The Comparative Statics of Sorting
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Online Appendix

D Nowhere Decreasing Optimizers

The space of matching cdf's is not a lattice, since the meet and the join are not defined for arbitrary matchings.[23] The matching problem (3) does not have a lattice constraint or an objective function that is quasi-supermodular in the control: standard monotone comparative static results (e.g. Milgrom and Shannon (1994)) do not apply. The next section presents a general comparative static result for single-crossing functions on partially ordered sets (*posets*) without assuming a well-defined meet or join.[24] We then apply this result to our sorting model to get a nowhere decreasing sorting result.

D.1 Nowhere Decreasing Optimizers for Arbitrary Posets

Let Z and Θ be posets. The correspondence $\varsigma : \Theta \to Z$ is *nowhere decreasing* if $z_1 \in \varsigma(\theta_1)$ and $z_2 \in \varsigma(\theta_2)$ with $z_1 \succeq z_2$ and $\theta_2 \succeq \theta_1$ imply $z_2 \in \varsigma(\theta_1)$ and $z_1 \in \varsigma(\theta_2)$.

Notably, any partial order $\succeq$ induces a complete (nowhere decreasing) order $\succeq^*$ such that $B \succeq^* A$ if $B = A$ or it is not true that $A \succeq B$. Since the domain of any complete order is a lattice, we can apply standard monotone logic, which we next do.

Theorem 3 (Nowhere Decreasing Optimizers)**.** *Let $F : Z \times \Theta \mapsto \mathbb{R}$, where Z and Θ are posets, and let $Z' \subseteq Z$. If $\max_{z \in Z'} F(z, \theta)$ exists for all θ and F is single crossing in (z, θ), then $\mathcal{Z}(\theta|Z') \equiv \arg\max_{z \in Z'} F(z, \theta)$ is nowhere decreasing in θ for all Z'. If $\mathcal{Z}(\theta|Z')$ is nowhere decreasing in θ for all $Z' \subseteq Z$, then $F(z, \theta)$ is single crossing.*

($\Rightarrow$): If $\theta_2 \succeq \theta_1$, $z_1 \in \mathcal{Z}(\theta_1)$, $z_2 \in \mathcal{Z}(\theta_2)$, and $z_1 \succeq z_2$, optimality and single crossing give:

$$F(z_1, \theta_1) \geq F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) \geq F(z_2, \theta_2) \quad \Rightarrow \quad z_1 \in \mathcal{Z}(\theta_2)$$

Now assume $z_2 \notin \mathcal{Z}(\theta_1)$. By optimality and single crossing, we get the contradiction:

$$F(z_1, \theta_1) > F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) > F(z_2, \theta_2) \quad \Rightarrow \quad z_2 \notin \mathcal{Z}(\theta_2)$$

[23] As shown in Proposition 4.12 in Müller and Scarsini (2006): If M dominates PAM2 and PAM4, then $M(2,1) \geq 1/3$ and $M(1,2) \geq 1/3$, but $M(1,1) = 0$ if NAM1 and NAM3 dominate M. So then $M(2,2) = 2/3$, but then NAM1 cannot PQD dominate M.

[24] This may be a known result. We include it for completeness, and as we cannot find any reference.

(⇐): If F is not single crossing, then for some $z_2 \succeq z_1$ and $\theta_2 \succeq \theta_1$, either: (i) $F(z_2,\theta_1) \geq F(z_1,\theta_1)$ and $F(z_2,\theta_2) < F(z_1,\theta_2)$; or, (ii) $F(z_2,\theta_1) > F(z_1,\theta_1)$ and $F(z_2,\theta_2) \leq F(z_1,\theta_2)$. Let $Z' = \{z_1, z_2\}$. In case (i), $z_2 \in \mathcal{Z}(\theta_1|Z')$ and $z_1 = \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_2 \notin \mathcal{Z}(\theta_2|Z')$. In case (ii), $z_2 = \mathcal{Z}(\theta_1|Z')$ and $z_1 \in \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_1|Z')$. □

D.2 Nowhere Decreasing Sorting

Sorting is nowhere decreasing in θ if the matching never falls in the PQD order. So for all $\theta_2 \succeq \theta_1$, if $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked $M_1 \succeq_{PQD} M_2$, then we have $M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$. We say that *weighted synergy is upcrossing*[25] in θ if the following is upcrossing in θ:

