange rate makes it difficult to estimate the mean *ex post* excess returns.

where their values (approximately) satisfy the optimality conditions derived in C.C1. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, tomorrow's policy (denoted by  $f^{next}$ ) is taken as given and solves for the optimal policy  $f$  today, which in turn is used to update the guess for tomorrow's policy. Convergence is achieved once  $\|f - f^{next}\| < \epsilon$ , where we set  $\epsilon = 10^{-5}$ . We then set  $\tilde{f} = f$ . In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow's policy. Following Garcia and Zangwill 1981, we can transform the complementarity conditions of our first order equilibrium conditions into equations. For more details on the time iteration procedure and how complementarity conditions are transformed into equations, see, for example, Brumm and Grill 2014. To come up with a starting guess for the consumption policy, we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function  $g$  which satisfies  $g(z^i, NBL(z^i)) = \bar{c} \forall i$  and use a reasonable guess for  $g(z^i, w_{max})$ .

#### C11. Calibration of Greek Output Process: Further Details

This annex contains the details of the model specification we use in our application to the Greek economic crisis. We use realized output levels in the Greek economy for 2009 to 2013 to estimate model GDP. We obtain the five states  $(z_{2009}, z_{2010}, z_{2011}, z_{2012}, z_{2013}) = (1-8.30\%, 1-15.90\%, 1-24.70\%, 1-32.7\%, 1-37.3\%) = (0.917, 0.841, 0.753, 0.676, 0.627)$ . In addition we have  $z_{2008} = 1$ . For 2014 we have estimated a value  $z_{2014} = 0.610$ . Remember that we have assumed that we start in  $z_{2008}$  and for each following year two scenarios are possible:

- 1) Return to trend growth of 3.6% from  $t + 1$  onwards (however, no return to level)
- 2) Observed fall in output relative to trend growth actually happening in  $t + 1$

Having defined the output process in terms of deviations from trend, the economy transitions from  $z_t$  to the absorbing state  $\bar{z}_t$  whenever the first scenario realizes.

The vector of states  $z$  for our model is therefore given by

$$\begin{aligned} z &= (z_{2008}, \bar{z}_{2009}, z_{2009}, \bar{z}_{2010}, z_{2010}, \bar{z}_{2011}, z_{2011}, \bar{z}_{2012}, z_{2012}, \bar{z}_{2013}, z_{2013}, z_{2014}) \\ &= (1, 0.917, 0.917, 0.841, 0.841, 0.753, 0.753, 0.676, 0.676, 0.627, 0.627, 0.610). \end{aligned}$$

To parametrize the state transition matrix we employ the following procedure: We use one year ahead OECD output forecasts to compute implied probabilities for both scenarios, i.e. we set the transition probabilities in our model such that the expected output value tomorrow is equal to the one year ahead OECD forecast. However, this is not possible for the years 2013 and 2014 where output is above the forecast. In this case, we set the probability for the second scenario to 99 per cent. This yields the following vector  $\pi$  of transition probabilities:

$$\begin{aligned}\pi &= (p_{2008}, p_{2009}, p_{2010}, p_{2011}, p_{2012}, p_{2013}) \\ &= (0.966, 0.732, 0.603, 0.445, 0.01, 0.01) .\end{aligned}$$

and the accompanying Markov transition matrix  $M$  is given by

$$(C32) \quad M = \begin{pmatrix} 0.966 & 0 & 0.034 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.732 & 0 & 0 & 0.268 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.603 & 0 & 0 & 0.397 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.445 & 0 & 0 & 0.555 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.010 & 0 & 0 & 0.990 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0.990 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \end{pmatrix} .$$

We compute the equilibrium by starting in 2014 and solving for the optimal policies backwards in time.