“(Reverse) Price Discrimination with Information Design” Online Appendix

Dong Wei   Brett Green

Remark. Equations and results introduced in this Online Appendix are numbered as OA.#. Equations and results with regular numbering refer to those in the printed manuscript.

I Optimality of Posted-Price Mechanisms with Uniform Information

We first prove an important lemma, which implies Proposition 1. This lemma per se has nothing to do with information design, but it implies that, *regardless of the value distribution induced by the public experiment,* the seller should later use some posted-price mechanism to maximize her profit; Proposition 1 thus follows.

Lemma OA.1. *Suppose that there is a single-unit indivisible object for sale. Let $V\subset\mathbb{R}$ be a bounded set where buyer's private value resides. For any selling mechanism, there is a posted-price mechanism that generates a weakly higher profit.*

This lemma follows directly from the next three claims. Let $\{x(v),t(v)\}_{v\in V}$ be a direct mechanism where $t(v)$ and $x(v)$ specify the buyer's payment and his probability of getting the good when he reports v.

Claim OA.1. *Let $\hat{V}$ be the convex hull of the closure of V.*[29] *A direct mechanism $\{x(\cdot),t(\cdot)\}$, defined on V, is incentive compatible, if and only if there is a direct mechanism $\{\hat{x}(\cdot),\hat{t}(\cdot)\}$, defined on $\hat{V}$, such that i) it is incentive compatible, and ii) $\hat{x}|_V=x$ and $\hat{t}|_V=t$.*

Proof. The “if” direction is obvious. So we focus on the “only if” direction.

Let $\{x(v),t(v)\}_{v\in V}$ be an IC direct mechanism defined on V. We would like to construct an extension, $\{\hat{x}(v),\hat{t}(v)\}_{v\in\hat{V}}$, which is IC on $\hat{V}$. We do such an extension in two steps. In Step 1, we extend the original mechanism to the closure V_c of V. In Step 2, we further extend it to the convex hull $\hat{V}$ of V_c.

Step 1: Extending $\{x(v),t(v)\}_{v\in V}$ to $\{x_c(v),t_c(v)\}_{v\in V_c}$, where V_c is the closure of V.

For any $v\in V_c-V$, there is a monotone sequence $v_n\in V$ such that $v_n\to v$. Because $\{x(v),t(v)\}$ is IC on V, $x(\cdot)$ and $t(\cdot)$ are nondecreasing functions, and thus $x(v_n)$ and $t(v_n)$ are bounded monotone sequences. Define $x_c(v)\equiv\lim_n x(v_n)$ and $t_c(v)\equiv\lim_n t(v_n)$.[30]

We verify that $\{x_c(\cdot),t_c(\cdot)\}$ constructed above is IC on V_c.

Case 1: $v\in V$.

Take any $v\in V$. Since $\{x(\cdot),t(\cdot)\}$ is IC on V, the buyer doesn't want to misreport any type in V. Also, by misreporting $\hat{v}\in V_c-V$, the buyer's payoff is

$$vx_c(\hat{v})-t_c(\hat{v})=\lim_n[vx(v_n)-t(v_n)]\leq vx(v)-t(v),$$

where the equality follows from our construction of $\{x_c(\cdot),t_c(\cdot)\}$, and the inequality follows from the IC of $\{x(\cdot),t(\cdot)\}$. Therefore, the buyer of type v does not want to misreport any type in V_c-V either.

[29] That is, $\hat{V}$ is the smallest compact interval that contains V.

[30] It is possible that different $\{v_n\}_n$ sequences lead to different limits for $x(v_n)$ and $t(v_n)$, and when that happens, our construction simply picks one of them.

Case 2: $v \in V_c - V$.

