Online Appendix for Dynamic Outside Options and Optimal Negotiation Strategies

Andrew McClellan*

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Appendix O.A Omitted Proofs

Proof of Lemma A.2

Proof. For $\delta \in [0,1]$, let $\overline{V}(x;\delta) := \max_{\tau,d_\tau} \mathbb{E}_x[e^{-r\tau}(d_\tau(u_A(1-\delta) - X_\tau) + X_\tau)]$, $\mathcal{C}^\delta := \{x : \overline{V}(x;\delta) > \max\{x, u_A(1-\delta)\}\}$, $\tau^\delta := \inf\{t : X_t \notin \mathcal{C}^\delta\}$ and $d^\delta_{\tau^\delta} := \mathbb{1}(u_A(1-\delta) \geq X_{\tau^\delta})$. By standard arguments (see Peskir and Shiryaev (2006)), $\mathcal{C}^\delta = (S^\delta, R^\delta)$ for some $S^\delta, R^\delta \in \mathcal{G}$, $\overline{V}(x;\delta)$ is continuous and decreasing in δ and $(\tau^\delta, d^\delta_{\tau^\delta})$ is in the arg max for $\overline{V}(x;\delta)$. Moreover, S^δ, R^δ are unique if $\mathcal{C}^\delta \neq \emptyset$. Unless stated otherwise, assume that $\mathcal{C}^0 \neq \emptyset$. Continuity of $\overline{V}$ implies $\mathcal{C}^\delta = \mathcal{C}^0$ for sufficiently small $\delta > 0$. Consider such δ.

Take $x < R^\delta$. We now show $\mathbb{P}_x(d^\delta_{\tau^\delta} = 1) > 0$. Suppose not, so $\mathbb{P}_x(d^\delta_{\tau^\delta} = 1) = 0$. If $x \in \mathcal{C}^\delta$, then $\overline{V}(x;\delta) = \mathbb{E}_x[e^{-r\tau^\delta} X_{\tau^\delta}] \leq x$, where the inequality follows by Doob's optional stopping theorem, contradicting $x \in \mathcal{C}^\delta$. If $x \notin \mathcal{C}^\delta$, then $x \leq S^\delta$, so $\mathbb{P}_x(d^\delta_{\tau^\delta} = 1) = 0$ implies $S^\delta > u_A(1-\delta)$. For $x' \in \mathcal{C}^\delta$, because $\mathbb{P}_{x'}(X_{\tau^\delta} \in \{R^\delta, S^\delta\}) = 1$, by $R^\delta > S^\delta > u_A(1-\delta)$, we have $\mathbb{P}_{x'}(d^\delta_{\tau^\delta} = 1) = 0$, which we have argued above cannot be. Thus, $\mathbb{P}_x(d^\delta_{\tau^\delta} = 1) > 0$ for all $x < R^\delta$.

Set $\overline{R} = R^0$. Take an optimal contract $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$. Let $\mathcal{H}_0 = \{h_t : \tau^* = t < \tau_+(\overline{R}),\ d^*_{\tau^*} = 0\}$. For the sake of contradiction, suppose the histories in $\mathcal{H}_0$ are realized with positive probability. Consider a new contract $(\tau, d_\tau, \alpha_\tau)$ which is identical to $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ except after $h_t \in \mathcal{H}_0$. After such histories, set $(\tau, d_\tau, \alpha_\tau)$ to use $(\tau^\delta, d^\delta_{\tau^\delta}, \alpha^\delta_{\tau^\delta})$ (where $\alpha^\delta_{\tau^\delta} = \delta$ with probability one) as its continuation contract. Because $(\tau^\delta, d^\delta_{\tau^\delta})$ maximizes A's utility given $\alpha^\delta_{\tau^\delta}$, $(\tau^\delta, d^\delta_{\tau^\delta}, \alpha^\delta_{\tau^\delta})$ satisfies DIR and A's continuation value after $h_t \in \mathcal{H}_0$ is

*University of Chicago Booth School of Business Email: andrew.mcclellan@chicagobooth.edu.

greater than his outside option. This change weakly increases A's continuation value at all earlier histories, so $(\tau, d_\tau, \alpha_\tau)$ satisfies DIR. Moreover, $(\tau, d_\tau, \alpha_\tau)$ strictly increases P's continuation value at $h_t \in \mathcal{H}_0$, since his continuation value is $\mathbb{E}_{X_t}[e^{-r\tau^\delta} d^\delta_{\tau^\delta} u_P(\delta)] > 0$ by $\mathbb{P}_{X_t}(d^\delta_{\tau^\delta} = 1) > 0$ as $X_t < R^0 = \overline{R}$ and $R^\delta = R^0$ by $\mathcal{C}^\delta = \mathcal{C}^0$, contradicting the optimality of $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$. Thus, $d^*_{\tau^*} = 1$ whenever $\tau^* < \tau_+(\overline{R})$. This construction implies there exists a DIR contract with strictly positive expected utility for P when $X_0 < \overline{R}$ and $J(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*}) > 0$ if $X_0 < \overline{R}$.

Next, we show it is without loss to focus on optimal contracts with $\tau^* \leq \tau_+(\overline{R})$ and $d^*_{\tau^*} = 0$ if $X_{\tau^*} \geq \overline{R}$. Any continuation contract at $\tau_+(\overline{R})$ which realizes a split in $(0, 1)$ with positive probability yields a strictly lower payoff than $\overline{V}(X_{\tau_+(\overline{R})}; 0) = X_{\tau_+(\overline{R})}$,[1] violating DIR. Therefore, any continuation contract $(\tau', d'_{\tau'}, \alpha'_{\tau'})$ of an optimal contract at $\tau_+(\overline{R})$ must have $\alpha'_{\tau'} = 0$ with probability one and deliver A a continuation value of $X_{\tau_+(\overline{R})}$, so P's continuation value is 0. It is therefore payoff equivalent to replace the continuation contract at $\tau_+(\overline{R})$ with taking the outside option; namely, it is without loss to assume any optimal contract $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ has $\tau^* \leq \tau_+(\overline{R})$ and $d^*_{\tau^*} = 0$ if $X_{\tau^*} \geq \overline{R}$.[2] Thus, $J(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*}) = 0$ if $X_0 \geq \overline{R}$.

Finally, suppose $\mathcal{C}^0 = \emptyset$. In this case, take $\overline{R} = \min\{x : \ x \geq u_A(1)\}$ if $\overline{X} > u_A(1)$ and $\overline{R} = \overline{X} + \epsilon$ otherwise. By analogous arguments as above, there exists no DIR contract with a strictly positive continuation value for P at $X_0 \geq \overline{R}$, so taking the outside option immediately is optimal. If $X_t < \overline{R}$, then reaching an immediate split with demand α such that $u_A(1 - \alpha) = X_t$ gives a positive expected utility to P, so it cannot be optimal to take the outside option at $X_t < \overline{R}$. □

Lemma O.A.1. *Any contract that satisfies DIR also satisfies $RDIR(c)$ for all $c \geq X_0$.*

Proof. Suppose $(\tau, d_\tau, \alpha_\tau)$ satisfies DIR and take any $c \geq X_0$. A's continuation value at $\tau_+(c)$ is $\mathbb{E}_c[e^{-r(\tau - \tau_+(c))}(d_\tau(u_A(1 - \alpha_\tau) - X_\tau) + X_\tau)|h_{\tau_+(c)}]$. DIR implies that this is (weakly) greater than c. Then $(\tau, d_\tau, \alpha_\tau)$ satisfies $RDIR(c)$ as

$$\begin{aligned}
&V(\tau, d_\tau, \alpha_\tau) - V(\tau \wedge \tau_+(c), d_\tau(c), \alpha_\tau) \\
&= \mathbb{E}\big[e^{-r\tau_+(c)}\big(\mathbb{E}_c[e^{-r(\tau - \tau_+(c))}(d_\tau(u_A(1 - \alpha_\tau) - X_\tau) + X_\tau)|h_{\tau_+(c)}] - c\big)\mathbb{1}(\tau_+(c) \leq \tau)\big] \\
&\geq 0.
\end{aligned}$$

□

[1]Note that if $\overline{V}(\overline{R}; 0) = u_A(1) > \overline{R}$, then, for $x \in \mathcal{C}^0$, we would have $\overline{V}(x; 0) = \mathbb{E}_x[e^{-r\tau^0} u_A(1)] < u_A(1)$, a contradiction of $x \in \mathcal{C}^\delta$. Thus, $\overline{V}(\overline{R}; 0) = \overline{R}$, which implies $\overline{V}(x; 0) = x$ for all $x \geq \overline{R}$.

