Online Appendix to "Redesigning the US Army's Branching Process: A Case Study in Minimalist Market Design"

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A Main Characterization Result

A.1 Proof of Theorem 1

Proof of Theorem 1: Fix $(\pi_b)_{b\in B} \in \Pi^{|B|}$ and $(\omega_b)_{b\in B} \in \prod_{b\in B} \omega_b$.

We first show that the mechanism ϕ^{MPCO} satisfies the five axioms. For the proofs of *individual rationality, non-wastefulness, no priority reversal,* and *respect for the price responsiveness scheme*, also fix $\succ \in \mathcal{Q}^{|I|}$.

Individual rationality: No cadet $i \in I$ ever makes a proposal to a branch b at a price $t \in T$ under the MPCO procedure, unless her preferences are such that $(b,t) \succ_i \emptyset$. Hence, the MPCO mechanism satisfies *individual rationality*.

Non-wastefulness: For any branch $b \in B$, unless there are already q contracts with distinct cadets on hold, it is not possible for the base-price contract of any given cadet to be rejected at any step of the MPCO procedure. Hence, the MPCO mechanism satisfies *non-wastefulness*.

No priority reversal: Suppose that $\phi_j^{MPCO}(\succ) \succ_i \phi_i^{MPCO}(\succ)$ for a pair of cadets $i,j \in I$. Since the MPCO mechanism is *individually rational*, $\phi_j^{MPCO}(\succ) \neq \emptyset$. Let branch $b \in B$ and price $t \in T$ be such that $\phi_j^{MPCO}(\succ) = (b,t)$. Let k be the final step of the MPCO procedure. Since $\phi_j^{MPCO}(\succ) \succ_i \phi_i^{MPCO}(\succ)$, cadet i has proposed the contract (i,b,t) to branch b at some step of the MPCO procedure, which is rejected by branch b (strictly speaking for the first time) either immediately or at a later step. Since the proposed contracts remain available until the termination of the MPCO procedure, the contract (i,b,t) is also rejected by branch b at the final Step k of the MPCO procedure. In contrast, since $\phi_j^{MPCO}(\succ) = (b,t)$, contract (j,b,t) is chosen by branch b at the final step k of the MPCO procedure. If the contract (j,b,t) is accepted as one of the first q_b^0 positions under the choice rule $\mathcal{C}_b^{MP}$, then $j \ \pi_b \ i$. Otherwise, if the contract (j,b,t) is accepted as one of the last q_b^f positions under the choice rule $\mathcal{C}_b^{MP}$, then $(j,b,t) \ \omega_b \ (i,b,t)$. In either case, we have $j \ \pi_b \ i$, proving that the MPCO mechanism satisfies *no priority reversal*.

Respect for the price responsiveness scheme: Let cadet $i \in I$ be such that $\phi_i^{MPCO}(\succ) = X_i = (b,t) \in B \times T^+$. Let price $t' \in T$ be such that $t' < t$. Let cadet $j \in I \setminus \{i\}$ be such that $(b,t') \succ_j \phi_j^{MPCO}(\succ)$. Then cadet j has proposed the contract (j,b,t') to branch b at some step of the MPCO procedure, which is rejected by branch b either immediately or at a later step. Let k be the final step of the MPCO procedure. Since the proposed contracts remain available until the termination of the procedure, the contract (j,b,t') is also rejected by branch b at the final Step k of the MPCO procedure. More specifically, it is rejected by the choice rule $\mathcal{C}_b^{MP}$ at the final Step k of the procedure both for the first q_b^0 positions using the baseline priority order π_b and for the last q_b^f positions using the price responsiveness scheme ω_b. In contrast, since $t \in T^+$ by assumption, contract (i,b,t) is chosen by branch b at the final Step k of the MPCO procedure using the price responsiveness scheme ω_b. Therefore, we have $(i,t)\ \omega_b\ (j,t')$, which in turn implies that cadet j does not have a legitimate claim for a price reduced version of $X_i = (b,t)$.

Let cadet $i \in I$ be such that $\phi_i^{MPCO}(\succ) = X_i = (b,t) \in B \times T$. Let price $t' \in T^+$ be such that $t' > t$. Let cadet $j \in I \setminus \{i\}$ be such that $(b,t') \succ_j \phi_j^{MPCO}(\succ)$ and $(j,t')\ \omega_b\ (i,t)$. Further assume that cadet i is the lowest π_b-priority cadet with an assignment of (b,t).

The relation $(b,t') \succ_j \phi_j^{MPCO}(\succ)$ implies that cadet j has proposed the contract (j,b,t') to branch b at some step of the MPCO procedure, which is rejected by branch b either immediately or at a later step. Let k be the final step of the MPCO procedure. Since the proposed contracts remain available until the termination of the procedure, the contract (j,b,t') is also rejected by branch b at the final Step k of the MPCO procedure. More specifically, it is rejected by the choice rule $\mathcal{C}_b^{MP}$ at the final Step k even for the last q_b^f positions using the price responsiveness scheme ω_b. Therefore, since by assumption we have $(j,t')\ \omega_b\ (i,t)$, cadet i must have received one of the first q^0 positions using the baseline priority ranking π_b. But since cadet i is the lowest π_b-priority cadet with an assignment of $(b,t) = (b,t^0)$, that means no cadet has received any of the last q_b^f positions at the base price of t^0. Therefore, since MPCO mechanism satisfies *non-wastefulness*,

$$\left| \{(k,t^+) \in I \times T^+ \ :\ (k,b,t^+) \in X_b\} \right| = q_b^f.$$

That is, the cap for flexible-price positions is already reached at branch b. As such, cadet j does not have a legitimate claim for a price increased version of $X_i = (b,t)$.

Since no cadet has a legitimate claim for either a priced reduced or a priced increased version of another cadet's assignment, the MPCO mechanism *respects the price responsiveness scheme*.

Strategy-proofness: The MPCO mechanism is a special case of the cumulative offer mechanism for *matching problems with slot-specific priorities* formulated in Kominers and Sönmez (2016). Hence *strategy-poofness* of the MPCO mechanism is a direct corollary of their Theorem 3, which proves *strategy-proofness* of the cumulative offer mechanism more broadly for matching problems with slot-specific priorities.

Uniqueness: We prove uniqueness via two lemmata.

Lemma 1. *Let $X, Y \in \mathcal{A}$ be two distinct allocations that respect the price responsiveness scheme and satisfy individual rationality, non-wastefulness and no priority reversal. Then there exists a cadet $i \in I$ who receives non-empty and distinct assignments under X and Y.*

Proof of Lemma 1: The proof is by contradiction. Fix $\succ \in \mathcal{Q}^{|I|}$. Let $X, Y \in \mathcal{A}$ be two distinct allocations that *respect the price responsiveness scheme* and satisfy *individual rationality, non-wastefulness* and *no priority reversal*. To derive the desired contradiction, suppose that, for any cadet $i \in I$,

$$X_i \neq Y_i \quad \Longrightarrow \quad X_i = \emptyset \ \text{ or } \ Y_i = \emptyset. \tag{1}$$

Pick any branch $b \in B$ such that $X_b \neq Y_b$. Let $j \in I$ be the highest π_b-priority cadet who is assigned to branch b either under X or under Y, but not both. W.l.o.g., let cadet j be assigned to branch b under allocation X but not under allocation Y. By relation (1),

$$Y_j = \emptyset. \tag{2}$$

Since allocation Y satisfies *non-wastefulness*, there exists a cadet $k \in I$ who is assigned to branch b under allocation Y but not under allocation X. By relation (1),

$$X_k = \emptyset, \tag{3}$$

and therefore, by the choice of cadet j, we have

$$j \ \pi_b \ k. \tag{4}$$

Let $X_j = (b, t)$ and $Y_k = (b, t')$. Since allocations X and Y satisfy *individual rationality*, we must have $t' \neq t$, for otherwise one of the allocations X, Y would fail *no priority reversal*. Moreover, since we either have $(j, t) \ \omega_b \ (k, t')$ or $(k, t') \ \omega_b \ (j, t)$, one of the two prices t, t' has to be equal to t^0, for otherwise one of the allocations X, Y would fail to *respect the price responsiveness scheme*. Therefore, by relation (4),

$$Y_k = (b, t') \quad \text{for some } t' \in T^+, \tag{5}$$

for otherwise (i.e., if $Y_k = (b, t^0)$) allocation Y would fail *no priority reversal*. Hence,

$$X_j = (b, t^0). \tag{6}$$

Since allocation Y *respects the price responsiveness scheme*, relations (2) and (5) imply

$$(k, t') \ \omega_b \ \underbrace{(j, t)}_{=(j,t^0)} . \tag{7}$$

Define

$$I^* = \{i \in I : X_i = (b, t_i^+) \ \text{ for some } t_i^+ \in T^+\}. \tag{8}$$

Since allocation Y *respects the price responsiveness scheme*,

$$|I^*| = q_b^f, \tag{9}$$

for otherwise cadet k would have a legitimate claim for a price increased version of cadet j's assignment $X_j = (b, t^0)$ by relations (3) and (7).

Since allocation X *respects the price responsiveness scheme* and

$$\underbrace{(b, t')}_{=Y_k} \succ_k \underbrace{X_k}_{=\emptyset},$$

for any $i \in I^*$, we have

$$(i, t_i^+) \; \omega_b \; (k, t'). \tag{10}$$

But since $Y_k = (b, t')$ for some $t' \in T^+$ by relation (5) and $|I^*| = q_b^f$ by relation (9), there exists a cadet $\ell \in I^*$ with $Y_\ell \notin \{(b, t_\ell^+) : t_\ell^+ \in T^+\}$. Therefore, by relations (1) and (8), we have

$$Y_\ell = \emptyset. \tag{11}$$

Since X satisfies *individual rationality* and $\ell \in I^*$, we have

$$(b, t_\ell^+) \;\succ_\ell\; \emptyset. \tag{12}$$

Therefore, by relations (5), (10), (11) and (12) allocation Y fails either *no priority reversal* or *respect for the price responsiveness scheme* (depending on whether $t_\ell^+ = t'$ or $t_\ell^+ \neq t'$), thus giving us the desired contradiction and completing the proof of Lemma (1). $\diamondsuit$

Lemma 2. *There can be at most one direct mechanism that respects the price responsiveness scheme and satisfies individual rationality, non-wastefulness, no priority reversal and strategy-proofness.*

Proof of Lemma 2: The proof of this lemma is inspired by a technique introduced by Hirata and Kasuya (2017). Towards a contradiction, suppose there exists two distinct direct mechanisms φ and ψ that *respect the price responsiveness scheme* and satisfy *individual rationality, non-wastefulness, no priority reversal* and *strategy-proofness*. Let the preference profile $\succ^* \in \mathcal{Q}^{|I|}$ be such that

1. $\varphi(\succ^*) \neq \psi(\succ^*)$, and
2. the aggregate number of acceptable contracts between all cadets is minimized among all preference profiles $\widetilde{\succ} \in \mathcal{Q}^{|I|}$ such that $\varphi(\widetilde{\succ}) \neq \psi(\widetilde{\succ})$.

Let $X = \varphi(\succ^*)$ and $Y = \psi(\succ^*)$. By Lemma 1, there exists a cadet $i \in I$ such that

1. $X_i \neq \emptyset$,
2. $Y_i \neq \emptyset$, and
3. $X_i \neq Y_i$.

Since both allocations X and Y satisfy *individual rationality*,

$$X_i \;\succ_i^*\; \emptyset \quad \text{and} \;\; Y_i \;\succ_i^*\; \emptyset.$$

W.l.o.g., assume

$$X_i \;\succ_i^*\; Y_i \;\succ_i^*\; \emptyset.$$

Construct the preference relation $\succ_i' \in \mathcal{Q}$ as follows:

If $X_i = (b, t^0)$ for some $b \in B$, then

$$(b, t^0) \ \succ'_i \ \emptyset \ \succ'_i \ (b', t') \qquad \text{for any } (b', t') \in B \times (T \setminus \{(b, t^0)\}).$$

Otherwise, if $X_i = (b, t^r)$ for some $b \in B$ and $r \in \{1, \ldots, h\}$, then

$$(b, t^0) \ \succ'_i \ \cdots \succ'_i \ (b, t^{r-1}) \ \succ'_i \ (b, t^r) \ \succ'_i \ \emptyset \ \succ'_i \ (b', t') \ \text{ for any } (b', t') \in B \times (T \setminus \{(b, t^0), \ldots, (b, t^r)\}).$$

Since $X_i \succ^*_i Y_i \succ^*_i \emptyset$ and $(b, t^0) \succ^*_i \ \cdots \ \succ^*_i \ (b, t^{r-1}) \ \succ^*_i \ (b, t^r)$, the preference relation $\succ'_i$ has strictly fewer acceptable contracts for cadet i than the preference relation $\succ^*_i$.

By *strategy-proofness* of the mechanism ψ, we have

$$\underbrace{\psi_i(\succ^*_i, \succ^*_{-i})}_{=Y_i} \ \succeq^*_i \ \psi_i(\succ'_i, \succ^*_{-i}),$$

and since no branch-price pair $(b', t') \in B \times T$ with $Y_i \succeq'_i (b', t')$ is acceptable under $\succ'_i$, by *individual rationality* of the mechanism ψ we have

$$\psi_i(\succ'_i, \succ^*_{-i}) = \emptyset. \tag{13}$$

Similarly, by *strategy-proofness* of the mechanism φ, we have

$$\varphi_i(\succ'_i, \succ^*_{-i}) \ \succeq'_i \ \underbrace{\varphi_i(\succ^*_i, \succ^*_{-i})}_{=X_i},$$

which in turn implies

$$\varphi_i(\succ'_i, \succ^*_{-i}) \neq \emptyset. \tag{14}$$

But then, by relations (13) and (14) we have

$$\varphi(\succ'_i, \succ^*_{-i}) \neq \psi(\succ'_i, \succ^*_{-i}),$$

giving us the desired contradiction, since between all cadets the preference profile $(\succ'_i, \succ^*_{-i})$ has strictly fewer acceptable contracts than the preference profile $\succ^*$. This completes the proof of Lemma 2. $\diamondsuit$

Since we have already shown that the MPCO mechanism satisfies all five axioms, Lemma 2 establishes the desired uniqueness result, thus concluding the proof of Theorem 1. $\square$

A.2 Independence of Axioms in Theorem 1

We establish the independence of the axioms in Theorem 1 by presenting five direct mechanisms. Each fails one of our five axioms and satisfies the other four. Our result shows that none of the axioms are redundant in Theorem 1 and each is important for the characterization of MPCO mechanism.