Take any $v \in V_c - V$. We first check that the buyer cannot profitably deviate by misreporting $\hat{v} \in V$. Let $\{v_n\}_n$ be the sequence in V used to construct $x_c(v)$ and $t_c(v)$. Because $\{x(\cdot), t(\cdot)\}$ is IC on V, we have

$$v_n x(\hat{v}) - t(\hat{v}) \leq v_n x(v_n) - t(v_n).$$

Taking the limit on both sides, we have

$$v x(\hat{v}) - t(\hat{v}) \leq v x_c(v) - t_c(v).$$

So the buyer gains nothing by misreporting $\hat{v} \in V$. Then, following the argument in Case 1, there is no profitable deviation by misreporting any $\hat{v} \in V_c - V$ either.

Step 2: Extending $\{x_c(v), t_c(v)\}_{v \in V_c}$ to $\{\hat{x}(v), \hat{t}(v)\}_{v \in \hat{V}}$, where $\hat{V}$ is the convex hull of V_c.

For any $v \in \hat{V} - V_c$, let $(\underline{v}, \bar{v})$ be such that i) $v \in (\underline{v}, \bar{v}) \subset \hat{V} - V_c$, and ii) $\underline{v}, \bar{v} \in V_c$.[31] Then, for any $v \in \hat{V} - V_c$, define

$$(\hat{x}(v), \hat{t}(v)) \equiv \begin{cases} (x_c(\underline{v}), t_c(\underline{v})), \text{ if } v x_c(\underline{v}) - t_c(\underline{v}) \geq v x_c(\bar{v}) - t_c(\bar{v}) \\ (x_c(\bar{v}), t_c(\bar{v})), \text{ if } v x_c(\underline{v}) - t_c(\underline{v}) < v x_c(\bar{v}) - t_c(\bar{v}) \end{cases}.$$

We verify that $\{\hat{x}(\cdot), \hat{t}(\cdot)\}$ constructed above is IC on $\hat{V}$. Note that our construction doesn't introduce new (x, t) pairs into the mechanism, so any $v \in V_c$ doesn't have an incentive to misreport. Now consider any $v \in \hat{V} - V_c$. Single-crossing of the buyer's payoff function implies that among the (x, t) pairs in $\{x_c(v), t_c(v)\}_{v \in V_c}$, his most preferred one is either $(x_c(\underline{v}), t_c(\underline{v}))$ or $(x_c(\bar{v}), t_c(\bar{v}))$. Since our construction of $\hat{x}(v)$ and $\hat{t}(v)$ exactly gives him that pair, he doesn't have an incentive to misreport either. $\square$

Claim OA.1 tells us that, when analyzing IC mechanisms, we can without loss think of the type space as a compact interval (because even when it is not a compact interval, artificially imposing additional IC constraints on the entire convex hull of it doesn't change the set of IC mechanisms on the original space.) **From now on, *for the purpose of imposing IC constraints,* we take V as a compact interval $[\underline{v}, \bar{v}]$, so that the revenue equivalence theorem applies** That is, a direct mechanism $\{x(\cdot), t(\cdot)\}$ is IC on V, if and only if $x(\cdot)$ is nondecreasing, and $t(v) = v x(v) - \int_{\underline{v}}^{v} x(s) ds - C$ where C is the utility of buyer $\underline{v}$.

Let us now introduce the following definition of a random price mechanism.

Definition 1. *A (indirect) mechanism is a* ***random price mechanism*** *if the seller posts a price that follows some probability distribution on V, and the buyer then optimally decides whether to buy at the realized price, with potential mixing when indifferent.*

Claim OA.2. *Any IC direct mechanism $\{x(\cdot), t(\cdot)\}$ that gives zero utility to buyer $\underline{v}$ can be implemented by a random price mechanism.*

Proof. Take any IC direct mechanism $\{x(\cdot), t(\cdot)\}$ that gives zero utility to buyer $\underline{v}$. This implies that $x(\cdot)$ is a nondecreasing function. Consider two different cases.

Case 1: $x(\cdot)$ **is right-continuous and satisfies** $x(\bar{v}) = 1$.

[31] Because V_c and $\hat{V}$ are both compact subsets of $\mathbb{R}$, $\hat{V} - V_c$ is constituted by a union of such open intervals.

