Online Appendix for "Disentangling Moral Hazard and Adverse Selection"

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1 Existence in the Relaxed Pure Adverse Selection Problem

To show existence, we will need the assumption that, for each θ, $\hat{C}(\cdot,\cdot,\theta)$ is strictly convex. For the canonical setting without moral hazard, $\hat{C}(a,u_0,\theta) = \varphi(u_0 + c(a,\theta))$, where $\varphi = u^{-1}$, and so this is immediate. The situation is more complicated in the decoupling program where $\hat{C} = C$ comes from the cost minimization step of the pure moral hazard problem. Although primitives for C convex in a are known (see Jewitt, Kadan, and Swinkels (2008) and Chade and Swinkels (2020) (CS)), ensuring convexity in (a,u_0) is harder. For the square-root utility case, one can show that all the assumptions are satisfied. Moreover, checking the convexity of a numerically generated C for any given set of primitives is straightforward. Finally, we have the following result, showing convexity on the relevant range as long as $\bar{u}$ is large enough.

Lemma 8 *Let $F \in \mathcal{C}^4$, let Assumption 6 hold, and let $\overline{a} < \infty$. Then for all $\bar{u}$ sufficiently large, $C(\cdot,\cdot,\theta)$ is strictly convex for each θ and for all (a,u_0) with $u_0 \geq \bar{u}$.*

Proof As in Lemma 6, C_{aa} and $C_{u_0u_0}$ are positive for u_0 sufficiently large. It remains only to show that for u_0 sufficiently large, the determinant $C_{aa}C_{u_0u_0} - (C_{au_0})^2$ is strictly positive. But,

$$\begin{aligned} C_{aa}C_{u_0u_0} - (C_{au_0})^2 &= \frac{C_{aa}}{\varphi' c_{aa}}\varphi' c_{aa}\frac{C_{u_0u_0}}{\varphi''}\varphi'' - \left(\frac{C_{au_0}}{\varphi'' c_a}\right)^2 (\varphi'' c_a)^2 \\ &=_s \frac{C_{aa}}{\varphi' c_{aa}} c_{aa}\frac{C_{u_0u_0}}{\varphi''} - \left(\frac{C_{au_0}}{\varphi'' c_a}\right)^2 \frac{\varphi''}{\varphi'} c_a^2, \end{aligned}$$

which converges to $c_{aa} > 0$, using that $\varphi''/\varphi' \to 0$ by Assumption 6. □

We are now ready to prove our existence and uniqueness result. Per Assumption 3, we will proceed with B linear with slope $\beta_1 > 0$. This is purely for convenience.

Proposition 2 *Let $\hat{C}$ be $\mathcal{C}^2$, let $\hat{C}(\cdot,\cdot,\theta)$ be strictly convex, let $\hat{C}_a(0,u_0,\theta) = 0$, and assume that there is $\varepsilon > 0$ such that $\hat{C}_a(\bar{a},u_0,\theta) > \beta_1 + \varepsilon$ for all (u_0,θ). Let $\overline{u}$ be in the interior of the range of u. Then a solution to the relaxed pure adverse selection problem*

$$\begin{aligned} \max_{\alpha,S} \quad & \int_{\underline{\theta}}^{\overline{\theta}} \left(B(\alpha(\theta)) - \hat{C}(\alpha(\theta), S(\theta), \theta)\right) h(\theta) d\theta \\ s.t. \quad & IC_S \end{aligned}$$

exists and is unique.

Proof Recall from Footnote 16 that the Hamiltonian of the problem is $\mathcal{H} = (B-\hat{C})h - \eta c_\theta$, where $\eta \leq 0$, and where strict concavity of $\mathcal{H}$ follows since (i) $\mathcal{H}_{aa} < 0$ since $B_{aa} = 0$, $c_{aa\theta} \leq 0$, and $\hat{C}_{aa} > 0$; (ii) $\mathcal{H}_{u_0u_0} < 0$ since $\hat{C}_{u_0u_0} > 0$; and (iii) $\mathcal{H}_{aa}\mathcal{H}_{u_0u_0} - \mathcal{H}^2_{au_0} > 0$ since $\hat{C}_{aa}\hat{C}_{u_0u_0} - \hat{C}^2_{au_0} > 0$.

Given the boundary conditions on $\hat{C}_a$, the optimality conditions are $\partial\mathcal{H}/\partial a = 0$, $\eta'(\theta) = -\partial\mathcal{H}/\partial S$, and $\eta(\bar{\theta}) = 0$, from which we obtain

$$B_a - \hat{C}_a = -\frac{c_{a\theta}}{h}\int_\theta^{\bar{\theta}} \hat{C}_{u_0} h, \tag{11}$$

plus IC_S. The concavity of $\mathcal{H}$ ensures that (11) plus IC_S are also sufficient. As a result, we will focus on them in our search for a solution (α, S) to the problem.

Define $a^*(s, z, \theta)$ as the solution in a to

$$B_a(a) - \hat{C}_a(a, s, \theta) = -\frac{c_{a\theta}(a,\theta)}{h(\theta)} z, \tag{12}$$

where a^* exists from the boundary conditions on $\hat{C}_a$, and is unique from the strict convexity of $\hat{C}$, the convexity of $-c_\theta$ in a, and $B_{aa} = 0$. We will then be done if we find a solution to the system of ordinary differential equations

$$\begin{bmatrix} S'(\theta) \\ Z'(\theta) \end{bmatrix} = \begin{bmatrix} g^S(S(\theta), Z(\theta), \theta) \\ g^Z(S(\theta), Z(\theta), \theta) \end{bmatrix}.$$

with boundary conditions $S(\underline{\theta}) = \overline{u}$ and $Z(\bar{\theta}) = 0$, where

$$\begin{bmatrix} g^S(S(\theta), Z(\theta), \theta) \\ g^Z(S(\theta), Z(\theta), \theta) \end{bmatrix} = \begin{bmatrix} -c_\theta(a^*(S(\theta), Z(\theta), \theta), \theta) \\ -C_{u_0}(a^*(S(\theta), Z(\theta), \theta), S(\theta), \theta)h(\theta) \end{bmatrix}.$$

Indeed if we take $\alpha(\theta) = a^*(S(\theta), Z(\theta), \theta)$ then $Z(\theta) = \int_\theta^{\bar{\theta}} C_{u_0}(\alpha(t), S(t), t)h(t)dt$. Hence, by definition of a^* and comparing (11) and (12), (α, S) satisfies the relevant conditions.

Define $u_{\max} = \bar{u} + (\overline{\theta} - \underline{\theta}) \max_{(a,\theta)\in[0,\overline{a}]\times[\underline{\theta},\bar{\theta}]}(-c_\theta(a, \theta))$. This is an upper bound on how high $S(\bar{\theta})$ could be if $S(\underline{\theta}) = \bar{u}$. Similarly, let

$$z_{\max} = (\overline{\theta} - \underline{\theta}) \max_{(a,s,\theta)\in[0,\overline{a}]\times[\overline{u},u_{\max}]\times[\underline{\theta},\overline{\theta}]} (C_{u_0}(a, s, \theta)h(\theta))$$

be an upper bound on how large $Z(\underline{\theta})$ can be if $Z(\bar{\theta}) = 0$. Choose $\delta \in [0, \varepsilon)$ such that $\overline{u} - \delta$ remains in the interior of the range of u, and let $R = [\overline{u}, u_{\max}] \times [0, z_{\max}]$ and $R_\delta = [\overline{u} - \delta, u_{\max} + \delta] \times [-\delta, z_{\max} + \delta]$, and define $R_{\delta/2}$ similarly. Then a^* is Lipschitz on $R_\delta \times [\underline{\theta}, \overline{\theta}]$, and hence so are g^S and g^Z.

Let $\zeta : \mathbb{R}^2 \to [0, 1]$ be a Lipschitz function such that $\zeta(s, z) = 1$ if $(s, z) \in R$ and $\zeta(s, z) = 0$ if

$(s,z) \notin R_{\delta/2}$. Write ζg^S for the function that is $\zeta(s,z)g^S(s,z,\theta)$ on R_δ, and zero otherwise, and similarly for ζg^Z. Then $(\zeta g^S, \zeta g^Z)$ is Lipschitz on $\mathbb{R}^2 \times [\underline{\theta}, \overline{\theta}]$. Thus, (see, for example, Theorems 2.3 and 2.6 in Khalil (1992)), there exist continuous functions $\hat{S}$ and $\hat{Z}$ such that $(\hat{S}(u_{\bar{\theta}}, \cdot), \hat{Z}(u_{\bar{\theta}}, \cdot))$ solves the system subject to *terminal* utility $u_{\bar{\theta}}$. That is, $\hat{S}$ and $\hat{Z}$ map $\mathbb{R} \times [\underline{\theta}, \overline{\theta}]$ into $\mathbb{R}$ such that $\hat{S}(u_{\bar{\theta}}, \bar{\theta}) = u_{\bar{\theta}}$, $\hat{Z}(\bar{s}, \bar{\theta}) = 0$, and

$$\begin{bmatrix} \hat{S}_\theta(u_{\bar{\theta}}, \theta) \\ \hat{Z}_\theta(u_{\bar{\theta}}, \theta) \end{bmatrix} = \begin{bmatrix} \left(\zeta g^S\right)(\hat{S}(u_{\bar{\theta}}, \theta), \hat{Z}(u_{\bar{\theta}}, \theta), \theta) \\ \left(\zeta g^Z\right)(\hat{S}(u_{\bar{\theta}}, \theta), \hat{Z}(u_{\bar{\theta}}, \theta), \theta) \end{bmatrix}.$$

Note that $\hat{S}(u_{\max}, \underline{\theta}) \geq \overline{u}$ since $\hat{S}_\theta \leq g^S = -c_\theta$, and by the definition of $u_{\max}$. Similarly, $\hat{S}(\overline{u}, \underline{\theta}) \leq \overline{u}$ since $\hat{S}_\theta \geq 0$. Hence, by continuity, there exists a terminal utility $u^* \in [\bar{u}, u_{\max}]$ such that the initial utility $\hat{S}(u^*, \underline{\theta})$ is equal to $\overline{u}$. But then, since $\hat{S}_\theta \geq 0$, $\hat{S}(u^*, \theta) \in [\bar{u}, u_{\max}]$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Similarly, since $\hat{Z}_\theta \leq 0$, and using the definition of $z_{\max}$, we have $\hat{Z}(u^*, \theta) \in [0, z_{\max}]$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Thus, $(\hat{S}(u^*, \theta), \hat{Z}(u^*, \theta)) \in R$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, and so since $\zeta = 1$ on R, the pair $(S(\cdot), Z(\cdot)) = (\hat{S}(u^*, \cdot), \hat{Z}(u^*, \cdot))$ satisfies the required conditions.

To see uniqueness, let $\left(\alpha^1, S^1\right)$ and $\left(\alpha^2, S^2\right)$ be optimal and differ on a positive measure set. Consider $\check{\alpha} = \left(\alpha^1 + \alpha^2\right)/2$, and note that since $c_{aa\theta} \leq 0$, $-c_\theta(\check{\alpha}, \theta) \leq \left(-c_\theta\left(\alpha^1, \theta\right) - c_\theta\left(\alpha^2, \theta\right)\right)/2$. Hence, $\check{S} = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\check{\alpha}(\tau), \tau)\, d\tau \leq (1/2)\left(S^1 + S^2\right)$. But then, because $B - C$ is strictly concave in a and u_0, and decreasing in u_0, $(\check{\alpha}, \check{S})$ is strictly more profitable than either $\left(\alpha^1, S^1\right)$ or $\left(\alpha^2, S^2\right)$, a contradiction. □

Lemma 9 *Under the conditions of Proposition 2, α is continuously differentiable.*

Proof For each θ, α is defined by $\eta(\alpha(\theta), \theta) + z(\theta) = 0$, where

$$\eta(a, \theta) = \frac{B_a - \hat{C}_a}{c_{a\theta}} h \leq 0 \text{ and } z(\theta) = \int_\theta^{\overline{\theta}} \hat{C}_{u_0} h \geq 0.$$

Consider any point (a, θ) with $\theta < \overline{\theta}$, where $\eta(a, \theta) + z(\theta) = 0$. Then, since $c_{a\theta} < 0$, $B_a - \hat{C}_a > 0$, and since $B_a - \hat{C}_a$ is strictly decreasing in a using $\hat{C}_{aa} > 0$ and $c_{aa\theta} \leq 0$, it follows that $\eta_a > 0$. And since η and z are continuous in θ, it follows that α is continuous in θ.