[2]The only time that there may exist a DIR contract that does not stop with probability one at or before $\tau_+(\overline{R})$ is if A is exactly indifferent between continuing and stopping at $\overline{R}$ in $\overline{V}(x; 0)$.

Proof of Lemma A.3

Proof. In order to write RDP in the notation of Altman (1999), we first describe an alternative way to specify a contract. We use a state $(H_t, X_t, M_t) \in \{0,1\} \times [\underline{X}, \overline{X}] \times [\underline{X}, \overline{X}]$ at time t where H_t will equal 1 if and only if P has not stopped prior to t (so $H_0 = 1$). An action in period t is $(a_t, d_t, \alpha_t) \in \{0,1\} \times \{0,1\} \times [0,1]$ where $a_t = 1$ if and only if stopping at time t (so $H_{t+\Delta} = H_t(1-a_t)$), d_t is an indicator for a split being made when stopping at t and α_t is the share of the surplus going to P when implementing a split at time t; we restrict the choice of (a_t, d_t, α_t) to all be 0 if $H_t = 0$.[3] A history at t takes the form $\tilde{h}_t = (H_0, X_0, M_0, a_0, d_0, \alpha_0, H_1, ..., \alpha_{t-1}, H_t, X_t, M_t)$ and a strategy maps each history into a distribution over (a_t, d_t, α_t).[4] By our restriction after Lemma A.2, we set $a_t = 1, d_t = 0, \alpha_t = 0$ whenever $X_t \geq \overline{R}$ and $H_t = 1$.

We now rewrite RDP in the form used in Altman (1999); namely, as the discounted sum of payoffs in $t \in \{0, \Delta, ...\}$, where the payoff in period t only depends on states (H_t, X_t, M_t) and actions (a_t, d_t, α_t). Take $(\tau, d_\tau, \alpha_\tau)$ and let (a_t, d_t, α_t) be the associated strategy. We first rewrite the objective function in our desired form:

$$\mathbb{E}[e^{-r\tau} d_\tau u_P(\alpha_\tau)] = \mathbb{E}[\sum_{t\in\{0,\Delta,...\}} e^{-rt} a_t d_t u_P(\alpha_t)].$$

Next, we rewrite $RDIR(X^n)$. As discussed in the Appendix, $RDIR(X^n)$ is equivalent to

$$\mathbb{E}[\big(e^{-r\tau}(d_\tau(u_A(1-\alpha_\tau) - X_\tau) + X_\tau) - e^{-r\tau_+(X^n)} X^n\big)\mathbb{1}(M_\tau \geq X^n)] \leq 0. \quad (1)$$

To rewrite (1) in our desired form, we do so separately for $\mathbb{E}[e^{-r\tau}(d_\tau(u_A(1-\alpha_\tau) - X_\tau) + X_\tau)\mathbb{1}(M_\tau \geq X^n)]$ and $\mathbb{E}[e^{-r\tau_+(X^n)} X^n \mathbb{1}(M_\tau \geq X^n)]$. The first is straightforward:

$$\begin{aligned}&\mathbb{E}[(e^{-r\tau}(d_\tau(u_A(1-\alpha_\tau) - X_\tau) + X_\tau)\mathbb{1}(M_\tau \geq X^n)] \\ &= \mathbb{E}[\sum_{t\in\{0,\Delta,...\}} e^{-rt} a_t\big(d_t(u_A(1-\alpha_t) - X_t) + X_t)\mathbb{1}(M_t \geq X^n)\big)].\end{aligned}$$

For $\mathbb{E}[e^{-r\tau_+(X^n)} X^n \mathbb{1}(M_\tau \geq X^n)]$, we first consider $X^n = X_0$. Then $\mathbb{E}[e^{-r\tau_+(X^n)} X^n \mathbb{1}(M_\tau \geq X^n)] = X_0$, which trivially takes our desired form. Next, take $X^n > X_0$, in which case

[3] Each contract $(\tau', d'_{\tau'}, \alpha'_{\tau'})$ is associated with a unique strategy (i.e., when $H_t = 1$, we have $a_t = \mathbb{1}(\tau' = t)$, $d_t = a_t d'_{\tau'}$ and $\alpha_t = a_t \alpha'_{\tau'}$) and each strategy induces a unique contract $(\tau', d'_{\tau'}, \alpha'_{\tau})$ (i.e., $\tau' = \inf\{t : a_t = 1\}$, $d'_{\tau'} = d_{\tau'}, \alpha'_\tau = \alpha_{\tau'}$).

[4] Although our baseline setting makes the randomizing device explicit in U, we keep it implicit here to match the notation of Altman (1999).

$\mathbb{P}(\tau_+(X^n) > 0) = 1$. We note that $\tau_+(X^n) = t$ if and only if $M_{t-\Delta} < X^n = X_t$ and $M_\tau \geq X^n$ if and only if $H_{\tau_+(X^n)} = 1$. We then have

$$\begin{aligned}
&\mathbb{E}[e^{-r\tau_+(X^n)} X^n \mathbb{1}(M_\tau \geq X^n)] \\
&\quad = \mathbb{E}[\sum_{t\in\{\Delta,\ldots\}} e^{-rt} H_t X^n \mathbb{1}(\tau_+(X^n) = t)] \\
&\quad = \mathbb{E}[\sum_{t\in\{\Delta,\ldots\}} e^{-rt} H_{t-\Delta}(1 - a_{t-\Delta}) X^n \mathbb{1}(M_{t-\Delta} < X^n)\mathbb{1}(X_t = X^n)] \\
&\quad = \mathbb{E}[\sum_{s\in\{0,\Delta,\ldots\}} e^{-r(s+\Delta)} H_s(1 - a_s) X^n \mathbb{1}(M_s < X^n)\mathbb{1}(X_{s+\Delta} = X^n)] \\
&\quad = \mathbb{E}[\sum_{s\in\{0,\Delta,\ldots\}} e^{-rs} H_s(1 - a_s)\mathbb{1}(M_s < X^n) e^{-r\Delta} X^n \mathbb{P}(X_{s+\Delta} = X^n|\tilde{h}_s)] \\
&\quad = \mathbb{E}[\sum_{s\in\{0,\Delta,\ldots\}} e^{-rs} H_s(1 - a_s)\mathbb{1}(M_s < X^n) e^{-r\Delta} X^n \mathbb{P}_{X_s}(X_\Delta = X^n)],
\end{aligned}$$

which takes our desired form.

We show a Slater condition holds, namely, that there exists a strategy under which all $RDIR(X^n)$ constraints are slack. For simplicity, we describe such a strategy using the contract it induces. Let $(\tau^0, d^0_{\tau^0}, \alpha^0_{\tau^0})$ be A's first-best contract (as defined in the proof of Lemma A.2). Thus, $\tau^0 = \inf\{t : X_t \notin (S^0, \overline{R})\}$ for some S^0. For some $\delta \in (0, 1)$, define a contract that in each period t takes the same action as $(\tau^0, d^0_{\tau^0}, \alpha^0_{\tau^0})$ would starting from X_t with probability δ and waits with probability $1 - \delta$. Taking $\delta \to 1$, A's continuation value at each t is arbitrarily close to his first-best continuation value, from which it is easy to see that all $RDIR(X^n)$ constraints are slack.[5]

By Theorem 8.4 of Altman (1999), there exists a solution to RDP that is stationary in (X, M).[6] Because the Slater condition holds, by Theorem 9.10 of Altman (1999)[7] there exists $\Lambda = (\lambda^0, ..., \lambda^N) \in \mathbb{R}^{N+1}_-$ such that the value of RDP is equal to $\mathcal{L}(\Lambda)$ and any solution $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ must satisfy complementary slackness conditions, namely $\lambda^n[V(\tau^* \wedge \tau_+(X^n), d^*_{\tau^*}(X^n), \alpha^*_{\tau^*}) - V(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})] = 0$ for all $n = 0, ..., N$. □

[5] If $X_0 > S^0$ (for S^0 as defined in Lemma A.2), then using $(\tau^0, d^0_{\tau^0}, \alpha^0_{\tau^0})$ would satisfy all $RDIR$ constraints with strict inequality. However, for $X_0 \leq S_0$, $(\tau^0, d^0_{\tau^0}, \alpha^0_{\tau^0})$ stops immediately, so $RDIR(X^n)$ constraints for $n > 0$ hold with equality. We get around this problem by only stopping with probability $\delta < 1$ at each $t < \tau_+(S^0 + \epsilon)$ so that $\tau_+(X^n)$ occurs prior to stopping with positive probability.