A.2.1 A mechanism that satisfies all axioms except individual rationality

Given any preference profile $\succ \in \mathcal{Q}^{|I|}$ and individual $i \in I$, let $\succ_i^0 \in \mathcal{Q}$ be the preference relation where the relative preference ranking of all branch-price pairs in $B \times T$ is the same as in $\succ_i$, and remaining unmatched (i.e. $\emptyset$) is the last choice. Define the direct mechanism ϕ^0 as, for any preference profile $\succ \in \mathcal{Q}^{|I|}$,

$$\phi^0(\succ) = \phi^{MPCO}(\ \succ^0\ ).$$

Mechanism ϕ^0 satisfies all axioms except *individual rationality*.

A.2.2 A mechanism that satisfies all axioms except non-wastefulness

Define the direct mechanism $\phi^{\emptyset}$ as, for any preference profile $\succ \in \mathcal{Q}^{|I|}$,

$$\phi^{\emptyset}(\succ) = \emptyset.$$

Mechanism $\phi^{\emptyset}$ satisfies all axioms except *non-wastefulness*.

A.2.3 A mechanism that satisfies all axioms except respect for the price responsiveness scheme

The individual-proposing deferred acceptance mechanism given in Online Appendix C.2 satisfies all axioms except *respect for the price responsiveness scheme*.[46]

A.2.4 A mechanism that satisfies all axioms except no priority reversal

We will assume that there are only two prices. In all other cases, assume that the outcome of mechanism ψ is same as the outcome of the MPCO mechanism. When there are two prices, t^0 and t^h, the outcome of the mechanism ψ is derived from the outcome of the MPCO mechanism as follows.

Fix a branch $b \in B$. Given any preference profile $\succ \in \mathcal{Q}^{|I|}$, let $i \in I$ be the lowest π_b-priority individual with $\mathsf{b}\big(\phi_i^{MPCO}(\succ)\big) = b$. Let the preference relation $\succ_i^{-b} \in \mathcal{Q}$ be constructed from $\succ_i$ by making branch-price pairs (b, t^0) and (b, t^h) unacceptable, but otherwise keeping the rest of the preference order same as in $\succ_i$. Let the outcome of the mechanism ψ be given as

- $\psi(\succ) = \phi^{MPCO}(\succ_{-i}, \succ_i^{-b})$ if all q_b^f flexible-price positions at branch b are awarded at the increased price t^h under both $\phi^{MPCO}(\succ_{-i}, \succ_i^{-b})$ and $\phi^{MPCO}(\succ)$, and
- $\psi(\succ) = \phi^{MPCO}(\succ)$ otherwise.

[46] More broadly the MPCO mechanism when implemented with a different profile of price responsiveness schemes than the underlying one also satisfies all axioms except *respect for the price responsiveness scheme*.

For any given branch $b \in B$, mechanism ψ derives its outcome mostly using the MPCO mechanism, except it "ignores" the lowest π_b-priority individual who receives a position at branch b under the MPCO mechanism provided that all flexible-price positions at branch b are awarded at the increased price t^h under the MPCO mechanism whether the lowest π_b-priority individual is being ignored or not. If in either scenario some of the q_b^f flexible-price positions are awarded at the base price t^0 or remain idle, then the outcome of the mechanism ψ is the same as the outcome of the MPCO mechanism.

Mechanism ψ satisfies all axioms except the *no priority reversal*. The detailed construction above assures that it does not also lose *respect for the price responsiveness scheme* or *strategy-proofness* due to the modification.

A.2.5 A mechanism that satisfies all axioms except strategy-proofness

We assume that there are only two prices. In all other cases, assume that the outcome of mechanism ψ is same as the outcome of the MPCO mechanism. The outcome of the mechanism φ is derived from the outcome of the MPCO mechanism as follows.

Fix a branch $b \in B$. Given any preference profile $\succ \in \mathcal{Q}^{|I|}$, let $i \in I$ be the lowest π_b-priority individual with $\mathsf{b}\big(\phi_i^{MPCO}(\succ)\big) = b$. If

1. $\phi_i^{MPCO}(\succ) = (b, t^0)$,
2. $(b, t^h) \succ_i \emptyset$, and
3. $\Big(\phi^{MPCO}(\succ) \setminus \big\{(i, b, t^0)\big\}\Big) \cup \big\{(i, b, t^h)\big\} \in \mathcal{A}$,

then let $\varphi(\succ) = \Big(\phi^{MPCO}(\succ) \setminus \big\{(i, b, t^0)\big\}\Big) \cup \big\{(i, b, t^h)\big\}$. Otherwise, i.e. if any of the three conditions fail, then let $\varphi(\succ) = \phi^{MPCO}(\succ)$.

Compared to the outcome of the MPCO mechanism, the mechanism φ simply increases the charged price for the lowest π_b-priority individual who receive a position at branch b under the MPCO mechanism if doing so is *feasible* and does not violate *individual rationality*.

Mechanism φ satisfies all axioms except *strategy-proofness*. The affected individual can profit by declaring the branch-price pair (b, t^h) as unacceptable under the mechanism φ. The detailed construction above assures that the mechanism does not also lose *individual rationality*, *no priority reversal*, or *respect for the price responsiveness scheme* due to the modification.

B Formal Analysis of USMA-2020 Mechanism

Section 4.3 presents the shortcomings of the USMA-2020 mechanism. In this section of the Online Appendix, we present a more in-depth analysis of the USMA-2020 mechanism to offer additional insight on why it resulted in a much more complex branching system than its predecessor USMA-2006 mechanism.

Since USMA-2020 is defined only when there is a single increased price, throughout this section, we assume that $T^+ = \{t^h\}$.

As with the USMA-2006 mechanism, truthful revelation of branch preferences is not a dominant strategy under the USMA-2020 mechanism, thereby making its formal analysis challenging. Fortunately, focusing on a simpler version of the model with a single branch is sufficient to illustrate and analyze the main challenges of the USMA-2020 mechanism.

Suppose we consider a single branch $b \in B$. When there is a single branch $b \in B$, there are only two preferences for any cadet $i \in I$. The base price contract (i, b, t^0) is by assumption preferred by cadet i to both its increased price version (i, b, t^h) and also to remaining unmatched. Therefore, the only variation in cadet i's preferences depends on whether the increased price contact (i, b, t^h) is preferred to remaining unmatched. For any cadet $i \in I$, $|\mathcal{Q}| = 2$ when there is a single branch $b \in B$, since

- indicating willingness to pay the increased price t^h under a quasi-direct mechanism can be naturally mapped to the preference relation where the increased price contact (i, b, t^h) is acceptable, whereas
- not doing so can be naturally mapped to the preference relation where the increased price contact (i, b, t^h) is unacceptable,

any quasi-direct mechanism can be interpreted as a direct mechanism. Therefore, unlike the general version of the model, the axioms of BRADSO-IC and elimination of strategic BRADSO are also well-defined for direct mechanisms when there is a single branch, and moreover, they are both implied by strategy-proofness.[47]

B.1 Single-Branch Mechanism ϕ^{MP} and Its Characterization

We next introduce a single-branch direct mechanism that is key for our analysis of the USMA-2020 mechanism. The main feature of this mechanism is its iterative subroutine (in Step 2), which determines how many flexible-price positions are assigned at the increased price and which cadets receive these positions.

Mechanism ϕ^{MP}

For any given profile of cadet preferences $\succ = (\succ_i)_{i \in I} \in \mathcal{Q}^{|I|}$, construct the allocation $\phi^{MP}(\succ)$ as follows:

Step 0. Let $I^0 \subset I$ be the set of q_b^0 highest π_b-priority cadets in I. For each cadet $i \in I^0$, finalize the assignment of cadet i as $\phi_i^{MP}(\succ) = (b, t^0)$.

[47] BRADSO-IC and elimination of strategic BRADSO together are equivalent to strategy-proofness when there is a single branch. Strategy-proofness of a single branch, called non-manipulability via contractual terms, also plays an important role in the analysis of Hatfield et al. (2021).

Step 1. Let $I^1 \subset I \setminus I^0$ be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. *Tentatively* assign each cadet in I^1 a position at the base price t^0. Relabel the set of cadets in I^1 so that cadet $i^1 \in I^1$ has the lowest π_b-priority in I^1, cadet $i^2 \in I^1$ has the second-lowest π_b-priority in I^1, …, and cadet $i^{q_b^f} \in I^1$ has the highest π_b-priority in I^1. Also relabel the lowest π_b-priority cadet in I^0 as $i^{q_b^f+1}$.

Step 2. This step determines how many positions are assigned at the increased price t^h.

Step 2.0. Let $J^0 \subset I \setminus (I^0 \cup I^1)$ be the set of cadets in $I \setminus (I^0 \cup I^1)$ who declared the position at the increased price t^h as acceptable:

$$J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^h) \succ_j \emptyset\}.$$

If

$$\left|\{j \in J^0 : (j, t^h) \; \omega_b \; (i^1, t^0)\}\right| = 0,$$

then finalize Step 2 and proceed to Step 3. In this case no position will be assigned at the increased price t^h.

Otherwise, if

$$\left|\{j \in J^0 : (j, t^h) \; \omega_b \; (i^1, t^0)\}\right| \geq 1,$$

then proceed to Step 2.1.

Step 2.$\boldsymbol{\ell}$. ($\boldsymbol{\ell = 1, \ldots, q_b^f}$) Let

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^h) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^h) \succ_{i^\ell} \emptyset. \end{cases}$$

If

$$\left|\{j \in J^\ell : (j, t^h) \; \omega_b \; (i^{\ell+1}, t^0)\}\right| = \ell,$$

then finalize Step 2 and proceed to Step 3.[48] In this case ℓ positions will be assigned at the increased price t^h.

Otherwise, if

$$\left|\{j \in J^\ell : (j, t^h) \; \omega_b \; (i^{\ell+1}, t^0)\}\right| \geq \ell + 1,$$

then proceed to Step 2.$(\ell+1)$, unless $\ell = q_b^f$, in which case finalize Step 2 and proceed to Step 3.

Step 3. Let Step 2.n be the final sub-step of Step 2 leading to Step 3. $\{i^1, \ldots, i^n\} \subset I^1$ is the set of cadets in I^1 who each lose their tentative assignment (b, t^0). For

[48] Since $J^\ell \supseteq J^{\ell-1}$ by construction, the fact that the procedure has reached Step 2.ℓ implies that the inequality $\left|\{j \in J^\ell : (j, t^h) \; \omega_b \; (i^{\ell+1}, t^0)\}\right| \geq \ell$ must hold.

each cadet $i \in I^1 \setminus \{i^1, \ldots, i^n\}$, finalize the assignment of cadet i as $\phi_i^{MP}(\succ) = (b, t^0)$.

For each cadet $i \in J^n$ with one of the n highest π_b-priorities in J^n, finalize the assignment of cadet i as $\phi_i^{MP}(\succ) = (b, t^h)$. Finalize the assignment of any remaining cadet as $\emptyset$.

The key step in the procedure is Step 2 where it is determined how many of the q_b^f flexible-price positions are to be awarded at the increased price t^h. To determine this number, the price responsiveness scheme ω_b is used to check

(1) whether there is at least one cadet with a lower baseline priority π_b than cadet i^1, who is willing to pay the increased price t^h and whose increased price contract has higher priority under the price responsiveness scheme ω_b than the base price contract of cadet i^1;

(2) whether there are at least two cadets each with a lower baseline priority π_b than cadet i^2, who are each willing to pay the increased price t^h and whose increased price contracts have higher priority under the price responsiveness scheme ω_b than the base price contract of cadet i^2;

$\vdots$

(q_b^f) whether there are at least q_b^f cadets each with a lower baseline priority π_b than cadet $i^{q_b^f}$, who are each willing to pay the increased price t^h and whose increased price contracts have higher priority under the price responsiveness scheme ω_b than the base price contract of cadet $i^{q_b^f}$.

Once the number of positions awarded through increased price t^h contracts is determined in this way, all other positions are assigned to the highest baseline priority cadets as base price contracts. The increased price contracts are awarded to the remaining highest baseline priority cadets who are willing to pay the increased price t^h.

Example 2. (Mechanics of Mechanism ϕ^{MP}) There is a single branch b with $q_b^0 = 3$ and $q_b^f = 3$. There are eight cadets, with their set given as $I = \{i^1, i^2, i^3, i^4, i^5, i^6, j^1, j^2\}$. The baseline priority order π_b is given as

$$i^6 \; \pi_b \; i^5 \; \pi_b \; i^4 \; \pi_b \; i^3 \; \pi_b \; i^2 \; \pi_b \; i^1 \; \pi_b \; j^1 \; \pi_b \; j^2,$$

and the price responsiveness scheme is the ultimate price responsiveness scheme $\overline{\omega}_b$. Cadet preferences are given as

$$\begin{aligned} (b, t^0) \; \succ_i \; (b, t^h) \; \succ_i \; \emptyset \qquad & \text{for any } i \in \{i^1, i^3, i^5, j^1\}, \text{ and} \\ (b, t^0) \; \succ_i \; \emptyset \; \succ_i \; (b, t^h) \qquad & \text{for any } i \in \{i^2, i^4, i^6, j^2\}. \end{aligned}$$

We next run the procedure for the mechanism ϕ^{MP}.