In this case, $x(\cdot)$ itself is a probability distribution function on V. Consider the random price mechanism in which the price distribution is defined by

$$\Pr(p \leq v) \equiv x(v),$$

with all buyer types buying the good with probability 1 when indifferent. It is easy to check that this random price mechanism induces an allocation rule equal to $x(\cdot)$ and gives zero utility to buyer $\underline{v}$.

Case 2: $x(\cdot)$ is not right-continuous or $x(\bar{v}) < 1$ (or both).

In this case, define $\bar{x}(\cdot)$ as

$$\bar{x}(v) \equiv \begin{cases} x(v_+) & \text{if } v < \bar{v} \\ 1, & \text{if } v = \bar{v} \end{cases}.$$

By construction, $\bar{x}(\cdot)$ is a probability distribution function on V. Consider the random price mechanism in which the price distribution is defined by

$$\Pr(p \leq v) \equiv \bar{x}(v),$$

with each buyer v buying with probability $\frac{x(v)-x(v_-)}{\bar{x}(v)-x(v_-)}$ when indifferent. It is easy to check that this random price mechanism induces an allocation rule equal to $x(\cdot)$ and gives zero utility to buyer $\underline{v}$. □

Claim OA.3. *For any random price mechanism, there is a deterministic posted-price mechanism that generates a weakly higher profit.*

Proof. Take any random price mechanism, and denote the price distribution by $\bar{x}$. Let $\pi(p)$ be the seller's profit from this mechanism if the realized price is p, taking account of the specific tie-breaking rule. The seller's expected profit is $\int_{\underline{v}}^{\bar{v}} \pi(p) d\bar{x}(p)$. It immediately follows (from a contradiction argument) that there must exist some $p^* \in V$ such that $\pi(p^*) \geq \int_{\underline{v}}^{\bar{v}} \pi(p) d\bar{x}(p)$. So a deterministic posted-price mechanism with price p^* generates a weakly higher profit. □

Proof of Lemma OA.1. By the revelation principle, we can without loss focus on IC direct mechanisms. By Claim OA.1, we can without loss impose IC constraints on the smallest compact interval that contains V. Then by Claims OA.2 and OA.3, for any (IC direct) mechanism, there is a deterministic posted-price mechanism that generates a weakly higher profit. □

II General Valuation Functions

Definition of Bounds Recall that $v(\theta,\omega) \equiv u(\theta,\omega) - \frac{1-G(\theta)}{g(\theta)} u_\theta(\theta,\omega) - c$, defined in (9). Notice that

$$v_\theta = u_\theta - \left(\frac{1-G}{g}\right)' u_\theta - \frac{1-G}{g} u_{\theta\theta} > 0, \tag{OA.1}$$

$$v_\omega = u_\omega - \frac{1-G}{g} u_{\theta\omega}, \tag{OA.2}$$

where the inequality follows from $u_\theta > 0$, $\left(\frac{1-G}{g}\right)' < 0$ and $u_{\theta\theta} \leq 0$. Recall also the definition of bounds $\bar{M}$ and $\underline{M}$ in (11) and (12).

An Auxiliary Benchmark Similar to Section 3.1, we first look at an auxiliary benchmark in which the quality is perfectly observable to both the seller and the buyer. In this case, a direct mechanism, $\{q(\theta,\omega),t(\theta,\omega)\}$, specifies that, if a buyer reports type θ and if the seller observes quality ω, then the buyer gets the good with probability $q(\theta,\omega)$ and makes a payment of $t(\theta,\omega)$ to the seller.

