Online Appendix for

Pricing Power in Advertising Markets: Theory and Evidence

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A. Proofs and Additional Theoretical Results

A.1. Preliminaries

Lemma 1. *(i) For any $V(\cdot)$, not necessarily monotone or submodular, in any equilibrium and for any owner $Z \in \mathcal{Z}$, the payments $p^*_{nZ}, p^*_{n'Z}$ made in equilibrium to owner Z by any two advertisers n, n' must coincide, $p^*_{nZ} = p^*_{n'Z}$.*

(ii) If $V(\cdot)$ is either monotone and submodular or strictly monotone and not necessarily submodular, then in any equilibrium each advertiser buys slots on all outlets.

Proof. Fix any equilibrium. First, we claim that because the advertisers are homogeneous, they must have the same equilibrium payoff. Suppose toward contradiction that advertiser n' has a strictly higher equilibrium payoff than advertiser n. Now, let advertiser n imitate the strategy of advertiser n'. Since ad slots are not scarce ($N \leq K$), advertiser n must obtain the equilibrium payoff of advertiser n', which is a contradiction. So all advertisers have the same equilibrium payoff, which we denote by W.

For part (i), suppose toward contradiction that in the equilibrium there exists some owner $Z \in \mathcal{Z}$ such that not all advertisers make the same total payment to Z. Let n be an advertiser whose equilibrium payment to owner Z is at least as much as any other advertiser's (and therefore strictly greater than some other advertiser's). Let $S \subseteq Z$ be the set of outlets owned by owner Z on which advertiser n buys slots. Let owner Z deviate by offering bundle $B = S$ at price $p^*_S - \varepsilon$ for any $\varepsilon > 0$, where p^*_S denotes the minimum price to buy slots on the set of outlets S at the on-path history in the supposed equilibrium. By subgame perfection, at every possible subgame, each advertiser must purchase a payoff-maximizing set of bundles. We claim that upon the deviation by owner Z, all advertisers buy bundle B offered by owner Z. To see why, note that by buying the bundle B upon the deviation (and imitating advertiser n's choices at the on-path history), any advertiser can obtain a payoff of $W + \varepsilon$, which is strictly greater than the payoff of W prescribed by the equilibrium, and therefore (because no prices have changed other than the price of bundle B) also strictly greater than the payoff from any other choice of bundles. For sufficiently small ε, the deviation increases the payment to owner Z from at least one advertiser other than n more than it decreases the payment from advertiser n, and therefore strictly increases the payoff to owner Z. This establishes that the deviation is profitable, and hence that there is a contradiction.

For part (ii), suppose toward contradiction that in the equilibrium there exists some advertiser n who does not buy a slot on some outlet $j \in Z$ owned by some owner $Z \in \mathcal{Z}$. Let $T \subset Z$ be the set of outlets owned by owner Z on which advertiser n buys slots, and let $R \subseteq \mathcal{J} \backslash Z$ be the set of outlets owned by other owners on which advertiser n buys slots. By part (i), all advertisers pay the same total amount to owner Z, which then must be p^*_T, the minimum price to buy slots on all outlets in T. The equilibrium payoff for each advertiser is then given by

$$W = V(T \cup R) - p_R^* - p_T^*,$$

where p_R^* is the minimum price to buy slots on all outlets in R. Let owner Z deviate by offering a single bundle Z with a price $p_T^* + \varepsilon$ for some $\varepsilon > 0$. We show that for ε small enough, every advertiser must purchase this bundle. First, note that $V(Z \cup R) - V(T \cup R) > 0$. This is clearly true if $V(\cdot)$ is strictly monotone. If $V(\cdot)$ is submodular and monotone, then we also have

$$V(Z \cup R) - V(T \cup R) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash (Z \backslash T)) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash \{j\}) = v_j > 0$$

where the strict inequality is due to our assumption that every outlet has positive incremental value. Thus, for ε small enough, we have

$$V(Z \cup R) - p_R^* - (p_T^* + \varepsilon) > V(T \cup R) - p_R^* - p_T^* = W.$$

Now, pick any such sufficiently small ε. Note again that by subgame perfection, at every possible subgame, each advertiser must purchase a payoff-maximizing set of bundles. Thus, upon this deviation, each advertiser buys the bundle Z at the price $p_T^* + \varepsilon$, because any strategy not doing so is a feasible strategy at the on-path history in the supposed equilibrium and therefore generates a payoff less than or equal to W. But this is then a profitable deviation for owner Z, and therefore a contradiction. $\square$

A.2. Proofs Omitted From the Main Text

Proof of Theorem 1. We first construct an equilibrium. Let each owner Z offer, at a price of $p_Z = v_Z$, a single bundle consisting of all outlets in Z. For any profile p of posted prices (including those off of the equilibrium path), and any set of outlets $S \subseteq \mathcal{J}$, let p_S denote the minimum price to buy slots on all the outlets in S. Now let every advertiser buy the same bundle $S^*(p)$, chosen arbitrarily from the set of solutions to the problem

$$\begin{aligned} &\max |S| \\ &s.t.\, S \in \underset{S \subseteq \mathcal{J}}{\operatorname{argmax}}\, V(S) - p_S \end{aligned}$$

By construction, $S^*(\cdot)$ constitutes an equilibrium strategy profile for the advertisers.

It remains to verify that no owner has a profitable deviation from the proposed profile p^*. Observe that if owner Z plays the proposed strategy, then each advertiser buys the bundle offered by owner Z regardless of the other owners' offerings. This is because for any $S \subseteq \mathcal{J} \backslash Z$, submodularity of $V(\cdot)$ implies

$$V(S \cup Z) - V(S) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash Z) = v_Z.$$

Fix any owner Z. Suppose all other owners $Z' \neq Z$ follow the proposed strategy. Fix any advertiser. By the above observation we know that the advertiser buys slots on all outlets not in Z regardless of the strategy of owner Z. It follows that the most that owner Z can extract from the advertiser is v_Z because for any $T \supseteq \mathcal{J} \backslash Z$, monotonicity of $V(\cdot)$ implies

$$v_Z = V(\mathcal{J}) - V(\mathcal{J} \backslash Z) \geq V(T) - V(\mathcal{J} \backslash Z)$$

where the right hand side is the most that the advertiser is willing to pay for slots on all the outlets in $T \backslash (\mathcal{J} \backslash Z)$. Therefore, there is no profitable deviation from the proposed strategy for owner Z. Since Z is an arbitrary owner, the proposed profile is an equilibrium.

To prove the second part of the statement, fix any equilibrium of the game, and denote its price profile by p^*. By Lemma 1(ii), all advertisers buy slots on all outlets in $\mathcal{J}$. Therefore, each advertiser pays p_Z^* to each owner Z. If $p_Z^* > v_Z$ for any owner Z, then any advertiser can profitably deviate by only buying slots on all outlets in $\mathcal{J} \backslash Z$. If $p_Z^* < v_Z$ for any owner Z, then, by the submodularity of $V(\cdot)$, owner Z can profitably deviate by offering a single bundle Z with a price $v_Z - \varepsilon$ for $\varepsilon > 0$ sufficiently small to extract $v_Z - \varepsilon > p_Z^*$ from each advertiser. Thus $p_Z^* = v_Z$ for all $Z \in \mathcal{Z}$.

Monotonicity and Submodularity of $V(\cdot)$ in the Viewer-level Model. Let $\mathcal{J}_F$ be the set of outlets whose owner is of format F. For each viewer $i \in \mathcal{I}$ and each set of outlets $S \subseteq \mathcal{J}$, let X_S^i count the number of outlets in S on which viewer i sees ads. For each $S \subseteq \mathcal{J}$ we can write

$$V(S) = |\mathcal{I}| \cdot \mathbb{E}\left[a_i u\left(X^i_{S\cap\mathcal{J}_1}, \ldots, X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right)\right] = |\mathcal{I}| \cdot \mathbb{E}\left[V_i(S)\right]$$

where the expectation is taken with respect to the population of viewers i and their random viewing behavior, and the value $V_i(\cdot)$ is based on the realized behavior of viewer i. To show that $V(\cdot)$ is monotone and submodular, it suffices to show that $V_i(S)$ is monotone and submodular for all i and realized $X^i_\cdot$, since averaging preserves monotonicity and submodularity.

To show monotonicity of $V_i(\cdot)$, pick some i and realized $X^i_\cdot$ and fix any $S \subseteq \mathcal{J}$ and $j \in \mathcal{J} \backslash S$. Let F be the format of the owner of outlet j. Then,

$$V_i(S \cup \{j\}) - V_i(S) = a_i \mathbf{1}_{i \to j}\left(u\left(X^i_{S\cap\mathcal{J}_1}, \ldots, X^i_{S\cap\mathcal{J}_F} + 1, \ldots, X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right) - u\left(X^i_{S\cap\mathcal{J}_1}, \ldots, X^i_{S\cap\mathcal{J}_F}, \ldots, X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right)\right)$$

where $i \to j$ denotes the event that viewer i sees ads on outlet j. This shows the monotonicity of $V_i(\cdot)$ as $u(\cdot)$ is monotone.

To show the submodularity of $V_i(\cdot)$, pick some i and realized $X^i_\cdot$ and fix any $S \subseteq T \subseteq \mathcal{J}$, and $j \in \mathcal{J} \backslash T$. Let F be the format of the owner of outlet j. Since $X^i_{S\cap\mathcal{J}_{F'}} \leq X^i_{T\cap\mathcal{J}_{F'}}$ for all formats F', we have

$$\begin{aligned}
V_i(S\cup\{j\}) - V_i(S) &= a_i \mathbf{1}_{i\to j}\left(u\left(X^i_{S\cap\mathcal{J}_1},\ldots,X^i_{S\cap\mathcal{J}_F}+1,\ldots,X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right) - u\left(X^i_{S\cap\mathcal{J}_1},\ldots,X^i_{S\cap\mathcal{J}_F},\ldots,X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right)\right) \\
&\geq a_i \mathbf{1}_{i\to j}\left(u\left(X^i_{S\cap\mathcal{J}_1},\ldots,X^i_{T\cap\mathcal{J}_F}+1,\ldots,X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right) - u\left(X^i_{S\cap\mathcal{J}_1},\ldots,X^i_{T\cap\mathcal{J}_F},\ldots,X^i_{S\cap\mathcal{J}_{|\mathcal{F}|}}\right)\right) \\
&\geq a_i \mathbf{1}_{i\to j}\left(u\left(X^i_{T\cap\mathcal{J}_1},\ldots,X^i_{T\cap\mathcal{J}_F}+1,\ldots,X^i_{T\cap\mathcal{J}_{|\mathcal{F}|}}\right) - u\left(X^i_{T\cap\mathcal{J}_1},\ldots,X^i_{T\cap\mathcal{J}_F},\ldots,X^i_{T\cap\mathcal{J}_{|\mathcal{F}|}}\right)\right) \\
&= V_i(T\cup\{j\})) - V_i(T)
\end{aligned}$$

where the second line follows because $u(\cdot)$ has decreasing differences in each argument, and the third line follows because $u(\cdot)$ is submodular. This shows the submodularity of $V_i(\cdot)$.

Proof of Corollary 1. Following the notation in the proof for the viewer-level model, we have that

$$\begin{aligned}
v_Z &= |\mathcal{I}| \cdot \mathbb{E}\left[a_i\left(u\left(X^i_{\mathcal{J}}\right) - u\left(X^i_{\mathcal{J}\backslash Z}\right)\right)\right] \\
&= |\mathcal{I}| \cdot \mathbb{E}\left[a_i \mathbf{1}_{i\to Z}\mathbf{1}_{i\not\to\mathcal{J}\backslash Z}\right] \\
&= \sum_i a_i \mathbb{E}\left[\mathbf{1}_{i\to Z}\right]\mathbb{E}\left[\mathbf{1}_{i\not\to\mathcal{J}\backslash Z}\right] \\
&= \sum_i a_i \eta_{iZ} \prod_{j\notin Z}\left(1-\eta_{ij}\right) \\
&= \sum_i a_i \eta_{iZ} \prod_{Z'\neq Z}\left(1-\eta_{iZ'}\right)
\end{aligned}$$

where we use the notation $\mathbf{1}_{i\to S}$ to denote the event that viewer i watches at least one outlet in the set S and the notation $\mathbf{1}_{i\not\to S}$ to denote the opposite. The second line follows because $u(M) = \mathbf{1}_{M>0}$, the third because of the independence of ad viewing across outlets, and the fourth and fifth by the definition of η_{iZ} and the structure of viewing behavior in the viewer-level model. The corollary then follows by Theorem 1.

Proof of Proposition 1. The setting of Proposition 1 is formally equivalent to a special case of the setting of Proposition 4 in Appendix A.3.1, in which $\beta_1 = u(1) - u(0) = 1 > 0 = u(2) - u(1) = \beta_2$ and, for each owner Z, we consider that it owns a single outlet with viewing probability $\eta_{gj} = \eta_{gZ}$. The desired result then follows by Proposition 4.

Proof of Proposition 2. The setting of Proposition 2 is formally equivalent to a special case of the setting of Proposition 5 in Appendix A.3.1, in which $\beta_1 = u(1) - u(0) = 1 > 0 = u(2) - u(1) = \beta_2$ and, for each owner Z, we consider that it owns a single outlet with viewing probability $\eta_{gj} = \eta_{gZ}$. The desired result then follows by Proposition 5.

Proof of Proposition 3. We proceed by showing that the setting of Proposition 3 is equivalent to a special case of the setting of Theorem 1. Let $\tilde{\mathcal{J}}$ be the set of outlets owned by owners in $\tilde{\mathcal{Z}}$. Let

$$\mathcal{J}^* = \left(\mathcal{J} \setminus \tilde{\mathcal{J}}\right) \cup \left(\tilde{\mathcal{J}} \times \mathcal{I}\right)$$

be an augmented set of outlets that includes viewer-specific instances of all outlets in $\tilde{\mathcal{J}}$. Let $\mathcal{Z}^* = \left(\mathcal{Z} \backslash \tilde{\mathcal{Z}}\right) \cup \left(\tilde{\mathcal{Z}} \times \mathcal{I}\right)$ be the corresponding augmented set of owners. Let $\mathcal{J}_i^* = \left(\mathcal{J} \setminus \tilde{\mathcal{J}}\right) \cup \left(\tilde{\mathcal{J}} \times \{i\}\right) \subseteq \mathcal{J}^*$ denote the set of outlets on which it is possible to show ads to viewer i. Let $V_i^*(\cdot)$ be the restriction of $V_i(\cdot)$ to $\mathcal{J}_i^*$, i.e.

$$V_i^*(S) = V_i\left(S \cap \mathcal{J}_i^*\right)$$

for all $S \subseteq \mathcal{J}^*$, where with slight abuse of notation we take $\{(j,i)\}$ as equivalent to $\{j\}$ when evaluating $V_i(\cdot)$. Let $V^*(\cdot) = \sum_{i \in \mathcal{I}} V_i^*(\cdot)$. Recall that both monotonicity and submodularity are preserved under restriction and addition. Observe that an instance of the game considered in Theorem 1 with primitives $\mathcal{J}^*, V^*(\cdot), \mathcal{Z}^*$ is equivalent to the game considered in Proposition 3. The conclusions of Proposition 3 therefore follow from Theorem 1 and the structure of the value function $V_i^*(\cdot)$.

Proof of Corollary 2. Following the notation in the proof of Corollary 1, for any owner Z in format 1 we have that

$$\begin{aligned}
v_Z &= |\mathcal{I}| \cdot \mathbb{E}\left[a_i \left(u\left(X^i_{\mathcal{J}_1}, X^i_{\mathcal{J}_2}\right) - u\left(X^i_{\mathcal{J}_1 \backslash Z}, X^i_{\mathcal{J}_2}\right)\right)\right] \\
&= |\mathcal{I}| \cdot \mathbb{E}\left[a_i \left(\mathbf{1}_{X^i_{\mathcal{J}_1} > 0} - \mathbf{1}_{X^i_{\mathcal{J}_1 \backslash Z} > 0} - \phi\left(\mathbf{1}_{X^i_{\mathcal{J}_1} X^i_{\mathcal{J}_2} > 0} - \mathbf{1}_{X^i_{\mathcal{J}_1 \backslash Z} X^i_{\mathcal{J}_2} > 0}\right)\right)\right] \\
&= |\mathcal{I}| \cdot \mathbb{E}\left[a_i \left(\mathbf{1}_{X^i_{\mathcal{J}_1} > 0} - \mathbf{1}_{X^i_{\mathcal{J}_1 \backslash Z} > 0} - \phi \mathbf{1}_{X^i_{\mathcal{J}_2} > 0}\left(\mathbf{1}_{X^i_{\mathcal{J}_1} > 0} - \mathbf{1}_{X^i_{\mathcal{J}_1 \backslash Z} > 0}\right)\right)\right] \\
&= |\mathcal{I}| \cdot \mathbb{E}\left[a_i \left(\mathbf{1}_{X^i_{\mathcal{J}_1} > 0} - \mathbf{1}_{X^i_{\mathcal{J}_1 \backslash Z} > 0}\right)\left(1 - \phi \mathbf{1}_{X^i_{\mathcal{J}_2} > 0}\right)\right] \\
&= \sum_i a_i \mathbb{E}\left[\mathbf{1}_{i \to Z}\right] \mathbb{E}\left[\mathbf{1}_{i \not\to \mathcal{J}_1 \backslash Z}\right]\left(1 - \phi \mathbb{E}\left[\mathbf{1}_{i \to \mathcal{J}_2}\right]\right)
\end{aligned}$$

$$= \sum_i a_i \eta_{iZ} \left(1 - \phi \left(1 - \prod_{Z' \in \mathcal{Z} \setminus F(Z)} (1 - \eta_{iZ'})\right)\right) \prod_{Z' \in F(Z) \setminus \{Z\}} (1 - \eta_{iZ'})$$

and

$$v_{iZ} = a_i \eta_{iZ} \left(1 - \phi \left(1 - \prod_{Z' \in \mathcal{Z} \setminus F(Z)} (1 - \eta_{iZ'})\right)\right) \prod_{Z' \in F(Z) \setminus \{Z\}} (1 - \eta_{iZ'}) .$$

The second line follows because $u(M_1, M_2) = \mathbf{1}_{M_1 > 0} + \mathbf{1}_{M_2 > 0} - \phi \mathbf{1}_{M_1 M_2 > 0}$ and the remaining lines follow from the independence of ad viewing across outlets, the structure of viewing behavior, and the definition of η_{iZ}. An analogous construction applies for any owner Z in format 2. The corollary then follows by Proposition 3.