Step 0: There are three base-price positions. The three highest π_b-priority cadets in the set I are i^6, i^5, and i^4. Let $I^0 = \{i^4, i^5, i^6\}$, and finalize the assignments of cadets in I^0 as $\phi^{MP}_{i^6}(\succ) = \phi^{MP}_{i^5}(\succ) = \phi^{MP}_{i^4}(\succ) = (b, t^0)$.
Step 1: There are three flexible-price positions. Three highest π_b-priority cadets in the set $I \setminus I^0$ are i^3, i^2, and i^1. Let $I^1 = \{i^1, i^2, i^3\}$, and the tentative assignment of each cadet in I^1 is (b, t^0). There is no need to relabel the cadets since cadet i^1 is already the lowest π_b-priority cadet in I^1, cadet i^2 is the second lowest π_b-priority cadet in I^1, and cadet i^3 is the highest π_b-priority cadet in I^1.
Step 2.0: The set of cadets in $I \setminus (I^0 \cup I^1) = \{j^1, j^2\}$ for whom the assignment (b, t^h) is acceptable is $J^0 = \{j^1\}$. Since

$$\underbrace{\left|\{j \in J^0 : (j, t^h)\ \overline{\omega}_b\ (i^1, t^0)\}\right|}_{=|J^0|=|\{j^1\}|=1} \geq 1,$$

we proceed to Step 2.1.
Step 2.1: Since $(b, t^h) \succ_{i^1} \emptyset$, we have $J^1 = J^0 \cup \{i^1\} = \{i^1, j^1\}$. Since

$$\underbrace{\left|\{j \in J^1 : (j, t^h)\ \overline{\omega}_b\ (i^2, t^0)\}\right|}_{=|J^1|=|\{i^1,j^1\}|=2} \geq 2,$$

we proceed to Step 2.2.
Step 2.2: Since $\emptyset \succ_{i^2} (b, t^h)$, we have $J^2 = J^1 = \{i^1, j^1\}$. Since

$$\underbrace{\left|\{j \in J^2 : (j, t^h)\ \overline{\omega}_b\ (i^3, t^0)\}\right|}_{=|J^2|=|\{i^1,j^1\}|=2} = 2,$$

we finalize Step 2 and proceed to Step 2.3.
Step 3: Step 2.2 is the last sub-step of Step 2. Therefore two lowest π_b-priority cadets in I^1, i.e cadets i^1 and i^2, lose their tentative assignments of (b, t^0). In contrast, the only remaining cadet in the set $I^1 \setminus \{i^1, i^2\}$, i.e cadet i^3 maintains her tentative assignment, which is finalized as $\phi^{MP}_{i^3}(\succ) = (b, t^0)$.

The two highest priority cadets in J^2 are i^1 and j^1. Their assignments are finalized as $\phi^{MP}_{i^1}(\succ) = \phi^{MP}_{j^1}(\succ) = (b, t^h)$. Assignments of the remaining cadets i^2 and j^2 are finalized as $\emptyset$. The final allocation is:

$$\phi^{MP}(\succ) = \begin{pmatrix} i^1 & i^2 & i^3 & i^4 & i^5 & i^6 & j^1 & j^2 \\ (b, t^h) & \emptyset & (b, t^0) & (b, t^0) & (b, t^0) & (b, t^0) & (b, t^h) & \emptyset \end{pmatrix}.$$

■

Our next result is the following characterization of the the single-branch direct mechanism ϕ^{MP}.

Proposition 2. *Suppose there is a single branch b. Fix a baseline priority order $\pi_b \in \Pi$ and a price responsiveness scheme $\omega_b \in \Omega_b$. A direct mechanism φ respects the price responsiveness scheme and* *satisfies individual rationality, non-wastefulness, no priority reversal and BRADSO-IC if and only if* $\varphi = \phi^{MP}$.

Since (i) a quasi-direct mechanism becomes a direct mechanism when there is a single branch, and (ii) strategy-proofness implies BRADSO-IC in this environment, Theorem 1 and Proposition 2 immediately imply the following result.

Corollary 2. *Suppose there is a single branch b. Fix a baseline priority order* $\pi_b \in \Pi$ *and a price responsiveness scheme* $\omega_b \in \Omega_b$. *Then, for any preference profile* $\succ \in \mathcal{Q}^{|I|}$,

$$\phi^{MP}(\succ) = \phi^{MPCO}(\succ).$$

The mechanism ϕ^{MP} is merely an alternative formulation of the MPCO mechanism that does not rely on the cumulative offer procedure when there is a single branch. This formulation is helpful for the single-branch equilibrium analysis of the USMA-2020 mechanism we present next.

B.2 Equilibrium Outcomes under the USMA-2020 Mechanism

While the USMA-2020 mechanism is not a direct mechanism in general, when there is a single branch it can be interpreted a direct mechanism. In this case, for any cadet $i \in I$ the first part of the message space $\mathcal{S}_i = \mathcal{P} \times 2^B$ becomes redundant, and the second part simply solicits whether branch b is acceptable by cadet i or not (analogous to a direct mechanism).

Our next result shows that when there is a single branch the truthful outcome of the direct mechanism ϕ^{MP} is the same as the unique Nash equilibrium outcome of the mechanism φ^{2020}.

Proposition 3. *Suppose there is a single branch b. Fix a baseline priority order* $\pi_b \in \Pi$, *a price responsiveness scheme* $\omega_b \in \Omega_b$, *and a preference profile* $\succ \in \mathcal{Q}^{|I|}$. *Then the strategic-form game induced by the mechanism* $(\mathcal{S}^{2020}, \varphi^{2020})$ *has a unique Nash equilibrium outcome that is equal to the allocation* $\phi^{MP}(\succ)$.[49]

Caution is needed when interpreting Proposition 3; if interpreted literally, this result can be misleading. What is more consequential for Proposition 3 is not the result itself, but rather its proof which constructs the equilibrium strategies of cadets. The proof provides insight into why the failure of BRADSO-IC, the presence of strategic BRADSO, and the presence of detectable priority reversals are all common phenomena under the real-life implementation of the USMA-2020 mechanism (despite the outcome equivalence suggested by Proposition 3).

Given the byzantine structure of the Nash equilibrium strategies even with a single branch, it is perhaps not surprising that reaching such a well-behaved Nash equilibrium is highly unlikely to be observed under the USMA-2020 mechanism. The following example illustrates the knife-edge structure of the Nash equilibrium strategies under the USMA-2020 mechanism.

[49]Using the terminology of the *implementation theory*, this result can be alternatively stated as follows: When there is a single branch, the mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$ implements the allocation rule ϕ^{MP} in Nash equilibrium.

Example 3. (Knife-Edge Nash Equilibrium Strategies)

To illustrate how challenging it is for the cadets to figure out their best responses under the USMA-2020 mechanism, we present two scenarios. The scenarios differ from each other minimally, but cadet best responses differ dramatically. Our first scenario is same as the one we presented in Example 2.

Scenario 1: There is a single branch b with $q_b^0 = 3$ and $q_b^f = 3$. There are eight cadets, $I = \{i^1, i^2, i^3, i^4, i^5, i^6, j^1, j^2\}$. The baseline priority order π_b is given as

$$i^6 \ \pi_b \ i^5 \ \pi_b \ i^4 \ \pi_b \ i^3 \ \pi_b \ i^2 \ \pi_b \ i^1 \ \pi_b \ j^1 \ \pi_b \ j^2 \quad \text{and}$$

and the price responsiveness scheme is the ultimate price responsiveness scheme $\overline{\omega}_b$. Cadet preferences are

$$\begin{array}{lll} (b,t^0) \ \succ_i \ (b,t^h) \ \succ_i \ \emptyset & & \text{for any } i \in \{i^1, i^3, i^5, j^1\}, \ \text{and} \\ (b,t^0) \ \succ_i \ \emptyset \ \succ_i \ (b,t^h) & & \text{for any } i \in \{i^2, i^4, i^6, j^2\}. \end{array}$$

Let s^* be a Nash equilibrium strategy for Scenario 1 under the USMA-2020 mechanism. Recall that when there is a single branch b, the message space for each cadet $i \in I$ is simply $\mathcal{S}_i = \{b, \emptyset\}$. We construct the Nash equilibrium strategies in several phases.

Phase 1: Consider cadets i^1 and j^1, each of whom prefers the increased price assignment (b,t^h) to remaining unmatched. Since there are six positions altogether and there are five higher π_b-priority cadets than either of these two cadets, at most one of them can receive a position (at any cost) unless each of them submit a strategy of b. And if one of them submits a strategy of $\emptyset$, the other one has a best response strategy of b assuring a position at the increased price rather than remaining unmatched. Hence, $s^*_{i^1} = s^*_{j^1} = b$ at any Nash equilibrium.

Phase 2: Consider cadet j^2 who prefers remaining unmatched to the increased price assignment (b,t^h). Since she is the lowest π_b-priority cadet, she cannot receive an assignment of (b,t^0) regardless of her strategy. In contrast, she can guarantee remaining unmatched with a strategy of $s_{j^2} = \emptyset$. While this does not at this point rule out a strategy of $s_{j^2} = b$ at Nash equilibrium (just yet), it means $\varphi^{2020}_{j^2}(s^*) = \emptyset$.

Phase 3: Consider cadet i^2 who prefers remaining unmatched to the increased price assignment (b,t^h). She is the fifth highest π_b-priority cadet, so she secures a position if she submits a strategy of $s_{i^2} = b$, but the position will have to be at the increased price t^h, since the lowest π_b-priority cadet j^2 is remaining unmatched from Phase 2, and therefore there cannot be three cadets with lower π_b-priority who receive an assignment of (b,t^h). But since cadet j^2 prefers remaining unmatched to the increased price assignment (b,t^h), she cannot receive an assignment of (b,t^h) at Nash equilibria. Hence, cadet i^2's Nash equilibrium strategy is $s^*_{i^2} = \emptyset$, and her Nash equilibrium assignment is $\varphi^{2020}_{i^2}(s^*) = \emptyset$.

Phase 4: Consider the remaining cadets i^3, i^4, i^5 and i^6. Since cadets i^2 and j^2 have to remain unmatched (from Phases 2 and 3) at Nash equilibria, they each receive a position at Nash equilibrium. Since only the two cadets i^1 and j^1 from Phases 1-3 have Nash equilibrium strategies of b, the lowest π_b-priority cadet of the four cadets i^3, i^4, i^5, i^6 who submit a strategy of b receives an assignment of (b, t^h). But this cannot happen at Nash equilibria since that particular cadet can instead submit a strategy of $\emptyset$ receiving a more preferred assignment of (b, t^0). Hence, $s^*_i = \emptyset$ and $\varphi^{2020}_i(s^*) = (b, t^0)$ for any $i \in \{i^3, i^4, i^5, i^6\}$.

The unique Nash equilibrium strategy s^* and its Nash equilibrium outcome $\varphi^{2020}(s^*)$ for Scenario 1 are given as:

Cadet	i^1	i^2	i^3	i^4	i^5	i^6	j^1	j^2
Nash equilibrium strategy	b	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	b	$\emptyset$
Nash equilibrium assignment	(b, t^h)	$\emptyset$	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^h)	$\emptyset$

Scenario 1 involves *BRADSO-IC* failures for cadets i^3 and i^5 whose Nash equilibrium strategies force them into hiding their willingness to pay the increased price t^h. Any deviation from her Nash equilibrium strategy by truthfully declaring her willingness to pay the increased price t^h will result in an detectable priority reversal for cadet i^5.

Scenario 2: This scenario differs from Scenario 1 in only the preferences of the lowest π_b-priority cadet j^2 and nothing else. Thus, cadet preferences for this scenario are given as:

$$\begin{array}{ll} (b, t^0) \succ'_i (b, t^h) \succ'_i \emptyset & \text{for any } i \in \{i^1, i^3, i^5, j^1, j^2\}, \text{ and} \\ (b, t^0) \succ'_i \emptyset \succ'_i (b, t^h) & \text{for any } i \in \{i^2, i^4, i^6\}. \end{array}$$

Let s' be a Nash equilibrium strategy for Scenario 2 under the USMA-2020 mechanism.

Phase 1: Identical to Phase 1 for Scenario 1, and thus $s'_{i^1} = s'_{j^1} = b$ at any Nash equilibrium.

Phase 2: Consider cadet i^2 who prefers remaining unmatched to the increased price assignment (b, t^h), and cadets i^3 and j^2, each of whom prefers the increased price assignment (b, t^h) to remaining unmatched. Since (i) there are six positions altogether, (ii) three cadets with higher π_b-priority than each one of i^2, i^3, and j^2, and (iii) $s'_{i^1} = s'_{j^1} = b$ from Phase 1, at most one of the cadets i^2, i^3, j^2 can receive an assignment of (b, t^0) if any. Therefore, submitting a strategy of $s_{i^3} = \emptyset$ is a best response for cadet i^3 only if both cadets i^2 and j^2 also submit a strategy of $\emptyset$ each. But this cannot happen in Nash equilibria, since it gives cadet j^2 a profitable deviation by submitting a strategy of $s_{j^2} = b$ and jumping ahead of cadets i^2 and i^3 securing her a position. Hence $s'_{i^3} = b$ and $\varphi^{2020}_{i^3}(s') = (b, t^h)$. When cadet i^3 joins the two cadets from Phase 1 each also submitting a strategy of b, this assures that exactly three positions will be assigned at the increased price t^h. Therefore a strategy of f $s_{i^2} = b$ assures assures cadet i^2 an assignment of (b, t^h), which cannot happen at Nash equilibrium. Therefore, $s'_{i^2} = \emptyset$ and $\varphi^{2020}_{i^2}(s') = \emptyset$. This not only assures that $\varphi^{2020}_{i^3}(s') = \varphi^{2020}_{i^1}(s') = \varphi^{2020}_{j^1}(s') = (b, t^h)$, but it also means that $s'_{j^2} = b$ at Nash equilibrium, for otherwise with two lower π_b-priority cadets with strategies of $\emptyset$, cadet i^3 would have an incentive to deviate himself and receiving the position at the base price rather than the increased price.

Phase 3: Consider the remaining cadets i^4, i^5 and i^6. Of all lower π_b-priority cadets, only the cadet i^2 and has Nash equilibrium strategies of $\emptyset$ from Phases 1 and 2. Therefore the lowest π_b-priority cadet of the three cadets i^4, i^5, i^6 who submit a strategy of $\emptyset$ receives an assignment of $\emptyset$. But this cannot happen at Nash equilibria since that particular cadet can instead submit a strategy of b and receive a more preferred assignment of (b, t^0) since three lower π_b-priority cadets already receive an assignment of (b, t^h) each from Phase 2. Therefore, regardless of their preferences $s'_{i^4} = s'_{i^5} = s'_{i^6} = b$, and $\varphi^{2020}_{i^4}(s') = \varphi^{2020}_{i^5}(s') = \varphi^{2020}_{i^6}(s')(b, t^0)$.

The unique Nash equilibrium strategy s' and its Nash equilibrium outcome $\varphi^{2020}(s')$ for Scenario 2 are given as:

Cadet	i^1	i^2	i^3	i^4	i^5	i^6	j^1	j^2
Nash equilibrium strategy	b	$\emptyset$	b	b	b	b	b	b
Nash equilibrium assignment	(b, t^h)	$\emptyset$	(b, t^h)	(b, t^0)	(b, t^0)	(b, t^0)	(b, t^h)	$\emptyset$

Not only does the Nash equilibrium strategies of cadets i^4 and i^6 involve strategic BRADSO in Scenario 2 and they have to declare willingness to pay the increased price t^h even though under their true preferences they do not, but any deviation from this Nash equilibrium strategy by declaring their unwillingness to pay the increased price t^h will result in detectable priority reversals for both cadets.