The fact that α is continuous implies that $S(\theta) = \bar{u} - \int_{\underline{\theta}}^{\theta} c_\theta(\alpha(s), s) ds$ is continuously differentiable. Hence, z is continuously differentiable, since the integrand $\hat{C}_{u_0}(\alpha(\theta), S(\theta), \theta) h(\theta)$ is continuous. But, η is continuously differentiable as well, and so, as $\eta_a > 0$, α is continuously differentiable by the Implicit Function Theorem. □

2 Other Omitted Proofs for Section III

Proposition 3 *Consider the pure adverse selection case in which $\hat{C}(a, u_0, \theta) = \varphi(u_0 + c(a, \theta))$. Assume that $c_{aa\theta\theta}$ and $c_{a\theta\theta\theta}$ exist. If h is log-concave and $-c_{a\theta}$ is log-convex in θ, then $\alpha' > 0$.*

Proof Note that the numerator of OC' in Section A.A3 rearranges to

$$-\frac{c_{a\theta\theta}}{c_{a\theta}}+\frac{h'}{h}+\left(\frac{c_{a\theta}\hat{C}_{u_0}+\hat{C}_{a\theta}-c_{\theta}\hat{C}_{au_0}}{c_{a\theta}\hat{C}_{u_0}}\right)\frac{\hat{C}_{u_0}h}{\int_{\theta}^{\bar{\theta}}\hat{C}_{u_0}h}>0.$$

We have $\hat{C}_a=\varphi' c_a$ and $\hat{C}_{u_0}=\varphi'$, and hence $\hat{C}_{a\theta}=\varphi'' c_\theta c_a+\varphi' c_{a\theta}$, and $\hat{C}_{au_0}=\varphi'' c_a$. From this, the term in parenthesis equals 2, and thus $\alpha'>0$ for any given θ if and only if for all θ,

$$z(\theta)\equiv-\frac{c_{a\theta\theta}}{c_{a\theta}}+\frac{h'}{h}+\frac{2\varphi' h}{\int_{\theta}^{\bar{\theta}}\varphi' h}>0. \tag{13}$$

Note that $z(\bar{\theta})>0$, since the first two terms are bounded while the last term diverges as θ goes to $\bar{\theta}$. Hence, by continuity, there is a smallest type $\theta_0\in[\underline{\theta},\bar{\theta})$ such that $z(\theta)>0$ for all $\theta>\theta_0$. We wish to show that $\theta_0=\underline{\theta}$. Towards a contradiction, assume that $\theta_0>\underline{\theta}$. Then $z(\theta_0)=0$, and $z'(\theta_0)\geq 0$ (since $z(\theta)>0$ for all $\theta>\theta_0$). We will show that these two properties cannot hold simultaneously under the stated assumptions on h and $c_{a\theta}$, yielding the desired contradiction.

Assume that $z(\theta_0)=0$ and consider $z'(\theta_0)$. The second term in (13) is decreasing in θ since h is log-concave. Note next that

$$\left(\frac{c_{a\theta\theta}}{-c_{a\theta}}\right)_\theta=\left(\frac{\partial}{\partial a}\frac{c_{a\theta\theta}}{-c_{a\theta}}\right)\alpha'+\frac{\partial}{\partial\theta}\frac{c_{a\theta\theta}}{-c_{a\theta}},$$

where we recall that $(\cdot)_\theta$ is the total derivative with respect to θ. When we evaluate this expression at $\theta=\theta_0$, the first term vanishes since $\alpha'(\theta_0)=0$, and the second term is negative since $-c_{a\theta}$ is log-convex in θ. Hence, a necessary condition for $z'(\theta_0)\geq 0$ is that $\left(\varphi' h/\int_\theta^{\bar{\theta}}\varphi' h\right)_\theta$ is positive at $\theta=\theta_0$, which holds if and only if

$$\varphi'' c_a\alpha' h\int_{\theta_0}^{\bar{\theta}}\varphi' h+\varphi' h'\int_{\theta_0}^{\bar{\theta}}\varphi' h+\varphi'^2h^2\geq 0$$

when evaluated at $\theta=\theta_0$. Since the first term vanishes at θ_0, we obtain $\varphi' h'\int_{\theta_0}^{\bar{\theta}}\varphi' h+\varphi'^2h^2\geq 0$, which holds if and only if

$$\frac{h'}{h}+\frac{\varphi' h}{\int_{\theta_0}^{\bar{\theta}}\varphi' h}\geq 0.$$

But this implies that

$$z(\theta_0)=-\frac{c_{a\theta\theta}}{c_{a\theta}}+\frac{h'}{h}+\frac{2\varphi' h}{\int_{\theta_0}^{\bar{\theta}}\varphi' h}>0,$$

contradicting that $z(\theta_0)=0$. Hence, $z(\theta_0)=0$ and $z'(\theta_0)\geq 0$ cannot hold simultaneously. $\square$

We now provide sufficient conditions for $\mu_a\geq 0$ and $\lambda_a\geq 0$, which pin down the sign of C_{au_0}

and $C_{a\theta}$. Let $\rho = (\varphi')^{-1}$ map $1/u'$ into u.

Lemma 10 *Let FOP hold and let $l_{xa} < 0$. Then, $\mu_a \geq 0$. If in addition f is log-concave in a and ρ is concave, then $\lambda_a \geq 0$, $C_{au_0} \geq 0$, and $C_{a\theta} \leq 0$.*

Proof From the first-order condition of the cost-minimization problem plus the binding participation and incentive constraints, we obtain the following system of equations in λ and μ:

$$\int \rho(\lambda + \mu l(x|a)) f(x|a) dx = c(a, \theta) + u_0$$
$$\int \rho(\lambda + \mu l(x|a)) f_a(x|a) dx = c_a(a, \theta).$$

Differentiating this system and manipulating (see *CS* for details),

$$\lambda_a = -\mu_a \int l\xi - \mu \int l_a \xi \quad \text{and} \quad \mu_a = \frac{1}{var_\xi(l)} \left(\frac{1}{\int \rho' f} \left(c_{aa} - \int \rho f_{aa} \right) - \mu \, cov_\xi(l_a, l) \right), \tag{14}$$

where ξ is the density with kernel $\rho'(\lambda + \mu l(\cdot|\alpha(\theta))) f(\cdot|\alpha(\theta))$ To see that $\mu_a > 0$, note that $c_{aa} - \int \rho f_{aa} \geq 0$ by *FOP*, while $cov_\xi(l_a, l) < 0$ under the assumption $l_{ax} < 0$. Turning to λ_a, notice that $\int l\xi =_s \int l\rho' f = \int \rho' f_a$, where we recall that $=_s$ indicates that the objects on either side have strictly the same sign. Now, $\int \rho' f_a$ is negative by Lemma 12, since f_a single-crosses zero from below, $\int f_a = 0$, and ρ' is positive and decreasing in x. Since $\mu_a \geq 0$, it follows that $\lambda_a \geq 0$ if $\int l_a \xi =_s \int l_a \rho' f \leq 0$. But this holds since f is log-concave in a, which is equivalent to $l_a \leq 0$.

Recall from the proof of Lemma 3 that $C_{au_0} = \lambda_a$ and $C_{a\theta} = \lambda_a c_\theta + \lambda c_{a\theta} + \mu_a c_{a\theta} + \mu c_{aa\theta}$. Thus, $C_{au_0} \geq 0$ since $\lambda_a \geq 0$, and, given that c_θ and $c_{a\theta}$ are negative, $C_{a\theta} \leq 0$ since both λ_a and μ_a are positive. □

3 Omitted Proofs for Section V

Here, we generalize Theorem 1 to the case that α is not continuously differentiable. Such α may arise when C has less structure than we have imposed thus far, or for example, if the principal is constrained in how many contracts she can offer.

Theorem 4 *Let (α, v) satisfy IC_{MH} and IC_S, let α satisfy IMC, and assume that for each θ, $\int v(x, \theta) f(x|\cdot) dx$ is concave. Then, (α, v) is feasible in $\mathcal{P}$.*

Proof We proceed in several steps. Denote by γ the generalized inverse of α (recall that α can jump up a countable number of times).

STEP 1. By IC_S, IR holds.

Step 2. From Lemma 2, it suffices to show that every deviation $(\theta_A, \hat{a}) \notin \mathbb{G}$ is dominated by some on-graph deviation. We focus on deviations with $\hat{a} > \alpha(\theta_A)$ (the other case is similar).

Let the agent's true type be θ_T. If $\theta_T \leq \theta_A$, then

$$\int v(x,\theta_A) f(x|\hat{a})dx - \int v(x,\theta_A) f(x|\alpha(\theta_A))dx \leq c(\hat{a},\theta_A) - c(\alpha(\theta_A),\theta_A) \leq c(\hat{a},\theta_T) - c(\alpha(\theta_A),\theta_T),$$

where the first inequality follows from the first-order condition IC_{MH}, from concavity of $\int vf$ in a, and from $\hat{a} > \alpha(\theta_A)$, and the second since c is submodular. But then, $(\theta_A, \alpha(\theta_A)) \in \mathbb{G}$ dominates $(\theta_A, \hat{a})$.

Step 3. If for any given $\tilde{\theta}$, $\hat{a} > \alpha(\tilde{\theta})$ and $\theta_A \leq \tilde{\theta}$, then $(\theta_A, \alpha(\tilde{\theta}))$ dominates $(\theta_A, \hat{a})$ for type $\tilde{\theta}$. To see this, consider any action $a \in [\alpha(\tilde{\theta}), \hat{a}]$. Then

$$\int v(x,\theta_A) f_a(x|a)dx \leq \int v(x,\theta_A) f_a(x|\alpha(\theta_A))dx = c_a(\alpha(\theta_A),\theta_A) \leq c_a(\alpha(\tilde{\theta}),\tilde{\theta}) \leq c_a(a,\tilde{\theta}),$$

where the first inequality follows from concavity of $\int vf$ in a, the equality follows by IC_{MH}, the second inequality follows by *IMC,* and the third by convexity of c in a. Hence, $\int v(x,\theta_A) f_a(x|a)dx - c_a(a,\tilde{\theta}) \leq 0$ for any $a \in [\alpha(\tilde{\theta}), \hat{a}]$, and so $(\theta_A, \alpha(\tilde{\theta}))$ dominates $(\theta_A, \hat{a})$ for type $\tilde{\theta}$.

From Step 2, and from Step 3 applied to $\tilde{\theta} = \theta_T$, we can restrict attention to deviations $(\theta_A, \hat{a})$ with $\theta_A \leq \theta_T$ and $\hat{a} \in (\alpha(\theta_A), \alpha(\theta_T)]$.

Step 4. Let $(\theta_A, \hat{a})$ be such that $\hat{a} > \alpha(\theta_A)$ and $(\gamma(\hat{a}), \hat{a}) \in \mathbb{G}$, that is, $\hat{a} = \alpha(\gamma(\hat{a}))$. We will show that

$$\int v(x,\gamma(\hat{a})) f(x|\hat{a})dx \geq \int v(x,\theta_A) f(x|\hat{a})dx \tag{15}$$

and hence, subtracting $c(\hat{a},\theta_T)$ from each side, $(\theta_A, \hat{a})$ is dominated for θ_T by $(\gamma(\hat{a}), \hat{a}) \in \mathbb{G}$.