[6] Writing RDP in terms of (a_t, d_t, α_t) as we did above, the results in Altman (1999) imply the existence of an optimal policy that is stationary in (H, X, M). However, because only the case when $H_t = 1$ is payoff relevant, this translates into a contract that is stationary in (X, M).

[7] Although the results in Chapter 9 of Altman (1999) are stated for a model without discounting, they can be adapted to one with discounting as shown in Chapter 10 of Altman (1999).

Lemma O.A.2. *There exists a solution $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ to RDP where, for some $S(\cdot), \gamma(\cdot), \alpha(\cdot)$, τ^* is an $\tau^{S(m),\gamma(m)}$ stopping time over $[\tau_+(m), \tau_+(m+\epsilon))$, $\alpha^*_{\tau^*} = \alpha(M_{\tau^*})$ and $S(\cdot), \gamma(\cdot), \alpha(\cdot)$ are constant over any $[m_1, m_2]$ such that $RDIR(m)$ is slack for all $m \in (m_1, m_2]$.*

Proof. Let $(\tau, d_\tau, \alpha_\tau)$ be an optimal contract that is stationary in (X, M). Suppose there exists $[m_1, m_2]$ such that, under $(\tau, d_\tau, \alpha_\tau)$, $RDIR(m)$ is slack for all $m \in (m_1, m_2]$ and $(\tau, d_\tau, \alpha_\tau)$ is not stationary over $[m_1, m_2]$—i.e., the corresponding $S(\cdot), \gamma(\cdot), \alpha(\cdot)$ are not all constant in m over $[m_1, m_2]$. If $(m, m) \notin \mathcal{R}$, then $\mathbb{P}(\tau_+(m) \leq \tau^*) = 0$ and $RDIR(m)$ binds. Therefore, $(m, m) \in \mathcal{R}$ for each $m \in [m_1, m_2]$.

Complementary slackness conditions imply $\lambda^n = 0$ for all $X^n \in (m_1, m_2]$; by the characterization of $\alpha(\cdot)$ in Lemma A.4, $\alpha(\cdot)$ is constant over $[m_1, m_2]$. Moreover, for each m, by the same arguments as in Lemma A.5, the value of S' for which stopping is first weakly optimal at $t \in [\tau_+(m), \tau_+(m+\epsilon))$ in $\mathcal{L}(\Lambda)$ is the same across all $m \in [m_1, m_2]$.[8] If stopping is strictly optimal at S', then $(S(m), \gamma(m)) = (S', 0)$ for all $m \in [m_1, m_2]$, in which case $(\tau, d_\tau, \alpha_\tau)$ is stationary over $[m_1, m_2]$, a contradiction. Therefore, stopping at S' is only weakly optimal. This implies that, for each $m \in [m_1, m_2]$, either $(S(m), \gamma(m)) = (S', 0)$, or $S(m) = S' - \epsilon$ and $\gamma(m) \in [0, 1)$. We abuse notation slightly by calling the $(S', 0)$-stopping threshold an $(S' - \epsilon, 1)$-stopping threshold, so $S(\cdot)$ is constant over $[m_1, m_2]$ and $\gamma(\cdot)$ must be non-constant over $[m_1, m_2]$. Thus, $\gamma(m') \neq \gamma(m' + \epsilon)$ for some $m' \in [m_1, m_2)$. Fix $m' = \max\{m \in [m_1, m_2] : \ \gamma(m) \neq \gamma(m + \epsilon) = \gamma(m'') \ \forall m'' \in [m + \epsilon, m_2]\}$.

Consider a change of the contract which moves $\gamma(m')$ and $\gamma(m'')$ closer together for all $m'' \in [m' + \epsilon, m_2]$. If $\gamma(m') < \gamma(m' + \epsilon)$, then, for all $m'' \in [m' + \epsilon, m_2]$, decrease $\gamma(m'')$ to $\gamma(m')$. Such a change must increase A's continuation value at $V(m', m'')$: $V(m', m'') = V(m', m')$ when $\gamma(m') = \gamma(m'')$, so if decreasing $\gamma(m'')$ lowered $V(m', m'')$, then we could increase $\gamma(m')$ to $\gamma(m'') = \gamma(m' + \epsilon)$ and increase $V(m', m')$, making both players better off, P strictly so.[9] Moreover, increasing $V(m', m'')$ increases $V(m'', m'')$, so all $RDIR$ constraints continue to hold after this change,[10] contradicting the optimality of our original contract. Therefore, $\gamma(m') > \gamma(m' + \epsilon)$.

For each $m'' \in [m' + \epsilon, m_2]$, increase $\gamma(m'')$ (by the same amount for all $m'' \in [m' + \epsilon, m_2]$ to keep all such $\gamma(m'')$ equal); such a change will decrease all $V(m'', m'')$ as well as $V(m', m')$, so we also decrease $\gamma(m')$ at a rate to keep $V(m', m')$ constant, proceeding in this way until

[8] When $\lambda^n = 0$, the continuation problem in $\mathcal{L}(\Lambda)$ at $(X_t, M_t) = (x, X^{n-1})$ is the same as at (x, X^n), so the decision of when it is optimal to stop is the same.

[9] Because $\alpha(\cdot)$ is decreasing, P strictly prefers stopping sooner, i.e., a higher γ.

[10] Because $S(\cdot), \gamma(\cdot), \alpha(\cdot)$ are constant across $[m' + m_2]$, so is $V(m', \cdot)$. We then have $V(m'', m'') = \mathbb{E}_{m''}[e^{-r\tau_-(m')} V(m', m'') \mathbb{1}(\tau_-(m') < \tau_+(m_2)) + e^{-r\tau_+(m_2)} V(m_2, m_2) \mathbb{1}(\tau_+(m_2) < \tau_-(m'))]$. Thus, any increase in $V(m', m'')$ increases $V(m'', m'')$.

either $\gamma(m') = \gamma(m'+\epsilon)$ or $V(m'', m'') = m''$ for some $m'' \in [m'+\epsilon, m_2]$ (i.e., $RDIR(m'')$ binds). In the either case, the new contract will still solve $\mathcal{L}(\Lambda)$[11] and satisfy all $RDIR$ constraints, and thus will be a solution to RDP. We can proceed iteratively in this way until we are left with a contract satisfying the desired properties. $\square$

Lemma O.A.3. *S^* is constant if $u_i(z) = z$ for both $i \in \{P, A\}$ and is strictly decreasing if u_A or u_P is strictly concave and $e^{-rt}Y_t$ is a strict supermartingale.*

Proof. Let $\hat{v}(y, m)$ be A's continuation value under $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ at $(Y_t, M_t) = (y, m)$ for $y > S^*(m)$; by standard arguments (Peskir and Shiryaev (2006)), $\hat{v}(y, m)$ is continuous in both arguments. Take any $y' < y \leq m < m'$ such that $y' > S^*(m)$. We then have[12]

$$\begin{aligned}\hat{v}(y, m) = \mathbb{E}_y[&e^{-r\tau_-(y')}\hat{v}(y', M_{\tau_-(y')} \vee m)\mathbb{1}(\tau_-(y') < \tau_+(m')) \\ &+ e^{-r\tau_+(m')}\hat{v}(m', m')\mathbb{1}(\tau_-(y') > \tau_+(m'))].\end{aligned} \tag{2}$$

Suppose $e^{-rt}Y_t$ is a strict supermartingale and $\hat{v}(y', \cdot)$ is not strictly increasing on some interval $[m, m']$. Without loss, we can take m and m' such that $\hat{v}(y', m) \geq \hat{v}(y', m'')$ for all $m'' \in [m, m']$. Because $M_{\tau_-(y')} \vee m''$ is increasing in m'', $\hat{v}(y', M_{\tau_-(y')} \vee m) \geq \hat{v}(y', M_{\tau_-(y')} \vee m'')$ (conditional on $\tau_-(y) < \tau_+(m')$) for $m'' \in [m, m']$ and (2) implies $\hat{v}(y, m) \geq \hat{v}(y, m'')$. This holds for all y, m such that $y' < y \leq m < m'$. Taking $y = m$, for all $m'' \in [m, m']$ we have $\hat{v}(m, m) \geq \hat{v}(m, m'')$. By Lemma A.7 (and taking the limit as $\Delta \to 0$), we have $\hat{v}(m, m) = m$. Thus, conditional on $\tau_-(m') < \tau_-(m)$, we have $\hat{v}(m, M_{\tau_-(m)} \vee m'') \leq m$. But then (2), after replacing y', m, y with m, m'', m'' respectively for some $m'' \in (m, m')$, and using $\hat{v}(m', m') = m'$, we have

$$\begin{aligned}\hat{v}(m'', m'') &= \mathbb{E}_{m''}[e^{-r\tau_-(m)}\hat{v}(m, M_{\tau_-(m)} \vee m'')\mathbb{1}(\tau_-(m) < \tau_+(m')) \\ &\qquad + e^{-r\tau_+(m')}\hat{v}(m', m')\mathbb{1}(\tau_-(m) > \tau_+(m'))] \\ &\leq \mathbb{E}_{m''}[e^{-r\tau_-(m)}m\mathbb{1}(\tau_-(m) < \tau_+(m')) + e^{-r\tau_+(m')}m'\mathbb{1}(\tau_-(m) > \tau_+(m'))] \\ &= \mathbb{E}_{m''}[e^{-r(\tau_-(m)\wedge\tau_+(m'))}Y_{\tau_-(m)\wedge\tau_+(m')}] \\ &< m''\end{aligned}$$

where the last inequality follows from Doob's optional stopping theorem and the fact that $e^{-rt}Y_t$ is a strict supermartingale. But this contradicts $\hat{v}(m'', m'') \geq m''$ by DIR.[13] Therefore, $\hat{v}(y', \cdot)$ must be strictly increasing if $e^{-rt}Y_t$ is a strict supermartingale.