Another key insight from this example is the dramatic difference between the Nash equilibrium strategies due to one minor change in the underlying economy, a preference change in the lowest base priority cadet. This minor change only affects the assignment of cadet i^3 by changing it from (b, t^0) to (b, t^h). It also changes the Nash equilibrium strategy of not only cadet i^3, and also all other higher π_b-priority cadets i^4, i^5, and i^6. Moreover, in addition to BRADSO-IC failures and the presence of strategic BRADSO under Nash equilibria, any deviation from these strategies result in detectable priority reversals. The fragility of our equilibrium strategies provides us intuition on the prevalence of these phenomena under the USMA-2020 mechanism. ■

Example 3 shows that while the failure of BRADSO-IC and the presence of strategic BRADSO can be observed at Nash equilibria of the USMA-2020 mechanism, the presence of detectable priority reversals is out-of-equilibrium behavior under complete information when there is a single branch. Example 1 in the main text further shows that if the complete information assumption is relaxed there can also be detectable priority reversals in the Bayesian equilibria of the USMA-2020 mechanism.

B.3 Proofs for Results in Online Appendix Section B

Proof of Proposition 2: Suppose there is only one branch $b \in B$, and fix a profile of cadet preferences $\succ \in \mathcal{Q}^{|I|}$. We first show that the direct mechanism ϕ^{MP} satisfies the five axioms.

Individual rationality: This axiom holds immediately under ϕ^{MP}, since no cadet $i \in I$ is considered for a position at the increased price t^h unless her submitted preferences is such that $(b, t^h) \succ_i \emptyset$.

Non-wastefulness: Since there is only one branch and we already established *individual rationality*, we can focus on cadets who consider a position at the base price to be acceptable. With this observation, *non-wastefulness* also holds immediately under ϕ^{MP}, since all positions are allocated at Steps 0 and 1 at the base price t^0 either as a final assignment or a tentative one. Tentative assignments from Step 1 may be altered later on by increasing their price to t^h and possibly changing their recipients, but not by leaving the position unassigned, hence assuring non-wastefulness.

No priority reversal: Under the mechanism ϕ^{MP}, each of the q_b^0 highest π_b-priority cadets are assigned a position at the base price t^0 at Step 0, and each of the next q_b^f highest π_b-priority cadets are tentatively assigned a position at the base price t^0 at Step 1. Tentative positions are lost in Step 2 only if there is excess demand from qualified cadets who are willing to pay the increased price t^h, and starting with the lowest π_b priority cadets with tentative assignments. That assures that, for any $i, j \in I$,

$$\phi_j^{MP}(\succ) = (b, t^0) \succ_i \phi_i^{MP}(\succ) \quad \Longrightarrow \quad j \; \pi_b \; i. \tag{15}$$

Moreover positions at the increased price t^h are offered to cadets with highest π_b priorities among those (i) who fail to receive a position at the base price t^0 and (ii) who declare the expensive assignment (b, t^h) as acceptable. Therefore, for any $i, j \in I$,

$$\phi_j^{MP}(\succ) = (b, t^h) \succ_i \phi_i^{MP}(\succ) = \emptyset \quad \Longrightarrow \quad j \; \pi_b \; i. \tag{16}$$

Relations (15) and (16) imply that mechanism ϕ^{MP} satisfies *no priority reversal*.

BRADSO-IC: Fix a cadet $i \in I$. For a given profile of preferences for all cadets except cadet i, whether cadet $i \in I$ receives an assignment of (b, t^0) under the mechanism ϕ^{MP} is independent of cadet i's preferences under the mechanism ϕ^{MP}: Cadets who are among the q_b^0 highest π_b-priority cadets in I always receive an assignment at the base price t^0; cadets who are not among the q highest π_b-priority cadets in I never receive an assignment at the base price t^0; and for any cadet i who has one of the highest q but not one of the highest q_b^0 priorities, whether she receives an assignment at the base price t^0 depends on how many lower π_b-priority cadets are both willing to pay the increased price t^h and also able to "jump ahead of" cadet i through the price responsiveness scheme. Hence if a cadet receives a position under ϕ^{MP} at the increased price t^h, changing her reported preferences can only result in losing the position altogether. Therefore mechanism ϕ^{MP} satisfies *BRADSO-IC*.

Respect for the price responsiveness scheme: The procedure for the mechanism ϕ^{MP} initially assigns all positions to the q_b highest π_b-priority cadets at the base price t^0, although the assignments of the q_b^f-lowest π_b-priority cadets among these awardees are only tentative. Step 2 of the procedure for mechanism ϕ^{MP} ensures that, if any cadet $j \in I$ loses her tentative assignment (b, t^0) from Step 1, then any cadet $i \in I$ who receives an assignment of (b, t^h) is such that $(i, t^h) \; \omega_b \; (j, t^0)$. Therefore,

$$\left.\begin{array}{l} \phi_i^{MP}(\succ) = (b, t^h), \text{ and} \\ (b, t^0) \succ_j \phi_j^{MP}(\succ) \end{array}\right\} \quad \Longrightarrow \quad (i, t^h) \; \omega_b \; (j, t^0). \tag{17}$$

Moreover, Step 2 of the same procedure also ensures that, for any $\ell \in \{1, \ldots, q_b^f\}$, the ℓ^{th} lowest π_b-priority cadet i^ℓ with a tentative assignment of (b, t^0) cannot maintain this tentative assignment, for as long as there are at least ℓ lower π_b-priority cadets who are both willing to pay the increased price t^h and also able to "jump ahead of" the cadet i^ℓ through the price responsiveness scheme. Therefore,

$$\left.\begin{array}{l} \phi_i^{DP}(\succ) = (b, t^0), \\ (b, t^+) \succ_j \phi_j^{DP}(\succ), \ \text{and} \\ (j, t^+) \ \omega_b^+ \ (i, t^0) \end{array}\right\} \quad \Longrightarrow \quad \left|\{i' \in I : \phi_{i'}^{DP}(\succ) = (b, t^+)\}\right| = q_b^f. \tag{18}$$

Relations (17) and (18) imply that mechanism ϕ^{MP} *respects the price responsiveness scheme*.

Uniqueness: We next show that mechanism ϕ^{MP} is the only mechanism that satisfies all five axioms.

Let the direct mechanism φ *respect the price responsiveness scheme* and satisfy *individual rationality, non-wastefulness, no priority reversal* and *BRADSO-IC*. We want to show that

$$\varphi(\succ) = \phi^{MP}(\succ).$$

If there are less than or equal to q cadets for whom the assignment (b, t^0) is acceptable under the preference profile $\succ$, all such cadets must receive an assignment of (b, t^0) by *individual rationality, non-wastefulness*, and *BRADSO-IC*. Since this is also the case under the allocation $\phi^{MP}(\succ)$, the result holds immediately for this case.

Therefore, w.l.o.g assume that there are strictly more than q cadets for whom the assignment (b, t^0) is acceptable under the preference profile $\succ$. Let I^0 be the set of q_b^0 highest π_b-priority cadets in I. By *non-wastefulness*, all positions are assigned under $\varphi(\succ)$. Since at most q_b^f positions can be awarded at the increased price t^h, at least q_b^0 positions has to be allocated at the base price t^0. Therefore,

$$\text{for any } i \in I^0, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{MP}(\succ) \tag{19}$$

by *no priority reversal*.

Let I^1 be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. Relabel the cadets in the set I^1 so that for any $\ell \in \{1, \ldots, q_b^f\}$, cadet i^ℓ is the ℓ^{th}-lowest π_b-priority cadet in I^1. Let

$$J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^h) \succ_j \emptyset\}.$$

By *individual rationality* and *no priority reversal*,

$$\text{for any } i \in I \setminus (I^0 \cup I^1 \cup J^0), \qquad \varphi_i(\succ) = \emptyset = \phi_i^{MP}(\succ). \tag{20}$$

By relations (19) and (20), the only set of cadets whose assignments are yet to be determined under

$\varphi(\succ)$ are cadets in $I^1 \cup J^0$. Moreover, by *no priority reversal*, cadets in J^0 can only receive a position at the increased price t^h. That is,

$$\text{for any } j \in J^0, \qquad \varphi_j(\succ) \neq (b, t^0). \tag{21}$$

For the next phase of our proof, we will rely on the sequence of individuals $i^1, \ldots, i^{q_b^f}$ and the sequence of sets $J^0, J^1, \ldots$ that are constructed for the Step 2 of the mechanism ϕ^{MP}. Here individual i^1 is the q^{th} highest π_b-priority cadet in set I, cadet i^2 is the $(q-1)^{\text{th}}$ highest π_b-priority cadet in set I, and so on. The starting element of the second sequence is $J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^h) \succ_j \emptyset\}$. Assuming Step 2.$n$ is the last sub-step of Step 2, the remaining elements of the latter sequence for $n \geq 1$ is given as follows: For any $\ell \in \{1, \ldots, n\}$,

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^h) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^h) \succ_{i^\ell} \emptyset \end{cases}$$

We have three cases to consider.

Case 1. $n = 0$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{MP}, we have

$$\left|\{j \in J^0 : (j, t^h)\ \omega_b\ (i^1, t^0)\}\right| = 0. \tag{22}$$

Therefore, by relations 20, 21, and condition (1) of the axiom *respect for the price responsiveness scheme*,

$$\text{for any } i \in I \setminus (I^0 \cup I^1), \qquad \varphi_i(\succ) = \emptyset = \phi_i^{MP}(\succ). \tag{23}$$

Hence by *non-wastefulness*,

$$\text{for any } i \in I^1, \qquad \varphi_i(\succ) \in \{(b, t^0), (b, t^h)\}. \tag{24}$$

But since φ satisfies *individual rationality*, relation (24) implies that $\varphi_i(\succ) = (b, t^0)$ for any $i \in I^1$ with $\emptyset \succ_i (b, t^h)$. Furthermore for any $i \in I^1$ with $(b, t^h) \succ_i \emptyset$, instead reporting the fake preference relation $\succ_i' \in \mathcal{Q}$ with $\emptyset \succ_i' (b, t^h)$ would guarantee cadet i an assignment of $\varphi_i(\succ_{-i}, \succ_i') = (b, t^0)$ due to the same arguments applied for the economy $(\succ_{-i}, \succ_i')$, and therefore by *BRADSO-IC* these cadets too must receive an assignment of (b, t^0) each. Hence

$$\text{for any } i \in I^1, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{MP}(\succ). \tag{25}$$

Relations (19), and (25) imply $\varphi(\succ) = \phi^{MP}(\succ)$, completing the proof for Case 1.■

Case 2. $n \in \{1, \ldots, q_b^f - 1\}$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{MP}, we have

$$\text{for any } \ell \in \{1, \ldots, n\}, \qquad \left|\{j \in J^{\ell-1} : (j, t^h)\ \omega_b\ (i^\ell, t^0)\}\right| \geq \ell, \tag{26}$$

and

$$\left|\{j \in J^n : (j, t^h)\ \omega_b\ (i^{n+1}, t^0)\}\right| = n. \tag{27}$$

Since mechanism φ satisfies condition (2) of the axiom *respect for the price responsiveness scheme*, *no priority reversal* and relation 26 imply

$$\text{for any } i \in \{i^1, \ldots, i^n\}, \qquad \varphi_i(\succ) \neq (b, t^0). \tag{28}$$

Therefore, by *non-wastefulness* and relations (19), (20), (21), and (28), at least n positions must be assigned at the increased price t^h.

Moreover, since mechanism φ satisfies *non-wastefulness, no priority reversal*, and condition (1) of the axiom *respect for the price responsiveness scheme*, relation (27) implies

$$\text{for any } i \in \{i^{n+1}, \ldots, i^{q_b^f}\}, \qquad \varphi_i(\succ) \in \{(b, t^0), (b, t^h)\}. \tag{29}$$

But since φ satisfies *individual rationality*, relation (29) implies that $\varphi_i(\succ) = (b, t^0)$ for any $i \in \{i^{n+1}, \ldots, i^{q_b^f}\}$ with $\emptyset \succ_i (b, t^h)$. Furthermore for any $i \in \{i^{n+1}, \ldots, i^{q_b^f}\}$ with $(b, t^h) \succ_i \emptyset$, instead reporting the fake preference relation $\succ'_i \in \mathcal{Q}$ with $\emptyset \succ'_i (b, t^h)$ would guarantee cadet i an assignment of $\varphi_i(\succ_{-i}, \succ'_i) = (b, t^0)$ due to the same arguments applied for the economy $(\succ_{-i}, \succ'_i)$, and therefore by *BRADSO-IC* these cadets must also receive an assignment of (b, t^0) each. Hence

$$\text{for any } i \in \{i^{n+1}, \ldots, i^{q_b^f}\}, \qquad \varphi_i(\succ) = (b, t^0) = \phi_i^{MP}(\succ). \tag{30}$$

Since we have already shown that at least n positions must be assigned at an increased price of t^h, relation (30) implies that exactly n positions must be assigned this cost, and therefore for any cadet $j \in J^n$ who is one of the n highest π_b-priority cadets in J^n,

$$\varphi_j(\succ) = (b, t^h) = \phi_i^{MP}(\succ) \tag{31}$$

by *no priority reversal*.

Relations (19), (30), and (31) imply $\varphi(\succ) = \phi^{MP}(\succ)$, completing the proof for Case 2. ■

Case 3. $n = q_b^f$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{MP}, we have

$$\text{for any } \ell \in \{1, \ldots, q_b^f\}, \qquad \left|\{j \in J^{\ell-1} : (j, t^h)\ \omega_b\ (i^\ell, t^0)\}\right| \geq \ell. \tag{32}$$

Since mechanism φ satisfies condition (2) of the axiom *respect for the price responsiveness scheme*, relation 32 implies

$$\text{for any } i \in \underbrace{\{i^1, \ldots, i^{q_b^f}\}}_{=I^1}, \qquad \varphi_i(\succ) \neq (b, t^0). \tag{33}$$

Therefore, by *non-wastefulness* and *no priority reversal*, exactly q_b^f positions must be assigned at the increased price t^h. Hence for any cadet $j \in J^{q_b^f}$ who is one of the q_b^f highest π_b-priority cadets in $J^{q_b^f}$,

$$\varphi_j(\succ) = (b, t^h) = \phi_i^{MP}(\succ) \tag{34}$$

by *no priority reversals*.