Subtract $\int v(x,\theta_A) f(x|\alpha(\theta_A))dx$ from each side of (15), and then use that $\int vf = S + c$ to arrive at the equivalent expression

$$S(\gamma(\hat{a})) + c(\hat{a},\gamma(\hat{a})) - (S(\gamma(\alpha(\theta_A))) + c(\alpha(\theta_A),\gamma(\alpha(\theta_A)))) \geq \int v(x,\theta_A) f(x|\hat{a})dx - \int v(x,\theta_A) f(x|\alpha(\theta_A))dx,$$

where in the second term on the *lhs*, we used that $\theta_A = \gamma(\alpha(\theta_A))$. Now, by Corollary 1, the *lhs* is increasing, and so, by Kolmogorov and Fomin (1970), Chapter 9, Section 33, Theorem 1, it is at least

$$\int_{\alpha(\theta_A)}^{\hat{a}} \left(\frac{\partial}{\partial a} \left(S(\gamma(a)) + c(a,\gamma(a)) \right) \right) da,$$

while by the Fundamental Theorem of Calculus, the *rhs* is equal to

$$\int_{\alpha(\theta_A)}^{\hat{a}} \left(\frac{\partial}{\partial a} \left(\int v(x,\theta_A) f(x|a)dx \right) \right) da = \int_{\alpha(\theta_A)}^{\hat{a}} \left(\int v(x,\theta_A) f_a(x|a)dx \right) da,$$

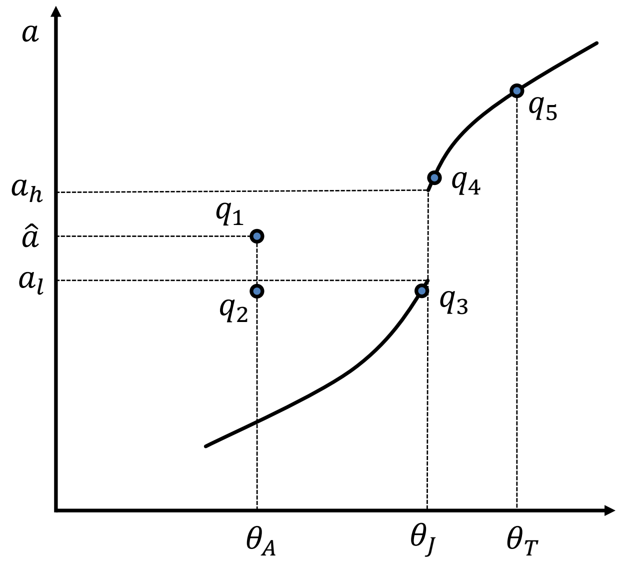

Figure 9: ***IMC***. Under *IMC*, a deviation by $\theta_J - \varepsilon$ to q_1 is dominated by one to q_2, which in turn is dominated by q_3, which, from the point of view of $\theta_J - \varepsilon$ is nearly as good as q_4. But then, from the point of view of θ_T, who has a lower incremental cost of effort, the (on-locus) point q_4 also nearly dominates q_1, and telling the truth and taking the recommended action is better yet.

and so it suffices that for all $a \in [\alpha(\theta_A), \hat{a}]$ at which $S(\gamma(a)) + c\,(a, \gamma(a))$ is differentiable,

$$\frac{\partial}{\partial a}\left(S(\gamma(a)) + c\,(a, \gamma(a))\right) \geq \int v(x, \theta_A) f_a(x|a) dx.$$

But, at points of differentiability,

$$\begin{aligned}\frac{\partial}{\partial a}\left(S(\gamma(a)) + c\,(a, \gamma(a))\right) &= \left(S'(\gamma(a)) + c_\theta\,(a, \gamma(a))\right)\gamma'(a) + c_a\,(a, \gamma(a)) \\ &= c_a\,(a, \gamma(a)) \\ &\geq c_a(\alpha(\theta_A), \theta_A),\end{aligned}$$

where the second equality follows since $S' = -c_\theta$ on $\mathbb{G}$ and since $\gamma' = 0$ where α jumps. To see the inequality, note that since $a > \alpha(\theta_A)$, it follows that $\gamma(a) \geq \theta_A$. Thus, if $(\gamma(a), a) \in \mathbb{G}$ then $c_a(a, \gamma(a)) \geq c_a(\alpha(\theta_A), \theta_A)$ by *IMC*. Otherwise, for all $\theta \in [\theta_A, \gamma(a)) \cup \{\theta_A\}$, $c_a(a, \theta) \geq c_a(\alpha(\theta), \theta) \geq c_a(\alpha(\theta_A), \theta_A)$, where the first inequality is by convexity of c in a, noting that $\theta < \gamma(a)$ implies $\alpha(\theta) < a$, and the second inequality is by *IMC*. But then, taking $\theta \uparrow \gamma(a)$, $c_a(a, \gamma(a)) \geq c_a(\alpha(\theta_A), \theta_A)$ as claimed.

Step 5. Let $(\theta_A, \hat{a})$ be such that $\hat{a} > \alpha(\theta_A)$ and $\hat{a} \neq \alpha(\gamma(\hat{a}))$. Then α jumps at $\theta_J = \gamma(\hat{a})$ from a_l to a_h with $\hat{a}$ within the jump. And, recalling $\hat{a} \in (\alpha\,(\theta_A)\,, \alpha(\theta_T)]$, $\theta_T \geq \theta_J$. See Figure 9.

For any $\varepsilon \in (0, \theta_J - \theta_A)$, note that

$$\int v(x,\theta_A) f(x|\hat{a}) dx - c(\hat{a}, \theta_J - \varepsilon) \leq \int v(x,\theta_A) f(x|\alpha(\theta_J - \varepsilon)) dx - c(\alpha(\theta_J - \varepsilon), \theta_J - \varepsilon)$$

by Step 3. That is, type $\theta_J - \varepsilon$ prefers to move from q_1 to q_2 in Figure 9. But, by Step 4,

$$\begin{aligned} &\int v(x,\theta_A) f(x|\alpha(\theta_J - \varepsilon)) dx - c(\alpha(\theta_J - \varepsilon), \theta_J - \varepsilon) \\ &\leq \int v(x,\theta_J - \varepsilon) f(x|\alpha(\theta_J - \varepsilon)) dx - c(\alpha(\theta_J - \varepsilon), \theta_J - \varepsilon) \\ &= S(\theta_J - \varepsilon) \end{aligned}$$

corresponding to the move by $\theta_J - \varepsilon$ from q_2 to q_3. Finally, by imitating type $\theta_J + \varepsilon$ (that is to say, at q_4), $\theta_J - \varepsilon$ obtains

$$S(\theta_J + \varepsilon) + c(\alpha(\theta_J + \varepsilon), \theta_J + \varepsilon) - c(\alpha(\theta_J + \varepsilon), \theta_J - \varepsilon).$$

But, since S is continuous, and since $\alpha(\theta_J + \varepsilon)$ is increasing and so has a well-defined and finite limit as $\varepsilon \downarrow 0$, it follows that for any given $\delta > 0$, and for ε small enough,

$$S(\theta_J - \varepsilon) \leq \delta + S(\theta_J + \varepsilon) + c(\alpha(\theta_J + \varepsilon), \theta_J + \varepsilon) - c(\alpha(\theta_J + \varepsilon), \theta_J - \varepsilon),$$

which is to say that $\theta_J - \varepsilon$ is hurt by at most δ by moving from q_3 to q_4. Combining, we have

$$\int v(x,\theta_A) f(x|\hat{a}) dx - c(\hat{a}, \theta_J - \varepsilon) \leq \delta + S(\theta_J + \varepsilon) + c(\alpha(\theta_J + \varepsilon), \theta_J + \varepsilon) - c(\alpha(\theta_J + \varepsilon), \theta_J - \varepsilon),$$

and so, since $\theta_J - \varepsilon < \theta_T$, and since c is submodular,

$$\begin{aligned} \int v(x,\theta_A) f(x|\hat{a}) dx - c(\hat{a}, \theta_T) &\leq \delta + S(\theta_J + \varepsilon) + c(\alpha(\theta_J + \varepsilon), \theta_J + \varepsilon) - c(\alpha(\theta_J + \varepsilon), \theta_T) \\ &\leq \delta + S(\theta_T), \end{aligned}$$

where the second inequality uses Lemma 2. But, $\delta > 0$ was arbitrary, and so θ_T prefers q_5, where he announces his true type and takes the recommended action to q_1, and we are done. □

4 Numerical Details for Section VI

4.1 Defining primitives

First, we define all the primitives of the model: the utility function, the effort cost function, output distribution, type distribution, principal's benefit from the effort, a minimum payment constraint, and a reservation utility.

Recall that ρ maps $1/u'$ into utility, and $\varphi \circ \rho$ maps $1/u'$ into income. Define

$$\gamma(a, \lambda, \mu) \equiv \int \rho\left(\lambda + \mu \ell(x|a)\right) f(x|a) dx$$

as the utility from income that the agent receives at the contract associated with λ, μ, and a, and

$$\gamma_a(a, \lambda, \mu) \equiv \int \rho\left(\lambda + \mu \ell(x|a)\right) f_a(x|a) dx$$

as the associated marginal incentive.

Let

$$\tilde{C}_a(a, \theta, \lambda, \mu) = \mu\left[c_{aa}(a, \theta) - \int \rho(\lambda + \mu \ell(x|a)) f_{aa}(x|a) dx\right] + \int \varphi(\rho(\lambda + \mu \ell(x|a))) f_a(x|a) dx.$$

Note that by the Envelope Theorem, if for given (a, s, θ) one finds the associated λ and μ satisfying (2) and (3), then $C_a(a, s, \theta) = \tilde{C}_a(a, \theta, \lambda, \mu)$. Similarly, recall that $C_{u_0}(a, s, \theta) = \lambda$, where λ is the associated participation constraint multiplier.

4.2 Plan of Attack

We first solve the problem assuming there is no exclusion. In particular, we characterize the surplus given to the highest type such that all types participate. Subsequently, we search over multiple values of the highest type's surplus to find the optimal exclusion threshold.

We will work with a discretization of Θ for tractability. Recall that in the solution to our decoupled problem, the surplus of the agent depends on the actions being assigned to all types *below* θ but the cost of changing the effort of the agent depends on the additional cost of providing an extra util to all types *above* θ. Motivated by this, in the Online Appendix, Section 1, we work with a system of two differential equations involving the objects $S(\theta)$ and $Z(\theta)$ where S corresponds to the surplus given to agent θ, corresponding to $-\int_{\underline{\theta}}^{\theta} c_\theta(\alpha(t), s) ds$, and Z is the "externality" term that captures the cost of giving utility to types above θ, corresponding to the expression $\int_{\theta}^{\overline{\theta}} C_{u_0}(\alpha(t), S(t), t) h(t) dt$. The boundary condition on S is evaluated at the lowest type, while the boundary condition on Z is evaluated at the highest type.

Our plan of attack is first to construct a numerical function that "locally" finds the relevant values of a, λ, and μ for given θ and for given values s and z of S and Z. Then, we use this function to solve the system of differential equations. We do this by guessing a value of surplus for the highest type. Then, we solve iteratively from top to bottom to find the surplus this implies for all lower types, particularly for the lowest type. If the lowest type utility is not equal to the outside option, we adjust the guess for the surplus value for the highest type and repeat.

4.3 Solving Locally

Following equation (12) in Section 1 of the Online Appendix, define

$$\chi(a, z, \theta, \lambda, \mu) \equiv B_a(a) - \tilde{C}_a(a, \theta, \lambda, \mu) + \frac{c_{a\theta}(a, \theta)}{h(\theta)} z. \tag{16}$$

The function χ represents the principal's first-order condition for the choice of recommended effort for type θ given the cost of implementing the action for the type in question, and the incremental surplus that her choice of action implies must be given to all types above. Let us first discuss how we take any given type of agent, the surplus that they receive s, and the cost of providing utility to higher types z and calculate what effort and associated multipliers the principal will choose. That is, for each vector (θ, s, z) we must find (λ, μ, a) such that

$$\gamma(a, \lambda, \mu) = c(a, \theta) + s, \quad \gamma_a(a, \lambda, \mu) = c_a(a, \theta), \text{ and } \chi(a, z, \theta, \lambda, \mu) = 0. \tag{17}$$

The first equation represents the participation constraint; the second is the agent's effort choice first-order condition (incentive compatibility), and the third is the principal's first-order condition with respect to the recommended effort. In a solution to the decoupled problem, all three must hold with equality[1].

To find the solution to (17), we define the numerical function "*solver*." *Solver* begins by using least squares to look for the root of this non-linear system of equations.[2] If the least squares approach does not find roots for the three constraints, the function *solver* sets $\lambda = 0$ and looks for roots of (3) and χ only using (μ, a). We check whether the problem is solved, assuming that (2) is slack. We then check whether (2) is indeed slack at this solution. Finally, if this second approach does not find the roots, the solution must be a corner effort level. We set $a = 1$ and find (λ, μ) that satisfies IC and IR. For this last step, we define the numerical function *multipliers*. The function *multipliers* uses least squares to find (λ, μ) that make (2) and (3) satisfied with equality while holding effort at the maximum level[3].

4.4 Solving the System

For numerical tractability, we discretize Θ with $n = 101$ equally spaced points. Recall that the boundary condition regarding participation binds at the lowest type, while the externality term

[1] Unless the participation constraint is slack, given the minimum payment constraint. In this case, $\gamma(a, \lambda, \mu) > c(a, \theta) + s$ and the other two expressions on (17) hold with equality. We discuss how to deal with such a case in more detail below.

[2] The numerical function *solver* requires initial guesses for (λ, μ, a). For clarity, they will be omitted in this algorithm description.

[3] In case we cannot find roots for (2) and (3) simultaneously, it must be the case that effort $a = 1$ and the participation constraint is slack. We then set $a = 1$, $\lambda = 0$, and look for μ that satisfies (3) with equality using a root finding function brentq.