[11] Because stopping and continuing are both optimal in $\mathcal{L}(\Lambda)$ at $(X_t, M_t) = (S'+\epsilon, m'')$ for $m'' \in [m_1, m_2]$, any (S', γ)-stopping threshold at such t will be optimal.

[12] The use of $\vee m$ in $M_{\tau_-(y')} \vee m$ captures the fact our expectation is set to start from $(Y_0, M_0) = (y, y)$ while we want the true value of M_t at $\tau_-(y)$ to be m if $M_{\tau_-(y)} < m$ when $(Y_0, M_0) = (y, y)$.

[13] The part of the proof of Theorem 1 establishing DIR holds does not rely on this lemma.

Fix some $m \in (Y_0, \overline{R}^*)$ and let $F(V)$ be P's continuation value from the optimal DIR contract delivering V continuation value to A when starting at m. It is easy to see that $F(\hat{v}(m, m'))$ is P's continuation value under the optimal contract when $(Y_t, M_t) = (m, m')$. Because we allowed for randomization devices in RDP, we can add a public randomization device to our continuous-time model without changing the structure of the optimal contract, in which case standard arguments imply $F(\cdot)$ is concave. Let $\Phi(S), \phi(S)$ be as defined in the text (for our choice of m). Consider the problem for P of choosing a fixed threshold and demand S, α and a continuation value V subject to delivering w expected utility to A:

$$\max_{S,\alpha,V} \ \phi(S)u_P(\alpha) + \Phi(S)F(V) \tag{3}$$
$$\text{subject to } w = \phi(S)u_A(1-\alpha) + \Phi(S)V.$$

For $w = \hat{v}(Y_0, m')$, the optimal choice of S, α, V above will be $S^*(m')$, $\alpha^*(m')$ and $\hat{v}(m, m')$.[14] Let $\alpha = 1 - u_A^{-1}(\frac{w - \Phi(S)V}{\phi(S)})$ be the value of α satisfying the constraint; for notational ease, we suppress the dependence of α on S, V, w. Then (3) is equal to $\max_{S,V} \ \phi(S)u_P(\alpha) + \Phi(S)F(V)$.

If u_P and u_A are linear, then (3) simplifies to $\max_{S,V} \ \phi(S) + \Phi(S)(V + F(V)) - w$. The optimal choice of S is clearly independent of w, which implies S^* is constant.

Suppose u_P or u_A is strictly concave and $e^{-rt}Y_t$ is a strict supermartingale. Because F is concave, it is differentiable almost everywhere. Without loss, consider an m' such that F is differentiable at $\hat{v}(m, m')$. The first-order condition for V is given by, after substituting in $\frac{\partial \alpha}{\partial V}$,

$$\Phi(S)(F'(V) + \frac{u_P'(\alpha)}{u_A'(1-\alpha)}) = 0, \tag{4}$$

and first-order condition for S is given by, after substituting in $\frac{\partial \alpha}{\partial S}$,

$$\phi'(S)(u_P(\alpha) + \frac{u_P'(\alpha)}{u_A'(1-\alpha)}u_A(1-\alpha)) + \Phi'(S)(F(V) + V\frac{u_P'(\alpha)}{u_A'(1-\alpha)}) = 0. \tag{5}$$

Suppose S^* is constant at m'. A higher m' translates into a higher w because $\hat{v}(Y_0, m')$ is strictly increasing in m'. Then the optimal S in (3) is constant in w at $w = \hat{v}(Y_0, m')$, in which case (5) must hold at this optimal S as we increase w. Thus, the derivative of the left-hand side of (5) with respect to w must equal 0. Taking this derivative and simplifying and using $F'(V) = -\frac{u_P'(\alpha)}{u_A'(1-\alpha)}$ by (4), we have

$$(\frac{\partial \alpha}{\partial w} + \frac{\partial \alpha}{\partial V}\frac{\partial V}{\partial w}))\big(\frac{u_P''(\alpha)}{u_P'(\alpha)} + \frac{u_A''(1-\alpha)}{u_A'(1-\alpha)}\big) \cdot \frac{u_P'(\alpha)}{u_A'(1-\alpha)}(\phi'(S)u_A(1-\alpha) + \Phi'(S)V) = 0. \tag{6}$$

[14] If they did not, then we can construct a strictly better contract that is equal to $(\tau^*, d^*_{\tau^*}, \alpha^*_{\tau^*})$ prior to reaching (Y_0, M_t) at which point it uses a continuation contract with a constant split threshold of S and demand $\alpha^*(m')$ that solves (3) before switching to the continuation contract that delivers $F(V)$.

By (5), $\frac{u_P'(\alpha)}{u_A'(1-\alpha)}(\phi'(S)u_A(1-\alpha)+\Phi'(S)V) = -(\phi'(S)u_P(\alpha)+\Phi'(S)F(V)) < 0$, where the inequality follows from the fact that because P always prefers to stop sooner, P must strictly prefer a higher S, namely $\phi'(S)u_P(\alpha)+\Phi'(S)F(V) > 0$. Because $\min\{u_P''(\alpha), u_A''(1-\alpha)\} < 0$ by strict concavity, for (6) to hold, it must be that $\frac{\partial\alpha}{\partial w} + \frac{\partial\alpha}{\partial V}\frac{\partial V}{\partial w} = 0$, namely the optimal choice of α is constant in w when the optimal S is also constant in w. But this implies that, for m in the region, call it $[m_1, m_2]$, over which S and α are constant (say at S', α'), we have $\hat{v}(Y_0, m) = \mathbb{E}[e^{-r\tau_-(S')}u_A(1-\alpha')\mathbb{1}(\tau_-(S') < \tau_+(m_2)) + e^{-r\tau_+(m_2)}\hat{v}(m_2, m_2)\mathbb{1}(\tau_-(S') > \tau_+(m_2))]$, which is constant in m, a contradiction. Therefore, S^* must be strictly decreasing. □

Comparative Statics Proofs

It is without loss to assume a unique $\arg\max_{y\in(\underline{Y},\overline{Y})} \sigma(y)$ exists and is above $\overline{R}^*$. If not, then we can increase $\sigma(y)$ for y sufficiently close to $\overline{Y}$ without changing the incentives to take the outside option at $\overline{R}^*$.[15] A similar argument holds for $\hat{Y}$ and $\hat{\sigma}(y)$. Let $\overline{R}^+$ be the max over the breakdown threshold in the optimal contract for Y and $\hat{Y}$. For the rest of the proof, we assume $\arg\max_{y\in(\underline{Y},\overline{Y})} \sigma(y) = \arg\max_{y\in(\underline{Y},\overline{Y})} \hat{\sigma}(y) > \overline{R}^+$ and $\sigma_0 = \max_{y\in(\underline{Y},\overline{Y})} \sigma(y) = \max_{y\in(\underline{Y},\overline{Y})} \hat{\sigma}(y)$.[16]

We will combine the proofs of Propositions 1 and 2 and so will assume throughout that $(\hat{\mu}, \hat{\sigma})$ are such that either $\hat{\mu} > \mu$ and $\hat{\sigma} = \sigma$, or $\hat{\mu} = \mu \le 0$ and $\hat{\sigma} > \sigma$. The proofs will look at the discrete-time versions of RDP for X approximating Y and $\hat{Y}$ in which we choose $\underline{X}, \overline{X}$ to be the same in both approximations.[17] Let $\Xi(x) := [\mu(x), \hat{\mu}(x)] \times [\sigma(x), \hat{\sigma}(x)]$ and $\Xi := \{(\tilde{\mu}, \tilde{\sigma}) : (\tilde{\mu}(x), \tilde{\sigma}(x)) \in \Xi(x)\ \forall x\}$. Throughout, when we condition in expectations on $(\tilde{\mu}, \tilde{\sigma})$, we mean that the transitions probabilities q_+, q_- for X are governed by $(\tilde{\mu}, \tilde{\sigma})$—namely, we replace (μ, σ) in the formulas for q_+, q_- with $(\tilde{\mu}, \tilde{\sigma})$.