Relations (19) and (34) imply $\varphi(\succ) = \phi^{MP}(\succ)$, completing the proof for Case 3, thus finalizing the proof of the theorem. ■ □

Proof of Proposition 3: Suppose that there is only one branch $b \in B$. Fixing the profile of cadet preferences $\succ \in \mathcal{Q}$, the baseline priority order π_b, and the price responsiveness scheme ω_b, consider the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. When there is only one branch, the first part of the message space becomes redundant and the second part contains only the two elements b and $\emptyset$. Hence, for any cadet $i \in I$, the message space of cadet $i \in I$ under the USMA-2020 mechanism is $\mathcal{S}_i^{2020} = \{\emptyset, b\}$.

For a given strategy profile $s \in \mathcal{S}^{2020}$, construct the priority order $\pi_b^+(s)$ as follows: For any $i, j \in I$,

1. $s_i = s_j \quad \Longrightarrow \quad i \ \pi_b^+(s)\ j \iff i\ \pi_b\ j,$
2. $s_i = b$ and $s_j = \emptyset \quad \Longrightarrow \quad i\ \pi_b^+(s)\ j \iff (i, t^h)\ \omega_b\ (j, t^0).$

Let $I^+(s)$ be the set of q_b highest $\pi_b^+(s)$-priority cadets in I.

For any cadet $i \in I$, the outcome of the USMA-2020 mechanism is given as,

$$\varphi_i^{2020}(s) = \begin{cases} \emptyset & \text{if } \ i \notin I^+(s), \\ (b, t^0) & \text{if } \ i \in I^+(s) \text{ and } s_i = \emptyset, \\ (b, t^0) & \text{if } \ i \in I^+(s) \text{ and } s_i = b \ \text{ and } \ \left|\{j \in I^+(s) : s_j = b \text{ and } i\ \pi_b\ j\}\right| \geq q_b^f, \\ (b, t^h) & \text{if } \ i \in I^+(s) \text{ and } s_i = b \ \text{ and } \ \left|\{j \in I^+(s) : s_j = b \text{ and } i\ \pi_b\ j\}\right| < q_b^f. \end{cases}$$

We first prove a lemma on the structure of Nash equilibrium strategies of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$.

Lemma 3. *Let s^* be a Nash equilibrium of the strategic-form game induced by the mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. Then, for any $i, j \in I$,*

$$\varphi_j^{2020}(s^*) \succ_i \varphi_i^{2020}(s^*) \quad \Longrightarrow \quad j\ \pi_b\ i.$$

Proof of Lemma 3: Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-

2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. Contrary to the claim suppose that, there exists $i, j \in I$ such that

$$\varphi_j^{2020}(s^*) \succ_i \varphi_i^{2020}(s^*) \quad \text{and} \quad i \; \pi_b \; j.$$

There are three possible cases, where in each case we reach a contradiction by showing that cadet i has a profitable deviation by mimicking the strategy of cadet j:

***Case 1*:** $\varphi_j^{2020}(s^*) = (b, t^0)$ and $\varphi_i^{2020}(s^*) = (b, t^h)$.

Since by assumption $\varphi_i^{2020}(s^*) = (b, t^h)$,

$$s_i^* = b.$$

Moreover the assumptions $\varphi_j^{2020}(s^*) = (b, t^0),\; \varphi_i^{2020}(s^*) \neq (b, t^0),$ and $i \; \pi_b \; j$ imply

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = \emptyset. \tag{35}$$

But then, relation (35) and the assumption $i \; \pi_b \; j$ imply that, for the alternative strategy $\hat{s}_i = \emptyset$ for cadet i,

$$i \in I^+(s_{-i}^*, \hat{s}_i),$$

and thus

$$\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0) \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy. This completes the proof for Case 1. ■

***Case 2*:** $\varphi_j^{2020}(s^*) = (b, t^0)$ and $\varphi_i^{2020}(s^*) = \emptyset$.

Since by assumption $\varphi_j^{2020}(s^*) = (b, t^0),\; \varphi_i^{2020}(s^*) = \emptyset,$ and $i \; \pi_b \; j$, we must have

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = b \quad \text{and} \quad \left|\{k \in I^+(s^*) : s_k^* = b \text{ and } j \; \pi_b \; k\}\right| \geq q_b^f, \tag{36}$$

and

$$s_i^* = \emptyset.$$

But then, relation (36) and the assumption $i \; \pi_b \; j$ imply that, for the alternative strategy $\hat{s}_i = b$ for cadet i,

$$i \in I^+(s_{-i}^*, \hat{s}_i) \quad \text{and} \quad \hat{s}_i = b \quad \text{and} \quad \left|\{k \in I^+(s_{-i}^*, \hat{s}_i) : s_k^* = b \text{ and } i \; \pi_b \; k\}\right| \geq q_b^f,$$

and thus

$$\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0) \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy. This completes the proof for Case 2. ■

***Case 3*:** $\varphi_j^{2020}(s^*) = (b, t^h)$ and $\varphi_i^{2020}(s^*) = \emptyset$.

Since by assumption $\varphi_j^{2020}(s^*) = (b, t^h)$,

$$j \in I^+(s^*) \quad \text{and} \quad s_j^* = b. \tag{37}$$

Moreover, since $\varphi_i^{2020}(s^*) = \emptyset$ by assumption,

$$i \notin I^+(s^*).$$

Therefore, since $i \; \pi_b \; j$ by assumption,

$$j \in I^+(s^*) \text{ and } i \notin I^+(s^*) \quad \Longrightarrow \quad s_i^* = \emptyset.$$

But then, again thanks to assumption $i \; \pi_b \; j$, the relation (37) implies that, for the alternative strategy $\hat{s}_i = b$ for cadet i,

$$i \in I^+(s^*_{-i}, \hat{s}_i),$$

and thus

$$\underbrace{\varphi_i^{2020}(s^*_{-i}, \hat{s}_i)}_{\in\{(b,t^0),(b,t^h)\}} \succ_i \varphi_i^{2020}(s^*),$$

contradicting s^* is a Nash equilibrium strategy,[50] completing the proof for Case 3, and concluding the proof of Lemma 3. ■ ◊

For the next phase of our proof, we rely on the construction in the Step 2 of the mechanism ϕ^{MP}: Let I^0 be the set of q_b^0 highest π_b-priority cadets in I, and I^1 be the set of q_b^f highest π_b-priority cadets in $I \setminus I^0$. Relabel the set of cadets in I^1, so that i^1 is the lowest π_b-priority cadet in I^1, i^2 is the second lowest π_b-priority cadet in I^1,..., and $i^{q_b^f}$ is the highest π_b-priority cadet in I^1. Note that, cadet i^1 is the q^{th} highest π_b-priority cadet in set I, cadet i^2 is the $(q-1)^{\text{th}}$ highest π_b-priority cadet in set I, and so on. Let $J^0 = \{j \in I \setminus (I^0 \cup I^1) : (b, t^h) \succ_j \emptyset\}$. Assuming Step 2.$n$ is the last sub-step of Step 2 of the mechanism ϕ^{MP}, for any $\ell \in \{1, \ldots, n\}$, let

$$J^\ell = \begin{cases} J^{\ell-1} & \text{if } \emptyset \succ_{i^\ell} (b, t^h) \\ J^{\ell-1} \cup \{i^\ell\} & \text{if } (b, t^h) \succ_{i^\ell} \emptyset \end{cases}$$

Recall that, under the mechanism ϕ^{MP}, exactly n cadets receive an assignment of (b, t^h). We will show that, the same is also the case under the Nash equilibria of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$.

Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. We have three cases to consider:

Case 1: $n = 0$

[50] Unlike the first two cases, in this case cadet i may even get a better assignment than cadet j (i.e. cadet i may receive an assignment of (b, t^0)) by mimicking cadet j's strategy.

Since by assumption $n = 0$ in this case,

$$\left\{j \in J^0 : (j, t^h)\ \omega_b\ (i^1, t^0)\right\} = \emptyset. \tag{38}$$

Towards a contradiction, suppose there exists a cadet $i \in I \setminus (I^0 \cup I^1)$ such that $i \in I^+(s^*)$. Since cadet i^1 is the q^{th} highest π_b-priority cadet in I, the assumption $i \in I^+(s^*)$ and relation (38) imply

$$i \notin J^0 \implies \emptyset \succ_i (b, t^h). \tag{39}$$

Moreover, since cadet i is not one of the q highest π_b-priority cadets in I,

$$i \in I^+(s^*) \implies s_i^* = b. \tag{40}$$

But this means cadet i can instead submit an alternative strategy $\hat{s}_i = \emptyset$, assuring that she remains unmatched, contradicting s^* is a Nash equilibrium. Therefore,

$$\text{for any } i \in I \setminus (I^0 \cup I^1), \qquad (i, t^h)\ \omega_b\ (i^1, t^0) \implies s_i^* = \emptyset, \tag{41}$$

which in turn implies

$$I^+(s^*) = I^0 \cup I^1. \tag{42}$$

Hence all cadets in $I^0 \cup I^1$ receive a position under $\varphi^{2020}(s^*)$. Next consider the lowest π_b-priority cadet $i \in I^0 \cup I^1$ such that $\varphi_i^{2020}(s^*) = (b, t^h)$. This can only happen if $s_i^* = b$. But this means cadet i can instead submit an alternative strategy $\hat{s}_i = \emptyset$, assuring that $\varphi_i^{2020}(s_{-i}^*, \hat{s}_i) = (b, t^0)$ by relation (41), contradicting s^* is a Nash equilibrium. Hence

$$\text{for any } i \in I^0 \cup I^1, \qquad \varphi_i^{2020}(s^*) = (b, t^0) = \phi_i^{MP}(\succ), \tag{43}$$

and therefore $\varphi^{2020}(s^*) = \phi^{MP}(\succ)$.

Finally observe that the strategy profile s' where $s_i' = \emptyset$ for any cadet $i \in I$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{MP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 1. ■

For any $\ell \in \{1, \ldots, n\}$, let $\overline{J^\ell}$ be the set of ℓ highest π_b-priority cadets in the set J^ℓ:

$$\overline{J^\ell} = \left\{j \in J^\ell :\ \left|\{i \in J^\ell :\ i\ \pi_b\ j\}\right| < \ell\right\}$$

Before proceeding with the next two cases, we prove the following lemma that will be helpful for both cases.

Lemma 4. *Suppose there are $n > 0$ positions allocated at the increased price t^h under the allocation $\phi^{MP}(\succ)$. Then, for any Nash equilibrium s^* of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$ and $\ell \in \{1, \ldots, n\}$,*

1. $\varphi^{2020}_{i^\ell}(s^*) = (b, t^h) \iff (b, t^h) \succ_{i^\ell} \emptyset$, *and*

2. $\varphi^{2020}_{i}(s^*) = (b, t^h)$ *for any* $i \in \overline{J^\ell}$.

Proof of Lemma 4: Let s^* be a Nash equilibrium of the strategic-form game induced by the USMA-2020 mechanism $(\mathcal{S}^{2020}, \varphi^{2020})$. First recall that,

$$\text{for any } j \in I \setminus (I^0 \cup I^1), \qquad \varphi^{2020}_j(s^*) \in \{(b, t^h), \emptyset\},$$

and therefore, since any cadet $j \in I \setminus (I^0 \cup I^1 \cup J^0)$ prefers remaining unmatched to receiving a position at the increased price t^h and she can assure remaining unmatched by submitting the strategy $s_j = \emptyset$,

$$\text{for any } j \in I \setminus (I^0 \cup I^1 \cup J^0), \qquad \varphi^{2020}_j(s^*) = \emptyset. \tag{44}$$

Also, by the mechanics of the Step 2 of the mechanism ϕ^{MP},

$$\text{for any } \ell \in \{1, \ldots, n\}, \qquad \left|\left\{j \in J^{\ell-1} : (j, t^h)\ \omega_b\ (i^\ell, t^0)\right\}\right| \geq \ell. \tag{45}$$

The proof of the lemma is by induction on ℓ. We first prove the result for $\ell = 1$.

Consider the highest π_b-priority cadet j in the set $\left\{j \in J^0 : (j, t^h)\ \omega_b\ (i^1, t^0)\right\}$. By relation 45, such a cadet exists.

First assume that $(b, t^h) \succ_{i^1} \emptyset$. In this case, $J^1 = J^0 \cup \{i^1\}$ and cadet i^1 is the highest π_b-priority cadet in J^1. Hence $\overline{J^1} = \{i^1\}$ in this case. Consider the Nash equilibrium strategies of cadet i^1 and cadet j. If $s^*_{i^1} = \emptyset$, then by relation (44) her competitor cadet j can secure himself an assignment of (b, t^h) by reporting a strategy of $s_j = b$, which would mean cadet i^1 has to remain unassigned, since by Lemma 3 no cadet in $I^0 \cup I^1$ can envy the assignment of cadet i^1 at Nash equilibria. In contrast, reporting a strategy of $s_{i^1} = b$ assures that cadet i^1 receives a position, which is preferred at any price to remaining unmatched by assumption $(b, t^h) \succ_{i^1} \emptyset$. Therefore, $s^*_{i^1} = b$, and hence

$$(b, t^h) \succ_{i^1} \emptyset \quad \Longrightarrow \quad \left\{ \begin{array}{l} \varphi^{2020}_{i^1}(s^*) = (b, t^h), \text{ and} \\ \varphi^{2020}_{i}(s^*) = (b, t^h) \quad \text{for any } i \in \overline{J^1} = \{i^1\}. \end{array} \right. \tag{46}$$

Next assume that $\emptyset \succ_{i^1} (b, t^h)$. In this case $J^1 = J^0$ and cadet j is the highest π_b-priority cadet in J^1. Hence $\overline{J^1} = \{j\}$ in this case. By Lemma 3, no cadet in $(I^0 \cup I^1) \setminus \{i^1\}$ can envy the assignment of cadet i^1 at Nash equilibria. Therefore, a strategy of $s_{i^1} = b$ means that cadet i receives an assignment of (b, t^h), which is inferior to remaining unmatched by assumption. Therefore $s^*_{i^1} = \emptyset$. Moreover reporting a strategy of $s_j = \emptyset$ means that cadet j remains unmatched, whereas reporting a strategy of $s_j = b$ assures that she receives an assignment of (b, t^h), which is preferred to remaining unmatched since $j \in J^0$. Therefore, $s^*_{i^1} = \emptyset$, and hence

$$\emptyset \succ_{i^1} (b, t^h) \quad \Longrightarrow \quad \left\{ \begin{array}{l} \varphi^{2020}_{i^1}(s^*) = \emptyset, \text{ and} \\ \varphi^{2020}_{i}(s^*) = (b, t^h) \quad \text{for any } i \in \overline{J^1} = \{j\}. \end{array} \right. \tag{47}$$

Relations (46) and (47) complete the proof for $\ell = 1$.