$z(\theta)$ depends on the efforts of all types above it.

To address this, we construct a numerical function called "*surplus*" that guesses the surplus of the highest type and iteratively calculates from top to bottom efforts, multipliers, surpluses, and externalities $\{(a_i, \lambda_i, \mu_i, s_i, z_i)\}_{i=0}^{n-1}$ for each type θ_i starting from the initial condition that z is zero for the highest type. The algorithm of *surplus* follows:

1. Set $s_{n-1} = u^*$ and $z_{n-1} = 0$.[4] Then, for i iteratively decreasing from $n-2$ to 0:

2. Given s_{i+1}, z_{i+1}, calculate the effort and multipliers of type θ_{i+1}. That is, let

$$(a_{i+1}, \lambda_{i+1}, \mu_{i+1}) = solver(\theta_{i+1}, s_{i+1}, z_{i+1}). \tag{18}$$

   The function *solver* also requires initial guesses for (a, λ, μ),. For speed, we use the ones calculated for the immediately higher type. For the highest type, we use an arbitrary guess.

3. Then, compute the surplus and externality of type θ_i. That is, let

$$s_i = s_{i+1} + \frac{1}{n-1} c_\theta(a_{i+1}, \theta_{i+1}), \tag{19}$$

   and

$$z_i = z_{i+1} + \frac{1}{n-1} \lambda_{i+1} h(\theta_{i+1}). \tag{20}$$

4. Finally, let

$$(a_0, \lambda_0, \mu_0) = solver(\theta_0, s_0, z_0).$$

Equation (18) describes the effort and multipliers for type θ_{i+1} given that he receives a surplus of s_{i+1}. Equation (19) computes the surplus of the type immediately below, and equation (20) computes the additional cost of providing one extra util to all types above θ_i. That is, for each surplus left to type θ_{i+1}, we use *solver* to find the effort, multipliers, and surplus of type θ_i. We run this set n times to arrive at $s_0(u^*)$. Since we are using Euler's method, standard results imply that the error term in this approximation versus the continuous system is of order $1/n$.

Note that for each guess $s_{n-1} = u^*$, the numerical function *surplus* outputs a vector of surpluses, including the surplus of the lowest type. Recall that in the solution to the decoupled problem, the participation of the lowest type must bind. Then, we search for the correct highest type's surplus guess that makes $s_0 = \bar{u}$. We do so by using the pre-built Python root finding function brentq[5]. Finally, we find the solution to the decoupled problem by evaluating our constructed function *surplus* at the correct u^* that makes $s_0(u^*) = \bar{u}$.

[4]Python indexes an array from 0 to $n-1$; hence $n-1$ corresponds to the highest type.

[5]For speed, if at some point when calculating the surpluses given an initial guess for s_{n-1}, some s_i falls below $\bar{u}$, then we stop the code because the initial guess for the high type's surplus $s_{n-1} = u^*$ was too low, and simply set s_0 to something below $\bar{u}$.

4.5 Checking *IMC*

Checking *IMC* simply requires verifying if the marginal effort cost of each type at their respective recommended effort is increasing in θ. That is, for each i from 0 to $n-2$ check whether

$$c_a(\alpha_i^*, \theta_i) \leq c_a(\alpha_{i+1}^*, \theta_{i+1}),$$

where α_i^* denotes the recommended efforts in the solution of the decoupled problem. If yes, then *IMC* is satisfied, and the solution to the decoupled problem solves the original one.

4.6 Optimal Exclusion

Given the solution to the procedure above, we have characterized the surplus to the highest type s_{n-1} that assures all types participate. Denote such surplus by $\bar{s}$. We then create a grid (with 101 equally spaced points) with surpluses from $\bar{u}$ to $\bar{s}$. Each level of such surplus, when assigned to the highest type, generates an exclusion cutoff $\theta_c(s)$. For instance, $s_{n-1} = \bar{s}$ generates no exclusion (i.e., $\theta_s = \underline{\theta}$), while $s_{n-1} = \bar{u}$ implies on excluding all types but the highest. We then evaluate the numerical function *surplus* at each s in the grid and compute its associated profits. The one that generates the highest profit is the optimal one, with the optimal associated exclusion cutoff.

4.7 Checking if C is convex

When looking for the optimal recommended effort for each type, we rely on the first-order condition regarding effort recommendation (4). For such a first-order condition to be sufficient, we need the function $C(a, s, \theta)$ to be convex in (a, s). To test whether such a condition is likely to be satisfied, we sample many pairs (a, s) and θ's to check if the resulting $C(a, s, \theta)$ violates convexity in (a, s) for any of the tuples sampled. If it violates, then the pure moral hazard cost is not convex. Otherwise, if our sample is large, the function will likely be convex.

Define the numerical function *multipliers*, which takes (a, s, θ) as given, and looks for a pair (λ, μ) that makes (2) and (3) hold. The approach is similar to before (using least squares), but now we hold a as exogenous. We compute the moral hazard cost using such multipliers and $\tilde{C}(a, \lambda, \mu)$ defined by

$$\tilde{C}(a, \lambda, \mu) \equiv \int \varphi(\rho(\lambda + \mu \ell(x|a))) f(x|a) dx.$$

Note that if for given (a, s, θ) one finds the associated λ and μ, then $C(a, s, \theta) = \tilde{C}(a, \lambda, \mu)$.

We sample *n-test* random tuples of $\{(a_1^i, s_1^i), (a_2^i, s_2^i), \theta^i\}_{i=1}^{n\text{-}test}$. For each i, we compute the average of pairs (a_1^i, s_1^i) and (a_2^i, s_2^i) and the multipliers associated with each pair of effort and surplus given θ^i. That is, let

$$(a^i_m, s^i_m) \equiv \left(\frac{a^i_1 + a^i_2}{2}, \frac{s^i_1 + s^i_2}{2}\right) \quad \forall i \in \{1, ..., n\text{-}test\},$$

and

$$(\lambda^i_j, \mu^i_j) \equiv multipliers(a^i_j, s^i_j, \theta^i) \quad \forall i \in \{1, ..., n - test\}, j \in \{1, 2, m\}.$$

Finally, we check whether convexity holds for each i. That is, we check if for each i

$$\tilde{C}(a^i_1, \lambda^i_1, \mu^i_1) + \tilde{C}(a^i_2, \lambda^i_2, \mu^i_2) \geq 2\tilde{C}(a^i_m, \lambda^i_m, \mu^i_m).$$

If this condition fails for any i, then C is not convex. Otherwise, C is likely to be convex. As a practical matter, we take 500 draws.

4.8 Numerical Application Parametrized by $\tau \in [0, 1]$

In Section VI.C, we state that one can parametrize the numerical application by $\tau \in [0, 1]$ and capture the pure moral hazard and the pure adverse selection cases when $\tau = 1$ and $\tau = 0$, respectively. To do so, let the agent's income utility be $u(w) = \sqrt{2w}$ and the disutility of effort c be given by

$$c(a, \theta, \tau) = \left(\frac{3}{2} - (1 - \tau)\left(\theta - \frac{1}{2}\right)\right) ae^{a^2 - 1}.$$

Thus $\tau = 0$ gives back the original disutility of effort used in Section VI.C, while $\tau = 1$ makes the disutility of effort independent of type, and we obtain a pure moral hazard problem. Similarly, parametrize the distribution of the signal in such a way that it becomes perfectly informative about effort at $\tau = 0$, which reduces to a pure adverse selection problem. To do this, let f be the density used in Section VI.C, and introduce the following density parametrized by τ:

$$g(x|a, \tau) = \frac{f(x|a)e^{-\frac{(x-a)^2}{\tau}}}{\int f(s|a)e^{-\frac{(s-a)^2}{\tau}} ds}.$$

This density satisfies *MLRP* and is degenerate at a when $\tau = 0$.[6]

[6]The only other change in comparison to the baseline example is setting $B(a) = 100\mathbb{E}[x|a]$. The outside option value $\bar{u}$, and distributions f and h remain as in the baseline example.

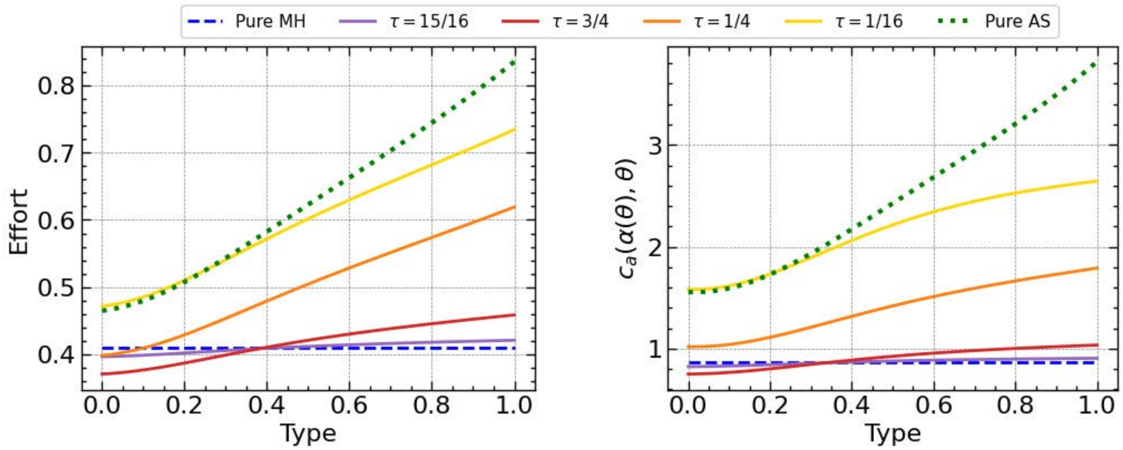

Figure 10: Convergence of Effort Schedule and *IMC* for Different θ's.

Figures 10 and 11 illustrate the optimal menu and verify *IMC* as τ varies. The left panel of Figure 10 shows how the effort schedules converge to pure moral hazard at $\tau = 1$ (flat effort level), and to the pure adverse selection case at $\tau = 0$. The right panel checks that we do not violate *IMC* as the values of τ change. In turn, Figure 11 depicts the compensation schemes for different values of θ as τ varies. The scheme is flat for each type under pure adverse selection when $\tau = 0$, and is increasing under pure moral hazard when $\tau = 1$.

Note that in Figure 10 the recommended efforts for different τ values are not consistently below the plotted pure moral hazard and pure adverse selection curves. One might wonder if this contradicts the intuition that, when decoupling holds, the distortions caused by moral hazard and adverse selection reinforce each other (see Section VI.C). It does not. In Section VI.C, we change whether the principal can directly observe efforts and/or types while holding fixed the primitives of the model, including c and f. In the example above, however, as we vary τ, we simultaneously affect both c and f. The explanation provided in Section VI.C would apply here if we fixed τ at some given value but varied what the principal could directly observe.

In this numerical application, the computation uses the fact that we have closed-form solutions for the multipliers λ and μ with a square root utility function. In particular, for a given (a, s, θ, τ) we get

$$\lambda(a, s, \theta, \tau) = s + c(a, \theta, \tau), \quad \text{and} \quad \mu(a, s, \theta, \tau) = \frac{c_a(a, \theta, \tau)}{\int l^2(x|a, \tau) f(x|a, \tau) dx}.$$

Hence, instead of searching for a triple of variables (a, λ, μ) that satisfy three equations — as described in Section 4 — one can look for the effort level that finds the root of (16) considering that λ and μ are a function of effort as described above.