Define

$$U(\theta,\hat{\theta}) \equiv \int_{\underline{\omega}}^{\bar{\omega}} \Big[u(\theta,\omega)q(\hat{\theta},\omega) - t(\hat{\theta},\omega)\Big] dF(\omega) = C(\hat{\theta}) + D(\theta,\hat{\theta}), \tag{OA.3}$$

$$V(\theta) \equiv U(\theta,\theta), \tag{OA.4}$$

where $C(\hat{\theta}) = -\int_{\underline{\omega}}^{\bar{\omega}} t(\hat{\theta},\omega)dF(\omega)$ and $D(\theta,\hat{\theta}) = \int_{\underline{\omega}}^{\bar{\omega}} u(\theta,\omega)q(\hat{\theta},\omega)dF(\omega)$. Here, $U(\theta,\hat{\theta})$ is buyer θ's interim payoff if he reports $\hat{\theta}$.

Claim OA.4. *In the auxiliary benchmark, truthful reporting of θ is optimal, i.e., $V(\theta) \geq U(\theta,\hat{\theta}), \forall \theta,\hat{\theta}$, only if*

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} D_1(s,s)ds, \tag{OA.5}$$

where $D_1(\theta,\hat{\theta}) = \frac{\partial D(\theta,\hat{\theta})}{\partial \theta}$. Moreover, if a mechanism satisfies (OA.5) *and that $D_1(\theta,\hat{\theta})$ is nondecreasing in $\hat{\theta}$, for all $\theta,\hat{\theta} \in \Theta$, then truthful report of θ is optimal.*

Proof. Suppose first that $V(\theta) \geq U(\theta,\hat{\theta}), \forall \theta,\hat{\theta}$. Recall that $D(\theta,\hat{\theta}) = \int_{\underline{\omega}}^{\bar{\omega}} u(\theta,\omega)q(\omega,\hat{\theta})dF(\omega)$. Since u is C^1 and $u_\theta(\theta,\omega)q(\omega,\hat{\theta})f(\theta) \leq \max_{(\theta,\omega)\in\Theta\times\Omega}[u_\theta(\theta,\omega)f(\theta)]$, by the Dominated Convergence Theorem $D_1(\theta,\hat{\theta}) = \int_{\underline{\omega}}^{\bar{\omega}} u_\theta(\theta,\omega)q(\omega,\hat{\theta})dF(\omega)$. Then, $|D_1(\theta,\hat{\theta})| \leq \int_{\underline{\omega}}^{\bar{\omega}} u_\theta(\theta,\omega)dF(\omega) \leq \max_{(\theta,\omega)\in\Theta\times\Omega} u_\theta(\theta,\omega)$. By Milgrom and Segal (2002, Theorem 2), $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} D_1(s,s)ds$.

Suppose now that (OA.5) is satisfied and that $D_1(\theta,\hat{\theta})$ is nondecreasing in $\hat{\theta}$, for all $\theta,\hat{\theta} \in \Theta$. We want to show that truthful reporting is optimal. Take any $\theta,\hat{\theta} \in \Theta$. We have

$$\begin{aligned} V(\theta) - U(\theta,\hat{\theta}) &= U(\theta,\theta) - U(\theta,\hat{\theta}) \\ &= V(\theta) - V(\hat{\theta}) + D(\hat{\theta},\hat{\theta}) - D(\theta,\hat{\theta}) \\ &= \int_{\hat{\theta}}^{\theta} D_1(s,s)ds - \int_{\hat{\theta}}^{\theta} D_1(s,\hat{\theta})ds \\ &= \int_{\hat{\theta}}^{\theta} [D_1(s,s) - D_1(s,\hat{\theta})]ds \\ &\geq 0, \end{aligned}$$

where the second line follows from equations (OA.3) and (OA.4), the third line follows from condition (OA.5), and the last line follows from D_1 being nondecreasing in its second variable. □

Claim OA.4 implies that the seller's profit maximization problem in the auxiliary benchmark can be written as

$$\max_{\{q(\theta,\omega),V(\underline{\theta})\}} -V(\underline{\theta})+\int_{\underline{\theta}}^{\bar{\theta}}\left[D(\theta,\theta)-\int_{\underline{\theta}}^{\theta}D_1(s,s)ds-c\int_{\underline{\omega}}^{\bar{\omega}}q(\theta,\omega)dF(\omega)\right]dG(\theta). \tag{OA.6}$$

s.t. $D_1(\theta,\hat{\theta})$ is nondecreasing in $\hat{\theta}$, and $V(\underline{\theta})\geq 0$