Next assume that the inductive hypothesis holds for $\ell = k < n$. We want to show that the result holds for $\ell = (k+1)$ as well.

By the inductive hypothesis,

$$\text{for any } i \in \overline{J^k}, \quad \varphi_i^{2020}(s^*) = (b, t^h). \tag{48}$$

By relation 45, there are at least $k+1$ cadets in the set $\{j \in J^k : (j, t^h)\ \omega_b\ (i^{k+1}, t^0)\}$. Therefore, since there are k cadets in the set $\overline{J^k}$, there is at least one cadet in the set

$$\{j \in J^k : (j, t^h)\ \omega_b\ (i^{k+1}, t^0)\} \setminus \overline{J^k}.$$

Let j be the highest π_b-priority cadet in this set.

First assume that $(b, t^h) \succ_{i^{k+1}} \emptyset$. In this case $J^{k+1} = J^k \cup \{i^{k+1}\}$ and cadet i^{k+1} is the highest π_b-priority cadet in J^{k+1}. Hence $\overline{J^{k+1}} = \overline{J^k} \cup \{i^{k+1}\}$ in this case. Consider the Nash equilibrium strategies of cadet i^{k+1} and cadet j. If $s^*_{i^{k+1}} = \emptyset$, then by relation (44) cadet j can secure herself an assignment of (b, t^h) by reporting a strategy of $s_j = b$, which would mean cadet i^{k+1} has to remain unassigned, since by Lemma 3 no cadet in $(I^0 \cup I^1) \setminus \{i^1, \ldots, i^k\}$ can envy the assignment of cadet i^{k+1} at Nash equilibria and by relation (48) all cadets in $\overline{J^k}$ receive an assignment of (b, t^h).[51] In contrast, reporting a strategy of $s_{i^{k+1}} = b$ assures that cadet i^{k+1} receives a position, which is preferred at any price to remaining unmatched by assumption $(b, t^h) \succ_{i^{k+1}} \emptyset$. Therefore, $s^*_{i^{k+1}} = b$, and hence

$$(b, t^h) \succ_{i^{k+1}} \emptyset \implies \begin{cases} \varphi_{i^{k+1}}^{2020}(s^*) = (b, t^h), \text{ and} \\ \varphi_i^{2020}(s^*) = (b, t^h) \text{ for any } i \in \overline{J^{k+1}} = \overline{J^k} \cup \{i^{k+1}\}. \end{cases} \tag{49}$$

Next assume that $\emptyset \succ_{i^{k+1}} (b, t^h)$. In this case $J^{k+1} = J^k$ and $\overline{J^{k+1}} = \overline{J^k} \cup \{j\}$. By Lemma 3, no cadet in $I^0 \cup I^1 \setminus \{i^1, \ldots, i^k\}$ can envy the assignment of cadet i^{k+1} at Nash equilibria. Therefore, since all cadets in $\overline{J^k}$ receive an assignment of (b, t^h) by relation (48), a strategy of $s_{i^{k+1}} = b$ means that cadet i^{k+1} receives an assignment of (b, t^h), which is inferior to remaining unmatched by assumption. Therefore $s^*_{i^{k+1}} = \emptyset$. Moreover reporting a strategy of $s_j = \emptyset$ means that cadet j remains unmatched, whereas reporting a strategy of $s_j = b$ assures that she receives an assignment of (b, t^h), which is preferred to remaining unmatched since $j \in J^k$. Therefore, $s^*_{i^{k+1}} = \emptyset$, and hence

$$\emptyset \succ_{i^{k+1}} (b, t^h) \implies \begin{cases} \varphi_{i^{k+1}}^{2020}(s^*) = \emptyset, \text{ and} \\ \varphi_i^{2020}(s^*) = (b, t^h) \quad \text{for any } i \in \overline{J^{k+1}} = \overline{J^k} \cup \{j\}. \end{cases} \tag{50}$$

Relations (49) and (50) complete the proof for $\ell = k+1$, and conclude the proof of Lemma 4. $\diamondsuit$

We are ready to complete prove the theorem for our last two cases:

[51] Since $\left|(I^0 \cup I^1) \setminus \{i^1, \ldots, i^k\}\right| = (q-k)$ and $\left|\overline{J^k}\right| = k$, this basically means cadets i^{k+1} and j are competing for a single position.

Case 2. $n \in \{1, \ldots, q_b^f - 1\}$

For this case, by the mechanics of the Step 2 of the mechanism ϕ^{MP},

$$\left|\{j \in J^n : (j, t^h)\ \omega_b\ (i^{n+1}, t^0)\}\right| = n. \tag{51}$$

Consider cadet i^{n+1}. There are $q - (n+1)$ cadets with higher π_b-priority, and by relation (51) there are n cadets in J^n whose increased price assignments have higher ω_b priority under the price responsiveness scheme than the base-price assignment for cadet i^{n+1}. For any other cadet $i \in I \setminus \Big(J^n \cup I^0 \cup \big(I^1 \setminus \{i^1, \ldots, i^{n+1}\}\big)\Big)$ with $(i, t^h)\ \omega_b\ (i^{n+1}, t^0)$, we must have $\emptyset \succ_i (b, t^h)$ since $J^n \supseteq J^0$. Therefore none of these individuals can receive an assignment of (b, t^h) under a Nash equilibrium strategy, and hence the number of cadets who can have higher $\pi_b^+(s^*)$-priority than cadet is i^{n+1} is at most $q - (n+1) + n = q - 1$ under any Nash equilibrium strategy. That is, cadet $i^{n+1} \in I^+(s^*)$ regardless of her submitted strategy, and therefore,

$$\varphi^{2020}_{i^{n+1}}(s^*) = (b, t^0), \tag{52}$$

since her best response $s^*_{i^{n+1}}$ to $s^*_{-i^{n+1}}$ results in an assignment of (b, t^0). Moreover, Lemma 3 and relation (52) imply that, for any cadet $i \in I^0 \cup \big(I^1 \setminus \{i^1, \ldots, i^{n+1}\}\big)$,

$$\varphi^{2020}_{i}(s^*) = (b, t^0). \tag{53}$$

Hence Lemma 4 and relations (52), (53) imply $\varphi^{2020}(s^*) = \phi^{MP}(\succ)$.

Finally, the strategy profile s' where $s'_i = b$ for any cadet $i \in J^n$ and $s'_j = \emptyset$ for any cadet $j \in I \setminus J^n$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{MP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 2. ■

Case 3. $n = q_b^f$

Since at most q_b^f positions can be assigned at the increased price t^h, Lemma 3 and Lemma 4 immediately imply $\varphi^{2020}(s^*) = \phi^{MP}(\succ)$.

Finally the strategy profile s' where $s'_i = b$ for any cadet $i \in J^{q_b^f} \cup I^0$ and $s'_j = \emptyset$ for any cadet $j \in I \setminus \big(J^n \cup I^0\big)$ is a Nash equilibrium, with an outcome $\varphi^{2020}(s') = \phi^{MP}(\succ)$, showing that there exists a Nash equilibrium completing the proof for Case 3, and the proof of the proposition. ■ □

Proof of Corollary 2: Since *BRADSO-IC* is implied by *strategy-proofness*, Corollary 2 is a direct implication of Theorem 1 and Proposition 2. □

C Formal Description of USMA-2006 Mechanism and Individual-Proposing Deferred Acceptance Algorithm

C.1 USMA-2006 Mechanism

The USMA-2006 mechanism is a quasi-direct mechanism with the following message space:

$$\mathcal{S}^{2006} = \left(\mathcal{P} \times 2^B\right)^{|I|}.$$

The following construction is useful to formulate the outcome function for the USMA-2006 mechanism:

Given an OML π and a strategy profile $s = (P_i, B_i)_{i \in I} \in \mathcal{S}^{2006}$, for any branch $b \in B$ construct the following adjusted priority order $\pi_b^+ \in \Pi$ on the set of cadets I. For any pair of cadets $i, j \in I$,

1. $b \in B_i$ and $b \in B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j \iff i \; \pi \; j$,
2. $b \notin B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j \iff i \; \pi \; j$, and
3. $b \in B_i$ and $b \notin B_j \quad \Longrightarrow \quad i \; \pi_b^+ \; j$.

Under the adjusted priority order π_b^+, any pair of cadets are rank ordered through the OML π if they have indicated the same willingness to pay the increased price for branch b, and through the ultimate price responsiveness scheme $\overline{\omega}_b$ (which gives higher priority to the cadet who has indicated to pay the increases price) otherwise.

Given an OML $\pi \in \Pi$ and a strategy profile $s = (P_i, B_i)_{i \in I} \in \mathcal{S}^{2006}$, the outcome $\varphi^{2006}(s)$ of the **USMA-2006 mechanism** is obtained with the following sequential procedure:

> ***Branch assignment***: At any step $\ell \geq 1$ of the procedure, the highest π-priority cadet i who is not tentatively on hold for a position at any branch applies to her highest-ranked acceptable branch b under her submitted branch preferences P_i that has not rejected her from earlier steps.[52]
>
> Branch b considers cadet i together with all cadets it has been tentatively holding both for its q_b^0 base-price positions and also for its q_b^f flexible-price positions, and
>
> 1. it tentatively holds (up to) q_b^0 highest π-priority applicants for one of its q_b^0 base-price positions,
> 2. among the remaining applicants it tentatively holds (up to) q_b^f highest π_b^+-priority applicants for one of its q_b^f flexible-price positions, and
> 3. it rejects any remaining applicant.

[52]The USMA-2006 mechanism can also be implemented with a variant of the algorithm where each cadet who is not tentatively holding a position simultaneously apply to her next choice branch among branches that has not rejected her application.

The procedure terminates when no applicant is rejected. Any cadet who is not tentatively on hold at any brach remains unmatched, and all tentative branch assignments are finalized.

Price assignment: For any branch $b \in B$,

1. any cadet $i \in I$ who is assigned one of the q_b^0 base-price positions at branch b is charged the base price t^0, and
2. any cadet $i \in I$ who is assigned one of the q_b^f flexible-price positions is charged
   (a) the increased price t^h if $b \in B_i$, and
   (b) the base price t^0 if $b \notin B_i$.

C.2 Individual-Proposing Deferred Acceptance Algorithm

The USMA-2020 mechanism was based on the individual-proposing deferred acceptance algorithm (Gale and Shapley, 1962). Given a ranking over branches, the individual-proposing deferred acceptance algorithm produces a matching as follows.

Individual-Proposing Deferred Acceptance Algorithm

Step 1: Each cadet applies to her most preferred branch. Each branch b tentatively assigns applicants with the highest priority until all cadets are chosen or all q_b slots as assigned and permanently rejects the rest. If there are no rejections, then stop.

Step k: Each cadet who was rejected in Step k-1 applies to her next preferred branch, if such a branch exists. Branch b tentatively assigns cadets with the highest priority until all all cadets are chosen or all q_b slots are assigned and permanently rejects the rest. If there are no rejections, then stop.

The algorithm terminates when there are no rejections, at which point all tentative assignments are finalized.

D Additional Results

D.1 Empirical Evidence on the Failure of Desiderata under the USMA-2006 and USMA-2020 Mechanisms

In this section, we report on the failure of BRADSO-IC, presence of strategic BRADSO, and presence of detectable priority reversals under the USMA-2006 and USMA-2020 mechanisms. We show that the USMA-2020 mechanism exacerbated challenges compared to the USMA-2006 mechanism. We use actual data submitted under these mechanisms and also simulated data generated from the MPCO mechanism for the USMA Class of 2021.

D.1.1 USMA-2006 and USMA-2020 Mechanisms in the Field

BRADSO-IC failures were much more common under USMA-2020 than under USMA-2006. Figure D1 shows that nearly four times (85 versus 22) as many cadets from the Class of 2020 (which used the USMA-2020 mechanism) were part of BRADSO-ICs than were cadets from the Classes of 2014 to 2019 (which used the USMA-2006 mechanism). Strategic BRADSOs must be more common under USMA-2020 because they are not possible under USMA-2006. For the Class of 2020, 18 cadets were part of strategic BRADSOs under the USMA-2020 mechanism. Importantly, fixing these instances ex-post would have required a change in branch assignments (rather than merely foregoing a BRADSO charge). Finally, nearly four times as many cadets were part of detectable priority reversals under the USMA-2020 mechanism than under the USMA-2006 mechanism (75 versus 20).

D.1.2 USMA-2006 and USMA-2020 Mechanisms with Simulated Data

Our comparison of prior mechanisms has so far been based on preferences submitted under those mechanisms. We can also use cadet preference data on branch-price pairs generated by the strategy-proof MPCO mechanism to simulate the outcome of USMA-2006 and USMA-2020 mechanisms under truthful strategies. This is valuable because for cadet preferences submitted under the USMA-2006 and USMA-2020 mechanisms, we could only measure detectable priority reversals (reported in Figure D1) and not all priority reversals.

To measure all priority reversals, we use preferences over branch-price pairs under the MPCO mechanism to construct a truthful strategy, denoted $s_i = (P_i, B_i)$, under a quasi-direct mechanism by using the branch rank ordering for P_i and assuming that if a cadet ever expresses a willingness to pay the increased price at a branch, then the cadet is willing to pay the increased price under B_i. Taking this constructed strategy as input, we then simulate the USMA-2006 and USMA-2020 mechanism using the branch capacities and priorities from the Class of 2021. Under the USMA-2006 mechanism simulation, there are 27 priority reversals and 20 are detectable priority reversals. Under the USMA-2020 mechanism simulation, there are 204 priority reversals and 197 are detectable priority reversals. This suggests that, in practice, detectable priority reversals likely constitute the majority of priority reversals among the Classes of 2014-2019, which used the USMA-2006 mechanism, and the Class of 2020, which used the USMA-2020 mechanism.

Using truthful strategies to evaluate the USMA-2006 and USMA-2020 mechanism, Figure D2 shows that there are nearly seven times as many BRADSO-IC failures under the USMA-2020 mechanism compared to the USMA-2006 mechanism (146 vs. 21) and seven times as many priority reversals under the USMA-2020 mechanism compared to the USMA-2006 mechanism (204 vs. 27). This pattern of behavior suggests that the comparison reported in Figure D1 potentially understates the dramatic increase in BRADSO-IC failures and priority reversals stemming from the adoption of the USMA-2020 mechanism because the Figure D1 comparison is based on strategies submitted under the message space of quasi-direct mechanisms and not underlying cadet preferences.