- $\int \phi_{12}(x,y|\theta)\lambda(x,y)dxdy$ for all nonnegative (measurable)[26] functions λ on $[0,1]^2$
- $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\lambda_{ij}$ for all positive weights $\lambda \in \mathbb{R}_+^{(n-1)^2}$

We first present the continuum analogue of the finite match output formula (5).[27]

Lemma 3 (Continuum Types). *Given type intervals $\mathcal{I} \equiv [0,1]$ and $\mathcal{J} \equiv (0,1]$, then:*

$$\int_{\mathcal{I}^2} \phi(x,y)M(dx,dy) = \int_{\mathcal{I}} \phi(x,1)G(dx) - \int_{\mathcal{J}} \phi_2(1,y)H(y)dy + \int_{\mathcal{J}^2} \phi_{12}(x,y)M(x,y)dxdy$$

Proof: If ψ is C^1 on $[0,1]$ and Γ is a cdf on $[0,1]$, integration by parts yields:

$$\int_{[0,1]} \psi(z)\Gamma(dz) = \psi(1)\Gamma(1) - \int_{(0,1]} \psi'(z)\Gamma(z)dz \tag{29}$$

where the interval $(0,1]$ accounts for the possibility that Γ may have a mass point at 0. Since $M(dx,y) \equiv M(y|x)G(dx)$ for a conditional matching cdf $M(y|x)$, we have:

$$M(x,y) \equiv \int_{[0,x]} M(y|x')G(dx') \tag{30}$$

By Theorem 34.5 in Billingsley (1995) and then in sequence (29), (30) and Fubini's

[25]Let Z be a partially ordered set. The function $\sigma : Z \mapsto \mathbb{R}$ is *upcrossing* if $\sigma(z) \geq (>)0$ implies $\sigma(z') \geq (>)0$ for $z' \succeq z$, *downcrossing* if $-\sigma$ is upcrossing. Similarly, σ is strictly upcrossing if $\sigma(z) \geq 0$ implies $\sigma(z') > 0$ for all $z' \succ z$, with strictly downcrossing defined analoguously.

[26]To save space, we henceforth assume measurable sets for integrals whenever needed.

[27]Equation (9) in Cambanis, Simons, and Stout (1976) reduces to our formula when output is C^2. We present our simpler proof for the C^2 case for completeness.

Theorem, (29), the objective function $\int_{[0,1]^2} \phi(x,y)M(dx,dy)$ in (3) equals:

$$\begin{aligned}
& \int_{[0,1]} \int_{[0,1]} \phi(x,y)M(dy|x)G(dx) \\
= & \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]} \int_{(0,1]} \phi_2(x,y)M(y|x)dyG(dx) \\
= & \int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]} \left[\phi_2(1,y)M(1,y) - \int_{(0,1]} \phi_{12}(x,y)M(x,y)dx\right] dy
\end{aligned}$$

which easily reduces to the desired expression, using $M(1,y) = H(y)$. □

Theorem 4. *Sorting is nowhere decreasing in θ if weighted synergy is upcrossing in θ, and thus if synergy is nondecreasing in θ. Also, if sorting is nowhere decreasing in θ for all type distributions G, H, then any rectangular synergy is upcrossing in θ.*

Proof of (a): First, $M' \succeq_{PQD} M$ iff $\lambda \equiv M' - M \geq 0$. As weighted synergy upcrosses:

$$\begin{array}{c}
\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M_{ij}) \geq (>)\ 0 \Rightarrow \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M_{ij}) \geq (>)\ 0 \\
\int_{(0,1]^2} \phi_{12}(\cdot|\theta)(M' - M) \geq (>)\ 0 \ \Rightarrow\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta')(M' - M) \geq (>)\ 0
\end{array} \tag{31}$$

Thus, match output is single crossing in (M,θ) by (5) (for finite types) and Lemma 3 for continuum types. Then the optimal matching $\mathcal{M}^*(\theta)$ (in the space of feasible matchings $\mathcal{M}(G,H)$) is nowhere decreasing in the state θ, by Theorem 3.