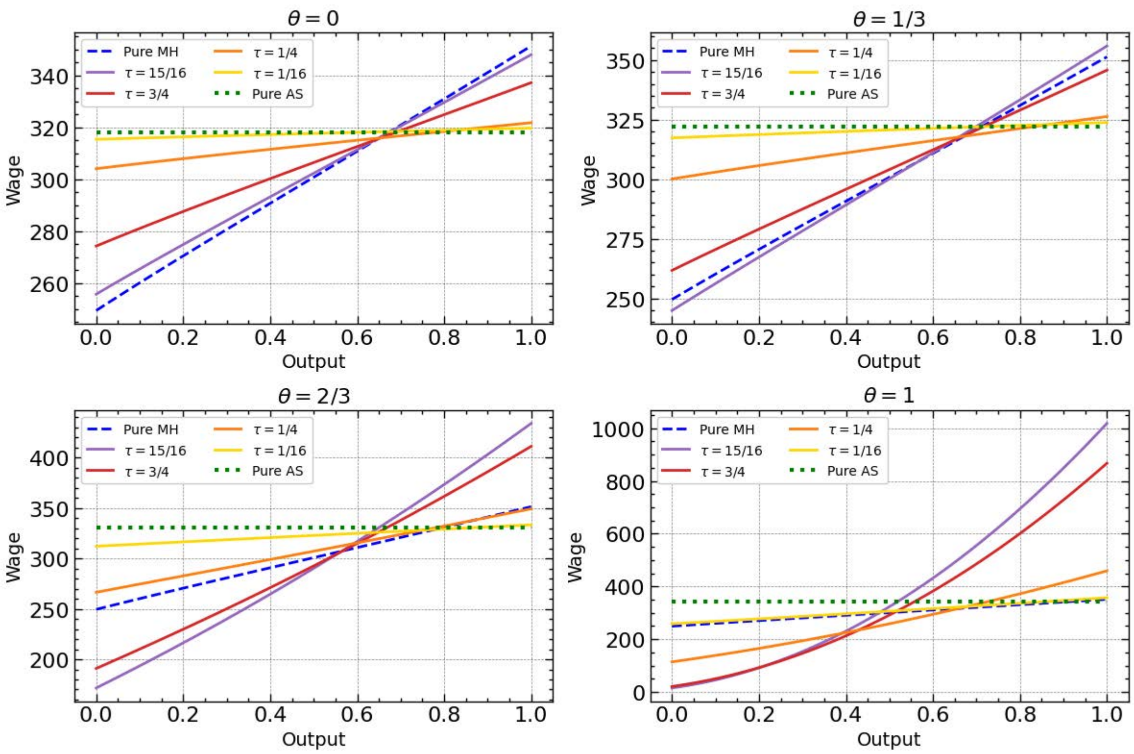

Figure 11: Compensation Schemes for Different θ's.

5 Omitted Proofs for Section VII.A

Proof of Lemma 5 Let T be any continuous distribution. By Theorem 1 in Cuadras (2002) specialized to our setting, for any C^2 function ζ of q,

$$\begin{aligned} cov_T(\zeta(q), q) &= \int \left(\int (T(\min(q, y)) - T(q)T(y))dy \right) \zeta'(q)dq \\ &= \int \left(\int_{\underline{l}}^{q} (T(y) - T(q)T(y))dy + \int_{q}^{\bar{l}} (T(q) - T(q)T(y))dy \right) \zeta'(q)dq \\ &= \int M_T(q)\zeta'(q)dq, \end{aligned}$$

where $M_T(q) = (1 - T(q)) \int_{\underline{l}}^{q} T(y)dy + T(q) \int_{q}^{\bar{l}} (1 - T(y))dy$, which is strictly positive on $(\underline{l}, \bar{l})$. Thus, since $var_T(q) = cov_T(q, q)$,

$$\frac{cov_T(q^2, q)}{var_T(q)} = \frac{2 \int M_T(q)qdq}{\int M_T(q)dq} = 2 \int m_T(q)qdq$$

where $m_T(\cdot)$ is the density given by $M_T(\cdot)/\int M_T(q)dq$. Since q is increasing, it is thus sufficient for the result that $m_{\hat{G}}/m_G$, or equivalently, $M_{\hat{G}}/M_G$ is increasing.

Now, $M_T(q) = T\left(\bar{l} - q - \int T\right) + \int_{\underline{l}}^q T = T(\mu_T - q) + \int_{\underline{l}}^q T > 0$, where μ_T is the expectation of q under T. Thus, $M_T' = t(\mu_T - q)$, and so,

$$\left(\frac{M_{\hat{G}}(q)}{M_G(q)}\right)_q =_s \frac{\hat{g}}{g}(\mu_{\hat{G}} - q)\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) - (\mu_G - q)\left(\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^q \hat{G}\right) \equiv Z(q).$$

We thus have

$$\begin{aligned}
Z' &= \left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q)\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) - \frac{\hat{g}}{g}\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) + \frac{\hat{g}}{g}(\mu_{\hat{G}} - q)\, g(\mu_G - q) \\
&\quad + \left(\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^q \hat{G}\right) - (\mu_G - q)\,\hat{g}(\mu_{\hat{G}} - q) \\
&= \left(\left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q) - \frac{\hat{g}}{g}\right)\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) + \left(\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^q \hat{G}\right).
\end{aligned}$$

where we note that since $\hat{g}/g$ is continuously differentiable, so is Z.

Consider first $q \in (\underline{l}, \mu_G)$. If $Z < 0$, then

$$\hat{G}(\mu_{\hat{G}} - q) + \int_{\underline{l}}^q \hat{G} > \frac{\hat{g}}{g}\frac{\mu_{\hat{G}} - q}{\mu_G - q}\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right),$$

and so

$$\begin{aligned}
Z' &> \left(\left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q) - \frac{\hat{g}}{g}\right)\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) + \frac{\hat{g}}{g}\frac{\mu_{\hat{G}} - q}{\mu_G - q}\left(G(\mu_G - q) + \int_{\underline{l}}^q G\right) \\
&=_s \left(\frac{\hat{g}}{g}\right)_q (\mu_{\hat{G}} - q) + \frac{\hat{g}}{g}\left(\frac{\mu_{\hat{G}} - q}{\mu_G - q} - 1\right) > 0,
\end{aligned}$$

noting that for $q < \mu_G$, $(\mu_{\hat{G}} - q)/(\mu_G - q) > 1$, and that $\hat{g}/g$ is increasing, and also recalling that the eliminated term is strictly positive except at the endpoints. But then, since $Z(\underline{l}) = 0$, Z is everywhere positive on $[\underline{l}, \mu_G]$. In particular, if $Z(\hat{q}) < 0$, then let $\tilde{q} \in [\underline{l}, \hat{q})$ be such that $Z(\tilde{q}) = 0$, and $Z(q) < 0$ on $(\tilde{q}, \hat{q}]$, where such a $\tilde{q}$ exists by continuity of Z. Then,

$$0 > Z(\hat{q}) - Z(\tilde{q}) = \int_{\tilde{q}}^{\hat{q}} Z'(q)\, dq > 0,$$

where the equality follows from the Fundamental Theorem of Calculus since Z is continuously differentiable. This is a contradiction. Similarly, $Z' < 0$ everywhere on $[\mu_{\hat{G}}, \bar{l})$, and so, since $Z(\bar{l}) = 0$, Z is everywhere positive on $[\mu_{\hat{G}}, \bar{l}]$. Finally, $Z(q) > 0$ on $[\mu_G, \mu_{\hat{G}}]$ since $\mu_{\hat{G}} - q$ and $-(\mu_G - q)$ are positive, with one of them strictly so. Thus, Z is everywhere positive. But then,

$M_{\hat{G}}/M_G$ is increasing, and we are done. □

Lemma 11 *Let f_L and f_H be strictly positive densities on $[0,1]$, with $skew_{F_L}(x) \leq 0$ and f_H/f_L increasing and concave. Let $f(x|a) = af_H + (1-a)f_L$ be the linear combination of f_L and f_H. Then $skew_F(l) \leq 0$ for all a.*

Proof For each a, since $\mathbb{E}_F(l) = 0$, and defining $r = f_H/f_L$,

$$skew_F(l) =_s \int l^3 f = \int \frac{f_a^3}{f^2} = \int \frac{(f_H - f_L)^3}{(af_H + (1-a)f_L)^2} dx = \int \frac{(r(x) - 1)^3}{(ar(x) + (1-a))^2} f_L(x) dx.$$

Differentiation shows that the last expression is decreasing in a. Hence, it is enough that $\int (r-1)^3 f_L(x) dx \leq 0$. But, since $\mathbb{E}_{F_L}[r] = 1$, $\int (r(x)-1)^3 f_L(x) dx =_s skew_{F_L}(r)$. Finally, since x is a convex increasing transformation of r, it follows from Theorem 3.1 in van Zwet (2012) that $skew_{F_L}(r) \leq skew_{F_L}(x)$, which is negative by assumption, and so we are done. □

In Footnote 28 we mentioned the two-outcome case. Since in this case $C(a, u_0, \theta) = a\varphi_h + (1-a)\varphi_l$, where $\varphi_i = \varphi(v_i)$, $i = l, h$, with $v_h = u_0 + c(a,\theta) + (1-a)c_a(a,\theta)$ and $v_l = u_0 + c(a,\theta) - ac_a(a,\theta)$, we obtain

$$C_a(a, u_0, \theta) = \varphi_h - \varphi_l + a(1-a) c_{aa} (\varphi_h' - \varphi_l').$$

Thus,

$$\begin{aligned} C_{aa}(a) &= \left(\varphi_h'(2-3a) - \varphi_l'(1-3a)\right) c_{aa} + a(1-a)\left(c_{aaa}(\varphi_h' - \varphi_l') + c_{aa}^2 \left(\varphi_h''(1-a) + \varphi_l'' a\right)\right) \\ &\geq \left(\varphi_l' + \varphi_h'(2-3a) - \varphi_l'(2-3a)\right) c_{aa} + a(1-a) c_{aaa} (\varphi_h' - \varphi_l') \\ &> (\varphi_h' - \varphi_l')(2-3a) c_{aa} + a(1-a) c_{aaa} (\varphi_h' - \varphi_l'), \end{aligned}$$

and so the first inequality in Footnote 28 is sufficient for $C_{aa} > 0$.

From Lemma 3 and (A5), strict *IMC* is guaranteed if

$$(c_{a\theta} C_{u_0} - C_{au_0} c_\theta + C_{a\theta}) c_{aa} < c_{a\theta} C_{aa}.$$

But, $C_{u_0} = a\varphi_h' + (1-a)\varphi_l'$, and so $C_{au_0} = \varphi_h' - \varphi_l' + a(1-a) c_{aa} (\varphi_h'' - \varphi_l'')$, and

$$\begin{aligned} C_{a\theta} &= \varphi_h'(c_\theta + (1-a) c_{a\theta}) - \varphi_l'(c_\theta - ac_{a\theta}) + a(1-a) c_{aa\theta}(\varphi_h' - \varphi_l') \\ &\quad + a(1-a) c_{aa}\left(\varphi_h''(c_\theta + (1-a) c_{a\theta}) - \varphi_l''(c_\theta - ac_{a\theta})\right). \end{aligned}$$

Substituting and manipulating, we want

$$\begin{aligned}&\left(c_{a\theta}\left(\varphi_l'+\varphi_h'\right)+a\left(1-a\right)\left[c_{aa}c_{a\theta}\left(\varphi_h''\left(1-a\right)+\varphi_l''a\right)+c_{aa\theta}\left(\varphi_h'-\varphi_l'\right)\right]\right)c_{aa}\\ &<c_{a\theta}c_{aa}\left(\varphi_h'(1-a)+\varphi_l'a\right)+c_{a\theta}\left(\left(1-2a\right)c_{aa}+a\left(1-a\right)c_{aaa}\right)\left(\varphi_h'-\varphi_l'\right)\\ &\quad+c_{a\theta}a\left(1-a\right)c_{aa}^2\left(\varphi_h''(1-a)+\varphi_l''a\right),\end{aligned}$$

or, $c_{aa}c_{a\theta}\varphi_l'+\left[a\left(1-a\right)\left(c_{aa}c_{aa\theta}-c_{a\theta}c_{aaa}\right)+\left(3a-1\right)c_{a\theta}c_{aa}\right]\left(\varphi_h'-\varphi_l'\right)<0$, and so, since $\varphi_l'>0$, it is sufficient that the term in square brackets is negative, or, equivalently, the second inequality in Footnote 28 holds.

6 Analysis for Section VIII

6.1 Linear Output Case and Necessity of *IMC*

We first analyze screening in the *linear-output case* with the addition of *IMC* as a constraint. To this end, for a given interval $[\theta_1,\theta_2]$, let $\phi(\theta)=0$ for $\theta<\theta_1$, $\phi(\theta)=\int_{\theta_1}^{\theta}\left(-c_{a\theta}/c_{aa}\right)d\tau$ for $\theta\in[\theta_1,\theta_2]$, and $\phi(\theta)=\int_{\theta_1}^{\theta_2}\left(-c_{a\theta}/c_{aa}\right)d\tau$ for $\theta>\theta_2$ (we will provide intuition shortly). We then have the following theorem that presents a general optimality condition that allows for ironing.

Theorem 5 (Optimality in Linear Probability Case) *Let $F_{aa}=0$, let $C(\cdot,\cdot,\theta)$ be convex, and let $\theta_1<\theta_2$ be such that IMC is slack immediately to the left of θ_1 and right of θ_2.*[7] *Then,*

$$\int_{\theta_1}^{\theta_2}\left(B_a-C_a\right)\frac{1}{c_{aa}}h=\int C_{u_0}\phi h. \tag{21}$$

At any point where IMC is slack, $B_a-C_a=\left(-c_{a\theta}/h\right)\int_{\theta}^{\bar{\theta}}C_{u_0}h$ as in OC.