Note that

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}}\left[D(\theta,\theta)-\int_{\underline{\theta}}^{\theta}D_1(s,s)ds-c\int_{\underline{\omega}}^{\bar{\omega}}q(\theta,\omega)dF(\omega)\right]dG(\theta)\\
=&\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\omega}}^{\bar{\omega}}\left(u(\theta,\omega)-c-\frac{1-G(\theta)}{g(\theta)}u_\theta(\theta,\omega)\right)q(\theta,\omega)g(\theta)dF(\omega)d\theta \qquad \text{(OA.7)}\\
=&\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\omega}}^{\bar{\omega}}v(\theta,\omega)q(\theta,\omega)g(\theta)dF(\omega)d\theta,
\end{aligned}$$

where the second line is obtained by integration by parts and substituting the expressions of D and D_1 into the equation, and the last line uses the definition of v in (9).

From (OA.1), (OA.2) and the assumption that $u_{\theta\omega}<\bar{M}$, we know that

$$v_\theta,v_\omega>0, \text{ whenever } v\geq 0. \tag{OA.8}$$

So for any given θ, there exists at most one $\omega\in\Omega$ such that $v(\theta,\omega)=0$. Wherever it exists, let $k(\theta)$ be such that $v(\theta,k(\theta))=0$, and let $k(\theta)\equiv\bar{\omega}$ ($\underline{\omega}$) if $v(\theta,\omega)$ is negative (positive) for all ω.

Condition (OA.8) implies that

$$\frac{dk(\theta)}{d\theta}=-\frac{v_\theta}{v_\omega}<0, \tag{OA.9}$$

for all θ s.t. $v(\theta,\cdot)$ admits a zero point in Ω.

The objective in (OA.7) suggests the following candidate solution:

$$q^{**}(\omega,\theta)=\begin{cases}1, & \text{if } \omega\geq k(\theta)\\ 0, & \text{if } \omega<k(\theta)\end{cases}. \tag{OA.10}$$

Under such a candidate solution, $D^*(\theta,\hat{\theta})=\int_{k(\hat{\theta})}^{\bar{\omega}}u(\theta,\omega)dF(\omega)$, so that $\frac{\partial^2 D^*}{\partial\theta\partial\hat{\theta}}=-u_\theta(\theta,\omega)\frac{dk(\hat{\theta})}{d\theta}>0$, which verifies that $D_1^*(\theta,\hat{\theta})$ is nondecreasing in $\hat{\theta}$. Hence, q^{**} defined in (OA.10) solves program (OA.6).

With an abuse of notation, define $\theta_1\equiv\inf_\Theta\{\theta:k(\theta)<\bar{\omega}\}$ and $\theta_2=\sup_\Theta\{\theta:k(\theta)>\underline{\omega}\}$. Since we have assumed $u(\bar{\theta},\bar{\omega})>c$ and $u(\underline{\theta},\underline{\omega})\leq c$, it is easy to check that $\underline{\theta}\leq\theta_1<\theta_2\leq\bar{\theta}$.[32] As in the additively separable case, all buyer types between θ_1 and θ_2 buy with probability strictly between 0 and 1.

Implementation As argued in Section 3.2, the profit obtained in the auxiliary benchmark in an upper bound on the seller's profit in our problem with endogenous information about quality, private signal realizations, and an RPIR constraint. We now show that this profit can be obtained in our setting with a recommendation mechanism, that is, a menu of price-experiment pairs with binary signals.

[32] That $u(\bar{\theta},\bar{\omega})>c$ ($u(\underline{\theta},\underline{\omega})\leq c$) implies that $k(\bar{\theta})<\bar{\omega}$ ($k(\underline{\theta})>\underline{\omega}$), so that θ_1 (θ_2) is well-defined.