Proof of (b): Assume two women (x_1, x_2) and men (y_1, y_2), and that $S(R|\theta)$ is not upcrossing in θ, i.e. for some $\theta'' \succeq \theta'$ and rectangle $R = (x_1, y_1, x_2, y_2)$, we have $S(R|\theta'') \leq 0 \leq S(R|\theta')$ with one inequality strict. These inequalities imply that NAM optimal at θ'' and PAM optimal at θ', and either NAM is uniquely optimal at θ'' or PAM is uniquely optimal at θ'. Either case precludes nowhere decreasing sorting. □

Easily, weighted synergy is upcrossing in θ if synergy is non-decreasing in θ. Thus:

Corollary 2 (Cambanis, Simons, and Stout (1976))**.** *Sorting is nowhere decreasing in θ if synergy is non-decreasing in θ.*

E Omitted Proofs for Economic Applications in §7

1. Diminishing Returns: Let $R(z|\theta) \equiv -z\psi''(z|\theta)/\psi'(z|\theta)$. Synergy is then:

$$\phi_{12}(x,y|\theta) = \psi'(xy|\theta)\left[\frac{\psi''(xy|\theta)xy}{\psi'(xy|\theta)} + 1\right] \equiv \psi'(xy|\theta)(1 - R(xy|\theta)) \tag{32}$$

By assumption $\psi' > 0$ and $R(xy|\theta)$ is decreasing in x, y, and $t = 1 - \theta$. Thus, synergy strictly upcrosses in x, y, and t. Further, $\psi'(xy|1-t)$ is LSPM in (x, y, t), since

$$[\log(\psi'(xy|1-t))]_x = \frac{y\psi''(xy|1-t)}{\psi'(xy|1-t)} = -x^{-1}R(xy|1-t)$$

is increasing in y and t by $R(z|\theta)$ decreasing in z and increasing in θ. Altogether, synergy (32) is the product of a strictly positive LSPM function and an increasing function; and thus, sorting increases in $t = 1 - \theta$ by Proposition 5, and so falls in θ.

2. Weakest to Strongest Link: We verify the premise of Proposition 4 to prove that sorting sorting increases in ρ for $\phi(x, y) = \psi(q(x, y))$ as in §7.2. Symmetric steps generalize this result for any $\psi'' < 0 < \psi'$, obeying $2\psi''(q) + q\psi'''(q) \leq 0$.

$$\phi_{12}(x, y) = \frac{q_1(x, y)q_2(x, y)}{q(x, y)}\left[(1 + \rho)(\alpha - 2\beta q(x, y)) - 2\beta q(x, y)\right] \tag{33}$$

Step 1. *Marginal rectangular synergy is strictly downcrossing in types.*

Proof: Since $q(x, y)$ increases in (x, y) and falls in ρ, the bracketed term in (33) falls in (x, y) and rises in ρ. Thus, synergy (33) is upcrossing in ρ and is strictly downcrossing in (x, y). Further, since $q_1(x, y)q_2(x, y)/q(x, y)$ is LSPM in (x, y) when $\rho \geq 0$, synergy is proportionately downcrossing in (x, y). So, marginal rectangular synergy is downcrossing in types, by Theorem 1. Finally, marginal rectangular synergy is strictly downcrossing in (x, y) by the proof logic after inequality (28) in Appendix C.5. □

Step 2. *Summed rectangular synergy is upcrossing in ρ.*

Proof: Since $\phi_{12}(x, y) = \phi_{12}(y, x)$, weighted synergy $\int_{[0,1]^2} \phi_{12}\hat{\lambda}$ is upcrossing in ρ for all weighting functions $\hat{\lambda}$, *iff* $\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dxdy$ is upcrossing in ρ for all weighting functions λ. Now use change of variable $y = kx$ to get:

$$\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dydx = 2\int_0^1 \int_0^1 x\phi_{12}(x, kx)\lambda(x, kx)dkdx$$

Let $x\phi_{12}(x, kx) = \sigma_A(k, \rho)\sigma_B(x, k, \rho)$, where $\sigma_A \equiv xq_1(x, kx)q_2(x, kx)/q(x, kx)$ and σ_B is the bracketed term in (33) evaluated at $y = kx$. Routine algebra yields $\sigma_A(k, \rho)$ LSPM in (k, ρ), while $\sigma_B(x, k, \rho)$ is decreasing in (x, k) and increasing in ρ. Altogether, $\sigma_A\sigma_B$ is proportionately upcrossing in (x, k, ρ). As synergy is also upcrossing in ρ by Step 1, so is weighted synergy, by Theorem 1 — as is summed rectangular synergy. □