Proof of Corollary 3. This is an immediate consequence of Corollary 2.

A.3. Extensions

A.3.1. Comparative Statics with General Diminishing Returns

Consider a special case of the viewer-level model in which there is a single format, every owner owns a single outlet, $a_i = a > 0$ for all $i \in \mathcal{I}$, and $\eta_{ij} = \eta_{i'j}$ for any $i, i' \in g$ for $g \in G$ and G a partition of $\mathcal{I}$. To ease notation, let $\beta_0 = 0$, and let

$$\beta_m := u(m) - u(m-1)$$

for $m \geq 1$, so that

$$u(M) = \sum_{m=0}^{M} \beta_m .$$

Let μ_g denote the size of viewers from group g. Let

$$\lambda_j = \sum_{g \in G} \mu_g \eta_{gj}, \quad \sigma_{gj} = \frac{\mu_g \eta_{gj}}{\lambda_j}$$

denote, respectively, the expected number of viewers seeing an ad on outlet j, and the share of this audience that comes from group g. Then p_j^* / λ_j is the equilibrium price per viewer charged by the owner for an ad slot. In this setting, we define a group $g \in G$ to be *less active* than group $h \in G$ if $\eta_{gj} \leq \eta_{hj}$ for all $j \in \mathcal{J}$.

Proposition 4. *Suppose that outlet $j \in \mathcal{J}$ draws a larger share of its audience from a less active group g and a smaller share of its audience from a more active group h than outlet $k \in \mathcal{J}$, in the sense that $\sigma_{gj} \geq \sigma_{gk}$ and $\sigma_{hj} \leq \sigma_{hk}$, and that the two outlets have equal total audience sizes,*

$\lambda_j = \lambda_k$, and equal shares of audience from groups other than g and h, $\sigma_{g'j} = \sigma_{g'k}$ for all $g' \neq g,h$. Then outlet j has a higher equilibrium price per viewer than outlet k, $p_j^*/\lambda_j \geq p_k^*/\lambda_k$, with strict inequality whenever $\prod_{l\neq j,k}\left(1-\eta_{gl}\right) > \prod_{l\neq j,k}(1-\eta_{hl})$, $\sigma_{gj} > \sigma_{gk}$, and $\beta_1 > \beta_2$.

Proof. With a slight abuse of notation, for any group g, let g denote both the group and a randomly sampled viewer from the group. By Theorem 1, we can write

$$p_j^* = a\sum_g \mu_g \mathbb{E}[\mathbf{1}_{g\to j}\beta_{X^g_{\mathcal{J}\setminus\{j\}}+1}]$$

where $g \to j$ denotes the event that a randomly sampled viewer g views ads on outlet j and $X^g_{\mathcal{J}\setminus\{j\}}$ counts the random number of outlets other than j on which viewer g views ads. For any $g' \neq g,h$, we have

$$\eta_{g'j} = \frac{\lambda_j\sigma_{g'j}}{\mu_{g'}} = \frac{\lambda_k\sigma_{g'k}}{\mu_{g'}} = \eta_{g'k}.$$

Therefore, for any $g' \neq g,h$, by independence and symmetry,

$$\mathbb{E}[\mathbf{1}_{g'\to j}\beta_{X^{g'}_{\mathcal{J}\setminus\{j\}}+1}] = \mathbb{E}[\mathbf{1}_{g'\to k}\beta_{X^{g'}_{\mathcal{J}\setminus\{k\}}+1}].$$

To prove that $p_j^*/\lambda_j \geq p_k^*/\lambda_k$, it then suffices to show

$$\mu_g\mathbb{E}[\mathbf{1}_{g\to j}\beta_{X^g_{\mathcal{J}\setminus\{j\}}+1} - \mathbf{1}_{g\to k}\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}] \geq \mu_h\mathbb{E}[\mathbf{1}_{h\to k}\beta_{X^h_{\mathcal{J}\setminus\{k\}}+1} - \mathbf{1}_{h\to j}\beta_{X^h_{\mathcal{J}\setminus\{j\}}+1}].$$

Because ad viewing is independent across outlets for any given viewer, we can write the above as

$$\mu_g\left[\eta_{gj}(1-\eta_{gk}) - \eta_{gk}(1-\eta_{gj})\right]\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}] \geq \mu_h\left[\eta_{hk}(1-\eta_{hj}) - \eta_{hj}(1-\eta_{hk})\right]\mathbb{E}[\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1}]$$

where $X^g_{\mathcal{J}\setminus\{j,k\}}$ counts the random number of outlets not in $\{j,k\}$ on which viewer g views ads. Since $\lambda_j = \lambda_k$, this reduces to

$$(\sigma_{gj} - \sigma_{gk})\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}] \geq (\sigma_{hk} - \sigma_{hj})\mathbb{E}[\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1}].$$

It follows easily from our assumptions that $\sigma_{gj} - \sigma_{gk} = \sigma_{hk} - \sigma_{hj} \geq 0$. So it suffices to show $\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}] \geq \mathbb{E}[\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1}]$. Since $\eta_{gj} \leq \eta_{hj}$ for all $j \in \mathcal{J}$ and viewing decisions are independent across outlets for both g and h, there exists a monotone coupling of the viewing decisions by g and h in the sense that $\mathbf{1}_{g\to j} \leq \mathbf{1}_{h\to j}$ for all $j \in \mathcal{J}$. Under this coupling, we have $X^g_{\mathcal{J}\setminus\{j,k\}} \leq X^h_{\mathcal{J}\setminus\{j,k\}}$ pointwise. The claim then follows directly by noting that β_m is non-increasing in m for $m \geq 1$ and by the definition of the viewer-level model.

To show strict inequality, suppose that $\prod_{l\neq j,k}\left(1-\eta_{gl}\right) > \prod_{l\neq j,k}(1-\eta_{hl})$, $\sigma_{gj} > \sigma_{gk}$, and

$\beta_1 > \beta_2$. Using integration by parts and the definition of β_m, we have

$$\begin{aligned}\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}] - \mathbb{E}[\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1}] &= \int_0^\infty \mathbb{P}(\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1} > s)ds - \int_0^\infty \mathbb{P}(\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1} > s)ds \\ &= \sum_{m=1}^{\infty} (\beta_m - \beta_{m+1}) \left(\mathbb{P}(X^g_{\mathcal{J}\setminus\{j,k\}} + 1 \le m) - \mathbb{P}(X^h_{\mathcal{J}\setminus\{j,k\}} + 1 \le m)\right) > 0\end{aligned}$$

where the strict inequality follows from the fact that each term in the summation is nonnegative, $\beta_1 > \beta_2$, and $\mathbb{P}(X^g_{\mathcal{J}\setminus\{j,k\}} = 0) = \prod_{l\neq j,k}(1-\eta_{gl}) > \prod_{l\neq j,k}(1-\eta_{hl}) = \mathbb{P}(X^h_{\mathcal{J}\setminus\{j,k\}} = 0)$. Since $\sigma_{gj} > \sigma_{gk}$, we then have $(\sigma_{gj} - \sigma_{gk})\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}] > (\sigma_{hk} - \sigma_{hj})\mathbb{E}[\beta_{X^h_{\mathcal{J}\setminus\{j,k\}}+1}]$ and hence $p^*_j/\lambda_j > p^*_k/\lambda_k$. $\square$

Proposition 5. *Suppose that outlet j has a larger audience than outlet k in the sense that for some $\delta \ge 1$, $\eta_{gj} = \delta\eta_{gk}$ for all $g \in G$. Then outlet j has a higher price per viewer than outlet k, $p^*_j/\lambda_j \ge p^*_k/\lambda_k$ with strict inequalty whenever $\eta_{gj}\prod_{l\neq j,k}(1-\eta_{gl}) > 0$, $\delta > 1$, and $\beta_1 > \beta_2$.*

Proof. We follow the same notation as in the proof of Proposition 4. By Theorem 1, we can write

$$\begin{aligned}p^*_j &= a\sum_g \mu_g \mathbb{E}\left[\mathbf{1}_{g\to j}\beta_{X^g_{\mathcal{J}\setminus\{j\}}+1}\right] \\ &= a\sum_g \mu_g \left(\eta_{gj}\eta_{gk}\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+2}\right] + \eta_{gj}(1-\eta_{gk})\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right]\right) \\ &= a\sum_g \mu_g \left(\eta_{gk}\eta_{gj}\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+2}\right] + \delta\eta_{gk}\left(1-\frac{1}{\delta}\eta_{gj}\right)\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right]\right) \\ &= a\sum_g \mu_g \left(\eta_{gk}\eta_{gj}\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+2}\right] + \eta_{gk}(1-\eta_{gj})\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right] + \eta_{gk}(\delta-1)\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right]\right) \\ &= a\sum_g \mu_g\eta_{gk} \left(\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}\right] + (\delta-1)\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right]\right) \\ &\ge a\sum_g \mu_g\eta_{gk} \left(\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}\right] + (\delta-1)\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}]\right) = \delta p^*_k = \frac{\lambda_j}{\lambda_k}p^*_k\end{aligned}$$

where we have used independence of ad viewing across outlets, $\eta_{gj} = \delta\eta_{gk}$, $\delta \ge 1$, $X^g_{\mathcal{J}\setminus\{k\}} \ge X^g_{\mathcal{J}\setminus\{j,k\}}$, and the fact that β_m is nonincreasing in m for $m \ge 1$.

To show strict inequality, suppose that $\eta_{gj}\prod_{l\neq j,k}(1-\eta_{gl}) > 0$, $\delta > 1$, and $\beta_1 > \beta_2$. For any group g, using integration by parts and the definition of β_m, we have

$$\begin{aligned}\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right] - \mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}] &= \int_0^\infty \mathbb{P}(\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1} > s)ds - \int_0^\infty \mathbb{P}(\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1} > s)ds \\ &= \sum_{m=1}^{\infty} (\beta_m - \beta_{m+1}) \left(\mathbb{P}(X^g_{\mathcal{J}\setminus\{j,k\}} + 1 \le m) - \mathbb{P}(X^g_{\mathcal{J}\setminus\{k\}} + 1 \le m)\right) > 0\end{aligned}$$

where the strict inequality follows from that each term in the summation is nonnegative, $\beta_1 > \beta_2$, and $\mathbb{P}\left(X^g_{\mathcal{J}\setminus\{j,k\}} = 0\right) - \mathbb{P}\left(X^g_{\mathcal{J}\setminus\{k\}} = 0\right) = \eta_{gj}\prod_{l\neq j,k}\left(1-\eta_{gl}\right) > 0$. Since $\delta > 1$, we then have $(\delta-1)\mathbb{E}\left[\beta_{X^g_{\mathcal{J}\setminus\{j,k\}}+1}\right] > (\delta-1)\mathbb{E}[\beta_{X^g_{\mathcal{J}\setminus\{k\}}+1}]$ and hence $p^*_j/\lambda_j > p^*_k/\lambda_k$. □

A.3.2. Endogenous Response of Viewers

There is a set of programs $\mathcal{K}$. Each outlet consists of K programs. Each program has one ad slot. Each advertiser that purchases an ad slot on a given outlet is randomly assigned to the slot in one of the outlet's programs. There is a set of viewers $\mathcal{I}$. Each viewer views a subset of programs. Whether a given viewer views a given program depends on whether that program carries an ad, but not on whether other programs do. Thus, as in Ambrus, Calvano, and Reisinger (2016, Section II), advertising may be a nuisance (or attraction) that affects a given outlet's audience, but advertising on one outlet does not drive audience to (or attract audience from) another. For each viewer that views its ad on $M \in \mathbb{N}$ distinct outlets, each advertiser gets value $u(M) \geq 0$ where $u(\cdot)$ is nondecreasing and exhibits decreasing differences.

Proposition 6. *There exists an equilibrium. In any equilibrium, all advertisers buy slots on all outlets, and the payment by each advertiser to each owner Z is given by $p^*_Z = v_Z$.*

Proof. By Theorem 1, it suffices to show that the value function $V(\cdot)$, in this setting, satisfies monotonicity and submodularity. Let $k \in \mathcal{K}$ denote a generic program and let $\mathcal{K}_j \subseteq \mathcal{K}$ be the programs associated with outlet j. For each viewer $i \in \mathcal{I}$, let $i \dashrightarrow k$ denote the event that viewer i watches program k in the scenario that program k carries an ad. This event may be random if viewing behavior is probabilistic. For a set of programs $\mathcal{K}' \subseteq \mathcal{K}$, let

$$X^i_{\mathcal{K}'} = \sum_{k\in\mathcal{K}'} \mathbf{1}_{i\dashrightarrow k}$$

be the (possibly random) number of programs in $\mathcal{K}'$ watched by i in the scenario that each program $k \in \mathcal{K}'$ carries an ad. Let

$$\mathcal{R} = \left\{\mathcal{K}' \subseteq \mathcal{K} : |\mathcal{K}' \cap \mathcal{K}_j| = 1 \text{ for all } j \in \mathcal{J}\right\}$$

be the set of all sets of representative programs, such that within each set there is one program for each outlet. For any set of outlets $S \subseteq \mathcal{J}$, let $\mathcal{K}_S = \cup_{j\in S}\mathcal{K}_j$. For each advertiser, the value of a set of outlets $S \subseteq \mathcal{J}$ can be written as

$$V(S) = |\mathcal{I}| \cdot \mathbb{E}\left[\frac{1}{|\mathcal{R}|}\sum_{\mathcal{K}'\in\mathcal{R}} u\left(X^i_{\mathcal{K}'\cap\mathcal{K}_S}\right)\right]$$

where the expectation is taken with respect to the population of viewers and the (possibly random) viewing behavior of each viewer. Because averaging preserves monotonicity and submodularity, it suffices to show that for any realized viewing behavior and representative programs $\mathcal{R}$, and any $\mathcal{K}' \in \mathcal{R}$, we have that the function

$$\tilde{V}_i(S) := u\left(X^i_{\mathcal{K}' \cap \mathcal{K}_S}\right)$$

is monotone and submodular. It is clear that $\tilde{V}_i(S)$ is monotone since $u(\cdot)$ is nondecreasing. For submodularity, note that for any $S \subseteq T \subseteq \mathcal{J}$ and any $j \in \mathcal{J} \backslash T$,

$$\begin{aligned}
\tilde{V}_i(S \cup \{j\}) - \tilde{V}_i(S) &= u\left(X^i_{\mathcal{K}' \cap \mathcal{K}_{S \cup \{j\}}}\right) - u\left(X^i_{\mathcal{K}' \cap \mathcal{K}_S}\right) \\
&\geq u\left(X^i_{\mathcal{K}' \cap \mathcal{K}_{T \cup \{j\}}}\right) - u\left(X^i_{\mathcal{K}' \cap \mathcal{K}_T}\right) \\
&= \tilde{V}_i(T \cup \{j\}) - \tilde{V}_i(T)
\end{aligned}$$

where the second line follows from the assumption that $u(\cdot)$ has decreasing differences. □

A.3.3. Rationing of Ad Slots

Suppose that we may have $N > K$ and assume that bundle prices can only take on values in the set $\{0, \Delta, 2\Delta, \cdots\}$ where $\Delta > 0$ is some fixed increment. For this extension, we allow for mixed strategies and assume that the advertisers make purchasing decisions sequentially (in a random order) rather than simultaneously.

Proposition 7. *There exists a subgame perfect Nash equilibrium, possibly in mixed strategies, and in any subgame perfect Nash equilibrium each owner Z earns an expected revenue per ad slot between $(v_Z - \Delta)/|Z|$ and $\sum_{j \in Z} V(\{j\})/|Z|$.*

Proof. We prove the second part of the statement first. Fix any subgame perfect equilibrium allowing for mixed strategies (SPEMS). Suppose, toward contradiction, that expected revenue per slot is strictly higher than $\sum_{j \in Z} V(\{j\})/|Z|$ for some owner $Z \in \mathcal{Z}$. Then the owner's expected total revenue is strictly higher than $K \sum_{j \in Z} V(\{j\})$. Thus with positive probability, the owner earns a realized revenue strictly higher than $K \sum_{j \in Z} V(\{j\})$. In any such event, there is at least one advertiser who buys slots on a set of outlets $B \subseteq Z$ and pays strictly more than $\sum_{j \in B} V(\{j\})$ to the owner. Let $S \subseteq \mathcal{J}$ be the set of outlets that this advertiser buys slots on. Since any non-negative submodular function is also sub-additive, we have that

$$V(S) - V(S \backslash B) \leq V(B) - V(\varnothing) \leq \sum_{j \in B} V(\{j\}).$$

So not buying anything from Z is a profitable deviation for the advertiser, implying a contradiction.