Consider the following recommendation mechanism. If a buyer reports θ, then he will be recommended to buy if and only if $q(\theta,\omega)=1$, i.e., $\omega \geq k(\theta)$. Moreover, the price he faces is

$$p^{**}(\theta)=\begin{cases} u(\theta_1,\bar{\omega},) & \text{if } \theta \in [\underline{\theta},\theta_1) \\ \frac{D^*(\theta,\theta)-\int_{\theta_1}^{\theta} D_1^*(s,s)ds}{1-F(k(\theta))}, & \text{if } \theta \in [\theta_1,\bar{\theta}] \end{cases}. \tag{OA.11}$$

Claim OA.5. *$p^{**}(\theta)$ is decreasing in θ.*

Proof. By its definition in (OA.11), p^{**} is constant on $[\underline{\theta},\theta_1]$ and $[\theta_2,\bar{\theta}]$, and is continuous at θ_2. For the possible case where $\theta_1>\underline{\theta}$, we first show that p^{**} is continuous at θ_1. Note that

$$\begin{aligned}\lim_{\theta\downarrow\theta_1} p^{**}(\theta) &= \lim_{\theta\downarrow\theta_1}\left[\frac{D^*(\theta,\theta)-\int_{\theta_1}^{\theta} D_1^*(s,s)ds}{1-F(k(\theta))}\right] \\ &= \frac{D_1^*(\theta_1,\theta_1)+D_2^*(\theta_1,\theta_1)-D_1^*(\theta_1,\theta_1)}{-f(k(\theta_1))k'(\theta_1)} \\ &= u(\theta_1,\bar{\omega}),\end{aligned}$$

where the second line follows from L'Hopital's rule, and the third line follows from $D_2^*(\theta,\theta) = -u(\theta,k(\theta))f(k(\theta))k'(\theta)$.

Now we show that p^{**} is strictly decreasing on (θ_1,θ_2). Taking derivative with respect to θ, we have

$$\begin{aligned}\frac{dp^*(\theta)}{d\theta} &= \frac{d}{d\theta}\left[\mathbb{E}[u(\theta,\omega)|\omega\geq k(\theta)]-\frac{\int_{\theta_1}^{\theta} D_1^*(s,s)ds}{1-F(k(\theta))}\right] \\ &= \frac{\int_{k(\theta)}^{\bar{\omega}} u_\theta(\theta,\omega)dF(\omega)}{1-F(k(\theta))}+\frac{f(k(\theta))\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_\omega(\theta,\omega)d\omega}{[1-F(k(\theta))]^2}k'(\theta) \\ &\quad -\left[\frac{D_1^*(\theta,\theta)}{1-F(k(\theta))}+\frac{f(k(\theta))\int_{\theta_1}^{\theta} D_1^*(s,s)ds}{[1-F(k(\theta))]^2}k'(\theta)\right] \\ &= -\frac{f(k(\theta))k'(\theta)}{[1-F(k(\theta))]^2}\left[\int_{\theta_1}^{\theta} D_1^*(s,s)ds-\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_\omega(\theta,\omega)d\omega\right],\end{aligned}$$

where the second line follows from the observation (analogous to Lemma A1) that

$$\frac{d}{dy}\mathbb{E}[u(\theta,x)|x\geq y]=\frac{f(y)\int_y^{\bar{x}}(1-F(x))u_x(\theta,x)dx}{(1-F(y))^2},$$

and the last line follows from $D_1^*(\theta,\theta)=\int_{k(\theta)}^{\bar{\omega}} u_\theta(\theta,\omega)dF(\omega)$.