Proof Consider a candidate solution, and let $[\theta_1,\theta_2]$ be any interval with the property that *IMC* is slack to the immediate right of θ_2 (as is automatic when $\theta_2=\bar{\theta}$) and immediate left of θ_1 (as is automatic when $\theta_1=\underline{\theta}$). Consider first shifting the action schedule up by an amount solving $c_a\left(\hat{\alpha}\left(\theta,\varepsilon\right),\theta\right)=c_a\left(\alpha\left(\theta\right),\theta\right)+\varepsilon$ on the interval $[\theta_1,\theta_2]$. That is, add a constant to c_a on this interval. Next, if ε is positive, set $c_a\left(\hat{\alpha}\left(\theta,\varepsilon\right),\theta\right)=c_a\left(\hat{\alpha}\left(\theta_2,\varepsilon\right),\theta_2\right)$ on an interval immediately to the right of θ_2 so as to reestablish *IMC*, where this interval will disappear as ε gets small, since *IMC* is strictly slack to the right of θ_2. Similarly, if ε is negative, then adjust $\hat{\alpha}$ on an arbitrarily small interval to the left of θ_1. Set surplus to change at rate given by ϕ. Since $\phi=0$ on $[\underline{\theta},\theta_1]$, IR continues to hold. The rate of change of the profit of the principal is

$$\int_{\theta_1}^{\theta_2}\left(B_a-C_a\right)\frac{1}{c_{aa}}h-\int C_{u_0}\phi h,$$

[7] These are automatic when, respectively, $\theta_1=\underline{\theta}$ or $\theta_2=\bar{\theta}$.

and so, since the schedule is optimal, we have (21).

As a sanity check, if IMC is slack at θ_2, then (21) holds on a neighborhood. Take ϕ as a function of θ_2, and differentiate both sides with respect to θ_2, to arrive at $(B_a - C_a)\,\frac{1}{c_{aa}}h = \int C_{u_0}\phi_{\theta_2}h$. But, $\phi_{\theta_2} = 0$ for $\theta < \theta_2$, and ϕ_{θ_2} is $-c_{a\theta}/c_{aa}$ evaluated at θ_2 for $\theta > \theta_2$, yielding OC. □

As usual, if IMC is slack then the action is distorted strictly downwards for all but the highest type. More generally, the lhs of (21) is proportional to a weighted expectation of $B_a - C_a$, and the rhs is strictly positive for any $\theta_1 < \theta_2 \leq \bar{\theta}$, and so effort is again distorted down in, but now in an expected sense. The idea behind (21) is to change actions on $[\theta_1, \theta_2]$ at rate $1/c_{aa}$. This changes c_a by a constant, and hence maintains IMC. By inspection, surplus then changes at rate ϕ if one holds fixed surplus at θ_1 and below. Condition (21) is that the benefit and costs of the perturbation are in balance.[8,9]

Note that while OC can be viewed as a generalization of the standard intuition of a screening problem, over regions where the problem is ironed, we are dealing with an implication that fundamentally depends on the two problems being present *simultaneously*. In particular, the fact that we need IMC as opposed to the usual (weaker) condition that α is increasing arises precisely because of the possibility that the agent might deviate *both* in his announcement and his action from the candidate equilibrium.

6.2 A Second Sufficient Condition: Single Crossing

In this section, we derive our second sufficient condition for incentive compatibility. The following lemma (Beesack (1957)) is central to our analysis.

Lemma 12 *(Beesack's inequality). Let $g : X \to \mathbb{R}$ be an integrable function with domain an interval $X \subseteq \mathbb{R}$. Assume that g is never first strictly positive and then strictly negative, and that $\int_X g(x)dx \geq 0$. Then, for any positive increasing function $h : X \to \mathbb{R}$ such that gh is integrable, $\int_X g(x)h(x)dx \geq 0$. If h is strictly increasing, and g is non-zero on some interval of positive length, then the inequality is strict. If $\int_X g(x)dx = 0$, then h need not be positive.*

First let us consider the case in which there are no jumps in the action schedule.

Theorem 6 *Let menu (α, S) be feasible in $\mathcal{P}_D$, and for each θ, let $v(\cdot, \theta) = v^{MH}(\cdot, \alpha(\theta), S(\theta), \theta)$. Let α be continuous, let v satisfy SCC, and let FOP hold. Then (α, v) is feasible in $\mathcal{P}$. Thus, if (α, S) is optimal in $\mathcal{P}_D$, then the associated (α, v) solves $\mathcal{P}$.*

[8] We believe that tools similar to those in the standard ironing literature (Guesnerie and Laffont (1984)) would allow us to further characterize where the ironed regions lay if the solution to $\mathcal{P}_D$ has a simple structure.

[9] Even over "pooling" regions where c_a is constant, effort is strictly increasing in θ, and so the optimal compensation scheme, which depends on $f(\cdot|a)$, is changing. The only exception in the linear case is that of two outcomes, where the compensation scheme is completely tied down by IC_{MH} and IR (or as in Castro-Pires and Moreira (2021), by IC_{MH} and limited liability).

So, if $\mathcal{P}_D$ with its associated v yields a solution satisfying SCC, then that solution is optimal in $\mathcal{P}$. For intuition, consider a type θ_T who contemplates a double deviation $(\theta_A, \hat{a})$, where $\hat{a} = \alpha(\hat{\theta})$ for some $\hat{\theta} > \theta_A$ and so, as in Figure 1, we are above the graph of α. We will show that the agent is better off, holding fixed the action at $\hat{a}$, to increase his announcement, sliding horizontally to the right until $\hat{\theta}$ is reached, and we are back on the graph, where θ_T is better off reporting the truth. In particular, consider any $\theta < \hat{\theta}$, and, consider the effect of a small increase in the announced type. Under SCC this increases the agent's income at high signals and lowers it at low signals. On the graph, the agent is indifferent about this trade-off by IC_A. But then, above the graph, where he is working harder, and thus more likely to attain high signals, the trade-off is profitable.

Proof of Theorem 6 Let menu (α, S) be feasible in $\mathcal{P}_D$, and for each θ, let $v(\cdot, \theta) = v^{MH}(\cdot, \alpha(\theta), S(\theta), \theta)$. By IC_S, IR holds. As in the proof of Theorem 1 it suffices to show that for any given θ_T, any deviation to $(\theta_A, \hat{a})$ with $\hat{a} > \alpha(\theta_A)$ is dominated by a deviation on $\mathbb{G}$. A symmetric argument holds for $\hat{a} < \alpha(\theta_A)$.

We claim that for any $\tilde{\theta}$ that the agent is contemplating announcing with $\hat{a} > \alpha(\tilde{\theta})$, the agent is better off by modifying his deviation so as to slightly raise θ from $\tilde{\theta}$. To see this, note that by Lemma 9, α and hence v are continuously differentiable in θ. But then, IC_A holds and so, $\int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta}))dx = 0$. Thus, since v_θ has sign pattern $-/+$ by hypothesis,

$$\int v_\theta(x, \tilde{\theta}) f(x|\hat{a})dx = \int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta})) \frac{f(x|\hat{a})}{f(x|\alpha(\tilde{\theta}))} dx \geq 0,$$

where we have used $MLRP$, $\hat{a} > \alpha(\tilde{\theta})$, and Beesack's inequality. Thus, the agent's expected utility is increasing in θ at $(\tilde{\theta}, \hat{a})$.

Hence, if there is a $\hat{\theta}$ such that $\alpha(\hat{\theta}) = \hat{a}$, then the agent is better off with deviation $(\hat{\theta}, \hat{a}) \in \mathbb{G}$. And, since α is continuous, there is such a $\hat{\theta}$ unless $\hat{a} > \alpha(\bar{\theta})$. So, finally, assume that $\hat{a} > \alpha(\bar{\theta})$. Then, by the previous paragraph, θ_T prefers $(\bar{\theta}, \hat{a})$ to $(\theta_A, \hat{a})$. But, since $\int vf$ is concave in a, and by IC_{MH}, $\bar{\theta}$ prefers $(\bar{\theta}, \alpha(\bar{\theta})) \in \mathbb{G}$ to $(\bar{\theta}, \hat{a})$. Since c is submodular, this holds *a fortiori* for θ_T. $\square$

Now let us consider the possibility of jumps in the action schedule. Such jumps are economically reasonable if, for example, the principal is constrained to a finite set of compensation schemes. To generalize Theorem 6 to this case, we need a regularity assumption. A menu (α, v) is *regular* if (i) everywhere that α is differentiable in θ, so is v; and (ii) for all θ, there are $\bar{v}(\cdot, \theta)$ and $\underline{v}(\cdot, \theta)$ such that as $\varepsilon \downarrow 0$, $v(\cdot, \theta + \varepsilon)$ converges uniformly to $\bar{v}$ and $v(\cdot, \theta - \varepsilon)$ converges uniformly to $\underline{v}$. This is more than we need, but simplifies the exposition.

Theorem 7 *Let (α, v) be regular and satisfy IC_{MH} and IC_S. Also, assume v satisfies SCC, and $\int vf$ is concave in a for each θ. Then (α, v) is feasible in $\mathcal{P}$.*

Proof We proceed in several steps.

Step 1. As before, IR holds, and we can fix θ_T, and consider $\hat{a} > \alpha(\theta_A)$, with the other case symmetric. As in the proof of Step 2 of Theorem 4, we can also take $\theta_T > \theta_A$. Our goal is to show that the agent is better off with some element of $\mathbb{G}$.

Step 2. For any $\tilde{\theta}$ with $\hat{a} > \alpha(\tilde{\theta})$, let us show that the agent is better off to raise his announcement slightly. Where v is differentiable in θ at $\tilde{\theta}$, this is as before. So, consider a jump point θ_J with endpoints a_l and a_h, and where $\hat{a} \geq a_h$. Letting S be the associated surplus function to (α, v), we claim

$$\int \bar{v}(x,\theta_J) f(x|a_h)dx - c(a_h,\theta_J) = S(\theta_J) \text{ and} \int \bar{v}(x,\theta_J) f_a(x|a_h)dx - c_a(a_h,\theta_J) = 0 \tag{22}$$

and similarly at a_l. To see the first equality, note that by definition, $\int v(x,\theta) f(x|\alpha(\theta))dx - c(\alpha(\theta),\theta) = S(\theta)$ for all $\theta > \theta_J$, and then use the definitions of a_h and $\bar{v}$, uniform convergence of $v(\cdot,\theta)$ to $\bar{v}(\cdot,\theta_J)$ as $\theta \downarrow \theta_J$, and continuity of S. The second equality similarly follows from IC_{MH}.

It is thus enough to show that

$$\int \left(\overline{v}(x,\theta_J) - \underline{v}(x,\theta_J)\right) f(x|a_h)dx \geq 0, \tag{23}$$

for then, since $\bar{v}(\cdot,\theta_J) - \underline{v}(\cdot,\theta_J)$ has sign pattern $-/+$, and since $f(\cdot|\hat{a})/f(\cdot|a_h)$ is increasing in x, we have by Lemma 12 that $\int \left(\overline{v}(x,\theta_J) - \underline{v}(x,\theta_J)\right) f(x|\hat{a})dx \geq 0$. Thus, the agent is again better off to raise the report of his type from just below θ_J to just above it. To show (23), note that

$$\begin{aligned}\int \bar{v}(x,\theta_J) f(x|a_h)dx - c(a_h,\theta_J) = S(\theta_J) &= \int \underline{v}(x,\theta_J) f(x|a_l)dx - c(a_l,\theta_J) \\ &\geq \int \underline{v}(x,\theta_J) f(x|a_h)dx - c(a_h,\theta_J),\end{aligned}$$

where the first two equalities use the first part of (22), and the inequality uses the second part of (22) and concavity of $\int vf$ in a. Comparing the outer terms and cancelling $c(a_h,\theta_J)$ gives (23).

Step 3. As in the proof of Theorem 6, if $\hat{a} > \alpha(\bar{\theta})$, then, using Step 2, the agent is better off with a deviation to $(\bar{\theta}, \alpha(\bar{\theta}))$.