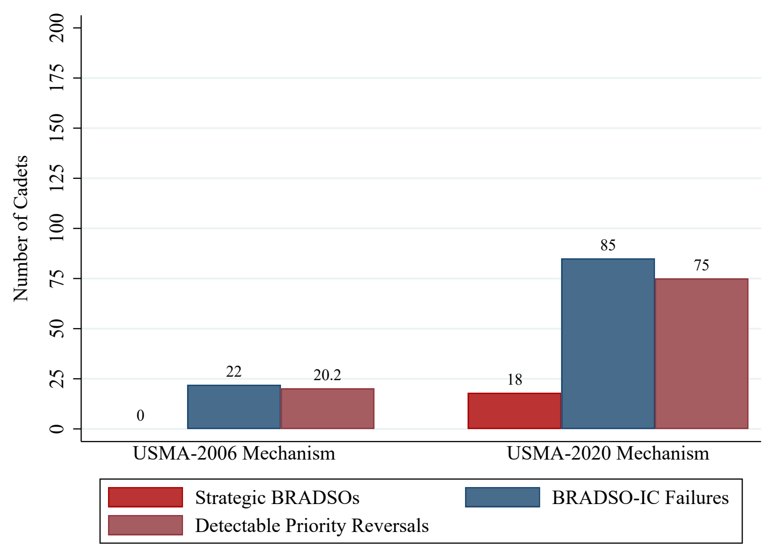
One reason the comparison between USMA-2006 and USMA-2020 in Figure D1 is not as striking as the comparison in Figure D2 is that, as we have presented in Section 4.3, many cadets were well-aware of the necessity to strategically make their increased price willingness choices under the USMA-2020 mechanism. Our analysis in Appendix B illustrates the perverse incentives in the USMA-2020 mechanism. For the Class of 2020, a dry-run of the mechanism where cadets submitted indicative rankings of branches and learned about their assignment took place. After observing their dry-run assignment, cadets were allowed to submit a final set of rankings under USMA-2020, and therefore had the opportunity to revise their strategies in response to this feedback. Figure D3 tabulates strategic BRADSOs, BRADSO-IC failures, and detectable priority reversals under indicative and final preferences. Final preferences result in fewer strategic BRADSOs, BRADSO-IC failures, and detectable priority reversals. This pattern is consistent with some cadets responding to the dry-run by ranking branch choices in response to these issues.

In general, cadets form their preferences over branches over time as they acquire more information about branches and their own tastes. Therefore, the change documented in Figure D3 may simply reflect general preference formation from acquiring information about branches, and not revisions to preferences in response to the specific mechanism. We briefly investigate this possibility by looking at the presence of strategic BRADSOs, BRADSO-IC failures, and priority reversals using data on the indicative and final preferences from the Class of 2021. This class participated in the strategy-proof MPCO mechanism. We take indicative and final cadet preferences under MPCO mechanism and construct truthful strategies, following the approach described above, for the USMA-2020 mechanism. Figure D4 shows that with preferences constructed from a strategy-proof mechanism, there are only modest differences in strategic BRADSOs, BRADSO-IC failures, and priority reversals between the indicative and final rounds. This comparison supports our claim that revisions of rank order lists in response to a dry-run of the USMA-2020 mechanism might understate the issues this mechanism created, and why these issues became so pronounced with the USMA-2020 mechanism relative to the USMA-2006 mechanism.

D.2 Cadet Utilization of the Richer message space of the MPCO Mechanism

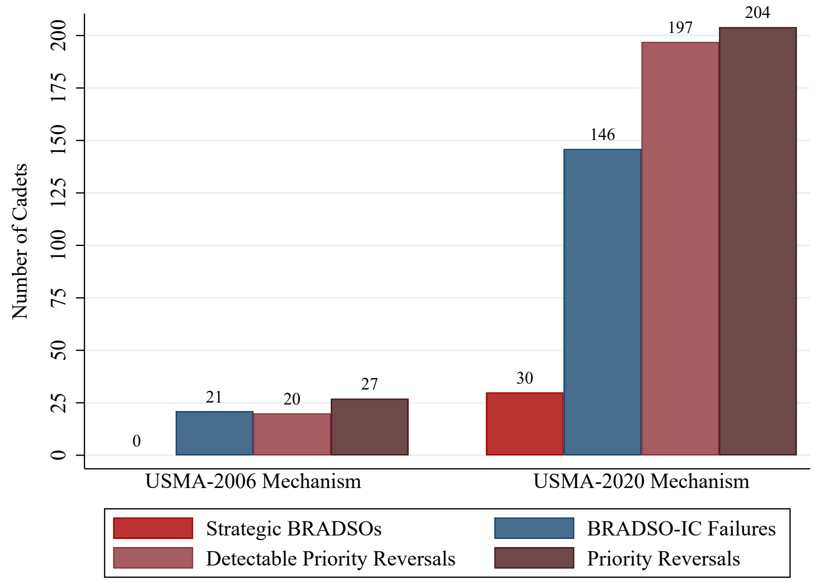
Preference data from the Class of 2021 confirm that cadets used the flexibility to express preferences over branch-price pairs. Figure D5 provides details on the extent to which cadets did not rank a branch with increased price immediately after the branch at base price. For each of 994 cadet first branch choices, 272 cadets rank that branch with increased price as their second choice and 36 cadets rank that branch with increased price as their third choice or lower. These 36 cadets would not have been able to express this preference under the message space of a quasi-direct mechanism like the USMA-2006 mechanism or the USMA-2020 mechanism. When we consider the next branch on a cadet's rank order list, cadets also value the flexibility of the new mechanism. For the branch that appears next on the rank order list, 78 cadets rank that branch with increased price as their immediate next highest choice and 24 cadets rank that branch with increased price two or more places below on their rank order list. These 24 cadets also would not have been able to express this preference under a quasi-direct mechanism.

Figure D1: Comparison of Outcomes of the USMA-2006 and USMA-2020 Mechanisms



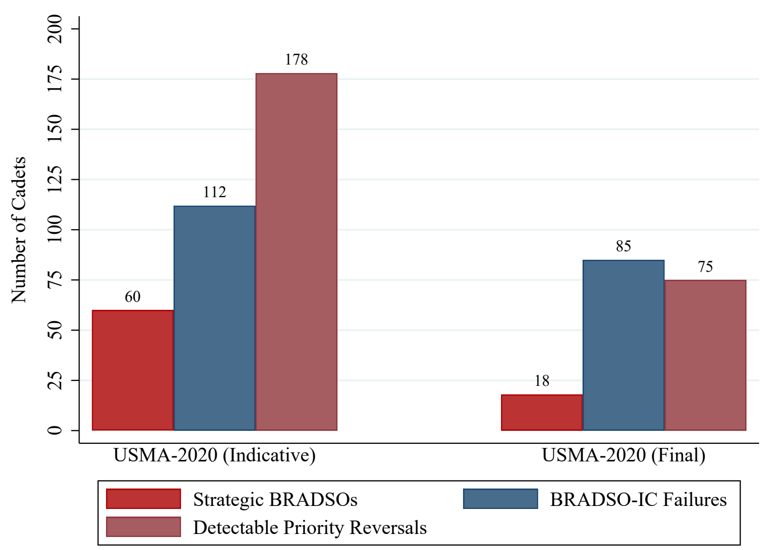
Notes. This figure reports Strategic BRADSOs, BRADSO-IC Failures, and Detectable Priority Reversals under the USMA-2006 and USMA-2020 Mechanisms. The first three columns correspond to outcomes under USMA-2006 Mechanism averaged over classes from 2014-2019. The last three columns correspond to outcomes under USMA-2020 Mechanism for the Class of 2020.

Figure D2: USMA-2006 and USMA-2020 Mechanism Performance under Truthful Strategies Simulated from Preference Data from Class of 2021



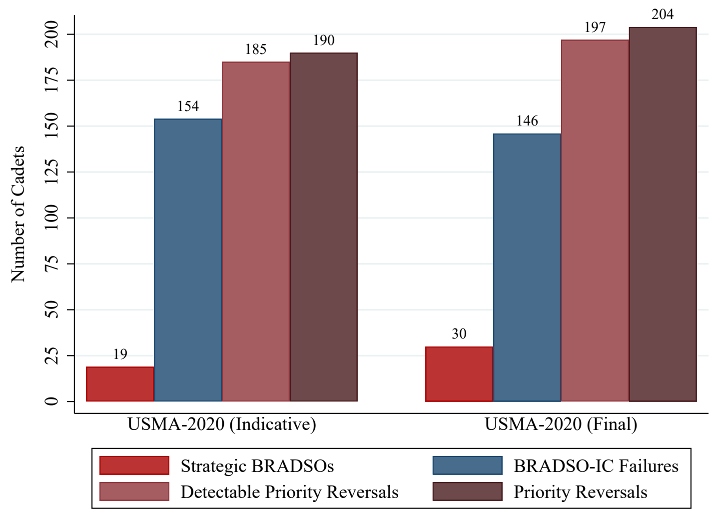
Notes. This figure uses data from the Class of 2021 to simulate the outcomes of the mechanisms USMA-2006 and USMA-2020. We use preferences over branch-price pairs under the MPCO mechanism to construct truthful strategies for USMA-2006 and USMA-2020 by assuming that willingness to BRADSO at a branch means the cadet's strategy under the USMA-2006 and USMA-2020 mechanisms has her willing to BRADSO. To compute Priority Reversals, we compare a cadet's outcome in either the USMA-2006 or USMA-2020 mechanism to a cadet's preference submitted under the MPCO mechanism. If a cadet prefers a higher ranked choice and has higher priority over a cadet who is assigned that choice, then the cadet is part of a Priority Reversal.

Figure D3: USMA-2020 Mechanism Performance Under Indicative and Final Strategies



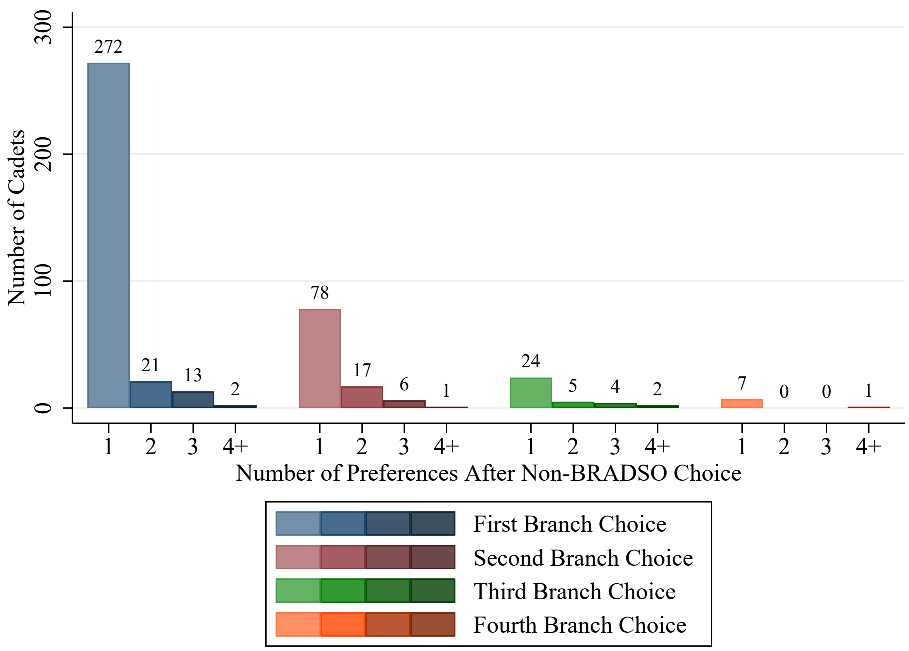
Notes. This figure reports on the number of Strategic BRADSOs, BRADSO-IC failures, and Detectable Priority Reversals under indicative strategies submitted in a dry-run of the USMA-2020 mechanism and final strategies of the USMA-2020 mechanism for the Class of 2020.

Figure D4: USMA-2020 Mechanism Performance under Truthful Strategies Simulated from Indicative and Final Preference Data from Class of 2021



Notes. USMA used the strategy-proof MPCO mechanism for the Class of 2021. This figure uses data from the indicative and final rounds from the Class of 2021 on cadet preferences, branch priorities, and branch capacities to simulate the outcome of the USMA-2020 mechanism. Since the message space of the mechanism USMA-2020 differs from that of the mechanism MPCO, cadet strategies that correspond to truthful branch-preferences and BRADSO willingness are are simulated from cadet preferences over branch-price pairs under the MPCO mechanism. Truthful strategies are constructed from Class of 2021 preferences by assuming that a preference indicating willingness to BRADSO at a branch means the cadet's strategy under the USMA-2006 and USMA-2020 mechanisms has her willing to BRADSO. USMA-2020 (Indicative) reports outcomes using strategies constructed from preferences submitted in the dry-run of the MPCO mechanism. USMA-2020 (Final) reports outcomes using strategies constructed from preferences submitted in the final run of the MPCO mechanism.

Figure D5: BRADSO Ranking Relative to Non-BRADSO Ranking by Class of 2021



Notes. This figure reports where in the preference list a branch is ranked with BRADSO relative to where it is ranked without BRADSO. A value of 1 (2 or 3) indicates that the branch is ranked with BRADSO immediately after (two places or three places after, respectively) the branch is ranked at base price. 4+ means that the a branch is ranked with BRADSO four or more choices after the branch is ranked at base price.

E Cadet Data and Survey Appendix

E.1 Data Appendix

Our data cover the West Point Classes of 2014 through 2021. We present two tables about data processing. The first table reports summary statistics on branches for the Class of 2020 and Class of 2021. The second table presents summary information about mechanism replication for the Classes of 2014-2021.

Table E1: Branches and Applications for Classes of 2020 and 2021

			Percent Correct	
	Total Applicants	Number Incorrect	Branch	BRADSDO
Applicant Class	(1)	(2)	(3)	(4)
2014	1006	29	97.1%	98.1%
2015	976	7	99.3%	99.8%
2016	951	11	98.8%	99.6%
2017	942	2	99.8%	100.0%
2018	962	9	99.1%	99.6%
2019	931	4	99.6%	100.0%
2020	1089	0	100.0%	100.0%
2021	994	0	100.0%	100.0%
All	7851	62	99.2%	99.6%

Notes. This table reports information on branches for the Class of 2020 and 2021. Number Assigned equals the capacity of the branch. Ranked First is the number of cadets ranking the branch as their highest rank choice. BRADSO Willing is the number of cadets who rank a BRADSO contract at the branch anywhere on their rank order list. Explosive Ordnance Disposal was not a branch option for the Class of 2020.