To show $dp^*/d\theta<0$, since $k'(\theta)<0$, we are done if $\int_{\theta_1}^{\theta} D_1^*(s,s)ds-\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_\omega(\theta,\omega)d\omega<0$

for all $\theta\in(\theta_1,\theta_2)$. Since $k(\theta_1)\leq\bar{\omega}$, these integrals are weakly less than 0 at $\theta=\theta_1$. For all $\theta\in[\theta_1,\theta_2]$,

$$
\begin{aligned}
&\frac{d}{d\theta}\left[\int_{\theta_1}^{\theta}D_1^*(s,s)ds-\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_\omega(\theta,\omega)d\omega\right]\\
&=D_1^*(\theta,\theta)-\left(\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_{\theta\omega}(\theta,\omega)d\omega-[1-F(k(\theta))]u_\omega(\theta,k(\theta))k'(\theta)\right)\\
&=[1-F(k(\theta))]\left[u_\theta(\theta,k(\theta))+k'(\theta)u_\omega(\theta,k(\theta))\right]\\
&=[1-F(k(\theta))][u_\theta(\theta,k(\theta))-v_\theta u_\omega(\theta,k(\theta))/v_\omega]\\
&=[1-F(k(\theta))]\left[\frac{-u_\theta\frac{1-G}{g}u_{\theta\omega}+u_\omega\left(\left(\frac{1-G}{g}\right)'u_\theta+\frac{1-G}{g}u_{\theta\theta}\right)}{u_\omega-\frac{1-G}{g}u_{\theta\omega}}\right]\\
&<0,
\end{aligned}
$$

where the second equality follows from integrating D_1^* by parts, the third and fourth equalities follow from (OA.8) and (OA.9), and the strict inequality at the end follows from $u_{\theta\omega}>\underline{M}$. Therefore, we have

$$\int_{\theta_1}^{\theta}D_1^*(s,s)ds-\int_{k(\theta)}^{\bar{\omega}}(1-F(\omega))u_\omega(\theta,\omega)d\omega<0,\text{ for all }\theta\in(\theta_1,\bar{\theta}),$$

as desired. □

Proof of Theorem 3. We now verify that this recommendation mechanism indeed allows the seller to obtain the same profit as in the auxiliary benchmark. First, by construction, a buyer in a recommendation mechanism can walk away from the transaction without paying anything after observing any signal realization, so it satisfies RPIR. Also, it is easy to verify that, *if every buyer type reports truthfully and follows the recommendation,* then $\{p^{**},q^{**}\}$ will generate the same profit as in the auxiliary benchmark. Therefore, it remains to check that under $\{p^{**},q^{**}\}$, every buyer type finds it optimal to report truthfully and follow the recommendation.

Note that buyer θ's payoff in the proposed recommendation mechanism if he reports truthfully and follows the recommendation is

$$V^{**}(\theta)\equiv C^{**}(\theta)+D^{**}(\theta,\theta),$$

where $C^{**}(\hat{\theta})=-p^{**}(\hat{\theta})\int_{\underline{\omega}}^{\bar{\omega}}q^{**}(\hat{\theta},\omega)dF(\omega)$ and $D^{**}(\theta,\hat{\theta})=\int_{\underline{\omega}}^{\bar{\omega}}u(\theta,\omega)q^{**}(\hat{\theta},\omega)dF(\omega)$. Buyer θ's payoff if he reports $\hat{\theta}$ and then follows the recommendation is

$$U^{**}(\theta,\hat{\theta})\equiv C^{**}(\hat{\theta})+D^{**}(\theta,\hat{\theta})$$

Buyer θ's payoff is 0 if he reports $\hat{\theta}$ and never buys the good regardless of the recommendation (i.e., disobeying the positive recommendation). Buyer θ's payoff is $\int_{\underline{\omega}}^{\bar{\omega}}u(\theta,\omega)dF(\omega)-p(\hat{\theta})$ if he reports $\hat{\theta}$ and always buys the good regardless of the recommendation (i.e., disobeying the negative recommendation).