Lemma O.A.4. *P's value of the negotiation is higher under $\hat{Y}$ than Y.*

Proof. Consider a version of RDP in which P can also choose the (μ, σ) governing governing X at each date, subject to $(\mu(X_t), \sigma(X_t)) \in \Xi(X_t)$ for all t. Formally, we let P choose a

[15] For $\delta > 0$, the expected length of time to reach $\overline{Y} - \delta$ goes to ∞ as $\delta \to 0$. Thus, it will never be optimal to continue until $\overline{Y} - \delta$ for sufficiently small δ, regardless of what happens to the evolution of Y above $\overline{Y} - \delta$.

[16] We make this assumption to ensure that, as we change $\sigma(\cdot)$, we are not also changing the step size ϵ, which is set equal to $\max_x \sigma(x)\sqrt{\Delta}$ in the random walk X approximating Y (and similar for $\hat{Y}$).

[17] The exact values of $\underline{X}, \overline{X}$ are not important in the proof of Theorem 1 other than that they converge to $\underline{Y}, \overline{Y}$ as $\Delta \to 0$.

function (μ^P, σ^P) that maps each history h_t into a choice in $\Xi(X_t)$ and consider the problem

$$\sup_{(\tau, d_\tau, \alpha_\tau),\ (\mu^P, \sigma^P)} \mathbb{E}[e^{-r\tau} d_\tau u_P(\alpha_\tau) | (\mu^P, \sigma^P)] \tag{7}$$

$$\text{subject to, } \forall n = 0, ..., N,$$

$$RDIR(X_n): \ \mathbb{E}[e^{-r(\tau \wedge \tau_+(X^n))}(d_\tau(X^n)(u_A(1-\alpha_\tau) - X_{\tau\wedge\tau_+(X^n)}) + X_{\tau\wedge\tau_+(X^n)})|(\mu^P, \sigma^P)]$$
$$\leq \mathbb{E}[e^{-r\tau}(d_\tau(u_A(1-\alpha_\tau) - X_\tau) + X_\tau)|(\mu^P, \sigma^P)].$$

Analogous arguments to those in the proof of Theorem 1 imply the optimal contract and choice of (μ^P, σ^P) are stationary in (X, M)[18] and, for some multipliers $(\lambda^0, ..., \lambda^N) \in \mathbb{R}^{N+1}_-$, they solve the Lagrangian

$$\max_{(\tau, d_\tau, \alpha_\tau),\ (\mu^P, \sigma^P)} \mathbb{E}\big[e^{-r\tau}\big(d_\tau u_P(\alpha_\tau) - \sum_{n=0}^{N} \lambda^n \mathbb{1}(M_\tau \geq X^n)\{d_\tau(u_A(1-\alpha_\tau) - X_\tau) + X_\tau\}\big)$$
$$+ \sum_{n=0}^{N} \lambda^n \mathbb{1}(M_\tau \geq X^n) e^{-r\tau_+(X^n)} X^n | (\mu^P, \sigma^P)\big].$$

We start by showing it is optimal to choose $(\hat{\mu}(X_t), \hat{\sigma}(X_t))$ at $t < \tau_+(X^1)$. As in Lemma A.5, let $\overline{u}(\lambda^0) = \max_\alpha \ u_P(\alpha^0) - \lambda^0 u_A(1-\alpha^0)$, which gives the value of stopping at $X_t = x$ for $t < \tau_+(X^1)$, and let $K(X^1)$ be the continuation value in our Lagrangian at $\tau_+(X^1)$. The value of the Lagrangian at $t < \tau_+(X^1)$ when $X_t = x$ is[19]

$$L^*(x) = \max_{\tau, (\mu^P, \sigma^P)} \mathbb{E}_x[e^{-r\tau}\overline{u}(\lambda^0)\mathbb{1}(\tau < \tau_+(X^1)) + e^{-r\tau_+(X^1)} K(X^1) \mathbb{1}(\tau \geq \tau_+(X^1))|(\mu^P, \sigma^P)].$$

Standard optimal stopping arguments imply $L^*(x) \geq \overline{u}(\lambda^0) > 0$ for all $x < X^1$. Let (μ^*, σ^*) be the optimal choice of (μ^P, σ^P). By the same arguments as in the proof of Lemma A.5, there exists (S^0, γ^0) such that τ^{S^0, γ^0} is an optimal stopping rule in $L^*(x)$ for all $x < X^1$.

Standard dynamic programming arguments imply that, if not stopping is weakly optimal at x (which is true for all $x > S^0$), then

$$L^*(x) = \max_{(\tilde{\mu}(x), \tilde{\sigma}(x)) \in \Xi(x)} e^{-r\Delta}\big[\frac{1}{2}((\frac{\tilde{\sigma}(x)}{\sigma_0})^2 + \frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})L^*(x+\epsilon)$$
$$+ \frac{1}{2}((\frac{\tilde{\sigma}(x)}{\sigma_0})^2 - \frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})L^*(x-\epsilon) + (1 - (\frac{\tilde{\sigma}(x)}{\sigma_0})^2)L^*(x)\big], \tag{8}$$

[18] Stationarity in (X, M) for (μ^P, σ^P) means the optimal (μ^P, σ^P) can be written as a function $(\tilde{\mu}(X_t, M_t), \tilde{\sigma}(X_t, M_t))$.

[19] If $t = 0$, we drop the constant $\lambda^0 X_0$ because it does not affect the optimal choice of τ or $(\tilde{\mu}, \tilde{\sigma})$.

with $(\mu^*(x), \sigma^*(x))$ in the arg max of (8).

Because stopping is optimal (at least weakly) at S^0, $L^*(S^)) = \overline{u}(\lambda^0)$. By $L^*(x) \geq \overline{u}(\lambda^0)$ for all $x < X^1$, we have $L^*(S^0) = \overline{u}(\lambda^0) \leq L^*(S^0 + \epsilon)$. Using this observation, we show $L^*(x) < L^*(x+\epsilon)$ for all $x \in (S^0, X_0]$. We proceed by induction (starting at $x = S^0 + \epsilon$), showing $L^*(x) < L^*(x+\epsilon)$ whenever $L^*(x-\epsilon) \leq L^*(x)$. Suppose not, so that, for some $x \in (S_0, X_0]$, $\max\{L^*(x-\epsilon), L^*(x+\epsilon)\} \leq L^*(x)$. Then (8) implies

$$\begin{aligned} L^*(x) &\leq e^{-r\Delta}\big[\frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 + \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})L^*(x) \\ &\quad + \frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 - \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})L^*(x) + (1 - (\frac{\sigma^*(x)}{\sigma_0})^2)L^*(x)\big] \\ &= e^{-r\Delta}L^*(x), \end{aligned}$$

a contradiction. We conclude $L^*(x) < L^*(x+\epsilon)$.

We now argue $(\hat{\mu}(x), \hat{\sigma}(x))$ is in the arg max of (8). If $x \leq S^0$, then $(\hat{\mu}(x), \hat{\sigma}(x))$ is weakly optimal (in fact any choice of $(\tilde{\mu}(x), \tilde{\sigma}(x))$ is optimal). Suppose for the rest of the proof that $x \in (S^0, X_0]$.

Consider the case in which $\sigma = \hat{\sigma}$ and $\hat{\mu} > \mu$. The derivative of the right-hand side of (8) with respect to $\tilde{\mu}(x)$ is $\frac{e^{-r\Delta}\sqrt{\Delta}}{2\sigma_0}[L^*(x+\epsilon) - L^*(x-\epsilon)] > 0$. Therefore, the uniquely optimal choice of $\tilde{\mu}(x)$ is $\hat{\mu}(x)$.