Next, toward contradiction suppose that the expected revenue per slot is strictly lower than $(v_Z - \Delta) / |Z|$ for some owner $Z \in \mathcal{Z}$. Then the expected total revenue is strictly lower than $K(v_Z - \Delta)$. Let the owner deviate by offering a single bundle Z with a price $\tilde{p}_Z = \lceil v_Z - \Delta \rceil$, where $\lceil x \rceil$ denotes the operator that rounds x up to the closest value in $\{0, \Delta, 2\Delta, \cdots\}$. Note that

$$v_Z - \Delta \leq \lceil v_Z - \Delta \rceil < v_Z.$$

Since $\tilde{p}_Z < v_Z$, by an argument analogous to the one in the proof of Theorem 1, submodularity of $V(\cdot)$ implies that, in any realization, the owner would be able to sell all the slots and secure revenue $K\tilde{p}_Z$. Because this is a profitable deviation for the owner, we have a contradiction.

To show the existence of a SPEMS, we construct an auxiliary finite game in normal form, apply the standard existence result, and then recover a SPEMS in the original game. Consider a simultaneous-move game between all the owners. Let

$$\mathcal{A}(Z) = \{0, \Delta, 2\Delta, \cdots, \lceil V(\mathcal{J}) \rceil, \infty\}^{|\mathcal{P}(Z)|}$$

be the set of pure strategies that an owner can choose from. Clearly, $\mathcal{A}(Z)$ is finite for any Z. For each pure strategy profile $p \in \mathcal{A}(Z)$, draw a random order for the advertisers and then let the advertisers, in that order, choose which slots to buy given the posted prices specified in p. Then, for each $p \in \mathcal{A}(Z)$, assign the resulting expected revenue (averaged over different orders) for owner Z as the payoff to owner Z in the auxiliary game. This constructs a finite normal-form game among the owners. Standard results then imply the existence of a Nash equilibrium, possibly in mixed strategies. Call this equilibrium $\mathcal{E}$. Now let each owner play the strategy prescribed by $\mathcal{E}$ in the original game. Evidently, this constructs a SPEMS for the original game. $\square$

A.3.4. Partially Increasing Returns

Theorem 1 relies on submodularity of $V(\cdot)$. Under strict monotonicity the conclusion of Theorem 1 obtains under a weakening of submodularity.

Proposition 8. *Suppose that $V(\cdot)$ is strictly monotone and that $V(S \cup Z) - V(S) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash Z)$ for all $Z \in \mathcal{Z}$ and $S \subseteq \mathcal{J} \backslash Z$. Then there exists an equilibrium, and in any equilibrium, all advertisers buy slots on all outlets, and the payment by each advertiser to each owner Z is given by $p_Z^* = v_Z$.*

Proof. We first construct an equilibrium. We use the same construction as in the proof of Theorem 1. When verifying the construction, the only properties of $V(\cdot)$ used in the proof of Theorem 1 are that $V(\cdot)$ is monotone and that for any $S \subseteq \mathcal{J} \backslash Z$,

$$V(S \cup Z) - V(S) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash Z)$$

which we assume.

To prove the second part of the statement, fix any equilibrium of the game. By Lemma 1(ii) and strict monotonocity of $V(\cdot)$, all advertisers buy slots on all outlets in $\mathcal{J}$. The rest follows analogously to the proof of Theorem 1. □

The decreasing differences condition on $V(\cdot)$ in the hypothesis of Proposition 8 is strictly weaker than submodularity. In particular, consider the following example.

Example 1. Owners are singletons, each of a set $\mathcal{I}$ of viewers $i \in \mathcal{I}$ sees ads on at least L outlets, each outlet has a strictly positive number of viewers, and an advertiser's value for viewer i seeing its ad M times is $a_i u(M) = a_i \sum_{m=0}^{M} \beta_m$ where $a_i > 0$ for all i, $\beta_0 = 0, \beta_m > 0$ for all m, β_m is non-increasing for all $m \geq L$, and $\beta_L \leq \min_{1 \leq m < L} \beta_m$.

Example 1 allows increasing returns to advertising for viewers receiving few impressions (as in, e.g., Dubé, Hitsch, and Manchanda 2005). Although Example 1 need not satisfy the hypotheses of Theorem 1, Example 1 does satisfy the hypotheses of Proposition 8.

Proposition 9. *The value function $V(\cdot)$ in Example 1 satisfies the hypotheses of Proposition 8.*

Proof. For any viewer $i \in \mathcal{I}$ and outlet $j \in \mathcal{J}$, let $i \rightarrow j$ denote the event that viewer i watches outlet j. For any viewer $i \in \mathcal{I}$ and set of outlets $S \subseteq \mathcal{J}$, let X_S^i denote the number of outlets in set S on which viewer i views ads.We have that

$$V(S) = |\mathcal{I}| \cdot \mathbb{E}\left[a_i u\left(X_S^i\right)\right] = |\mathcal{I}| \cdot \mathbb{E}\left[V_i(S)\right]$$

where the expectation is taken with respect to the population of viewers and their (possibly random) viewing behavior. To show strict monotonicity, observe that for any $j \in \mathcal{J}$ and any $S \subseteq \mathcal{J} \backslash \{j\}$,

$$V_i(S \cup \{j\}) - V_i(S) = a_i \mathbf{1}_{i \rightarrow j} \beta_{X_S + 1}$$

is weakly positive, and strictly so with positive probability. To show that

$$V(S \cup Z) - V(S) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash Z)$$

for all $Z \in \mathcal{Z}$, $S \subseteq \mathcal{J} \backslash Z$, because owners are singletons, it suffices to show that for any j and any

$S \subseteq \mathcal{J} \setminus \{j\}$, we have

$$V_i(S \cup \{j\}) - V_i(S) = a_i \mathbf{1}_{i \to j} \beta_{X_S+1} \geq a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J}\setminus\{j\}}+1} = V_i(\mathcal{J}) - V_i(\mathcal{J} \setminus \{j\}).$$

To see the above, consider the event $i \to j$. Then, $X_{\mathcal{J}\setminus\{j\}} + 1 = X_{\mathcal{J}} \geq L$. Note that $X_S + 1 = X_{S\cup\{j\}} \leq X_{\mathcal{J}}$. If $X_{S\cup\{j\}} \geq L$, then $\beta_{X_{S\cup\{j\}}} \geq \beta_{X_{\mathcal{J}}}$ because β_m is non-increasing for $m \geq L$. If $X_{S\cup\{j\}} < L$, then we have

$$\beta_{X_{\mathcal{J}}} \leq \beta_L \leq \min_{1 \leq m < L} \beta_m \leq \beta_{X_{S\cup\{j\}}}.$$

So in either case, the claimed inequality holds. □

The setting of Example 1 continues to satisfy the hypotheses of Proposition 8 if a small number of viewers view fewer than L outlets.

Example 2. Owners are singletons. There is a set of viewers $\overline{\mathcal{I}}$. There is a subset of viewers $\mathcal{I} \subseteq \overline{\mathcal{I}}$ such that (i) each viewer $i \in \mathcal{I}$ sees ads on at least L outlets and (ii) each viewer $i \in \mathcal{I}$ sees ads on both of any pair of outlets $\{j,k\} \in \mathcal{J}^2$ with strictly positive probability. An advertiser's value for viewer i seeing its ad M times is $a_i u(M) = a_i \sum_{m=0}^{M} \beta_m$ where $a_i \in (0, \overline{a})$ for all i, $\beta_0 = 0, \beta_m > 0$ for all m, β_m is strictly decreasing for all $m \geq L$, and $\beta_L < \min_{1 \leq m < L} \beta_m$.

Proposition 10. *There exists $\overline{\varepsilon} > 0$ such that the value function $V(\cdot)$ in Example 2 satisfies the hypotheses of Proposition 8 as long as the share of viewers not in $\mathcal{I}$ is no more than $\overline{\varepsilon}$.*

Proof. Strict monotonicity follows by the same argument as in the proof of Proposition 9. Now, fix any $j \in \mathcal{J}$, any $S \subset \mathcal{J} \setminus \{j\}$, and any $k \in \mathcal{J} \setminus (S \cup \{j\})$. Because each viewer $i \in \mathcal{I}$ sees ads on both of any pair of outlets $\{j,k\} \in \mathcal{J}^2$ with strictly positive probability, following the notation in the proof of Proposition 9 we have that

$$\mathbb{P}\left(i \to j, X_S^i < X_{\mathcal{J}\setminus\{j\}}^i | i \in \mathcal{I}\right) > 0.$$

Thus, since β_m is strictly decreasing for all $m \geq L$, and $\beta_L < \min_{1 \leq m < L} \beta_m$, we have

$$\mathbb{E}\left[a_i \mathbf{1}_{i \to j} \beta_{X_S^i+1} | i \in \mathcal{I}\right] > \mathbb{E}\left[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J}\setminus\{j\}}^i+1} | i \in \mathcal{I}\right].$$

Now let

$$\tau := \min_{j \in \mathcal{J}, S \subset \mathcal{J}\setminus\{j\}} \left(\mathbb{E}\left[a_i \mathbf{1}_{i \to j} \beta_{X_S^i+1} | i \in \mathcal{I}\right] - \mathbb{E}\left[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J}\setminus\{j\}}^i+1} | i \in \mathcal{I}\right]\right) > 0.$$

Let $\overline{\beta} = \max_{1 \leq m < L} \beta_m$. Let

$$\overline{\varepsilon} = \frac{\tau}{\tau + \overline{a}\overline{\beta}} > 0.$$

We claim that if $\mathbb{P}(i \notin \mathcal{I}) \leq \overline{\varepsilon}$ then $V(\cdot)$ satisfies that $V(S \cup Z) - V(S) \geq V(\mathcal{J}) - V(\mathcal{J} \backslash Z)$ for all $Z \in \mathcal{Z}, S \subseteq \mathcal{J} \backslash Z$. Because owners are singletons, it suffices to consider any $j \in \mathcal{J}$ and any $S \subset \mathcal{J} \backslash \{j\}$. Note that

$$\begin{aligned}
V(S \cup \{j\}) - V(S) &\geq |\overline{\mathcal{I}}| \cdot \mathbb{P}(i \in \mathcal{I}) \mathbb{E}[a_i \mathbf{1}_{i \to j} \beta_{X_S+1} | i \in \mathcal{I}] \\
&\geq |\overline{\mathcal{I}}| \cdot \mathbb{P}(i \in \mathcal{I}) \left(\mathbb{E}[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J} \backslash \{j\}}+1} | i \in \mathcal{I}] + \tau \right) \\
&\geq |\overline{\mathcal{I}}| \cdot \left(\mathbb{P}(i \in \mathcal{I}) \mathbb{E}[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J} \backslash \{j\}}+1} | i \in \mathcal{I}] + \mathbb{P}(i \notin \mathcal{I}) \overline{a} \overline{\beta} \right) \\
&\geq |\overline{\mathcal{I}}| \cdot \left(\mathbb{P}(i \in \mathcal{I}) \mathbb{E}[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J} \backslash \{j\}}+1} | i \in \mathcal{I}] + \mathbb{P}(i \notin \mathcal{I}) \mathbb{E}[a_i \mathbf{1}_{i \to j} \beta_{X_{\mathcal{J} \backslash \{j\}}+1} | i \notin \mathcal{I}] \right) \\
&= V(\mathcal{J}) - V(\mathcal{J} \backslash \{j\}),
\end{aligned}$$

where the second inequality follows from the construction of τ and the third inequality follows from the fact that $\mathbb{P}(i \notin \mathcal{I}) \leq \overline{\varepsilon}$. □

Lastly, we show that analogues of Propositions 1 and 2 hold in a special case of Example 1 that is analogous to the setting of Section I.B.

Proposition 11. *Consider a special case of Example 1 with $a_i = a$ for all i, and further impose the structure in Section I.B, where for each group $g \in G$, there is a set $\mathcal{L}_g \subseteq \mathcal{J}$ such that $|\mathcal{L}_g| \geq L$ and $\eta_{gj} = 1$ for all $j \in \mathcal{L}_g$. Then the conclusions of Propositions 1 and 2 hold.*

Proof. We follow the same arguments and notation as in the proofs of Propositions 4 and 5. By Propositions 8 and 9, we have only to characterize the incremental value for each owner Z.

For the conclusion of Proposition 1, recall that we use a monotone coupling in the proof of Proposition 4. Under that coupling we have $X^g_{\mathcal{J} \backslash \{j,k\}} \leq X^h_{\mathcal{J} \backslash \{j,k\}}$ pointwise, where we recall that $X^g_{\mathcal{J} \backslash \{j,k\}}$ counts the random number of outlets not in $\{j,k\}$ on which viewer g sees ads. It follows that $X^g_{\mathcal{J} \backslash \{j,k\}}, X^h_{\mathcal{J} \backslash \{j,k\}} \geq L-2$ and so $X^g_{\mathcal{J} \backslash \{j,k\}}+1, X^h_{\mathcal{J} \backslash \{j,k\}}+1 \geq L-1$. Since $\beta_L \leq \min_{1 \leq m < L} \beta_m \leq \beta_{L-1}$ and β_m is non-increasing in m for $m \geq L$, β_m is non-increasing in m for $m \geq L-1$. Since $X^g_{\mathcal{J} \backslash \{j,k\}}+1 \leq X^h_{\mathcal{J} \backslash \{j,k\}}+1$ pointwise, we have $\beta_{X^g_{\mathcal{J} \backslash \{j,k\}}+1} \geq \beta_{X^h_{\mathcal{J} \backslash \{j,k\}}+1}$ pointwise and so $\mathbb{E}\left[\beta_{X^g_{\mathcal{J} \backslash \{j,k\}}+1}\right] \geq \mathbb{E}\left[\beta_{X^h_{\mathcal{J} \backslash \{j,k\}}+1}\right]$, which concludes the proof as before.

For the conclusion of Proposition 2, recall that $X^g_{\mathcal{J} \backslash \{k\}}$ counts the random number of outlets other than k on which viewer g sees ads. So $X^g_{\mathcal{J} \backslash \{k\}}+1 \geq X^g_{\mathcal{J} \backslash \{j,k\}}+1 \geq L-1$. Because β_m is non-increasing in m for $m \geq L-1$, we have $\beta_{X^g_{\mathcal{J} \backslash \{j,k\}}+1} \geq \beta_{X^g_{\mathcal{J} \backslash \{k\}}+1}$ pointwise and so $\mathbb{E}\left[\beta_{X^g_{\mathcal{J} \backslash \{j,k\}}+1}\right] \geq \mathbb{E}\left[\beta_{X^g_{\mathcal{J} \backslash \{k\}}+1}\right]$, which concludes the proof as before. □

A.3.5. Heterogeneous Advertisers

Suppose now that each advertiser $n \in \mathcal{N}$ has a monotone and submodular value function $V_n(\cdot)$. If outlets can post advertiser-specific prices, then the result is parallel to that in Theorem 1, in the sense that the equilibrium price of owner Z's bundle to advertiser n is given by $v_{n,Z} = V_n(\mathcal{J}) - V_n(\mathcal{J} \setminus Z)$. If outlets cannot post advertiser-specific prices, then incremental pricing holds if heterogeneity among the advertisers is sufficiently small compared to the incremental value of a single outlet. Let $\underline{v}_Z = \min_{n \in \mathcal{N}} v_{n,Z}$ and $\bar{v}_Z = \max_{n \in \mathcal{N}} v_{n,Z}$ denote the minimum and maximum values of $v_{n,Z}$, respectively, with respect to n. Let $\varphi(Z) = \min_{n \in \mathcal{N}, j \in Z} V_n((\mathcal{J} \backslash Z) \cup \{j\}) - V_n(\mathcal{J} \backslash Z)$ denote the minimal incremental value of any one of owner Z's outlets. In the special case where Z is a single-outlet owner, $\varphi(Z) = \underline{v}_Z$.

Proposition 12. *Suppose that heterogeneity in the value functions $V_n(\cdot)$ is small in the sense that $\bar{v}_Z - \underline{v}_Z \leq \frac{1}{N}\varphi(Z)$ for all $Z \in \mathcal{Z}$. Then there exists an efficient equilibrium, and in any efficient equilibrium, all advertisers buy slots on all outlets, and $p_Z^* = \underline{v}_Z$ for all $Z \in \mathcal{Z}$.*

Proof. As in the proof of Theorem 1, we first construct an equilibrium. For any profile p of posted prices (including those off of the equilibrium path), let p_S denote the minimum price to buy slots on all the outlets in S. Now let every advertiser n buy the bundle $S_n^*(p)$, chosen arbitrarily from the set of solutions to the problem

$$\max |S|$$
$$s.t.\, S \in \operatorname*{argmax}_{S \subseteq \mathcal{J}} V_n(S) - p_S$$

By construction, $(S_1^*(\cdot), \ldots, S_N^*(\cdot))$ constitutes an equilibrium strategy profile for the advertisers.