Therefore, buyer θ finds it optimal to report truthfully and follow the recommendation if and only if

$$V^{**}(\theta) \geq \max\left\{U^{**}(\theta,\hat{\theta}),\, 0,\, \int_{\underline{\omega}}^{\bar{\omega}} u(\theta,\omega)dF(\omega) - p^{**}(\hat{\theta})\right\},\, \forall \hat{\theta} \in [\underline{\theta},\bar{\theta}]. \tag{OA.12}$$

Indeed, $V^{**}(\theta) \geq U^{**}(\theta,\hat{\theta})$ because V^{**} by construction satisfies the envelope condition $V^{**}(\theta) = \int_{\underline{\theta}}^{\theta} D_1^{**}(s,s)ds$; $V^{**}(\theta) \geq 0$ because $V^{**}(\theta)$ is increasing and $V^{**}(\underline{\theta}) = 0$ by construction. It remains to verify $V^{**}(\theta) \geq \max_{\hat{\theta}} \int_{\underline{\omega}}^{\bar{\omega}} u(\theta,\omega)dF(\omega) - p^{**}(\hat{\theta})$. To do so, we first show that, $\mathbb{E}[u(\theta,\omega)|\omega < k(\theta)] - p^{**}(\theta) \leq 0$ for all θ; that is, when recommended not to buy, each type finds it optimal to follow. Note that, for $\theta \in [\theta_1,\bar{\theta}]$,

$$\begin{aligned}
\mathbb{E}[u(\theta,\omega)|\omega < k(\theta)] - p^{**}(\theta) &\leq u(\theta,k(\theta)) - p^{**}(\theta) \\
&= u(\theta,k(\theta)) - \left[\mathbb{E}[u(\theta,\omega)|\omega \geq k(\theta)] - \frac{\int_{\theta_1}^{\theta} D_1^{*}(s,s)ds}{1-F(k(\theta))}\right] \\
&= \frac{1}{1-F(k(\theta))}\left(\int_{\theta_1}^{\theta} D_1^{*}(s,s)ds - \int_{k(\theta)}^{\bar{\omega}} (1-F(\omega))u_w(\theta,\omega)d\omega\right) \\
&\leq 0
\end{aligned} \tag{OA.13}$$

where the last inequality follows from our analysis of the terms in the parenthesis in the proof of Claim OA.5. Any $\theta \in [\underline{\theta},\theta_1)$ are always recommended not to buy, which they optimally follow because type θ_1 does so.

Now, recall that $\min_{\hat{\theta}} p(\hat{\theta}) = p(\bar{\theta})$ by Claim OA.5. So for any $\theta \in [\underline{\theta},\bar{\theta}]$, we have

$$\begin{aligned}
\max_{\hat{\theta}} \mathbb{E}_{\omega}[u(\theta,\omega)] - p(\hat{\theta}) &= \mathbb{E}_{\omega}[u(\theta,\omega)] - p(\bar{\theta}) \\
&= [1-F(k(\bar{\theta}))]\left[\mathbb{E}_{\omega}[u(\theta,\omega)|\omega \geq k(\bar{\theta})] - p(\bar{\theta})\right] + F(k(\bar{\theta}))\left[\mathbb{E}_{\omega}[u(\theta,\omega)|\omega < k(\bar{\theta})] - p(\bar{\theta})\right] \\
&\leq [1-F(k(\bar{\theta}))]\left[\mathbb{E}_{\omega}[u(\theta,\omega)|\omega \geq k(\bar{\theta})] - p(\bar{\theta})\right] + F(k(\bar{\theta}))\left[\mathbb{E}_{\omega}[u(\bar{\theta},\omega)|\omega < k(\bar{\theta})] - p(\bar{\theta})\right] \\
&\leq [1-F(k(\bar{\theta}))]\left[\mathbb{E}_{\omega}[u(\theta,\omega)|\omega \geq k(\bar{\theta})] - p(\bar{\theta})\right] \\
&= U(\theta,\bar{\theta})
\end{aligned}$$

where the first inequality is from $\theta \leq \bar{\theta}$, and the second one from (OA.13). So, (OA.12) is satisfied. $\square$