Now consider the case in which $\hat{\sigma} > \sigma$ and $\mu = \hat{\mu} \leq 0$. The derivative of the right-hand side of (8) with respect to $\tilde{\sigma}(x)$ is $\frac{2e^{-r\Delta}\tilde{\sigma}(x)}{\sigma_0^2}[\frac{1}{2}L^*(x+\epsilon) + \frac{1}{2}L^*(x-\epsilon) - L^*(x)]$. Therefore, $\tilde{\sigma}(x) = \hat{\sigma}(x)$ is strictly optimal if and only if $\frac{1}{2}L^*(x+\epsilon) + \frac{1}{2}L^*(x-\epsilon) > L^*(x)$. Rearranging terms in (8), we get

$$\begin{aligned} L^*(x) &= \frac{e^{-r\Delta}\frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 + \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})}{1 - e^{-r\Delta}(1 - (\frac{\sigma^*(x)}{\sigma_0})^2)}L^*(x+\epsilon) + \frac{e^{-r\Delta}\frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 - \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})}{1 - e^{-r\Delta}(1 - (\frac{\sigma^*(x)}{\sigma_0})^2)}L^*(x-\epsilon) \\ &< \frac{\frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 + \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})}{(\frac{\sigma^*(x)}{\sigma_0})^2}L^*(x+\epsilon) + \frac{\frac{1}{2}((\frac{\sigma^*(x)}{\sigma_0})^2 - \frac{\mu^*(x)\sqrt{\Delta}}{\sigma_0})}{(\frac{\sigma^*(x)}{\sigma_0})^2}L^*(x-\epsilon) \\ &= \frac{1}{2}L^*(x+\epsilon) + \frac{1}{2}L^*(x-\epsilon) + \frac{\mu^*(x)\sigma_0\sqrt{\Delta}}{2(\sigma^*(x))^2}[L^*(x+\epsilon) - L^*(x-\epsilon)] \\ &\leq \frac{1}{2}L^*(x+\epsilon) + \frac{1}{2}L^*(x-\epsilon), \end{aligned}$$

where the final inequality follows from $\mu^*(x) \leq 0$ and $L^*(x+\epsilon) > L^*(x-\epsilon)$. We conclude $\hat{\sigma}(x)$ is the unique optimal choice of $\tilde{\sigma}(x)$.

The above argument shows $(\hat{\mu}(X_t), \hat{\sigma}(X_t))$ is the optimal choice at $t < \tau_+(X^1)$. We can repeat the above arguments at $\tau_+(X^1)$ to conclude $(\hat{\mu}(X_t), \hat{\sigma}(X_t))$ is also the optimal choice at $t \in [\tau_+(X^1), \tau_+(X^2))$. Proceeding in this way, we conclude $(\hat{\mu}, \hat{\sigma})$ is P's optimal choice of (μ^P, σ^P).

The value of our problem in (7) is clearly at least as large as the value in RDP when X is the discrete-time approximation to Y since (μ, σ) is a feasible choice of (μ^P, σ^P) in (7). Moreover, because $(\hat{\mu}, \hat{\sigma})$ is the optimal choice of (μ^P, σ^P), the value of (7) is equal to the value in RDP when X is the discrete-time approximation to $\hat{Y}$. Taking the limit as $\Delta \to 0$ yields our desired conclusion. □

All that is left to show is that $\hat{\alpha}^* \geq \alpha^*$. Let $\alpha(m)$, $(S(m), \gamma(m))$ and $\hat{\alpha}(m)$, $(\hat{S}(m), \hat{\gamma}(m))$ be P's demand function and thresholds in the solution to RDP under the discrete-time approximations to Y and $\hat{Y}$ respectively.

We now show $\hat{\alpha}(X_0) \geq \alpha(X_0)$. We adopt the convention that if taking the outside option immediately is optimal, then P's demand is 0. Thus, $\hat{\alpha}(X_0) \geq \alpha(X_0)$ clearly holds if taking the outside option immediately is optimal in RDP when X approximates Y. Moreover, Lemma O.A.4 implies that if taking the outside option immediately is optimal under the discrete-time approximation to $\hat{Y}$, then it is also optimal under the discrete-time approximation to Y, in which case $\hat{\alpha}(X_0) = \alpha(X_0) = 0$. We henceforth assume it is not optimal to immediately take the outside option in the RDP for X approximating Y or $\hat{Y}$.

Suppose $S(X_0) = X_0$. By Lemma A.7, $X_0 = V(X_0, X_0)$ and $V(X_0, X_0) = u_A(1 - \alpha(X_0))$ when $S(X_0) = X_0$; thus, $\alpha(X_0) = 1 - u_A^{-1}(X_0)$. Similarly, $\hat{\alpha}(X_0) = 1 - u_A^{-1}(X_0)$ if $\hat{S}(X_0) = X_0$, in which case we have $\hat{\alpha}(X_0) = \alpha(X_0)$. If $\hat{S}(X_0) < X_0$, then $\hat{\alpha}(X_0) > 1 - u_A^{-1}(X_0)$; otherwise, if $\hat{\alpha}(X_0) \leq 1 - u_A^{-1}(X_0)$ and $\hat{S}(X_0) < X_0$, then because $\hat{\alpha}$ is decreasing, $\hat{\alpha}(M_\tau) \leq \hat{\alpha}(X_0)$ and P would be better off immediately implementing a split that gives him $1 - u_A^{-1}(X_0)$ share of the pie. Thus, $\hat{\alpha}(X_0) \geq \alpha(X_0)$ whenever $S(X_0) = X_0$.

Now suppose $\hat{S}(X_0) = X_0$, which implies $\hat{\alpha}(X_0) = 1 - u_A^{-1}(X_0)$. It is straightforward from the arguments in Lemma O.A.4 that $\hat{S}(X_0) = X_0$ implies $S(X_0) = X_0$ so $\alpha(X_0) = 1 - u_A^{-1}(X_0)$, in which case $\alpha(X_0) = \hat{\alpha}(X_0)$. We therefore focus on Y and $\hat{Y}$ for which $\max\{\hat{S}(X_0), S(X_0)\} < X_0$.

We now prove several supporting Lemmas before showing $\hat{\alpha}(X_0) \geq \alpha(X_0)$.

Lemma O.A.5. $u_A(1 - \alpha(m)) < m + \epsilon$.

Proof. If $u_A(1 - \alpha(m)) \geq m + \epsilon$, then, because $V(m, m) = m$, A would be better off taking a split giving him $1 - \alpha(m)$ immediately at $\tau_+(m)$. Doing so would strictly increase P's expected utility: because $\alpha(m)$ is decreasing, $J(m, m) = \mathbb{E}_m[e^{-r\tau} d_\tau u_P(\alpha(M_\tau))] \leq \mathbb{E}_m[e^{-r\tau} d_\tau u_P(\alpha(m))] < u_P(\alpha(m))$, contradicting the optimality of our original contract. □

Our next Lemma will show that, under the optimal contract in *RDP* for X approximating Y, A prefers X to be governed by $(\hat{\mu}, \hat{\sigma})$ rather than (μ, σ). Fix any $m < \overline{R}$ and, for $x \leq m$ define $\tilde{V}(x, \tilde{\mu}, \tilde{\sigma})$ to be

$$\tilde{V}(x, \tilde{\mu}, \tilde{\sigma}) = \mathbb{E}_x[e^{-r\tau^{S(m),\gamma(m)}} u_A(1-\alpha(m))\mathbb{1}(\tau_+(m+\epsilon) > \tau^{S(m),\gamma(m)}) \\ + e^{-r\tau_+(m+\epsilon)}(m+\epsilon)\mathbb{1}(\tau_+(m+\epsilon) \leq \tau^{S(m),\gamma(m)})|(\tilde{\mu}, \tilde{\sigma})].$$

We note that $\tilde{V}(X_t, \mu, \sigma)$ is A's continuation value in *RDP* for X approximating Y at $t \in [\tau_+(m), \tau_+(m+\epsilon))$.

Lemma O.A.6. *For $x > S(m)$, $\tilde{V}(x, \mu, \sigma) < \tilde{V}(x, \hat{\mu}, \hat{\sigma})$.*

Proof. Let $\tilde{V}^*(x) = \max_{(\tilde{\mu},\tilde{\sigma})\in\Xi} \tilde{V}(x, \tilde{\mu}, \tilde{\sigma})$. The lemma follows immediately if we can show that $(\hat{\mu}, \hat{\sigma})$ is the strictly optimal choice in $\tilde{V}^*$.