Now, let each owner Z offer a single bundle consisting of all outlets in Z with a price $p_Z = \underline{v}_Z$. We only need to check that each owner has no profitable deviation. Observe that if $p_Z = \underline{v}_Z$ is offered by some owner Z and there is no proper subset $W \subset Z$ being offered, then any advertiser will buy the bundle Z regardless of the prices posted by owners other than Z. This is because for any $S \subseteq \mathcal{J} \backslash Z$, submodularity of $V_n(\cdot)$ implies that

$$V_n(S \cup Z) - V_n(S) \geq V_n(\mathcal{J}) - V_n(\mathcal{J} \backslash Z) \geq \min_{n' \in \mathcal{N}} V_{n'}(\mathcal{J}) - V_{n'}(\mathcal{J} \backslash Z) = \underline{v}_Z.$$

Fix an owner $Z \in \mathcal{Z}$ and suppose all other players follow the proposed strategy. We claim that offering a single bundle Z with a price $\underline{v}_Z$ is an optimal strategy for owner Z. To see this, consider two cases.

Case 1: Suppose Z offers some set of bundles $\mathcal{B}_Z$ such that every advertiser buys a slot on every outlet in Z. Then the minimal price to buy all outlets in Z must be no more than $\underline{v}_Z$ because otherwise there is one advertiser who can profitably deviate by simply not buying anything in $\mathcal{B}_Z$. Hence in this case the owner cannot do better than simply offering the bundle Z with a price $\underline{v}_Z$.

Case 2: Suppose Z offers some set of bundles $\mathcal{B}_Z$ such that there exist at least one outlet $j \in Z$ and one advertiser $\tilde{n} \in \mathcal{N}$ such that advertiser $\tilde{n}$ does not buy a slot on outlet j. The total revenue that owner Z extracts is no more than

$$\max_{n,j} \left\{ \sum_{n' \neq n} v_{n',Z} + v_{n,Z \setminus \{j\}} \right\}$$

where $v_{n,Z\setminus\{j\}} = V_n(\mathcal{J}) - V_n(\mathcal{J} \setminus (Z \setminus \{j\}))$ is the incremental value to advertiser n of the owner's outlets excluding outlet j. We also have that

$$\begin{aligned}
\max_{n,j} \left\{ \sum_{n' \neq n} v_{n',Z} + v_{n,Z \setminus \{j\}} \right\} \leq & \max_{n,j} \left\{ N\overline{v}_Z - \left(v_{n,Z} - v_{n,Z\setminus\{j\}} \right) \right\} \\
= & N\overline{v}_Z - \min_{n,j} \left\{ V_n \left((\mathcal{J} \setminus Z) \cup \{j\} \right) - V_n \left(\mathcal{J} \setminus Z \right) \right\} \\
= & N\overline{v}_Z - \varphi(Z) \\
\leq & N\overline{v}_Z - N(\overline{v}_Z - \underline{v}_Z) = N\underline{v}_Z
\end{aligned}$$

where we have used the assumption that $\overline{v}_Z - \underline{v}_Z \leq \frac{1}{N}\varphi(Z)$. Hence the owner also cannot do better than simply offering the bundle Z with a price $\underline{v}_Z$.

Thus the construction is an equilibrium. The outcome is efficient because all advertisers buy slots on all outlets.

To prove the second part of the statement, fix any efficient equilibrium. All outlets must sell N slots, because the preferences for each player are quasilinear in money and thus the total surplus is maximized only if all potential trades are realized (recall $K \geq N$). Then by the argument in Case 1, we know that $p_Z^* \leq \underline{v}_Z$ for all $Z \in \mathcal{Z}$. Moreover, p_Z^* cannot be strictly lower than $\underline{v}_Z$ for any owner Z, because if this were the case then it would be a profitable deviation for owner Z to offer a single bundle Z with a price $\underline{v}_Z - \varepsilon$ for $\varepsilon > 0$ small enough. Hence $p_Z^* = \underline{v}_Z$ for all $Z \in \mathcal{Z}$. □

Note that the hypothesis of Proposition 12 restricts the incremental values rather than the level of $V_n(\cdot)$, in the sense that it allows for $V_n(\cdot) = V(\cdot) + c_n$ for any c_n that preserves non-negativity.[1] The restriction on incremental values becomes more demanding as the number of advertisers, N, grows large.

[1]When there are two or more owners, it also allows for $V_n(\varnothing) = V(\varnothing)$ and $V_n(\mathcal{J}') = V(\mathcal{J}') + c_n$ for $\varnothing \neq \mathcal{J}' \subseteq \mathcal{J}$, where $c_n \geq 0$.

A.3.6. Unbundled Pricing

Suppose that each owner $Z \in \mathcal{Z}$ is endowed with a partition $\mathcal{F}_Z$ of Z such that owner Z can offer a bundle $B \subseteq Z$ if and only if there exists $C \in \mathcal{F}_Z$ such that $B \subseteq C$. For example, if $\mathcal{F}_Z$ partitions owner Z's outlets into singletons, then the owner cannot bundle together ads on different outlets. Denote by $v_B^S = V(S) - V(S \setminus B)$ the incremental value of bundle $B \subseteq S$ relative to set $S \subseteq \mathcal{J}$. We refine the notion of equilibrium by assuming that, when indifferent, owners break ties in favor of offering fewer bundles, and each advertiser breaks ties by favoring owners according to a prespecified ordering.

Proposition 13. *In any equilibrium satisfying the tie-breaking rule, each bundle sold has a price of $p_B^* = v_B^S$, where $S \subseteq \mathcal{J}$ is the set of all outlets sold.*

Proof. Let $\mathcal{O}_n$ denote advertiser n's tie-breaking ordering over owners; that is, if indifferent among one or more sets of bundles, advertiser n chooses in a manner that maximizes the payoffs of the owners according to a lexicographic preference over owners defined by $\mathcal{O}_n$.

If in some equilibrium advertiser n' obtains a strictly greater payoff than advertiser n, advertiser n can improve their payoff by mimicking the strategy of advertiser n'. It follows that in any equilibrium all advertisers must obtain the same payoff.

Now, fix some equilibrium with advertiser payoff W and pricing profile p. For any set $S \subseteq \mathcal{J}$, let p_S be the minimum price to buy slots on all the outlets in S in the equilibrium.

We first prove that, analogous to Lemma 1(i), in the equilibrium all advertisers pay the same total amount to any given owner Z. Suppose toward contradiction that there exists some owner $Z \in \mathcal{Z}$ such that not all advertisers make the same total payment to Z. Let n be an advertiser whose payment to owner Z in the supposed equilibrium is at least as large as any other advertiser's, and strictly larger than some other advertiser's. Let S_C be the set of outlets that advertiser n buys slots on in the cell $C \in \mathcal{F}_Z$. Let owner Z deviate by offering the bundles $\mathcal{B} := \{S_C : S_C \neq \varnothing, C \in \mathcal{F}_Z\}$ with prices $\{p_{S_C} - \varepsilon : S_C \neq \varnothing, C \in \mathcal{F}_Z\}$ for some $\varepsilon > 0$. Note that $|\mathcal{B}| \geq 1$ since advertiser n pays a positive amount to owner Z. By buying all the bundles offered in this deviation of Z (and imitating advertiser n's choices at the on-path history in the supposed equilibrium), any advertiser can obtain a payoff of $W + \varepsilon|\mathcal{B}|$. Because any set of outlets an advertiser wants to buy slots on after this deviation is also a valid choice at the on-path history, if an advertiser does not buy all the bundles in $\mathcal{B}$, then the advertiser gets at most $W + \varepsilon(|\mathcal{B}| - 1)$. Hence after this deviation, all advertisers buy the bundles in $\mathcal{B}$ offered by Z. Because this is a profitable deviation for owner Z when ε is small enough, we have a contradiction.

We next prove that every advertiser must buy the same set of bundles from any given owner Z and that owner Z offers at most one bundle from each cell in $\mathcal{F}_Z$. Fix any owner $Z \in \mathcal{Z}$ and any advertiser $n \in \mathcal{N}$. Let S_C be the set of outlets that advertiser n buys slots on in the cell $C \in \mathcal{F}_Z$. Suppose that owner Z offers the bundles $\mathcal{B} := \{S_C : S_C \neq \varnothing, C \in \mathcal{F}_Z\}$ with prices $\left\{p_{S_C} : S_C \neq \varnothing, C \in \mathcal{F}_Z\right\}$. We claim that owner Z weakly increases their payoff with this strategy compared to any equilibrium strategy. For each advertiser, this change restricts the set of possible choices while keeping at least one choice that maintains the equilibrium payoff (imitating the choice of advertiser n at the on-path history in the equilibrium). Because at the on-path history all advertisers pay the same total amount to Z, this change can only decrease owner Z's payoff if there is some advertiser n' (not necessarily different from n) who now breaks ties in favor of some owner that ranks higher than Z in $\mathcal{O}_{n'}$. However, that choice must also be made at the on-path history by advertiser n' due to the tie-breaking rule. But then advertiser n' pays strictly less than advertiser n to owner Z in the equilibrium, which contradicts what we previously showed.

Because an owner chooses to offer fewer bundles when indifferent, the above observation implies that every advertiser must buy the same set of bundles from any given owner Z and that owner Z offers at most one bundle from each cell in $\mathcal{F}_Z$. (Otherwise, owner Z may simply pick an advertiser n who buys the smallest number of bundles from Z and offer the set of bundles $\mathcal{B}$ as defined above to strictly decrease the total number of bundles offered without decreasing payoff.) Then all advertisers buy slots on the same set of outlets (say S) and any owner Z offers $\mathcal{B}_Z := \{S \cap B : S \cap B \neq \varnothing, B \in \mathcal{F}_Z\}$ as the available bundles.

Therefore, in the second stage, the set of feasible bundles that advertisers can choose

$$\{S \cap B : S \cap B \neq \varnothing, B \in \mathcal{F}_Z\}_{Z \in \mathcal{Z}}$$

must be a partition of the set S. For any bundle B offered by any owner Z, by rationality of the advertisers,

$$p_B \leq V(S) - V(S \backslash B) = v_B^S.$$

Next we show that $p_B \geq v_B^S$. Suppose toward contradiction that there exist some owner Z and some bundle $B' \in \mathcal{F}_Z$, $B' \subseteq S$ such that $p_{B'} < v_{B'}^S$. Consider the following deviation. Let owner Z offer all bundles in $\mathcal{B}_Z$ as in the equilibrium but change the price for each bundle B to $\tilde{p}_B = v_B^S - \varepsilon$ for some $\varepsilon > 0$. We claim that after this deviation, all advertisers continue buying slots on the same outlets from owner Z as in the equilibrium. Indeed, if an advertiser stops buying some bundle $B \in \mathcal{B}_Z$, then the advertiser can only choose $S' \subseteq S \backslash B$ since the set of available bundles is a partition of S. But submodularity of $V(\cdot)$ implies

$$V\left(S' \cup B\right) - V\left(S'\right) \geq V(S) - V(S \backslash B) = v_B^S > \tilde{p}_B.$$

Therefore owner Z can extract $v_B^S - \varepsilon$ for each bundle $B \in \mathcal{B}_Z$ from each advertiser. For ε sufficiently small, this is then a profitable deviation for owner Z since in the equilibrium we have

$p_B \leq v_B^S$ for all $B \in \mathcal{B}_Z$ and $p_{B'} < v_{B'}^S$ for some bundle $B' \in \mathcal{B}_Z$. We therefore have contradiction. □

A.3.7. Bargaining between Owners and Advertisers

Suppose that rather than simultaneously posting prices, owners bargain with advertisers a la Nash-in-Nash (Lee, Whinston, and Yurukoglu 2021). We follow the notation in Lee, Whinston, and Yurukoglu (2021). For each owner Z and each advertiser n, let

$$\mathcal{C}_{Zn} := \{(B,p) : B \subseteq Z, p \in \mathbb{R}_+\}$$

be the contract space, with an element denoted by $\mathbb{C}_{Zn}$. For a contract $\mathbb{C}_{Zn}$, let $B(\mathbb{C}_{Zn})$ and $p(\mathbb{C}_{Zn})$ denote the associated bundle and price. Let $\mathbb{C}_0 = \{(\varnothing, 0)\}$ denote the null contract. For a given set of contracts $\mathbb{C} := \{\mathbb{C}_{Zn}\}_{Z \in \mathcal{Z}, n=1,\ldots,N}$, owner Z's payoff is given by

$$\Pi_Z(\mathbb{C}) = \sum_n p(\mathbb{C}_{Zn})$$

and advertiser n's payoff is given by

$$\Pi_n(\mathbb{C}) = V\left(\bigcup_{Z \in \mathcal{Z}} B(\mathbb{C}_{Zn})\right) - \sum_{Z \in \mathcal{Z}} p(\mathbb{C}_{Zn}).$$

Given the set of contracts $\mathbb{C}_{-Zn}$ excluding pair (Z,n), let

$$\mathcal{C}_{Zn}^{+}(\mathbb{C}_{-Zn}) = \{\mathbb{C}_{Zn} \in \mathcal{C}_{Zn} : \Pi_n(\{\mathbb{C}_{Zn}, \mathbb{C}_{-Zn}\}) - \Pi_n(\{\mathbb{C}_0, \mathbb{C}_{-Zn}\}) \geq 0\}$$

be the set of contracts between Z and n that give non-negative gains from trade to owner Z and advertiser n (note that only the constraint for the advertiser is relevant as any contract would give non-negative gains from trade to owner Z). Recall that a set of contracts $\hat{\mathbb{C}}$ is a *Nash-in-Nash equilibrium* if:

(i) For all Z, n such that $\hat{\mathbb{C}}_{Zn} \neq \mathbb{C}_0$,

$$\hat{\mathbb{C}}_{Zn} \in \underset{\mathbb{C}_{Zn} \in \mathcal{C}_{Zn}^{+}(\hat{\mathbb{C}}_{-Zn})}{\operatorname{argmax}} [\Pi_Z(\{\mathbb{C}_{Zn}, \hat{\mathbb{C}}_{-Zn}\}) - \Pi_Z(\{\mathbb{C}_0, \hat{\mathbb{C}}_{-Zn}\})]^{\xi_Z} [\Pi_n(\{\mathbb{C}_{Zn}, \hat{\mathbb{C}}_{-Zn}\}) - \Pi_n(\{\mathbb{C}_0, \hat{\mathbb{C}}_{-Zn}\})]^{1-\xi_Z},$$

where $\xi_Z \in [0,1]$ denotes the bargaining weight for owner Z.

(ii) For all Z, n such that $\hat{\mathbb{C}}_{Zn} = \mathbb{C}_0$, there is no contract in $\mathcal{C}_{Zn}^{+}(\hat{\mathbb{C}}_{-Zn})$ that gives strictly positive gains from trade to both Z and n.

Proposition 14. *If all owners have identical bargaining weights, there exists a Nash-in-Nash equilibrium in which all advertisers buy slots on all outlets, and the payment by each advertiser to each owner Z is proportional to v_Z. If $V(\cdot)$ is strictly monotone, then this outcome is unique.*

Proof. We first show that $\hat{\mathbb{C}} := \{(Z, \xi_Z v_Z)\}$ is a Nash-in-Nash equilibrium. Condition (ii) clearly holds. For (i), note that

$$\Pi_Z(\{\mathbb{C}_{Zn}, \hat{\mathbb{C}}_{-Zn}\}) - \Pi_Z(\{\mathbb{C}_0, \hat{\mathbb{C}}_{-Zn}\}) = p(\mathbb{C}_{Zn})$$

and

$$\Pi_n(\{\mathbb{C}_{Zn}, \hat{\mathbb{C}}_{-Zn}\}) - \Pi_n(\{\mathbb{C}_0, \hat{\mathbb{C}}_{-Zn}\}) = V(B(\mathbb{C}_{Zn}) \cup (\mathcal{J} \backslash Z)) - V(\mathcal{J} \backslash Z) - p(\mathbb{C}_{Zn}).$$

Because $V(\cdot)$ is monotone, a solution to the Nash bargaining problem is given by $B(\mathbb{C}_{Zn}) = Z$ and $p(\mathbb{C}_{Zn}) = \xi_Z(V(\mathcal{J}) - V(\mathcal{J} \backslash Z))$. This proves that $\hat{\mathbb{C}}$ is a Nash-in-Nash equilibrium.

For uniqueness, suppose that $V(\cdot)$ is strictly monotone and fix any Nash-in-Nash equilibrium $\tilde{\mathbb{C}}$. For any Z and n, regardless of $\tilde{\mathbb{C}}_{-Zn}$, given that $V(\cdot)$ is strictly monotone, any solution to the Nash bargaining problem must have $B(\mathbb{C}_{Zn}) = Z$. Therefore, for any Z and n, any solution to the Nash bargaining problem must have $p(\mathbb{C}_{Zn}) = \xi_Z(V(\mathcal{J}) - V(\mathcal{J} \backslash Z))$, proving the claim.