Step 4. Let us now complete the proof. If there is a $\hat{\theta}$ such that $\alpha(\hat{\theta}) = \hat{a}$, then by Step 2, the agent is better off with deviation $(\hat{\theta}, \alpha(\hat{\theta})) \in \mathbb{G}$, and we are done. Suppose instead that for some θ_J there is a jump at θ_J such that $\hat{a} \in [a_l, a_h]$. Assume first that $\theta_J > \theta_T$. Then, by Step 2, and using that by Step 1, $\theta_T > \theta_A$, we have $\int v(x,\theta_T) f(x|\hat{a})dx \geq \int v(x,\theta_A) f(x|\hat{a})dx$, and so type θ_T prefers the deviation $(\theta_T, \hat{a})$ to $(\theta_A, \hat{a})$. But, by concavity of $\int vf$ in a, $(\theta_T, \alpha(\theta_T))$ is better still and we are back on $\mathbb{G}$. So, assume $\theta_J \leq \theta_T$. Define $s_1 = \int v(x,\theta_A) f(x|\hat{a})dx - c(\hat{a},\theta_T)$, $s_2 = \int \underline{v}(x,\theta_J) f(x|\hat{a})dx - c(\hat{a},\theta_T)$, $s_3 = \int \bar{v}(x,\theta_J) f(x|a_h)dx - c(a_h,\theta_T)$, and $s_4 = S(\theta_T)$. These

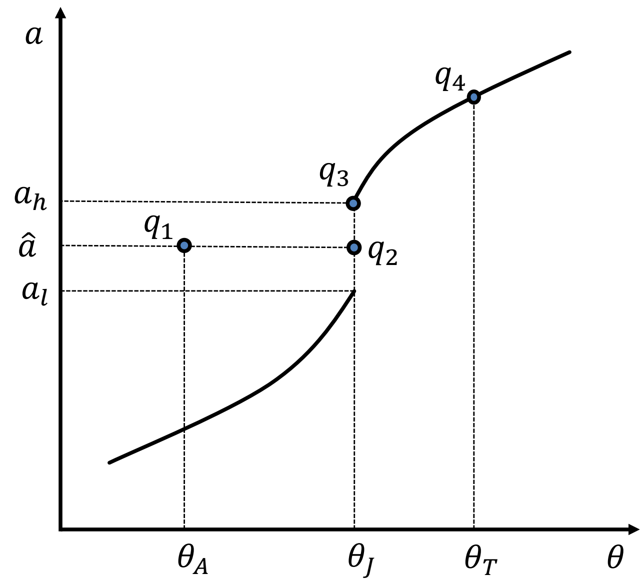

Figure 12: ***SCC***. Under *SCC*, a deviation by θ_T to q_1 is dominated by one to q_2. But that deviation in turn is dominated by a deviation to q_3 and, since q_3 is on locus, it is dominated by telling the truth and taking the recommended action at point q_4.

are the expected utilities for type θ_T at the points q_i, $i = 1,2,3,4$, in Figure 12, where q_2 reflects a limit from the left, and q_3 from the right.

By Lemma 2 and (22), we have $s_4 \geq s_3$, while by Step 2, $s_2 \geq s_1$. It remains only to show that $s_3 \geq s_2$. Note that

$$\int \bar{v}(x,\theta_J)f(x|a_h)dx - c(a_h,\theta_J) = S(\theta_J) = \int \underline{v}(x,\theta_J)f(x|a_l)dx - c(a_l,\theta_J)$$
$$\geq \int \underline{v}(x,\theta_J)f(x|\hat{a})dx - c(\hat{a},\theta_J)$$

where the two equalities follow from the first part of (22) and the inequality by the second part (22) and by concavity of $\int vf$ in a. But then, since $\theta_T \geq \theta_J$ and since c is submodular,

$$s_3 = \int \bar{v}(x,\theta_J)f(x|a_h)dx - c(a_h,\theta_T) \geq \int \underline{v}(x,\theta_J)f(x|\hat{a})dx - c(\hat{a},\theta_T) = s_2,$$

and we are done. Thus, the agent is better off at $q_4 \in \mathbb{G}$ than at q_1, and we are done. □

6.3 Common Values

Let us now consider the common-values case in which the type of the agent directly enters the conditional density of the signal. The next assumption imposes further conditions on f.

Assumption 8 *Each of f_a/f and f_θ/f is increasing in x, with $F_{a\theta} \leq 0$.*

The following is an example that satisfies Assumption 8.

Example 1 *Let g be a continuous parametrized family of densities on $[\underline{x}, \overline{x}]$ satisfying strict* MLRP*, where for each $a \in [0,1]$, $g(\cdot|a)$ is strictly increasing, and where $g(\underline{x}|\cdot)$ is bounded away from zero. Let r be a continuous strictly positive function on $[\underline{x}, \overline{x}]$, and define*

$$f(x|a,\theta) = \frac{r(x)g^\theta(x|a)}{\int r(s)g^\theta(s|a)ds}.$$

Let $\bar{\theta}$ be such that $(f_\theta/f)\theta + 1 > 0$ for all (x,a,θ) with $\theta \in [0,\overline{\theta}]$.[10] *Then, Assumption 8 holds. To see this, note that since $\log f = \log r + \theta \log g - \log \int rg^\theta$, we have $f_\theta/f = \log g - (\int rg^\theta (\log g)/\int rg^\theta)$, and so $(f_\theta/f)_x = g_x/g > 0$. Similarly, $f_a/f = (\theta g_a/g) - (\theta \int rg^{\theta-1}g_a/\int rg^\theta)$, and so $(f_a/f)_x = \theta(g_a/g)_x > 0$ for $\theta \in (0,\bar{\theta}]$ since g satisfies strict* MLRP *by assumption. It remains to show that $F_{a\theta} \leq 0$. But,*

$$f_a = f\theta\left(\frac{g_a}{g} - \frac{\int rg^\theta \frac{g_a}{g}}{\int rg^\theta}\right) = f\theta\left(\frac{g_a}{g} - \int \frac{g_a}{g}f\right) = f\theta\left(\frac{g_a}{g} - \gamma\right),$$

where $\gamma = \int (g_a/g)f$. Note that $\gamma_\theta = \int (g_a/g)f_\theta = \int (g_a/g)_x(-F_\theta) > 0$, where the inequality follows using that g satisfies strict MLRP*, and that since f_θ/f is strictly increasing, $-F_\theta > 0$ on $(\underline{x}, \bar{x})$. Now,*

$$f_{a\theta} = \left(\left(\frac{f_\theta}{f}\theta + 1\right)\left(\frac{g_a}{g} - \gamma\right) - \theta\gamma_\theta\right)f.$$

To show that $F_{a\theta} \leq 0$, it is enough to show that $f_{a\theta}(\cdot|a,\theta)$ single-crosses zero from below, using that $F_{a\theta}(x|a,\theta) = \int_{\underline{x}}^x f_{a\theta}ds$, and that $F_{a\theta}(\underline{x}|a,\theta) = F_{a\theta}(\bar{x}|a,\theta) = 0$. But, since $\gamma_\theta \geq 0$, and since by assumption $(f_\theta/f)\theta + 1 > 0$, it follows that at any point where $f_{a\theta}(s|a,\theta) = 0$, both $(f_\theta/f)\theta + 1$ and $(g_a/g) - \gamma$ are positive and strictly increasing in x, and the result follows.

Note that except for the presence of θ in f, the first-order conditions IC_{MH} and IC_A are the same. But, now $S'(\theta) = \int v(x,\theta)f_\theta(x|\alpha(\theta),\theta)dx - c_\theta(\alpha(\theta),\theta)$, since as the agent's type changes, there is a direct effect through f_θ. This in hand, we generalize Theorem 6 to this case.

Proposition 4 *Let FOP and Assumption 8 hold, let (α, S) solve $\mathcal{P}_D$, and let (α, v) be its associated menu. If v satisfies SCC, then (α, v) solves $\mathcal{P}$.*

Define the expected utility to type θ for compensation scheme $\hat{v}$ and action a, given a and $\hat{v}$ as $U(\theta, a, \hat{v}) = \int \hat{v}(x)f(x|a,\theta)dx - c(a,\theta)$. Note that $U_a = \int \hat{v}f_a - c_a = \int \hat{v}_x(-F_a) - c_a$, and hence if $\hat{v}$ is increasing, then $U_{a\theta} = \int \hat{v}_x(-F_{a\theta}) - c_{a\theta} \geq 0$ since $F_{a\theta} \leq 0$, and since c is submodular.

[10] Such a $\bar{\theta} > 0$ exists since the expression is strictly positive at $\theta = 0$, x and a come from compact sets, and f_θ/f is continuous.

Lemma 13 *Let* (α, v) *solve* $\mathcal{P}_D$*, and let* v *satisfy SCC. Let* $\tilde{S}(\theta_T, \hat{\theta}) = U(\theta_T, \alpha(\hat{\theta}), v(\cdot, \hat{\theta}))$ *be the value to type* θ_T *of imitating* $\hat{\theta}$*'s action and announcement. Then,* $\tilde{S}(\theta_T, \cdot)$ *is single-peaked at* θ_T *for all* θ_T*.*

Proof By the analogue to Lemma 9, α and v are continuously differentiable. To show single-peakedness, it is enough to show that for $\hat{\theta} < \theta_T$, $\tilde{S}_{\hat{\theta}}(\theta_T, \hat{\theta}) \geq 0$, where the case $\hat{\theta} > \theta_T$ is symmetric.

Choose $\hat{\theta} < \theta_T$. Then,

$$\tilde{S}_{\hat{\theta}}(\theta_T, \hat{\theta}) = \int v_\theta(x, \hat{\theta}) f(x|\alpha(\hat{\theta}), \theta_T) dx + \alpha'(\hat{\theta}) U_a(\theta_T, \alpha(\hat{\theta}), v(\cdot, \hat{\theta})).$$

By IC_A, $\int v_\theta(x, \hat{\theta}) f(x|\alpha(\hat{\theta}), \hat{\theta}) dx = 0$, and so, since $f(\cdot|\alpha(\hat{\theta}), \theta_T)/f(\cdot|\alpha(\hat{\theta}), \hat{\theta})$ is increasing, and since v_θ is $-/+$, the first term on the *rhs* is positive using Beesack's Inequality. The second term is positive using that $\alpha' \geq 0$, that $U_a(\hat{\theta}, \alpha(\hat{\theta}), v(\cdot, \hat{\theta})) = 0$, and that $U_{a\theta} \geq 0$. □

Proof of Proposition 4 Consider a type θ_T, and deviation $(\hat{\theta}, \hat{a})$. We focus on the case where $\hat{\theta} \leq \theta_T$, and then appeal to symmetry. Given Lemma 13, the key, as before, is to show that there is $(\theta, \alpha(\theta)) \in \mathbb{G}$ that θ_T prefers to $(\hat{\theta}, \hat{a})$.

Assume first that $\hat{a} \leq \alpha(\hat{\theta})$. Then, since $U_a(\hat{\theta}, \alpha(\hat{\theta}), v(\cdot, \hat{\theta})) = 0$, it follows from *FOP* that for any $a \in [\hat{a}, \alpha(\hat{\theta})]$, $U_a(\hat{\theta}, a, v(\cdot, \hat{\theta})) \geq 0$, and so, since $U_{a\theta} \geq 0$, the deviation $(\hat{\theta}, \hat{a})$ is dominated for θ_T by $(\hat{\theta}, \alpha(\hat{\theta})) \in \mathbb{G}$.

Assume next that $\hat{a} > \alpha(\hat{\theta})$. We will show that, holding fixed $\hat{a}$, type θ_T is better off to increase his announced type until he reaches either the graph or θ_T. In the latter case, using IC_{MH} and *FOP*, $(\theta_T, \alpha(\theta_T))$ is better still.