We first prove that $\tilde{V}^*(x) < \tilde{V}^*(x+\epsilon)$ for $x \in [S(m), m]$ by induction. If $\tilde{V}(S(m)+\epsilon, \mu, \sigma) \leq u_S(1-\alpha(m))$, then, in *RDP* for X approximating Y, P would be better off using a $(S(m)+\epsilon, 0)$-stopping threshold between $[\tau_+(m), \tau_+(m+\epsilon))$ rather than the $(S(m), \gamma(m))$-stopping threshold because switching weakly increases A's expected utility and strictly increases P's expected utility,[20] contradicting the optimality of using $(S(m), \gamma(m))$. Thus, $\tilde{V}^*(S(m)+\epsilon) \geq \tilde{V}(S(m)+\epsilon, \mu, \sigma) > u_S(1-\alpha(m))$ Because $\tilde{V}^*(S(m)) = u_A(1-\alpha(m))$, we have $\tilde{V}^*(S(m)+\epsilon) > \tilde{V}^*(S(m))$.

For the sake of contradiction, suppose there exists $x' \in (S(m), m]$ such that $\tilde{V}^*(x') \geq \tilde{V}^*(x'+\epsilon)$. Let x be the lowest such x', which implies $\tilde{V}^*(x) \geq \max\{\tilde{V}^*(x-\epsilon), \tilde{V}^*(x+\epsilon)\}$ and $\tilde{V}^*(x) \geq \tilde{V}^*(S(m)+\epsilon)$. Let $\zeta(x) = \mathbb{1}(x = S(m)+\epsilon)\gamma(m)$, which gives the probability of implementing a split at t with $(X_t, M_t) = (x, m)$ and $x > S(m)$. Then

$$\tilde{V}^*(x) = \max_{(\tilde{\mu}(x),\tilde{\sigma}(x))\in\Xi(x)} \zeta(x)u_A(1-\alpha(m)) \\ + (1-\zeta(x))e^{-r\Delta}\Bigg[\frac{1}{2}(\frac{\tilde{\sigma}(x)^2}{\sigma_0^2} + \frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})\tilde{V}^*(x+\epsilon) \\ + \frac{1}{2}(\frac{\tilde{\sigma}(x)^2}{\sigma_0^2} - \frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})\tilde{V}^*(x-\epsilon) + (1-\frac{\tilde{\sigma}(x)^2}{\sigma_0^2})\tilde{V}^*(x)\Bigg]. \quad (9)$$

Using $u_A(1-\alpha(m)) < \tilde{V}^*(S(m)+\epsilon) \leq \tilde{V}^*(x)$ and $\tilde{V}^*(x) \geq \max\{\tilde{V}^*(x-\epsilon), \tilde{V}^*(x+\epsilon)\}$, (9)

[20] P strictly benefits from immediately implementing a split with demand $\alpha(M_s)$ at dates s with $(X_s, M_s) = (S(m)+\epsilon, m)$ because $\alpha(M_t)$ is only decreasing over time.

implies

$$\tilde{V}^*(x) \leq \max_{(\tilde{\mu}(x),\tilde{\sigma}(x))\in\Xi(x)} e^{-r\Delta}\Big[\frac{1}{2}(\frac{\tilde{\sigma}(x)^2}{\sigma_0^2}+\frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})\tilde{V}^*(x) + \frac{1}{2}(\frac{\tilde{\sigma}(x)^2}{\sigma_0^2}-\frac{\tilde{\mu}(x)\sqrt{\Delta}}{\sigma_0})\tilde{V}^*(x)+(1-\frac{\tilde{\sigma}(x)^2}{\sigma_0^2})\tilde{V}^*(x)\Big]$$
$$= e^{-r\Delta}\tilde{V}^*(x),$$

a contradiction. We conclude $\tilde{V}^*(x) < \tilde{V}^*(x+\epsilon)$ for $x \in [S(m), m]$. Analogous arguments to those in Lemma O.A.4 imply $(\hat{\mu}, \hat{\sigma})$ is strictly optimal in V^*. □

Our next Lemma looks at properties of the optimal-stopping rule in a problem analogous to our Lagrangian $\mathcal{L}(\Lambda)$. Define functions η_P, η_A giving P and A's expected utility for a fixed (S, γ, α) and $(\tilde{\mu}, \tilde{\sigma})$ when holding fixed their continuation value at $\tau_+(X^1)$:

$$\eta_P(S,\gamma,\alpha,\tilde{\mu},\tilde{\sigma},\tilde{J}) = \mathbb{E}[e^{-r\tau^{S,\gamma}}u_P(\alpha)\mathbb{1}(\tau_+(X^1) > \tau^{S,\gamma}) + e^{-r\tau_+(X^1)}\tilde{J}\mathbb{1}(\tau_+(X^1)\leq\tau^{S,\gamma})|(\tilde{\mu},\tilde{\sigma})],$$
$$\eta_A(S,\gamma,\alpha,\tilde{\mu},\tilde{\sigma}) = \mathbb{E}[e^{-r\tau^{S,\gamma}}u_A(1-\alpha)\mathbb{1}(\tau_+(X^1) > \tau^{S,\gamma}) + e^{-r\tau_+(X^1)}X^1\mathbb{1}(\tau_+(X^1)\leq\tau^{S,\gamma})|(\tilde{\mu},\tilde{\sigma})].$$

Let $\overline{\eta}$ maximize (over S, γ, α) a weighted sum of η_P, η_A for some $\tilde{\lambda} \leq 0$:

$$\overline{\eta}(\tilde{\lambda},\tilde{\mu},\tilde{\sigma},\tilde{J}) = \max_{S,\gamma,\alpha} \eta_P(S,\gamma,\alpha,\tilde{\mu},\tilde{\sigma},\tilde{J}) - \tilde{\lambda}\eta_P(S,\gamma,\alpha,\tilde{\mu},\tilde{\sigma},\tilde{J}).$$

Letting λ^0 be the multiplier on $RDIR(X_0)$ and $J(X^1, X^1)$ is P's continuation value at (X^1, X^1) in RDP for X approximating Y, because A's continuation value at $\tau_+(X^1)$ is equal to X^1 in RDP, $\overline{\eta}(\lambda^0, \mu, \sigma, J(X^1, X^1))$ is equal to $\mathcal{L}(\Lambda)$ (after dropping the constant $\lambda^0\mathbb{1}(M_\tau \geq X^0)e^{-r\tau_+(X^0)})X^0$, which is realized at $t = 0$ regardless of the choice of $(\tau, d_\tau, \alpha_\tau)$ and so is decision irrelevant)..

Next we look at how the optimal thresholds in $\overline{\eta}$ depend with $\tilde{\mu}, \tilde{\sigma}, \tilde{J}$. Let $\mathcal{S}(\tilde{\lambda}, \tilde{\mu}, \tilde{\sigma}, \tilde{J})$ be the set of (S, γ) in the arg max of $\overline{\eta}(\tilde{\lambda}, \tilde{\mu}, \tilde{\sigma}, \tilde{J})$. Let J and $\hat{J}$ be P's continuation value at $\tau_+(X^1)$ under the solution to RDP when X approximates to Y and $\hat{Y}$ respectively. By the arguments in Lemma O.A.4, we know $J < \hat{J}$.[21]

Lemma O.A.7. *If $(S, \gamma) \in \mathcal{S}(\lambda, \mu, \sigma, J)$ and $(S', \gamma') \in \mathcal{S}(\lambda, \hat{\mu}, \hat{\sigma}, \hat{J})$, then $S' < S$, or $S' = S$ and $0 = \gamma' \leq \gamma$.*

[21]Because A's continuation contract at $\tau_+(m)$ is equal to m, the optimal continuation contract at $\tau_+(m)$ is equal to the optimal contract if $X_0 = m$, and so Lemma O.A.4 implies P's continuation value at $\tau_+(m)$ is higher under the discrete-time approximation to $\hat{Y}$ than under the discrete-time approximation to Y. Moreover, it is clear from the proof that this inequality is strict whenever it is not optimal to immediately stop.

Proof. Fix some $\lambda \le 0$ and let $\overline{u}(\lambda) = \max_\alpha \; u_P(\alpha) - \lambda u_A(1-\alpha)$. Define

$$L(x;\tilde{\mu},\tilde{\sigma},\tilde{J}) = \sup_\tau \; \mathbb{E}_x[e^{-r\tau}\overline{u}(\lambda)\mathbb{1}(\tau < \tau_+(X^1)) + e^{-r\tau_+(X^1)}(\tilde{J} - \lambda X^1)\mathbb{1}(\tau_+(X^1) \le \tau)|(\tilde{\mu},\tilde{\sigma})]. \tag{10}$$

Because $\tau^{S,\gamma}$ is a feasible choice above for all (S,γ), $L(X_0;\tilde{\mu},\tilde{\sigma},\tilde{J}) \ge \overline{\eta}(\lambda,\tilde{\mu},\tilde{\sigma},\tilde{J})$. By the same arguments as in Lemma A.5, there exists (S,γ) such that $\tau^{S,\gamma}$ solves $L(X_0;\tilde{\mu},\tilde{\sigma},\tilde{J})$, so $L(X_0;\tilde{\mu},\tilde{\sigma},\tilde{J}) = \overline{\eta}(\lambda,\tilde{\mu},\tilde{\sigma},\tilde{J})$. Standard optimal stopping results imply stopping is optimal in (10) when $X_t = x$ if and only if $L(x;\tilde{\mu},\tilde{\sigma},\tilde{J}) = \overline{u}(\lambda)$.