Lastly, observe that when $\xi_Z = \xi$ for all $Z \in \mathcal{Z}$, the payments to each owner under $\hat{\mathbb{C}}$ are proportional to v_Z. □

A.3.8. Auctioning of Advertising Slots

Suppose that rather than simultaneously posting prices, owners simultaneously set reserve prices for each of their bundles, and then conduct simultaneous first-price auctions.

Proposition 15. *If owners simultaneously set reserve prices and then conduct simultaneous first-price auctions, there exists an equilibrium, and in any equilibrium all advertisers buy slots on all outlets, and the payment by each advertiser to each owner Z is given by v_Z.*

Proof. Fix any profile of announced reserve prices $p := \{p_B : B \subseteq Z, Z \in \mathcal{Z}\}$. For any advertiser, bidding strictly above the reserve price for any bundle B is strictly dominated by bidding at the reserve price p_B, because in both cases the advertiser is guaranteed to win the bundle (as $K \geq N$). Thus, for any bundle B, each advertiser either bids at the reserve price for that bundle, or bids below the reserve price and loses the auction. Therefore, after eliminating the strictly dominated strategies for the advertisers, this game is strategically equivalent to the pricing game of our main model. Hence, the claim follows directly from Theorem 1. □

Now, in addition, we further characterize equilibrium in a model where owners conduct auctions, advertising slots are scarce, and advertisers' valuations are heterogeneous. Suppose that each owner owns one outlet, and identify each owner with the outlet j that the owner owns. Suppose further that each outlet has K slots, where $K < N$ (so ad slots are scarce). The advertisers are heterogeneous, with value functions given by $\alpha_n V(\cdot)$ where $V(\cdot)$ is monotone and submodular with $V(\varnothing) = 0$. We order the advertisers so that $\alpha_1 > \alpha_2 > \cdots > \alpha_N > 0$. We assume that

$$\text{(A1)} \qquad \alpha_{K+1} V(\{j\}) < \alpha_K (V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\})) \; \forall j \in \mathcal{J}.$$

Each owner runs a uniform price auction in which the K slots are allocated to the K highest bidders at a price equal to the $(K+1)^{th}$ highest bid, with ties broken in favor of advertisers with higher α_n. The auctions happen simultaneously. Each advertiser simultaneously submits bids to every auction. We take an equilibrium to be a Nash equilibrium in pure strategies. We say an equilibrium is *owner-optimal* if there is no other equilibrium that gives weakly higher payoffs to all owners and strictly higher payoff to at least one owner. We say an equilibrium is *efficient* if the equilibrium allocation maximizes total surplus among all possible allocations.

Proposition 16. *Suppose that Assumption* (A1) *holds. Then, there exists an efficient owner-optimal equilibrium, and in every efficient owner-optimal equilibrium, for every owner j, the clearing price of auction j is $\alpha_K(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\}))$.*

Proof. Consider the following strategy profile: in every auction j, each advertiser $n \leq K$ bids $\alpha_n(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\}))$; advertiser $K+1$ bids $\alpha_K(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\}))$; and each advertiser $n > K+1$ bids 0.

We show that this is an equilibrium. Fix any j and any advertiser n with $n \leq K$. Because this is a $(K+1)$-th price auction, the advertiser cannot influence the price it pays conditional on winning. Regardless of the choices the advertiser makes on other auctions, the advertiser weakly prefers to win auction j at the price of $\alpha_K(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\}))$ rather than to lose the auction at that price because for any $S \subseteq \mathcal{J} \setminus \{j\}$, we have

$$\alpha_n(V(S \cup \{j\}) - V(S)) \geq \alpha_K(V(S \cup \{j\}) - V(S)) \geq \alpha_K(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\})),$$

where we have used submodularity of $V(\cdot)$. Therefore, advertiser n has no profitable deviation.

Next, fix any j and any advertiser n with $n > K$. Note that advertiser n loses every auction under the proposed strategy profile. Also note that to win auction j, advertiser n has to pay $\alpha_K(V(\mathcal{J}) - V(\mathcal{J} \setminus \{j\}))$. However, regardless of the choices the advertiser makes on other auctions, the advertiser strictly prefers not to win auction j at this price, because for any $S \subseteq \mathcal{J} \setminus \{j\}$, we have

$$\alpha_n\left(V(S\cup\{j\})-V(S)\right)\leq\alpha_n V(\{j\})\leq\alpha_{K+1}V(\{j\})<\alpha_K\left(V(\mathcal{J})-V(\mathcal{J}\setminus\{j\})\right),$$

where we have used submodularity of $V(\cdot)$ and Assumption (A1). Therefore, advertiser n has no profitable deviation.

Now, we show that this equilibrium is owner-optimal. Suppose toward contradiction that there is another equilibrium that gives some owner j a strictly higher payoff and all other owners weakly higher payoffs. Fix any such equilibrium $\mathcal{E}'$. By the argument above, no advertiser n with $n>K$ would want to win auction j at a price strictly higher than $\alpha_K(V(\mathcal{J})-V(\mathcal{J}\setminus\{j\}))$. Therefore, the K winning bidders in auction j must be advertisers $1,\ldots,K$. However, at a price strictly higher than $\alpha_K(V(\mathcal{J})-V(\mathcal{J}\setminus\{j\}))$ for auction j, advertiser K must lose some auction $j'\neq j$, because otherwise the advertiser can profitably deviate to losing auction j. Then, since there are K winners in auction j', there must be an advertiser n' with $n'>K$ who wins auction j'. For owner j' to have a weakly higher payoff in equilibrium $\mathcal{E}'$ than in the original equilibrium, the clearing price in auction j' must be weakly higher than $\alpha_K(V(\mathcal{J})-V(\mathcal{J}\setminus\{j'\}))$. But then advertiser n' can profitably deviate to losing auction j' by the argument above, which is a contradiction.

We claim that the allocation of ad slots to advertisers under this equilibrium is the unique efficient allocation. To see this, fix any efficient allocation x and suppose toward contradiction that x differs from the allocation under the equilibrium. Then, it must be that some advertiser $n\leq K$ is not allocated to an ad slot on some outlet j, which means that some advertiser $n'>K$ is allocated to an ad slot on outlet j. Consider an allocation $\tilde{x}$ that is the same as x except that it allocates the ad slot on outlet j to n instead of n'. We claim that this change strictly increases the total surplus. Indeed, let S be the set of outlets whose slots are assigned to advertiser n under allocation x, and similarly define S' for advertiser n'. Then,

$$\begin{aligned}\alpha_n\left(V(S\cup\{j\})-V(S)\right)&\geq\alpha_K\left(V(\mathcal{J})-V(\mathcal{J}\setminus\{j\})\right)\\&>\alpha_{K+1}V(\{j\})\geq\alpha_{n'}\left(V(S')-V(S'\setminus\{j\})\right),\end{aligned}$$

where we have used submodularity of $V(\cdot)$ and Assumption (A1). Therefore,

$$\alpha_n V(S\cup\{j\})+\alpha_{n'}V(S'\setminus\{j\})>\alpha_n V(S)+\alpha_{n'}V(S'),$$

and hence $\tilde{x}$ gives a strictly higher total surplus than x, which is a contradiction.

Finally, fix any efficient owner-optimal equilibrium. By efficiency and the argument above, the winning bidders in every auction must be advertisers $1,\ldots,K$. If there is any auction j in which the clearing price is strictly higher than $\alpha_K(V(\mathcal{J})-V(\mathcal{J}\setminus\{j\}))$, then advertiser K can profitably deviate to losing auction j. Therefore, in every auction j, the clearing price must be weakly lower than $\alpha_K(V(\mathcal{J}) - V(\mathcal{J}\setminus\{j\}))$. Now, if there is any auction j' in which the clearing price is strictly lower than $\alpha_K(V(\mathcal{J}) - V(\mathcal{J}\setminus\{j'\}))$, the equilibrium cannot be owner-optimal, because we have just shown an equilibrium that has clearing prices equal to $\alpha_K(V(\mathcal{J}) - V(\mathcal{J}\setminus\{j\}))$ for all j. Thus, in every efficient owner-optimal equilibrium, the clearing price in every auction j must be exactly $\alpha_K(V(\mathcal{J}) - V(\mathcal{J}\setminus\{j\}))$. □

A.3.9. Competition between Advertisers

We consider a setting in which each owner owns a single outlet and advertisers make purchasing decisions sequentially in random order. We modify the value function $V(\cdot)$ as follows. Let advertiser n's value for buying ads on the set of outlets S_n be $V(S_n, \vec{S}_{-n})$, where $\vec{S}_{-n} \in \mathcal{J}^{N-1}$ is the vector of sets bought by other advertisers. We say $\vec{S}_{-n} \leq \vec{S}'_{-n}$ if each entry of the vector is smaller in the set-inclusion order. Since all owners are single-outlet owners, we use j to denote both an outlet and the owner associated with the outlet. Let $\vec{\mathcal{J}}$ be the vector of length $N-1$ with $\mathcal{J}$ in each entry, and $\overrightarrow{\mathcal{J}\setminus\{j\}}$ be the vector of length $N-1$ with $\mathcal{J}\setminus\{j\}$ in each entry. We impose two assumptions:

(A2) $\quad V(\mathcal{J},\vec{S}) - V(\mathcal{J}\setminus\{j\},\vec{S}) \geq V(\mathcal{J},\vec{S}') - V(\mathcal{J}\setminus\{j\},\vec{S}')$ for any $\vec{S} \leq \vec{S}'$ and j;

(A3) $\quad V(\mathcal{J},\overrightarrow{\mathcal{J}\setminus\{j\}}) - V(\mathcal{J}\setminus\{j\},\overrightarrow{\mathcal{J}\setminus\{j\}}) \leq \left(1+\frac{1}{N}\right)\left(V(\mathcal{J},\vec{\mathcal{J}}) - V(\mathcal{J}\setminus\{j\},\vec{\mathcal{J}})\right)$ for any j.

Let $\tilde{v}_j = V(\mathcal{J},\vec{\mathcal{J}}) - V(\mathcal{J}\setminus\{j\},\vec{\mathcal{J}})$ denote the modified incremental value of outlet j in this setting.

Proposition 17. *Suppose that $V(\cdot,\vec{S})$ is monotone and submodular for any $\vec{S}$, and that $V(\cdot,\cdot)$ satisfies* (A2) *and* (A3). *Then there exists an equilibrium in which all advertisers buy slots on all outlets, and the price for outlet j is $p_j^* = \tilde{v}_j$.*

Proof. We construct an equilibrium as follows. Let each owner j announce price $\tilde{v}_j$. For each profile of prices p announced (including off-the-equilibrium-path histories), the subgame in the second stage is a finite extensive-form game and hence admits an equilibrium by backward induction. When doing the backward induction, if an advertiser is indifferent between different sets of outlets to buy slots on, we pick one with the largest cardinality. Now we verify that no owner has a profitable deviation.

Observe that at any history, if $p_j = \tilde{v}_j$ is offered by an owner $j \in \mathcal{J}$, then any advertiser will buy a slot on outlet j regardless of p_{-j} and what other advertisers do. This is because for any $S \subseteq \mathcal{J}\setminus\{j\}$ and any $\vec{S}_{-n} \leq \vec{\mathcal{J}}$,

$$V(S\cup\{j\},\vec{S}_{-n}) - V(S,\vec{S}_{-n}) \geq V(\mathcal{J},\vec{S}_{-n}) - V(\mathcal{J}\setminus\{j\},\vec{S}_{-n}) \geq V(\mathcal{J},\vec{\mathcal{J}}) - V(\mathcal{J}\setminus\{j\},\vec{\mathcal{J}})$$

where we have used submodularity of $V(\cdot, \vec{S}_{-n})$ and Assumption (A2). Further, when all other advertisers buy slots on all outlets, the incremental value for an advertiser to buy a slot on some outlet j is exactly $V(\mathcal{J}, \vec{\mathcal{J}}) - V(\mathcal{J}\backslash\{j\}, \vec{\mathcal{J}})$. Therefore, at the proposed price profile, for any outlet j, each advertiser is indifferent between buying and not buying a slot on outlet j in the proposed equilibrium.

Now fix any owner $j \in \mathcal{J}$. Suppose all other players follow the proposed strategy. Owner j is selling N slots by announcing price $\tilde{v}_j$ and clearly has no incentive to decrease the price. Consider the deviation of raising the price. By the earlier observation, all advertisers would continue buying slots on outlets in $\mathcal{J}\backslash\{j\}$. Therefore, by (A2), owner j cannot extract more than $V(\mathcal{J}, \overrightarrow{\mathcal{J}\backslash\{j\}}) - V(\mathcal{J}\backslash\{j\}, \overrightarrow{\mathcal{J}\backslash\{j\}})$ from any advertiser. Further, we claim that at least one advertiser would stop buying the slot on outlet j after the price increase, because if not, the last advertiser to move can profitably deviate by buying only the slots on the outlets in $\mathcal{J}\backslash\{j\}$. Thus there are at most $N-1$ advertisers buying a slot on outlet j. Hence owner j's revenue is at most

$$(N-1)\left(V(\mathcal{J}, \overrightarrow{\mathcal{J}\backslash\{j\}}) - V(\mathcal{J}\backslash\{j\}, \overrightarrow{\mathcal{J}\backslash\{j\}})\right) \leq (N-1)\left(1+\frac{1}{N}\right)\left(V(\mathcal{J}, \vec{\mathcal{J}}) - V(\mathcal{J}\backslash\{j\}, \vec{\mathcal{J}})\right) \leq N\tilde{v}_j$$

where the first inequality is due to (A3). So there is no profitable deviation for owner j. Since this holds for any owner, the construction is an equilibrium. □

A.3.10. Incentive to Invest in Content

Suppose that there is a set of viewers $\mathcal{I}$. Each viewer $i \in \mathcal{I}$ is attracted to each owner Z's content with probability $\alpha_{iZ} \in [0,1]$. If a viewer i is attracted to owner Z's content, the viewer sees ads on outlets $j \in Z$ with probability $\eta_{ij} \in (0,1)$, independently across j, and other details follow the reach-only model. Prior to the game specified in Section I, each owner simultaneously announces a choice of α_{iZ} for all viewers i, paying a content cost $\sum_{i\in I} C_{iZ}(\alpha_{iZ})$ where $C_{iZ}(0) = C'_{iZ}(0) = 0$ and $C'_{iZ}(1) > a_i$, for $C'_{iZ}(\cdot)$ the first derivative of $C_{iZ}(\cdot)$. For a given investment profile $\{(\alpha_{iZ})_{i\in\mathcal{I}}\}_{Z\in\mathcal{Z}}$, a viewer i, and an owner Z, let $V_i^Z(\cdot\,;\alpha)$ denote the value function induced by the viewing probabilities of viewer i conditional on viewer i being attracted to owner Z.

Proposition 18. *Suppose the investment profile $\{(\alpha_{iZ})_{i\in\mathcal{I}}\}_{Z\in\mathcal{Z}}$ is an equilibrium. Then,*

$$C'_{iZ}(\alpha_{iZ}) = V_i^Z(\mathcal{J};\alpha) - V_i^Z(\mathcal{J}\backslash Z;\alpha) = a_i\eta_{iZ}\prod_{Z'\neq Z}(1-\alpha_{iZ'}\eta_{iZ'}).$$

Proof. For a given investment profile α, by Theorem 1, the equilibrium payments in the subgame are given by

$$p_Z^*(\alpha) = \sum_i \alpha_{iZ}\left(V_i^Z(\mathcal{J};\alpha) - V_i^Z(\mathcal{J}\backslash Z;\alpha)\right).$$

So when making investment choices, owner Z maximizes the objective

$$\sum_i \alpha_{iZ}\left(V_i^Z(\mathcal{J};\alpha)-V_i^Z(\mathcal{J}\backslash Z;\alpha)\right)-\sum_i C_{iZ}(\alpha_{iZ}),$$

which is differentiable in $(\alpha_{iZ})_{i\in\mathcal{I}}$, and where $V_i^Z(\mathcal{J};\alpha)-V_i^Z(\mathcal{J}\backslash Z;\alpha)$ does not depend on α_{iZ}. For owner Z, the first order condition for α_{iZ} is given by

$$C'_{iZ}(\alpha_{iZ})=V_i^Z(\mathcal{J};\alpha)-V_i^Z(\mathcal{J}\backslash Z;\alpha).$$

Since $C'_{iZ}(1)>a_i\geq V_i^Z(\mathcal{J};\alpha)-V_i^Z(\mathcal{J}\backslash Z;\alpha)$ for all α, i, and Z, in any equilibrium no owner Z will choose $\alpha_{iZ}=1$ for any viewer i. Then, since $C'_{iZ}(0)=0$ and $\eta_{iZ}>0$, in any equilibrium no owner Z will choose $\alpha_{iZ}=0$ for any viewer i. Hence, in any equilibrium, the above first order condition must hold for all i and all Z. The form of $V_i^Z(\mathcal{J};\alpha)-V_i^Z(\mathcal{J}\backslash Z;\alpha)$ follows by Corollary 1. $\square$

In the (unattainable) limiting case where $\alpha_{iZ}=1$ for all $Z\in\mathcal{Z}$, we have that $C'_{iZ}(1)=p^*_{iZ}$ for all Z.

