Diversity Balance in Centralized Public School Admissions
Online Appendix

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Proof of Proposition 1

Consider the following choice rule $\mathcal{C}^r$: For each $A \subseteq \mathcal{A}$,

1. **Stage 1.** Select up to r_t of highest priority applicants of each type t. Let $A' \subseteq A$ denote the set of all selected applicants.
2. **Stage 2.** From the remaining applicants $A \setminus A'$, select highest priority applicants up to the capacity. Let $\mathcal{C}^r(A)$ be the set of chosen applicants after these two stages.

Now fix an arbitrary regular application order profile $\triangleright$. To prove Proposition 1, it is sufficient to show that $\mathcal{C}^\triangleright = \mathcal{C}^r$. This would establish that any regular application order gives the same choice rule, i.e., $\mathcal{C}^r$.

Consider an arbitrary $A \subseteq \mathcal{A}$. From the definitions of $\mathcal{C}^\triangleright$ and $\mathcal{C}^r$, is immediate that both choice rules are non-wasteful. Without loss of generality, suppose that $|A| < q$. Since both $\mathcal{C}^\triangleright$ and $\mathcal{C}^r$ are non-wasteful,

$$|\mathcal{C}^\triangleright(A)| = q = |\mathcal{C}^r(A)|.$$

Let A' be the set of applicants selected at Stage 1 of the implementation of $\mathcal{C}^r$. First, we show that $A' \subseteq \mathcal{C}^\triangleright(A)$. Consider an arbitrary $a \in A'$. By description of $\mathcal{C}^r$, a is one of the $r_{\tau(a)}$ highest priority type-$\tau(a)$ applicants in $A_{\tau(a)}$. Therefore, she is one of the $r_{\tau(a)}$ highest $\succ_{\tau(a)}$ priority applicants in $A_{\tau(a)}$. Hence, in the implementation of $\mathcal{C}^\triangleright$, a first applies to $s_{\tau(a)}$ and is never rejected by the school. This establishes that $a \in \mathcal{C}^\triangleright(A)$.

Now, for the sake of contradiction, suppose $\mathcal{C}^\triangleright(A) \neq \mathcal{C}^r(A)$. Since $|\mathcal{C}^\triangleright(A)| = q = |\mathcal{C}^r(A)|$, the set $\mathcal{C}^r(A) \setminus \mathcal{C}^\triangleright(A)$ is non-empty. Consider an applicant a in this set. Since $A' \subseteq \mathcal{C}^\triangleright(A)$, it should be that $a \notin A'$. Therefore, a is selected at the second stage of $\mathcal{C}^r$'s implementation. By description of $\mathcal{C}^r$, a is one of the $q - \sum_{t \in T} \min\left\{|A_t|, r_t\right\}$ highest priority applicants in $A \setminus A'$. This contradicts that a is not chosen by $\mathcal{C}^\triangleright(A)$.

Proof of Theorem 1

Consider the regular reserves rule $\mathcal{C}^r$. That $\mathcal{C}^r$ is reserves-respecting and non-wasteful is immediate from its definition. Let $\mathcal{C} \neq \mathcal{C}^r$ be an arbitrary reserves-respecting and non-wasteful choice rule, that is not the regular reserves rule. To establish Theorem 1, it is sufficient to show that $\mathcal{C}$ is not priority violations minimal in the class of reserves-respecting and non-wasteful choice rule. Since at least one priority violations minimal rule exists, this would imply that the regular reserves rule $\mathcal{C}^r$ is the unique priority violations minimal choice rule in the class of reserves-respecting and non-wasteful choice rules.

Let us define another axiom.

Axiom 1 (Within-type priority compatibility)**.** *A choice rule $\mathcal{C}$ is* ***within-type priority compatible*** *if for any priority violation instance (a, a'), $\tau(a) \neq \tau(a')$.*

By definition, the regular rule $\mathcal{C}^r$ is within-type priority compatible. We will study two cases:

Case 1. $\mathcal{C}$ is not within-type priority compatible.

If $\mathcal{C}$ is not within-type priority compatible, then for some subset A, $\mathcal{C}$ creates a priority violation instance (a, a') with $\tau(a) = \tau(a')$. Consider another choice rule $\mathcal{C}'$ that differs from $\mathcal{C}$ by that it swaps the assignments of a and a' when choosing from subset A, and otherwise it agrees with $\mathcal{C}$. Then, $\mathcal{C}'$ creates strictly less priority violations than $\mathcal{C}$. Moreover, because $\tau(a) = \tau(a')$, $\mathcal{C}'$ is reserves-respecting and non-wasteful. Hence, $\mathcal{C}$ is not priority violations minimal in the class of reserves-respecting and non-wasteful choice rules.

Case 2. $\mathcal{C}$ is within-type priority compatible.

To show that $\mathcal{C}$ is not priority violations minimal in the class of reserves-respecting and non-wasteful choice rule, it is sufficient to show that $\mathcal{C}^r$ creates strictly less priority violations than $\mathcal{C}$.

We will prove a stronger result that for any $A \subseteq \mathcal{A}$,

$$a \in \mathcal{C}(A) \setminus \mathcal{C}'(A) \text{ and } a' \in \mathcal{C}'(A) \setminus \mathcal{C}(A) \text{ implies } a \succ a'.$$

Consider an arbitrary $A \subseteq \mathcal{A}$, such that $\mathcal{C}^r(A) \neq \mathcal{C}(A)$.

Since both $\mathcal{C}^r$ and $\mathcal{C}$ are non-wasteful,

$$\left|\mathcal{C}^r(A)\right| = q = \left|\mathcal{C}(A)\right|.$$

Thus, $\mathcal{C}^r(A) \neq \mathcal{C}(A)$ implies that there are $a, a' \in A$ such that

$$a \in \mathcal{C}^r(A) \setminus \mathcal{C}(A) \text{ and } a' \in \mathcal{C}(A) \setminus \mathcal{C}^r(A).$$

We want to show that $a \succ a'$.

Since $\mathcal{C}$ is reserves-respecting and $a \notin \mathcal{C}(A)$, there should be at least $r_{\tau(a)}$ applicants in $\mathcal{C}(A) \cap A_{\tau(a)}$. Moreover, since $\mathcal{C}$ is within-type priority compatible, all $r_{\tau(a)}$ highest priority applicants in $A_{\tau(a)}$ are in $\mathcal{C}(A)$. Therefore, a is not one of the $r_{\tau(a)}$ highest priority type-$\tau(a)$ applicants in $A_{\tau(a)}$. Since $\mathcal{C}^r$ is reserves-respecting and within-type priority compatible, we can use similar arguments to establish that a' is not one of the $r_{\tau(a')}$ highest priority applicants in $A_{\tau(a')}$.

Consider the two-stage implementation of $\mathcal{C}^r$ described in Proposition 1. Since neither a not a' are one of the $r_{\tau(a)}$ and $r_{\tau(a')}$ highest priority applicants of their respective types in A, it should be that neither applicant is selected at Stage 1 of the implementation of $\mathcal{C}^r$. Since a is selected over a' at Stage 2 of the implementation of $\mathcal{C}^r$, we conclude that $a \succ a'$.