E.2 Cadet Survey Questions and Answers

In September 2019, the Army administered a survey to West Point cadets in the Class of 2020. This survey asked two questions related to assignment mechanisms, one on cadet understanding of USMA-2020 and the other on cadet preferences over assignment mechanisms. This section reports the questions and the distribution of survey responses.

Question 1. *What response below best describes your understanding of the impact of volunteering to BRADSO for a branch in this year's branching process?*

A. I am more likely to receive the branch, but I am only charged a BRADSO if I would have failed to receive the branch had I not volunteered to BRADSO. (43.3% of respondents)

B. I am charged a BRADSO if I receive the branch, regardless of whether volunteering to BRADSO helped me receive the branch or not. (9.5% of respondents)

Table E2: Mechanism Replication Rate

			Percent Correct	
	Total Applicants	Number Incorrect	Branch	BRADSDO
Applicant Class	(1)	(2)	(3)	(4)
2014	1006	29	97.1%	98.1%
2015	976	7	99.3%	99.8%
2016	951	11	98.8%	99.6%
2017	942	2	99.8%	100.0%
2018	962	9	99.1%	99.6%
2019	931	4	99.6%	100.0%
2020	1089	0	100.0%	100.0%
2021	994	0	100.0%	100.0%
All	7851	62	99.2%	99.6%

Notes. This table reports the replication rate of the USMA assignment mechanism across years. The USMA-2006 mechanism is used for the Classes of 2014-2019, USMA-2020 mechanism is used for the Class of 2020, and the multi-price Cumulative Offer mechanism is used for the Class of of 2021. Number incorrect are the number of cadets who obtain a different assignment under our replication. Branch percent correct is the number of branch assignments that we replicate. BRADSO percent correct is the number of BRADSO assignments we replicate.

C. I am more likely to receive the branch, but I may not be charged a BRADSO if many cadets who receive the same branch not only rank below me but also volunteer to BRADSO. (38.8% of respondents)

D. I am more likely to receive the branch, but I do not know how the Army determines who is charged a BRADSO. (6.7% of respondents)

E. I am NOT more likely to receive the branch even though I volunteered to BRADSO. (1.8 percent of respondents)

38.8% of cadets selected the correct answer (answer C). 43.3% of cadets believed that the 2020 mechanism would only charge a BRADSO if required to receive the branch (answer A)

Question 2. *A cadet who is charged a BRADSO is required to serve an additional 3 years on Active Duty. Under the current mechanism, cadets must rank order all 17 branches and indicate if they are willing to BRADSO for each branch choice. For example:*

- **Current Mechanism Example**:
  - **1: AV/BRADSO, 2: EN, 3: CY**

Under an alternative mechanism, cadets could indicate if they prefer to receive their second branch choice without a BRADSO charge more than they prefer to receive their first branch choice with a BRADSO charge. For example:

- **Alternative Mechanism Example**:
  - **1: AV, 2: EN, 3: AV/BRADSO, 4: CY**

When submitting branch preferences, which mechanism would you prefer?

- A. Current Mechanism (21.4% of respondents)
- B. Alternative Mechanism (49.7% of respondents)
- C. Indifferent (24.2% of respondents)
- D. Do Not Understand (4.8% of respondents)

F Applications Beyond the US Army's Branching System

The individual-proposing deferred acceptance (DA) algorithm of Gale and Shapley (1962) plays a prominent role in several market design applications, in particular for priority-based resource allocation (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Our model is perhaps one of the most natural extensions of this approach for settings where the priorities of individuals can be increased with a costly action for a subset of positions at each institution. Based on Theorem 1, we believe that the MPCO mechanism is a natural counterpart of DA for such settings. Therefore, while our paper is mainly motivated by the Army's 2020 branching reform, our model in Section 3 and main characterization result in Theorem 1 have other direct applications.

In this section, we present a direct application of our analysis in the context of a school choice policy widely deployed in the recent history of China and several other potential applications.

F.1 High School Seat Purchasing Policies in China

In many cities in China, the priority ranking of students at public high schools mainly depends on their exam scores. Motivated by a departure from this policy in several Chinese cities between 1990s and 2015, Zhou and Wang (2021) present an extension of the school choice model by Abdulkadiroğlu and Sönmez (2003). In their application, students gain increased priority for a subset of seats at each school by paying higher tuition levels. Zhou and Wang (2021) refer to this policy as the *Ze Xiao (ZX) policy*.

Cities that deployed the ZX policy used a scoring-based price responsiveness scheme we formulated in Section 3.2.3. Parallel to our main application on the US Army's branching process, the cities of Shanghai and Tianjin used a single level of increased tuition for the *ZX positions*.[53] The empirical analysis in Zhou and Wang (2021) is for a city where the ZX policy is more involved

[53] Zhou and Wang (2021) present the following details for Shanghai and Tianjin: "Shanghai is one of the cities that discontinued the ZX policy immediately after the announcement from the Ministry of Education in 2012. The total percentage of ZX students was restricted within 15% for each school in 2011, which is the percentage for ZX policy in the previous year. The ZX tuition in Shanghai was charged according to the type of school. In district-level key high

with four prices: A base price of 1,600 yuan, and three layers of increased price (3,333 yuan, 5000 yuan, and 6000 yuan). Zhou and Wang (2021) describes the scoring-based price responsiveness scheme used in this city until 2014 as follows:

> "Three levels of the higher tuition paid by ZX [increased price] students are based on their exam scores. A ZX student pays a total of 3,333.3 yuan annually if her score is within 10 points of the school's cut-off, 5,000 yuan if it is within 11–20 points, and 6,000 yuan per year if it is within 21–30 points."

This practice is equivalent to boosting the merit score of a student by 10 points if she is willing to pay a tuition of 3,333.3 yuans, by 20 points if she is willing to pay a tuition of 5,000 yuans, and by 30 points if she is willing to pay a tuition of 6,000 yuans. That is, for any school $b \in B$, the scoring rule $S^b : \{t^0, t^1, t^2, t^3\} \rightarrow \mathbb{Z}^+$ and the price responsiveness scheme ω_b^S for the empirical application in Zhou and Wang (2021) are given as follows:

For any school $b \in B$ and $t \in \{t^0, t^1, t^2, t^3\}$,

$$S^b(t) = 10(t - t^0).$$

Given a list of merit scores $(m_i)_{i \in I}$ and a high school $b \in B$, for any two student-tuition pairs $(i, t), (j, t') \in I \times \{t^0, t^1, t^2, t^3\}$,

$$(i, t) \; \omega_b^S \; (j, t') \quad \iff \quad m_i + S^b(t) > m_j + S^b(t').$$

By Theorem 1, MPCO is the only direct mechanism for this application that *respects the price responsiveness scheme* and satisfies *individual rationality, non-wastefulness, no priority reversals* and *strategy-proofness*. The interpretation and desirability of all axioms except *respect for the price responsiveness scheme* is from standard arguments in the literature. So let us explore to what extent the axiom *respect for the price responsiveness scheme* is desirable in this setting.

Consider an allocation $X \in \mathcal{A}$ and a student $i \in I$ such that $X_i = (b, t) \in B \times \{t^1, t^2, t^3\}$. Suppose there exists a student $j \in I \setminus \{i\}$ who has a legitimate claim for a price-reduced version of student $i's$ assignment (b, t). Then, there exists a tuition level $t' < t$ such that

$$(b, t') \succ_j X_j \quad \text{and} \quad (j, t') \; \omega_b^S \; (i, t),$$

or equivalently,

$$(b, t') \succ_j X_j \quad \text{and} \quad (m_j - m_i) > 10(t - t').$$

The last pair of relations directly contradict the city's ZX policy, because the difference between

schools, the basic tuition for students was 2,400 Yuan/year, whereas the ZX tuition was 6,000 Yuan/year before 2011 and 4,266 Yuan/year in 2011. For the city-level key high schools, the basic tuition was 3,000 Yuan/year, whereas the ZX tuition was 10,000 Yuan/year before 2011 and 7,000 Yuan/year in 2011. For the boarding schools, the basic tuition was 4,000 Yuan/year, whereas the ZX tuition was 13,333 Yuan/year before 2011 and 9,333 Yuan/year in 2011.

[$\cdots$] Tianjin canceled its ZX policy in 2015. Before 2015, the ZX tuition was standardized across all general high schools at 8,000 Yuan/year, which was a fourfold increase in the basic tuition (2,000 Yuan/year)."

merit scores is too large to justify to award the seat to student i at a higher tuition than t' while the higher merit-score student j is eager to receive the seat at this tuition level.

Next, consider an allocation $X \in \mathcal{A}$ and a student $i \in I$ such that $X_i = (b, t) \in B \times \{t^0, t^1, t^2\}$. Suppose there exists a student $j \in I \setminus \{i\}$ who has a legitimate claim for a price-elevated version of student $i's$ assignment (b, t). Then, there exists a tuition level $t' > t$ such that

$$(b, t') \succ_j X_j, \qquad (j, t') \; \omega_b^S \; (i, t), \quad \text{and} \qquad \left| \left\{ (k, t^+) \in I \times \{t^1, t^2, t^3\} \; : \; (k, b, t^+) \in X_b \right\} \right| < q_b^f,$$

or equivalently

$$(b, t') \succ_j X_j, \qquad (m_i - m_j) < 10(t' - t), \qquad \text{and} \qquad \left| \left\{ (k, t^+) \in I \times \{t^1, t^2, t^3\} \; : \; (k, b, t^+) \in X_b \right\} \right| < q_b^f.$$

The last triple of relations directly contradicts the city's ZX policy, because the difference between merit scores is not large enough to justify awarding the seat to student i at a lower tuition level than t' while student j is eager to receive the seat at a tuition level of t' and doing so is feasible.

Therefore, the axiom *respect for the price responsiveness scheme* is also highly plausible in this setting, thus making MPCO a highly desirable mechanism in this setting.

F.2 Talent Alignment and Retention in Priority-Based Assignment Markets

1. *Diplomat / Foreign Service Officer Placement*

   Each year, thousands of applicants compete for diplomatic positions at more than 285 U.S. embassies and consulates around the world. Prioritization is based on scores on the foreign service officer test, with additional points given for applicants based on veterans or disability status and foreign language ability (State Department, 2019). In this market, a price responsiveness scheme where willingness to work for an extended tour in exchange for a priority boost could help manage retention and talent alignment.

2. *Civil Service Placement*

   Governments around the world use centralized systems to place personnel into positions. For example, Khan et al. (2019) describe the use of a centralized assignment mechanism to assign property tax inspectors in Pakistan. They designed a policy where priority was determined by past performance as an inspector. In such a policy, a price responsiveness scheme, where a willingness to sign an extended service commitment generates increased priority in the assignment, could help manage retention and talent alignment.

   Bar et al. (2021) describe the process used to assign police officers to positions in other districts in Chicago. The priority is based on officer seniority. A challenge in this setting is the lack of demand for working in unsafe neighborhoods and oversubscription in safe neighborhoods. The officer assignment board may be able to use this oversubscription to increase retention by awarding desirable positions to officers who are willing to extend their time in a posting in exchange for higher priority.

3. *Centralized Teacher Assignment*

   Centralized schemes are used in teacher placement in several countries including in Czech Republic, France, Germany, Mexico, Peru, Portugal, Turkey, and Uruguay (Combe et al., 2022a,b). In these markets, teachers priority is often based on seniority. The central administration aspires to assign teachers respecting their preferences, while at the same time avoiding a surplus of inexperienced teachers in disadvantaged areas. Ajzenman et al. (2020) and Bertoni et al. (2021) use data to describe Peru's national teacher selection process. In that system, teachers can rank up to 5 schools and performance on a standardized test is used for prioritization. Since there is oversubscription in advantaged regions of the country, a price responsiveness scheme where lower performing teachers can buy priority by extending their service commitment could cause some more experienced teachers to be assigned to less advantaged regions.

4. *Other Military Sectors: Marines Corps and Air Force*

   Centralized placement is also widespread in the military, aside from the United States Army. Graduates of the U.S. Air Force Academy obtain their career field using a centralized mechanism where cadets rank fields (Armacost and Lowe, 2005). The Air Force judges success of their placement process based on retention-related outcomes and an Airman's fit (NASEM, 2021). Likewise, the U.S. Marine Corps struggles with turnover of marines, and a 2021 manpower report describes creating a digital talent marketplace to address this retention concern and balance the needs of units (United States Marine Corps, 2021). Both of these markets are situations where the flexibility of a price responsiveness scheme may facilitate a balance between talent alignment and retention.

F.3 Priority-Based Assignment with or without Amenities

Some of our earlier examples use a price-responsiveness scheme as a tool to manage retention-related outcomes. Here we describe two examples where the mechanism could unbundle the assignment into an assignment under two terms to manage resource constraints. First, nearly 15,000 officers and 500 units in the Army participate in the Army Talent Alignment Process each year (United States Army, 2019b).[54] Starting in 2019, this system used officer preferences and a version of the deferred acceptance algorithm for placement into units (Davis et al., 2023; Greenberg et al., 2020). In this market, when an officer is assigned outside the U.S., they must reside in government-controlled military family housing if it is available. However, not all officers may wish to bring their families abroad and may not require this housing. Hence, the system could offer job assignment with and without family housing, with the base price corresponding to housing and the increased price corresponding to no family housing. In places where there is scarcity of family housing options, a price responsiveness scheme could allow an officer who is willing

[54] The cadet-branch assignment process is used to determine a new officer's occupation. The Army Talent Alignment Process is used to match officers to specific jobs at later points in their Army career.

to forego family housing to buy priority for a position over an officer who needs family housing. The same concept could apply for college admissions, where a student can be assigned with the right to on-campus housing or without the right to on-campus housing.

Second, consider student assignment at K-12 as in Abdulkadiroğlu and Sönmez (2003). In that framework, students are assigned schools, and each position at a school is identical. However, a school position can be offered to a student under different terms. For example, for kindergarten and pre-kindergarten, a school can sometimes offer a full-day or half-day option. These two terms correspond to the base price and the increased price. A price responsiveness scheme where an applicant can buy priority if she is willing be assigned a half-day option is an instrument that would allow certain lower priority applicants to access a sought-after school for a half-day that they could not otherwise access. It is possible to envision similar ideas, like offering options for a school with an early start time or late start time (a common way to manage overcrowding), or offering a school with meal or without meal service. A price responsiveness scheme in these cases would allow applicants willing to take the increased cost option (e.g., starting school early for some or attending school without free breakfast) in exchange for increased priority. If these ideas are used within the context of a centralized mechanism, then our axioms are natural and imply that the MPCO is the only possible mechanism.