A.4. Alternative Models with Declining Audience

In this section, we present two alternative models for ad markets, and show that these models cannot generate rising ad revenues with declining audience, even though they can generate rising prices.

A.4.1. Heterogeneous Advertisers with Additively Separable Values

Here we interpret declining audience as a decline in advertisers' separable valuations for outlets. Consider a special case of the extension in A.3.5 with heterogeneous advertisers in which each owner owns a single outlet $j\in\mathcal{J}$ and each advertiser $n\in\mathcal{N}$ has value $V_n(\cdot)$ such that

$$V_n(S)=\sum_{j\in S}V_n(\{j\})$$

for any $S\subseteq\mathcal{J}$.

Proposition 19. *Consider two markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$, one in which the advertisers' values are V_n and the other in which the advertisers' values are $\tilde{V}_n$. Suppose that $\tilde{V}_n\leq V_n$ for all advertisers n. Then the total ad revenue in any equilibrium of market $\tilde{\mathcal{M}}$ must be weakly lower than the total ad revenue in any equilibrium of market $\mathcal{M}$.*

Proof. We show that for any outlet j, its ad revenue must be lower in market $\tilde{\mathcal{M}}$. Note that because each advertiser has an additively separable value function, the pricing problem for outlet j, regardless of how other outlets price, is equivalent to the pricing problem where the N advertisers have values $V_n(\{j\})$. Now, note that for any price p,

$$p \cdot \sum_n \mathbf{1}_{V_n(\{j\}) \geq p} \geq p \cdot \sum_n \mathbf{1}_{\tilde{V}_n(\{j\}) \geq p}.$$

Thus, for any price p,

$$\sup_{p'} \left\{ p' \cdot \sum_n \mathbf{1}_{V_n(\{j\}) \geq p'} \right\} \geq p \cdot \sum_n \mathbf{1}_{\tilde{V}_n(\{j\}) \geq p}.$$

Therefore, we have

$$\sup_{p'} \left\{ p' \cdot \sum_n \mathbf{1}_{V_n(\{j\}) \geq p'} \right\} \geq \sup_{p} \left\{ p \cdot \sum_n \mathbf{1}_{\tilde{V}_n(\{j\}) \geq p} \right\}.$$

The left-hand side is the ad revenue for outlet j in any equilibrium of market $\mathcal{M}$, and the right-hand side is the ad revenue for outlet j in any equilibrium of market $\tilde{\mathcal{M}}$. The claim follows. □

Proposition 19 does not preclude that the equilibrium price for any outlet j can be higher in market $\tilde{\mathcal{M}}$ than in market $\mathcal{M}$. For example, suppose that there are two advertisers such that $V_1 = V_2 - \varepsilon > 0$ in market $\mathcal{M}$, and $\tilde{V}_1 = 0$ and $\tilde{V}_2 = V_2$ in market $\tilde{\mathcal{M}}$. For $\varepsilon > 0$ small enough, the equilibrium price for each outlet is greater in market $\tilde{\mathcal{M}}$ than in market $\mathcal{M}$.

A.4.2. *Falling Supply of Ad Slots*

Here we interpret declining audience as a decrease in the total supply of ad slots K. Suppose that the advertisers are heterogeneous. Specifically, each advertiser n values the number of total impressions at b_n per impression, and has a budget constraint c_n. We use perfect competition as a solution concept, so that a price per impression p^* is an equilibrium if and only if

$$D(p^*) = K$$

where $D(\cdot)$ is the aggregate market demand for ad slots, with $D(p) = \sum_{n \in \mathcal{N}} D_n(p) = \frac{1}{p} \sum_{n \in \mathcal{N}} c_n \mathbf{1}_{b_n \geq p}$ for any price $p \geq 0$.

Proposition 20. *Consider two markets $\mathcal{M}$ and $\tilde{\mathcal{M}}$, one in which the supply of ad slots is K and the other in which the supply of ad slots is $\tilde{K}$. Suppose that $\tilde{K} < K$. Then the total ad revenue in any equilibrium of market $\tilde{\mathcal{M}}$ must be weakly lower than the total ad revenue in any equilibrium of market $\mathcal{M}$, although the price per impression in any equilibrium of market $\tilde{\mathcal{M}}$ must be weakly greater than the price per impression in any equilibrium of market $\mathcal{M}$.*

Proof. At any equilibrium price p^* for market $\mathcal{M}$, we have that

$$\frac{1}{p^*}\sum_n c_n \mathbf{1}_{b_n \geq p^*} = K\,.$$

At any equilibrium price $\tilde{p}^*$ for market $\tilde{\mathcal{M}}$, we have that

$$\frac{1}{\tilde{p}^*}\sum_n c_n \mathbf{1}_{b_n \geq \tilde{p}^*} = \tilde{K}\,.$$

It follows by inspection that $\tilde{p}^* \geq p^*$ because $\tilde{K} < K$. We also observe that revenues are weakly lower in market $\tilde{\mathcal{M}}$ because

$$\tilde{p}^*\tilde{K} = \sum_n c_n \mathbf{1}_{b_n \geq \tilde{p}^*} \leq \sum_n c_n \mathbf{1}_{b_n \geq p^*} = p^* K$$

and $\tilde{p}^* \geq p^*$. $\square$

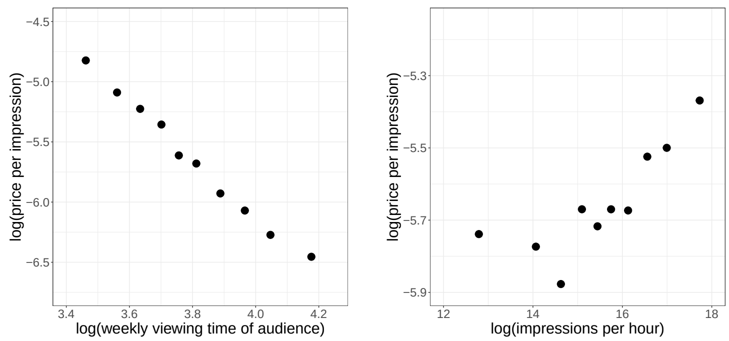
B. Additional Empirical Results

Appendix Figure 1: **Sensitivity to Alternative Samples**

Price per impression vs.

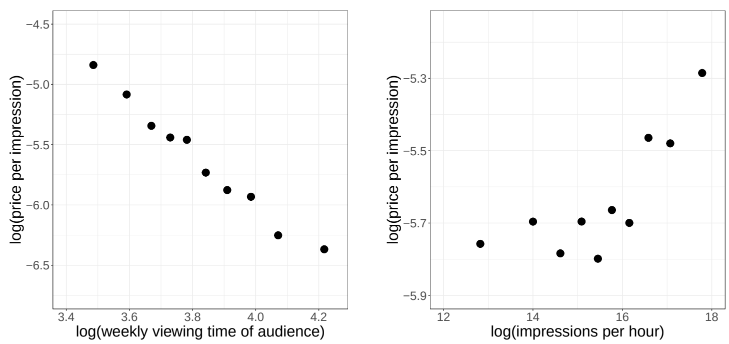
audience activity | audience size

Baseline

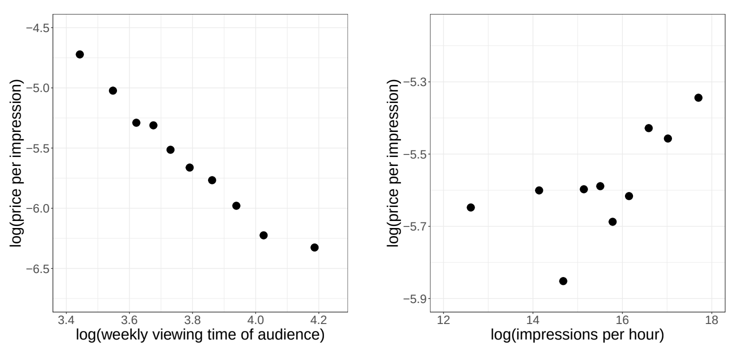


Alternative sample years

2014



2016



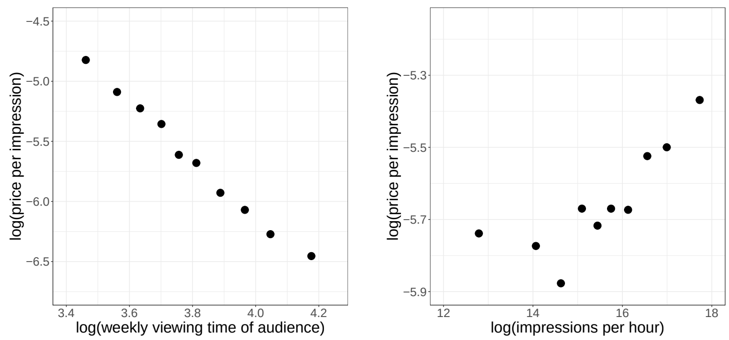
Notes: Within a given row, both plots are based on the same regression specification. The row labeled "Baseline" corresponds to the main specification in the paper, with the plot "Price per impression vs. audience activity" corresponding to Panel B of Figure 2 and the plot "Price per impression vs. audience size" corresponding to Panel B of Figure 4. The rows under the header "Alternative sample years" present results for alternative sample years.

Appendix Figure 2: **Sensitivity to Alternative Outlet Definitions**

Price per impression vs.

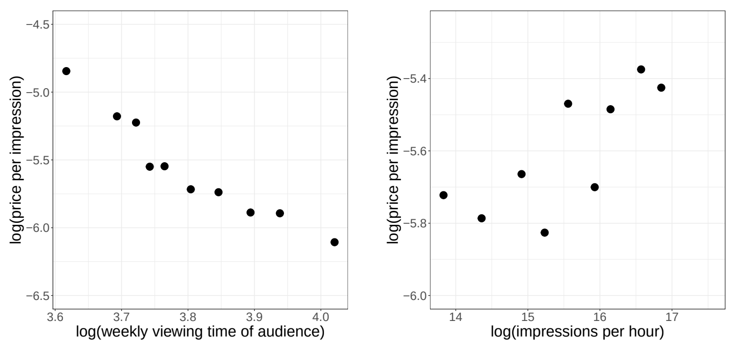
audience activity | audience size

Baseline

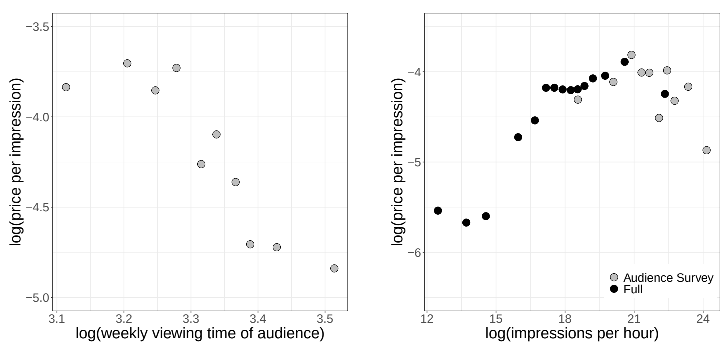


Alternative definition of outlet

Network



Broadcast program



Notes: Within a given row, both plots are based on the same regression specification. The row labeled "Baseline" corresponds to the main specification in the paper, with the plot "Price per impression vs. audience activity" corresponding to Panel B of Figure 2 and the plot "Price per impression vs. audience size" corresponding to Panel B of Figure 4. The rows under the header "Alternative definition of outlet" consider different outlet definitions. In the row labeled "Network" an outlet j is a network. In the "Network" row, the "Price per impression vs. audience activity" specification includes controls for the share of total impressions that are to adults and for indicators of deciles of audience size, and the "Price per impression vs. audience size" specification includes controls for the share of total impressions that are to adults and for indicators of deciles of audience activity. In the row labeled "Broadcast program" an outlet j is a broadcast program, with bins corresponding to 15 quantiles of the full sample of broadcast programs (3055 programs) colored black and bins corresponding to deciles of the subsample of broadcast programs included in the audience survey (173 programs) colored gray. In the "Broadcast program" row, the "Price per impression vs. audience activity" specification includes controls for the share of total impressions that are to adults and for indicators of deciles of audience size, and the "Price per impression vs. audience size" specification includes a control for the share of total impressions that are to adults.

Appendix Figure 3: **Sensitivity to Alternative Controls**

Price per impression vs.

audience activity | audience size

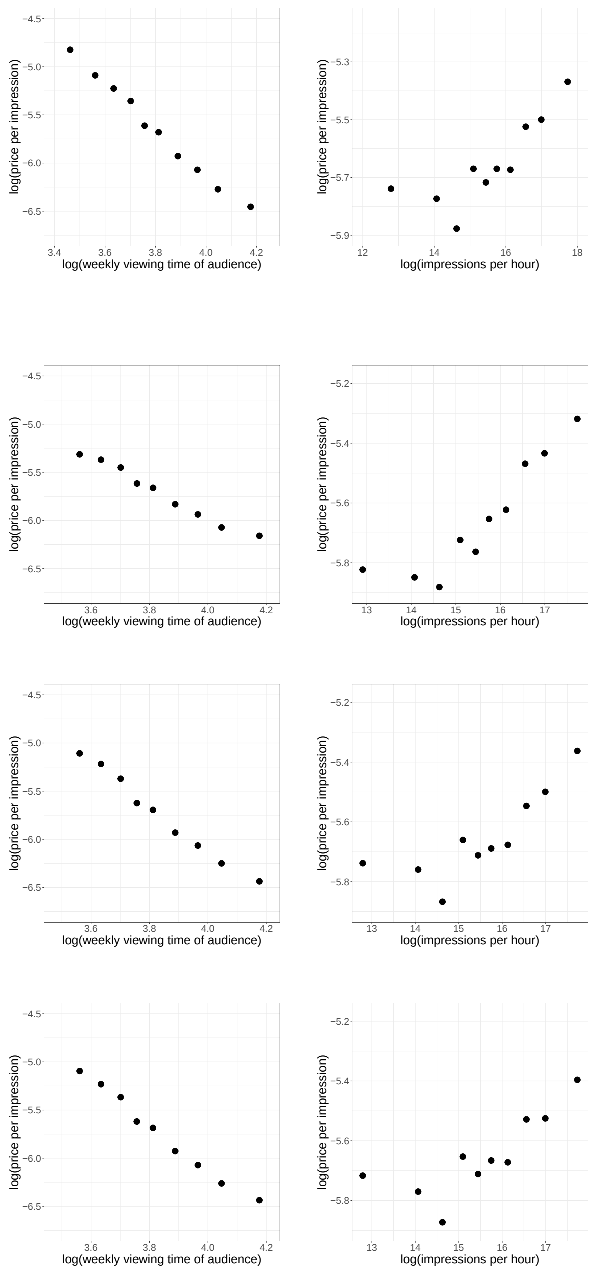
Baseline

Alternative controls

Income

Attentiveness

Attitude



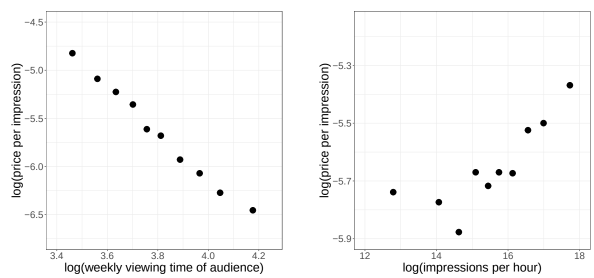
Notes: Within a given row, both plots are based on the same regression specification. The row labeled "Baseline" corresponds to the main specification in the paper, with the plot "Price per impression vs. audience activity" corresponding to Panel B of Figure 2 and the plot "Price per impression vs. audience size" corresponding to Panel B of Figure 4. The rows under the header "Alternative controls" consider different sets of control variables. The row labeled "Income" adds controls for indicators for deciles of the average household income of adult impressions. The row labeled "Attentiveness" adds controls for indicators for deciles of the time-weighted average attentiveness of the outlet's viewers, where a viewer's attentiveness is the viewer's average self-reported attentiveness across broadcast and cable programs, coded as some (0.5), most (0.75), or full (1), and measured for each program relative to the mean among all respondents who rate the program. The row labeled "Attitude" adds controls for indicators for deciles of the time-weighted average of viewers' attitudes toward television advertising, where a viewer's attitude toward advertising is measured as the first principal component of the viewer's responses (on a five-point scale) to a series of eight questions about TV advertising.

Appendix Figure 3: **Sensitivity to Alternative Controls (continued)**

Price per impression vs.