Let $b = \max\{x : \; L(x;\tilde{\mu},\tilde{\sigma},\tilde{J}) = \overline{u}(\lambda)\}$; if stopping is strictly optimal at b, then $\mathcal{S}(\lambda,\tilde{\mu},\tilde{\sigma},,\tilde{J}) = \{(b,0)\}$. Otherwise, stopping is only weakly optimal at b and, by analogous arguments to those in Lemma A.5, stopping is strictly optimal at any $x < b$, so $\mathcal{S}(\lambda,\tilde{\mu},\tilde{\sigma},\tilde{J}) = \{(b,0)\} \cup \{(b-\epsilon,\gamma) : \; \gamma \in [0,1)\}$.

$L(x;\tilde{\mu},\tilde{\sigma},\tilde{J})$ is strictly increasing in $\tilde{J}$ for all $x > b$ and $x = b$ if stopping is not strictly optimal when $X_t = b$. Therefore, b must be weakly decreasing in $\tilde{J}$ and if $(b-\epsilon,\gamma) \in \mathcal{S}(\lambda,\hat{\mu},\hat{\sigma},\tilde{J})$ for some $\gamma \in (0,1)$, then $(b-\epsilon,0)$ will be strictly optimal upon any sufficiently small increase in $\tilde{J}$. By analogous arguments to those in the proof of Lemma O.A.4, $L(x;\tilde{\mu},\tilde{\sigma},\tilde{J})$ is strictly increasing in $\tilde{\mu}$ and in $\tilde{\sigma}$ if $\tilde{\mu} \le 0$, so the same conclusions apply upon any small increase in $\tilde{\mu}$, or in $\tilde{\sigma}$ when $\tilde{\mu} \le 0$. Our desired results follow from these comparative statics on $\mathcal{S}$. □

Lemma O.A.8. $\hat{\alpha}(X_0) \ge \alpha(X_0)$.

Proof. Let $\lambda, \hat{\lambda} \le 0$ be the multipliers on $RDIR(X_0)$ in RDP when using the discrete-time approximation to Y and $\hat{Y}$, respectively. Because $\overline{\eta}(\lambda,\mu,\sigma,J)$ is equivalent to $\mathcal{L}(\Lambda)$ prior to $\tau_+(X^1)$, $\alpha(X_0),(S(X_0),\gamma(X_0))$ must solve $\overline{\eta}(\lambda,\mu,\sigma,J)$, so $(S(X_0),\gamma(X_0)) \in \mathcal{S}(\lambda,\mu,\sigma,J)$. Similarly, $\hat{\alpha}(X_0),(\hat{S}(X_0),\hat{\gamma}(X_0))$ must solve $\overline{\eta}(\hat{\lambda},\hat{\mu},\hat{\sigma},\hat{J})$, so $(\hat{S}(X_0),\hat{\gamma}(X_0)) \in \mathcal{S}(\hat{\lambda},\hat{\mu},\hat{\sigma},\hat{J})$.

For the sake of contradiction, suppose $\hat{\lambda} < \lambda$, which, by the characterization of α in Lemma A.4, implies $\arg\max_\alpha u_P(\alpha) - \hat{\lambda} u_A(1-\alpha) = \hat{\alpha}(X_0) < \alpha(X_0) = \arg\max_\alpha u_P(\alpha) - \lambda u_A(1-\alpha)$. Take any $(S',\gamma') \in \mathcal{S}(\lambda,\hat{\mu},\hat{\sigma},\hat{J})$. By Lemma O.A.7 and $(S(X_0),\gamma(X_0)) \in \mathcal{S}(\lambda,\mu,\sigma,J)$, either $S' < S(X_0)$, or $S' = S(X_0)$ and $0 = \gamma' \le \gamma(X_0)$. Because $u_P(\alpha(X_0)) > u_P(\hat{\alpha}(X_0)) \ge \hat{J}$ and $\tau^{S(X_0),\gamma(X_0)} \le \tau^{S',\gamma'}$, for demand $\alpha(X_0)$ P's utility is higher under $(S(X_0),\gamma(X_0))$ than (S',γ'),[22] namely,

$$\eta_P(S(X_0),\gamma(X_0),\alpha(X_0),\hat{\mu},\hat{\sigma},\hat{J}) \ge \eta_P(S',\gamma',\alpha(X_0),\hat{\mu},\hat{\sigma},\hat{J}).$$

[22] P will be better off stopping immediately at $\tau^{S(X_0),\gamma(X_0)}$ because it guarantees him a payoff $u_P(\alpha(X_0))$ that is higher than what he can receive if continuing; namely, discounted values of either $u_P(\alpha(X_0))$ or $\hat{J}$.

Optimality of (S', γ') in $\overline{\eta}(\lambda, \hat{\mu}, \hat{\sigma}, \hat{J})$ then implies that A's utility (S', γ') must be weakly higher than from the $(S(X_0), \gamma(X_0))$-stopping threshold, namely,

$$\eta_A(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}) \geq \eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \hat{\mu}, \hat{\sigma}). \tag{11}$$

$(S', \gamma') \in \mathcal{S}(\lambda, \hat{\mu}, \hat{\sigma}, \hat{J})$ and $\alpha(X_0) = \arg\max_\alpha u_P(\alpha) - \lambda u_A(1-\alpha)$ imply

$$\begin{aligned}
&\eta_P(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}, \hat{J}) - \lambda \eta_A(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}) \\
&= \overline{\eta}(\lambda, \hat{\mu}, \hat{\sigma}, \hat{J}) \\
&\geq \eta_P(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}, \hat{J}) - \lambda \eta_A(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}),
\end{aligned}$$

while $(\hat{S}(X_0), \hat{\gamma}(X_0)) \in \mathcal{S}(\hat{\lambda}, \hat{\mu}, \hat{\sigma}, \hat{J})$ and $\hat{\alpha}(X_0) = \arg\max_\alpha \; u_P(\alpha) - \hat{\lambda} u_A(1-\alpha)$ imply

$$\begin{aligned}
&\eta_P(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}, \hat{J}) - \hat{\lambda} \eta_A(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}) \\
&= \overline{\eta}(\hat{\lambda}, \hat{\mu}, \hat{\sigma}, \hat{J}) \\
&\geq \eta_P(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}, \hat{J}) - \hat{\lambda} \eta_A(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}).
\end{aligned}$$

Adding these two inequalities together and simplifying, we get

$$\eta_A(S', \gamma', \alpha(X_0), \hat{\mu}, \hat{\sigma}) \leq \eta_A(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}).$$

Combining this inequality with (11), we get

$$\eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \hat{\mu}, \hat{\sigma}) \leq \eta_A(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}). \tag{12}$$

Note that $\tilde{V}(X_0, \cdot, \cdot) = \eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \cdot, \cdot)$ when $m = X_0$ in $\tilde{V}$ and, by $\tilde{V}(X_0, \mu, \sigma) = V(X_0, X_0)$, we have $V(X_0, X_0) = \eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \mu, \sigma)$. Using (12) and Lemmas O.A.6 and A.7, we have

$$\begin{aligned}
X_0 = V(X_0, X_0) &= \eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \mu, \sigma) \\
&< \eta_A(S(X_0), \gamma(X_0), \alpha(X_0), \hat{\mu}, \hat{\sigma}) \\
&\leq \eta_A(\hat{S}(X_0), \hat{\gamma}(X_0), \hat{\alpha}(X_0), \hat{\mu}, \hat{\sigma}).
\end{aligned} \tag{13}$$

But the last line in (13) is A's expected utility under the optimal contract in the relaxed problem RDP when we use the discrete-time approximation for $\hat{Y}$, contradicting Lemma A.7, which shows A's continuation value is equal to X_0 at $t = \tau_+(X_0) = 0$. Therefore, $\hat{\lambda} \geq \lambda$, which implies $\alpha(X_0) \leq \hat{\alpha}(X_0)$. □

We can apply the same arguments at $\tau_+(X^1)$, $\tau_+(X^2)$,... to conclude $\hat{\alpha}(m) \geq \alpha(m)$ for each m. Taking the continuous-time limits of our discrete-time approximations, we get $\hat{\alpha}^* \geq \alpha^*$.

References

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