3. Nowhere Decreasing Sorting in Kremer and Maskin (1996):
We prove (13): *sorting is nowhere decreasing in θ and nowhere increasing in $\varrho = -\rho$.*

Step 1. *PAM is not optimal if* $\varrho > (1-2\theta)^{-1}$*, and is uniquely optimal for* $\varrho < (1-2\theta)^{-1}$.

Proof: In a unisex model, PAM is optimal iff the symmetric rectangular synergy $S(x,x,y,y)$ is globally positive. Its sign is constant along any ray $y = kx$, and proportional to:

$$s(k) \equiv 2^{\frac{1-2\theta}{\varrho}}(1+k) - 2k^{\theta}(1+k^{\varrho})^{\frac{1-2\theta}{\varrho}} \tag{34}$$

Since $s(1) = s'(1) = 0$, $s''(1) \propto (1+\varrho(2\theta-1))$, and $\theta \in [0,1/2]$, we have $s(k) < 0$ close to $k = 1$ precisely when $\varrho > (1-2\theta)^{-1} \geq 1$. In this case, the symmetric rectangular synergy is negative in a cone around the diagonal, and PAM fails.

Conversely, posit $\varrho < (1-2\theta)^{-1}$. Then $s(k) > 0$ for all $k \in [0,1]$. Since $S(x,x,y,y)$ is symmetric about $y = x$, it is globally positive and PAM is uniquely optimal. □

Step 2. *If* $\varrho \geq (1-2\theta)^{-1}$ *then weighted synergy is upcrossing in* θ*, downcrossing in* ϱ.

Proof: Change variables $y = kx$. If $\Delta(k) = \int_0^1 \lambda(x,kx)dx$, weighted synergy is

$$\int\int \phi_{12}(x,y)\lambda(x,y)dydx = 2\int_0^1\int_0^1 x\phi_{12}(x,kx)\lambda(x,kx)dkdx = \int_0^1 \sigma(k,\theta,\varrho)\Delta(k)dk$$

where $\sigma = \sigma_A\sigma_B$ for $\sigma_A \equiv 2k^{\theta-1}(1+k^{\varrho})^{\frac{1-2\theta-2\varrho}{\varrho}}$ and $\sigma_B \equiv \theta(1-\theta)(1+k^{2\varrho}) + (1-\varrho+2\theta(\theta-1+\varrho))k^{\varrho}$. As $\varrho \geq (1-2\theta)^{-1}$, $\sigma_A > 0$ is LSPM in (k,θ,ϱ), σ_B is increasing in $(\theta,-k,-\varrho)$ for $k \in [0,1]$. So $\sigma = \sigma_A\sigma_B$ is proportionately downcrossing in (k,θ) and $(k,-\varrho)$. Weighted synergy is upcrossing in θ, downcrossing in ϱ, by Theorem 1. □

Step 3. *Sorting is nowhere decreasing in* θ *and nowhere increasing in* ϱ.

Proof: Pick $\theta'' > \theta'$. If $\varrho < (1-2\theta'')^{-1}$, then PAM is uniquely optimal at θ'' (Step 1) and sorting increases from θ' to θ''. If $\varrho \geq (1-2\theta'')^{-1}$, then $\varrho > (1-2\theta')^{-1}$ and weighted synergy is upcrossing on $[\theta',\theta'']$ (Step 2) and sorting is non-decreasing (Proposition 4).

Now pick any θ and $\varrho'' > \varrho'$. If $\varrho' < (1-2\theta)^{-1}$, then PAM is uniquely optimal at ϱ' (Step 1) and sorting is decreasing from ϱ' to ϱ''. If, instead, $\varrho' \geq (1-2\theta)^{-1}$, then, necessarily, $\varrho'' > (1-2\theta)^{-1}$, weighted synergy is downcrossing from ϱ' to ϱ'' (Step 2) and sorting is non-increasing in ϱ, by Proposition 4. □