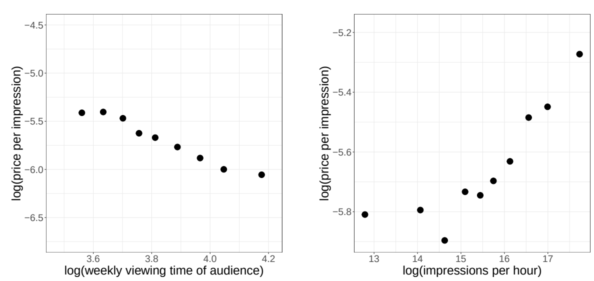
audience activity | audience size

Baseline



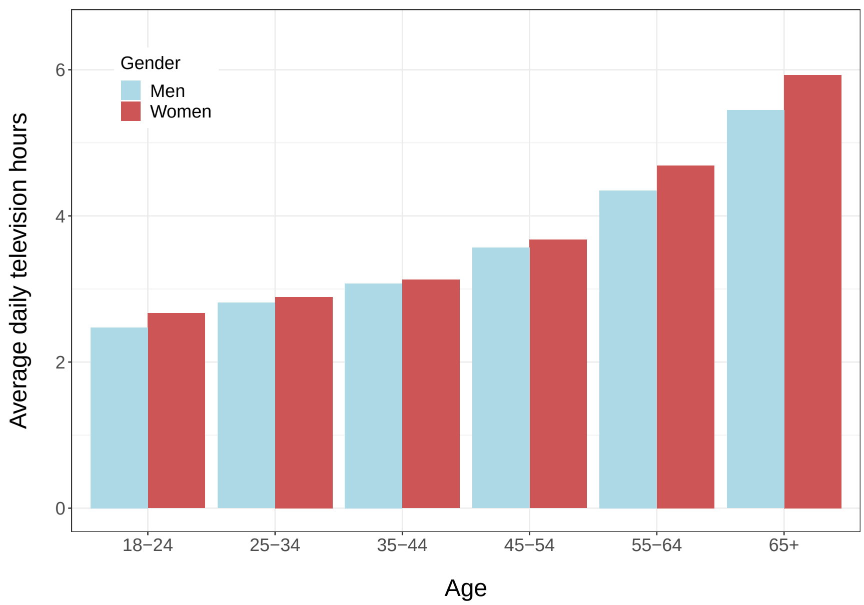
Alternative controls

Industry



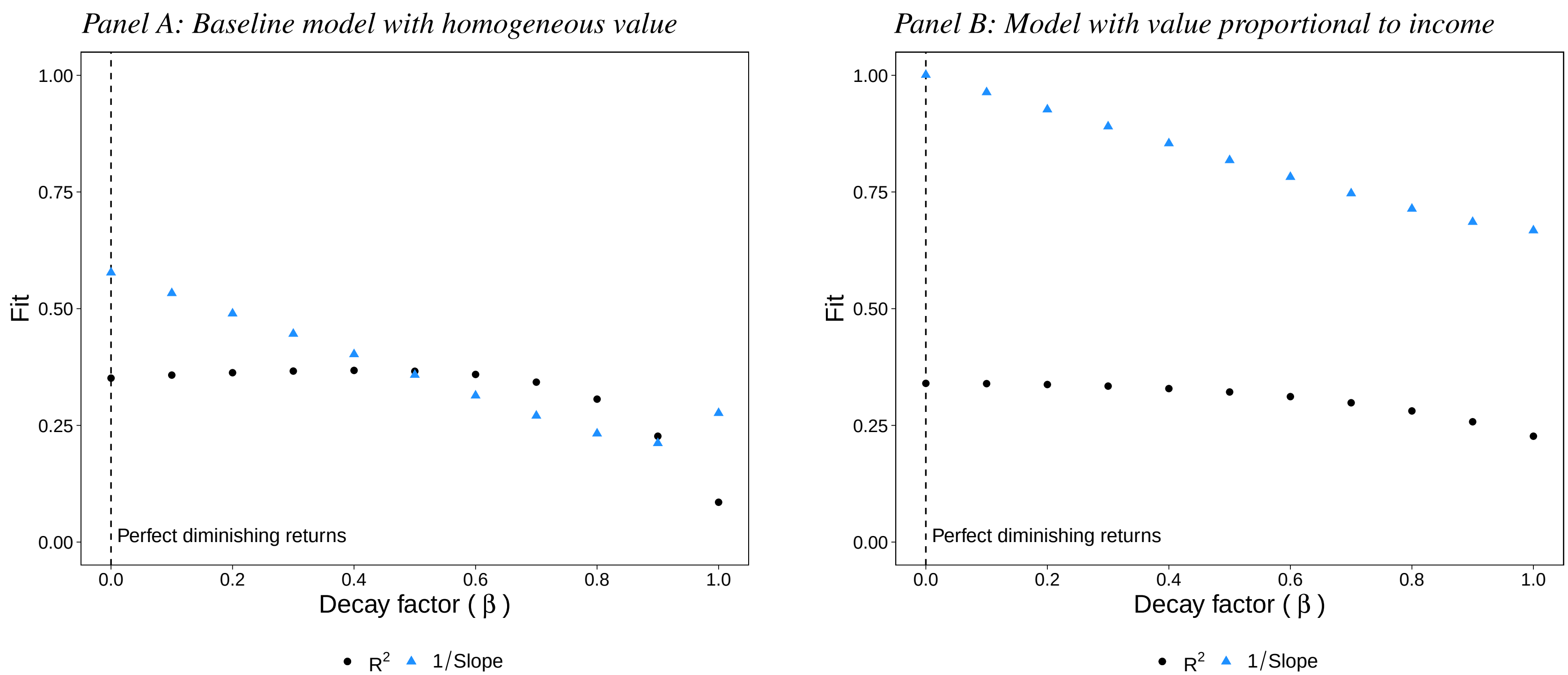
Notes: Within a given row, both plots are based on the same regression specification. The row labeled "Baseline" corresponds to the main specification in the paper, with the plot "Price per impression vs. audience activity" corresponding to Panel B of Figure 2 and the plot "Price per impression vs. audience size" corresponding to Panel B of Figure 4. The rows under the header "Alternative controls" consider different sets of control variables. The row labeled "Industry" adds controls for the share of the outlet's adult impressions that are to ads whose advertisers are in each of 11 industry categories: automotive; business and consumer services; business supplies; drugs and remedies; entertainment; food and drink; home and garden; insurance and real estate; retail; travel; and other.

Appendix Figure 4: **Average Television Viewing Hours Per Day by Age and Gender**



Notes: The figure shows the average daily viewing hours spent on television across age groups by gender.

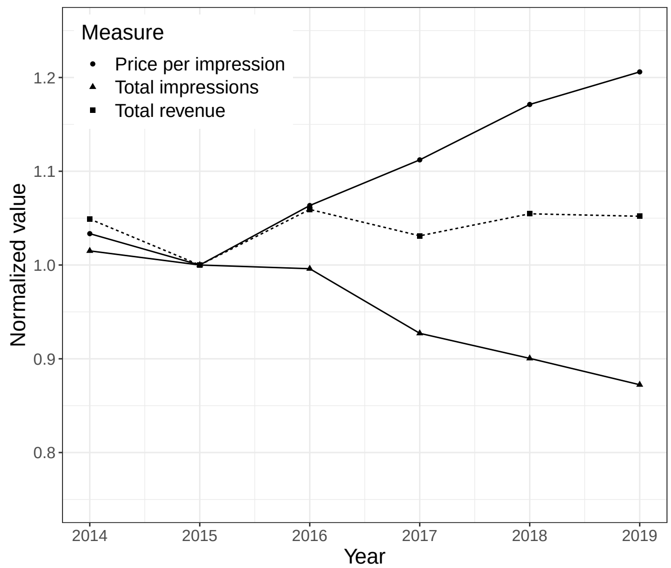
Appendix Figure 5: **Fit of Quantitative Model Under Imperfect Diminishing Returns**



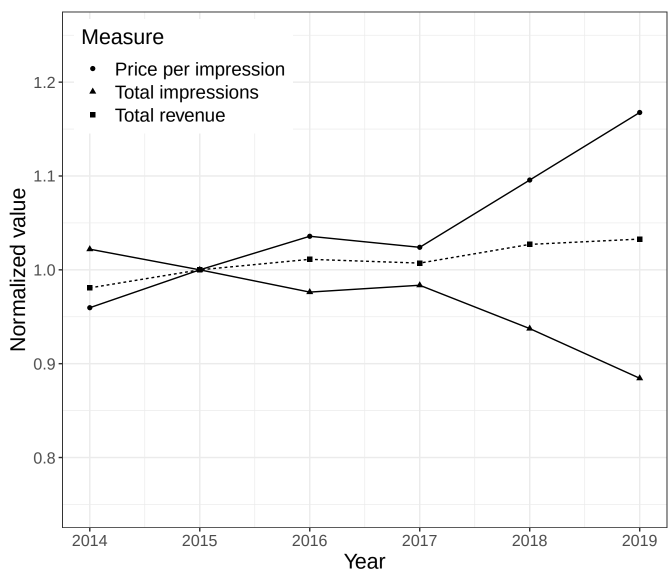
Notes: In each plot we depict the fit of a model-based prediction of advertising prices (y-axis) as a function of the extent of diminishing returns from advertising assumed in the model (x-axis). To produce the plot, we parameterize the viewer-level model with a single format such that $u(M) = \sum_{m=1}^{M} \beta_m$ where $\beta_m = \beta^{m-1}$ for $m \in \{1, \ldots, \overline{M}\}$, $\beta_m = 0$ for $m > \overline{M}$, and $\beta \in [0,1]$. Here, β describes the extent of diminishing returns, with $\beta = 0$ denoting the special case of the reach-only model and $\beta = 1$ denoting the special case of no diminishing returns up to the $\overline{M}^{th}$ impression. We assume that $\overline{M} = 10$ and calculate η_{ij} as described in Section IV. We calculate the log(price per viewer) implied by the model for each value $\beta \in \{0, 0.1, \ldots, 1\}$ depicted on the x-axis. We then regress the log(price per impression) of a 30-second spot observed in the data, as described in Section II.A, against the log(price per viewer) predicted by the model and depict on the y-axis the R^2 of the regression and the multiplicative inverse of the estimated slope. Panel A uses log(price per viewer) predicted from the baseline model in which advertisers' value of a first impression is homogeneous across viewers. Panel B uses log(price per viewer) predicted from the model in which advertisers' value of a first impression is proportional to a viewer's income. The unit of analysis for the regression is an owner Z, and all variables in the regression are residualized with respect to the share of the owner's impressions that are to adults.

Appendix Figure 6: **Observed and Predicted Television Advertising Revenues, Alternate Estimates of Impressions**

Panel A: Observed trends

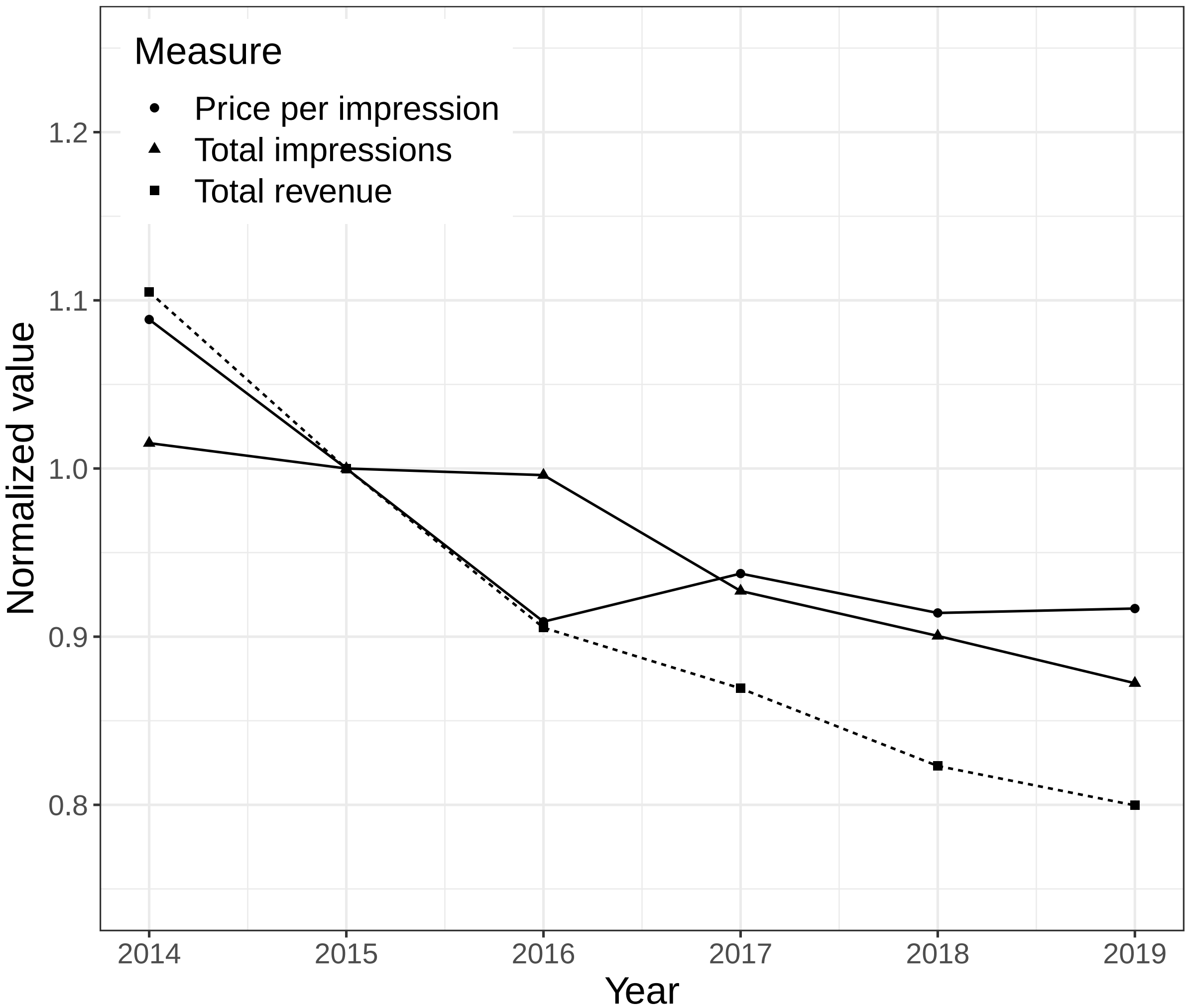


Panel B: Predicted trends



Notes: Each plot depicts trends in the television advertising market over the sample period. We plot trends in total revenue, total impressions, and price per impression (total revenues divided by total impressions), all normalized relative to their 2015 value. In Panel A, all series are as observed in the data, as described in Section II.A, and revenue is deflated to 2015 dollars using the US Consumer Price Index (Organization for Economic Co-operation and Development 2022). In Panel B, the trends in revenue and impressions are predicted by the baseline model in which advertisers' value of a first impression is homogeneous across viewers, as described in Section IV.

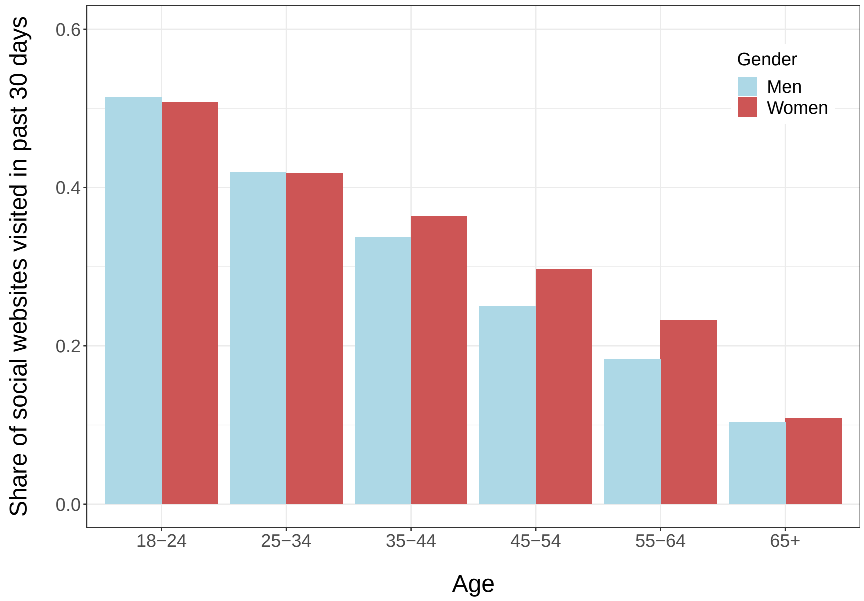
Appendix Figure 7: **Predicted Television Advertising Revenues, Strong Cross-Format Diminishing Returns**



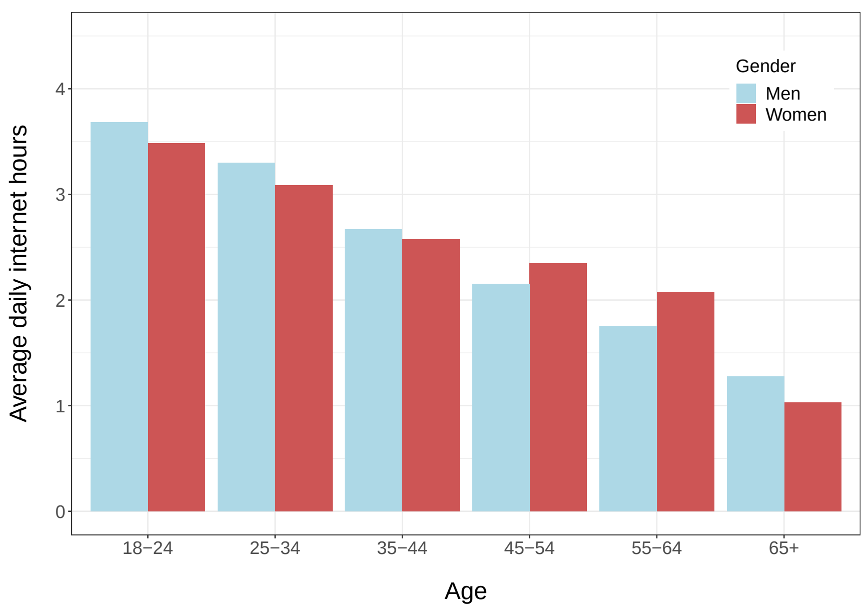
Notes: The plot depicts trends in total revenue, total impressions, and price per impression (total revenues divided by total impressions), all normalized relative to their 2015 value. Total revenue is is predicted by the cross-format reach-only model defined in Section V where $\phi = 1$ such that diminishing returns operate just as strongly between as within formats. Total impressions are as observed in the data, as described in Section II.A. Price per impression is calculated as the ratio of the predicted revenue to total impressions.

Appendix Figure 8: **Measures of Online Activity by Age and Gender**

Panel A: Share of social media sites visited in the past 30 days



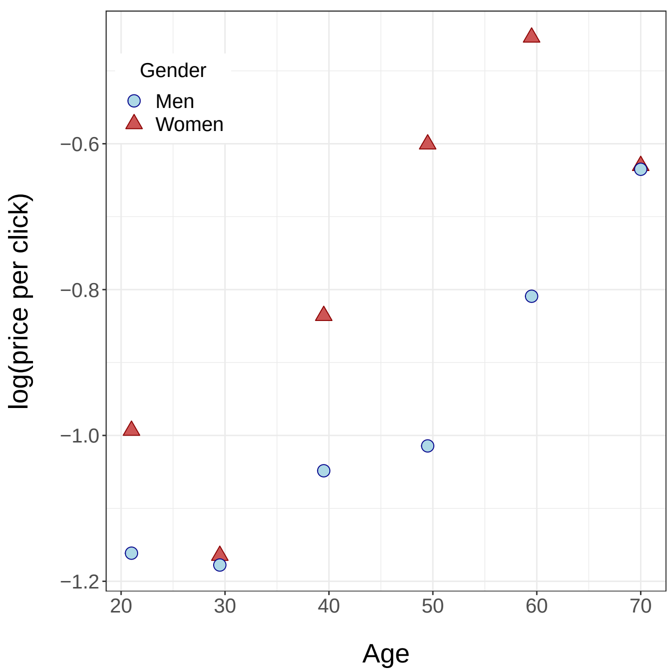
Panel B: Average internet hours per day



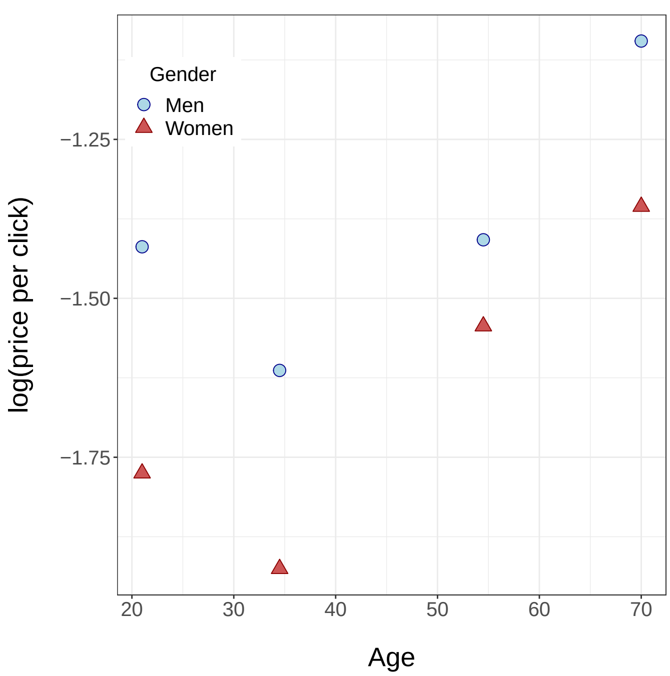
Notes: Panel A shows the average share of five social media sites (Facebook, Instagram, Reddit, Twitter and YouTube) visited in the past 30 days across age groups by gender. Panel B shows average daily hours spent on the internet across age groups by gender.

Appendix Figure 9: **Demographic Premia (Per Click) and Viewing Time on Facebook**

Panel A: Data from our experiment



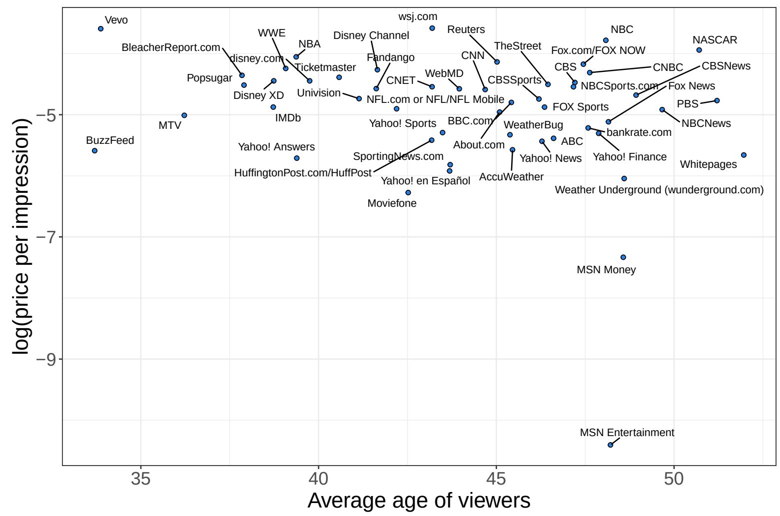
Panel B: Data from Allcott et al. (2020b)



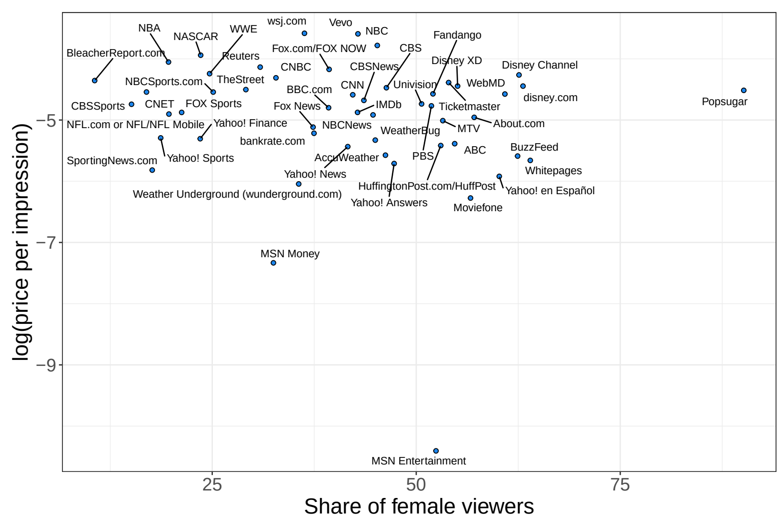
Notes: The plot shows the log(price per click) for advertisement sets targeted to a given gender and age group. In Panel A, the data are taken from our own experiment, and the groups are {Men, Women} $\times$ {18-24, 25-34, 35-44, 45-54, 55-64, 65+}. In Panel B, the the data are taken from Allcott et al. (2020b), and the groups are {Men, Women}$\times${18-24, 25-44, 45-64, 65+}. In both panels, the y-axis value is the log(price per click) for advertisement sets targeting the given group, and the x-axis value is the midpoint of the age range for the given group, treating 70 as the midpoint for ages 65+.

Appendix Figure 10: **Advertising Prices and Demographics of Digital Platforms**

Panel A: Average age



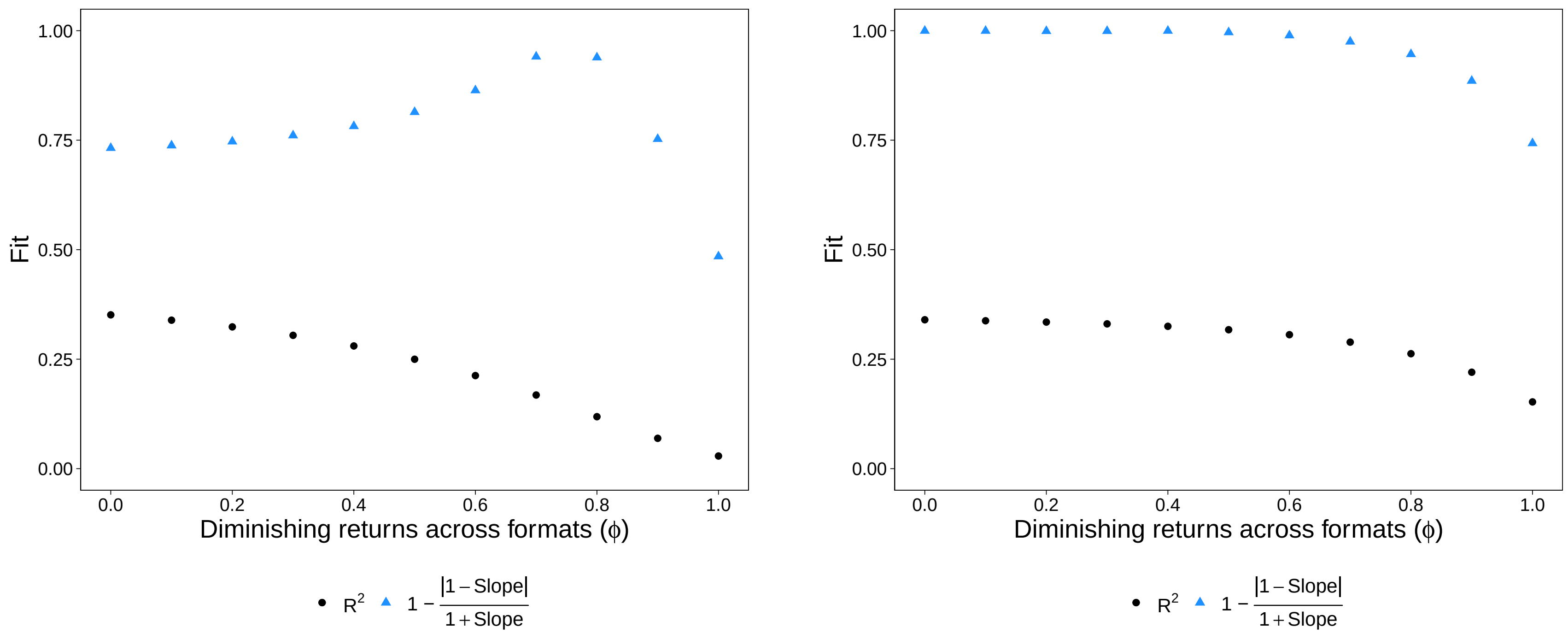
Panel B: Share female



Notes: Each plot is a scatterplot of the log(price per impression) of display advertising on a platform against the demographic characteristics of the platform's viewers. We construct the price per impression by computing the ratio of total revenue to total impressions across all display ads on the platform reported in AdIntel 2017 (The Nielsen Company 2022). The sample of platforms is the set of platforms that AdIntel 2017 (The Nielsen Company 2022) classifies as Entertainment, Finance, Information/Reference, News/Commentary, Spanish, Sports, Technology, or Weather, excluding some platforms such as those that focus primarily on direct sales of products or services. The x-axis shows the average age (Panel A) or share female (Panel B) of those who report visiting the platform in the previous 30 days in GfK MRI's 2017 Survey of the American Consumer (GfK Mediamark Research and Intelligence 2019).

Appendix Figure 11: **Fit of Quantitative Model of Television Prices with Social Media Competition**

Panel A: Baseline model with homogeneous value *Panel B: Model with value proportional to income*



Notes: In each plot we depict the fit of a model-based prediction of advertising prices (y-axis) as a function of the strength of cross-format diminishing returns (x-axis). To produce the plot, for each television owner Z and for each value $\phi \in \{0, 0.1, ..., 1\}$ of the parameter governing the strength of cross-format diminishing returns, we calculate the predicted log(price per viewer) implied by the cross-format reach-only model defined in Section V. For each value of ϕ, we then regress the log(price per impression) of a 30-second spot observed in the data, as described in Section II.A, against the log(price per viewer) predicted by the model and depict on the y-axis the R^2 of the regression and one minus the percent deviation of the slope from one. Panel A uses log(price per viewer) predicted from the baseline model in which advertisers' value of a first impression is homogeneous across viewers. Panel B uses log(price per viewer) predicted from the model in which advertisers' value of a first impression is proportional to a viewer's income. The unit of analysis for the regression is an owner Z, and all variables in the regression are residualized with respect to the share of the owner's impressions that are to adults.

Appendix Table 1: **Restrictiveness and Completeness of Quantitative Economic Model**

Panel A: Restrictiveness

Economic model:	*Homogeneous values*	*Value proportional to income*
	(1)	(2)
MSE of log(simulated price per impression) w.r.t.		
....constant model	0.6650	0.6650
....economic model	0.7190	0.8200
Restrictiveness of economic model	1.0800	1.2330
Number of owners	33	33
Number of viewers	21506	21506

Notes: The table evaluates the restrictiveness (Fudenberg, Gao, and Liang 2023) and completeness (Fudenberg et al. 2022) of the quantitative model of log(price per viewer) described in Section IV. Column (1) uses log(price per viewer) predicted from the baseline model in which advertisers' value of a first impression is homogeneous across viewers. Column (2) uses log(price per viewer) predicted from the model in which advertisers' value of a first impression is proportional to a viewer's income. The constant model predicts log(price per impression) with its mean across owners. In calculating the mean squared error, we represent both the observed and predicted values in terms of deviation from the mean across owners.

To evaluate restrictiveness (Panel A), in each of 10,000 replicates, we randomly draw values of each owner's log(price per impression), independently uniform over the support of the observed log(price per impression). Restrictiveness is the ratio of the mean, across replicates, of the mean squared error of the economic model of log(price per viewer) described in Section IV, and the mean squared error of the constant model, with respect to the random draws.

Appendix Table 1: (continued): **Restrictiveness and Completeness of Quantitative Economic Model**

Panel B: Completeness

Economic model:	*Homogeneous values*	*Value proportional to income*
	(1)	(2)
MSE of log(observed price per impression) w.r.t.		
....constant model	0.4570	0.4570
....owner-level regression model	0.4360	0.4240
....viewer-level lasso model	0.4910	0.4910
....economic model	0.3290	0.3090
Completeness of economic model w.r.t.		
....owner-level regression model	6.0480	4.4490
....viewer-level lasso model	—	—
Number of owners	33	33
Number of viewers	21506	21506

Notes (continued): The table evaluates the restrictiveness (Fudenberg, Gao, and Liang 2023) and completeness (Fudenberg et al. 2022) of the quantitative model of log(price per viewer) described in Section IV.

To evaluate completeness (Panel B), we consider two comparison models. In the first comparison model, we estimate a linear regression of the observed log(price per impression) of each owner Z on the log value, $\ln(\lambda_Z)$, of total impressions, the log value, $\ln\left(\frac{\sum_{i\in\mathcal{I}} a_i \eta_{iZ}}{\sum_{i\in\mathcal{I}} a_i}\right)$, of total weighted impressions, and the log value, $\ln\left(\frac{\sum_{i\in\mathcal{I}} a_i \prod_{Z'\neq Z}(1-\eta_{iZ'})}{\sum_{i\in\mathcal{I}} a_i}\right)$, of the weighted fraction of viewers not seeing an ad on other owners' networks.

In the second comparison model, for each viewer i, we estimate a linear regression of the observed log(price per impression) of each owner Z on the log value, $\ln(\eta_{iZ})$, of the viewer's probability of seeing an ad on that owner's networks, as well as the log values, $\ln\left(1-\eta_{i(Z)}\right)$, of the viewer's probability of not seeing an ad on each of the other owners' networks, indexed in descending order. When a value inside a logarithm is zero we replace it with its minimum across all owners Z for the given viewer i, and we include indicators for imputed values in the regression. We estimate the model via lasso using 10-fold cross-validation to choose the penalty, and for each viewer i we hold out one randomly chosen target owner whose data is excluded from the estimation sample and for which we predict the log(price per impression) from the final lasso fit. In the 1.9 percent of cases where there is insufficient variation in the regressors to estimate the model, we use the mean of the dependent variable as the lasso fit. For each owner Z, we take the weighted mean predicted log(price per impression) across all viewers i for which the given owner is the target, and treat this mean as the lasso-predicted log(price per impression) for the given owner. When comparing to the model with homogeneous values we use uniform weights; when comparing to the model with value proportional to household income we use household income as the weight.

Completeness is the ratio of the improvement in mean squared error between the economic model of log(price per viewer) described in Section IV and the comparison model, each evaluated relative to the constant model. We treat completeness as undefined when a given model has higher mean squared error than the constant model.

Appendix References Not Appearing in Main Text

The Nielsen Company. 2022. "Ad Intel dataset." The Nielsen Company [publisher], Kilts Center for Marketing, University of Chicago Booth School of Business [distributor]. `https://www.chicagobooth.edu/research/kilts/datasets/nielsenIQ-nielsen`. (accessed March 2022).

GfK Mediamark Research and Intelligence. 2019. "Survey of the American Consumer, Respondent-Level Data for 2017." GfK Mediamark Research and Intelligence. `https://www.mrisimmons.com/our-data/national-studies/survey-american-consumer/`. (accessed March 2022).