So, consider, any $\tilde{\theta} < \theta_T$ at which $\hat{a} > \alpha(\tilde{\theta})$. Using the analogue to Lemma 9, $\int v_\theta(x, \tilde{\theta}) f(x|\alpha(\tilde{\theta}), \tilde{\theta}) dx = 0$. Let us show that $\int v_\theta(x, \tilde{\theta}) f(x|\hat{a}, \theta_T) dx \geq 0$. Since v_θ is $-/+$ by assumption, and using Beesack's Inequality, it is enough that

$$\frac{f(x|\hat{a}, \theta_T)}{f(x|\alpha(\tilde{\theta}), \tilde{\theta})} = \frac{f(x|\hat{a}, \theta_T)}{f(x|\alpha(\tilde{\theta}), \theta_T)} \frac{f(x|\alpha(\tilde{\theta}), \theta_T)}{f(x|\alpha(\tilde{\theta}), \tilde{\theta})}$$

increases in x. But, since each of f_a/f and f_θ/f are increasing in x, $f(x|\hat{a}, \theta_T)/f(x|\alpha(\tilde{\theta}), \tilde{\theta})$ is the product of positive increasing functions, and so is increasing, and we are done. □

6.4 Optimal Exclusion

For any θ_c, let $\hat{\alpha}(\cdot,\theta_c)$ and $\hat{S}(\cdot,\theta_c)$ be defined on $[\theta_c,\bar{\theta}]$ by

$$(\hat{\alpha}(\cdot,\theta_c),\hat{S}(\cdot,\theta_c)) = \arg\max_{(\alpha,S)} \int_{\theta_c}^{\bar{\theta}} \left(B(\alpha(\theta)) - C(\alpha(\theta),S(\theta),\theta)\right) h(\theta)d\theta$$
$$s.t.\;\; S(\theta) = \bar{u} - \int_{\theta_c}^{\theta} c_\theta(\alpha(\tau),\tau)d\tau \;\forall\; \theta \geq \theta_c,$$

noting that relative to $\mathcal{P}_D$, we have replaced $S(\underline{\theta}) = \bar{u}$ by $S(\theta_c) = \bar{u}$. This is the unique solution to the principal's relaxed problem subject to excluding types below θ_c.

Proposition 5 *Assume decoupling is valid, and that $C(\cdot,\cdot,\theta)$ is convex for each θ.*[11] *Interior cutoff level θ_c is optimal only if $B - C = (-c_\theta/h)\int_{\theta_c}^{\bar{\theta}} C_{u_0}h$, evaluated at θ_c and $(\hat{\alpha}(\cdot,\theta_c),\hat{S}(\cdot,\theta_c))$. If $(-c_\theta/h)$ is decreasing in θ and $C_{u_0a} > 0$, then this condition is sufficient as well.*

At the optimal cutoff, $B - C$ is strictly positive. Necessity is both simple and intuitive. The direct benefit of adding types near θ is given by the *lhs* of the equation. The *rhs* reflects that including additional types increases the information rent paid to types above the cutoff. Sufficiency is more involved. They key is that the convexity of $C(\cdot,\cdot,\theta)$ implies that profits are strictly quasiconcave in the cutoff.

Proof of Proposition 5 Extend $\hat{\alpha}$ to have domain $[\underline{\theta},\bar{\theta}]$ by taking $\hat{\alpha}(\theta,\theta_c) = \hat{\alpha}(\theta_c,\theta_c)$ for $\theta < \theta_c$. Let $\tilde{S}(\theta,\theta_c,\theta^*) = \bar{u} - \int_{\theta_c}^{\theta} c_\theta(\hat{\alpha}(\tau,\theta^*),\tau)d\tau$ be the surplus the agent receives if the principal uses action schedule $\hat{\alpha}(\cdot,\theta^*)$, but excludes types below θ_c. The value to the principal of choosing cut-off θ_c but implementing action schedule θ^* is then

$$Z(\theta_c,\theta^*) = \int_{\theta_c}^{\bar{\theta}} \Big(B(\hat{\alpha}(\theta,\theta^*)) - C(\hat{\alpha}(\theta,\theta^*),\tilde{S}(\theta,\theta_c,\theta^*),\theta)\Big) h(\theta)d\theta.$$

To see necessity, differentiate $Z(\theta_c,\theta_c)$, noting that $\tilde{S}_{\theta_c}(\theta,\theta_c,\theta^*) = c_\theta(\hat{\alpha}(\theta_c,\theta^*),\theta_c)$, and that $Z_{\theta^*}(\theta_c,\theta^*)|_{\theta^*=\theta_c} = 0$ by the Envelope Theorem to obtain

$$(Z(\theta_c,\theta_c))_{\theta_c} = -\left(B(\hat{\alpha}(\theta_c,\theta_c)) - C(\hat{\alpha}(\theta_c,\theta_c),\bar{u},\theta_c\right) h(\theta_c)$$
$$- c_\theta(\hat{\alpha}(\theta_c,\theta_c),\theta_c)\int_{\theta_c}^{\bar{\theta}} C_{u_0}(\hat{\alpha}(\theta,\theta_c),\tilde{S}(\theta,\theta_c,\theta_c),\theta)h(\theta)d\theta.$$

Setting this equal to zero and rearranging yields the claimed necessary condition.

For sufficiency, let us show that if $(-c_\theta/h)$ is decreasing in θ and $C_{u_0a} > 0$, then $Z(\theta_c,\theta_c)$ is

[11] See Online Appendix, Section 1 for a discussion and primitives.

strictly quasiconcave in θ_c. But, by (6.4),

$$(Z(\theta_c,\theta_c))_{\theta_c\theta_c} = \left(-\left(B_a - C_a\right)h - c_{\theta a}\int_{\theta_c}^{\bar{\theta}} C_{u_0}h\right)(\hat{\alpha})_{\theta_c} + C_\theta h - (B-C)\,h' - c_{\theta\theta}\int_{\theta_c}^{\bar{\theta}} C_{u_0}h$$

$$+ c_\theta C_{u_0}h - c_\theta \int_{\theta_c}^{\bar{\theta}} \left(C_{u_0}(\hat{\alpha}(\theta,\theta_c),\tilde{S}(\theta,\theta_c,\theta_c),\theta)\right)_{\theta_c} h(\theta)d\theta$$

The first term is zero using the *FOC* with respect to the implemented action at θ_c. It is immediate that $C_\theta h < 0$, and $c_\theta C_{u_0}h < 0$. And, where $(Z)_{\theta_c} = 0$, $B - C = -(c_\theta/h)\int_{\theta_c}^{\bar{\theta}} C_{u_0}h$ from the necessary condition, and so

$$-\left(B-C\right)h' - c_{\theta\theta}\int_{\theta_c}^{\bar{\theta}} C_{u_0}h = \left(\frac{c_\theta}{h}\int_{\theta_c}^{\bar{\theta}} C_{u_0}h\right)h' - c_{\theta\theta}\int_{\theta_c}^{\bar{\theta}} C_{u_0}h =_s \left(\frac{-c_\theta}{h}\right)_\theta \leq 0$$

by assumption. So, to show that $Z(\theta_c,\theta_c)$ is strictly quasiconcave in θ_c, and since $-c_\theta > 0$, it would be sufficient to show $k(\theta_c) \leq 0$, where

$$k(\theta) = \int_\theta^{\bar{\theta}} \left(C_{u_0}(\hat{\alpha}(\tau,\theta_c),\tilde{S}(\tau,\theta_c,\theta_c),\tau)\right)_{\theta_c} h(\tau)d\tau.$$

We will in fact show that $k(\theta) \leq 0$ for all $\theta \in \left[\theta_c,\bar{\theta}\right]$. Since $k(\bar{\theta}) = 0$, it is enough that whenever $k > 0$, $k' > 0$. But, $k' =_s -(C_{u_0}(\hat{\alpha}(\theta,\theta_c),\tilde{S}(\theta,\theta_c,\theta_c),\theta))_{\theta_c}$, and it suffices that

$$C_{u_0 a}\hat{\alpha}_{\theta_c}(\theta,\theta_c) + C_{u_0 u_0}(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} < 0.$$

Let us first show that $(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} \leq 0$. Fix any $\theta_H > \theta_L$. Then, we claim that $\tilde{S}(\cdot,\theta_H,\theta_H) \leq \tilde{S}(\cdot,\theta_L,\theta_L)$. To see this, note first that

$$\tilde{S}(\theta_H,\theta_H,\theta_H) = \bar{u} < \bar{u} - \int_{\theta_L}^{\theta_H} c_\theta\left(\hat{\alpha}\left(\tau,\theta_L\right),\tau\right)d\tau = \tilde{S}(\theta_H,\theta_L,\theta_L).$$

So, assume that at some point $\tilde{\theta}$, $\tilde{S}(\tilde{\theta},\theta_H,\theta_H) = \tilde{S}(\tilde{\theta},\theta_L,\theta_L) = \tilde{u}$. Then, we claim, $(\hat{\alpha}(\cdot,\theta_H),\tilde{S}(\cdot,\theta_H,\theta_H))$ and $(\hat{\alpha}(\cdot,\theta_L),\tilde{S}(\cdot,\theta_L,\theta_L))$ coincide for all $\theta > \tilde{\theta}$. In particular, each must on $[\tilde{\theta},\bar{\theta}]$ equal the (unique) solution to

$$\max_{(\alpha,S)} \int_{\tilde{\theta}}^{\bar{\theta}} (B-C)h$$

$$s.t. \;\; S(\theta) = \tilde{u} - \int_{\tilde{\theta}}^{\theta} c_\theta(\alpha(\tau),\tau)d\tau,$$

since otherwise one could paste this solution together with the relevant solution below $\tilde{\theta}$ for a strict increase in profits. Hence, $\tilde{S}(\cdot,\theta_H,\theta_H) \leq \tilde{S}(\cdot,\theta_L,\theta_L)$ and so $(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} \leq 0$.

Now, from the optimality of $\hat{\alpha}(\cdot,\theta_c)$, we have that for all θ_c,

$$B_a(\hat{\alpha}(\theta,\theta_c)) - C_a(\hat{\alpha}(\theta,\theta_c),\tilde{S}(\theta,\theta_c,\theta_c),\theta) = \\ -\frac{1}{h(\theta)} c_{a\theta}(\hat{\alpha}(\theta,\theta_c),\theta) \int_{\theta}^{\bar{\theta}} C_{u_0}(\hat{\alpha}(\tau,\theta_c),\tilde{S}(\tau,\theta_c,\theta_c),\tau) h(\tau)\, d\tau,$$

and hence, differentiating by θ_c, and recalling that $B_{aa} = 0$,

$$\left(-C_{aa} + \frac{1}{h} c_{aa\theta} \int_{\theta}^{\bar{\theta}} C_{u_0}\right) \hat{\alpha}_{\theta_c}(\theta,\theta_c) = C_{au_0}(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} - \frac{1}{h} c_{a\theta} k(\theta)$$

and so, since $k(\theta) > 0$ by assumption,

$$\left(-C_{aa} + \frac{1}{h} c_{aa\theta} \int_{\theta}^{\bar{\theta}} C_{u_0}\right) \hat{\alpha}_{\theta_c}(\theta,\theta_c) > C_{au_0}(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c}$$

and so, since the term in the large parentheses is strictly negative,

$$\hat{\alpha}_{\theta_c}(\theta,\theta_c) < \frac{C_{au_0}}{-C_{aa} + \frac{1}{h} c_{aa\theta} \int_{\theta}^{\bar{\theta}} C_{u_0}} (\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} \leq \frac{C_{au_0}}{-C_{aa}} (\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c}$$

where we use that $c_{aa\theta} \leq 0$, and that $(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} \leq 0$. But then, since $C_{u_0a} > 0$, we have

$$C_{u_0a}\hat{\alpha}_{\theta_c}(\theta,\theta_c) + C_{u_0u_0}(\tilde{S}(\theta,\theta_c,\theta_c))_{\theta_c} < \left(C_{u_0a}\frac{C_{au_0}}{-C_{aa}} + C_{u_0u_0}\right)(\tilde{S}(\theta,\theta_c,\theta_c)_{\theta_c} \leq 0,$$

where the inequality follows since the bracketed term is positive by the convexity of C in a and u_0, and we are done. □

6.5 Random Mechanisms

Consider a setting in which first the agent announces a type, and then, based on the announcement, the principal randomizes over pairs $(a,\hat{v})$ consisting of a compensation scheme and recommended action. The agent needs to be willing to report his type honestly given the lottery he faces, and to follow the recommended action for each realized pair $(a,\hat{v})$.

Proposition 6 *Let $C(\cdot,\cdot,\theta)$ be convex for each θ, and assume decoupling is valid. Then, the solution (α,v) associated with $\mathcal{P}_D$ remains optimal even if randomization is allowed.*

The key to the proof is to consider any randomized solution to the relaxed screening problem $\mathcal{P}_D$. Now, replace actions by their expectations. Because $-c_\theta$ is convex in effort, this menu requires less surplus to be given to the agent than the surplus in the randomized mechanism. And, since $B - C$ is concave, replacing actions and surplus by their expectations raises the value of the objective function.

Proof of Proposition 6 A randomized mechanism is a map σ that for each θ generates a distribution $\sigma(\cdot|\theta)$ over triples $(\hat{a}, \hat{s}, \hat{v})$ consisting of a recommended action $\hat{a}$, a surplus $\hat{s}$, and a compensation scheme $\hat{v}$, where $\hat{s} = \int \hat{v}(x) f(x|\hat{a})dx - c(\hat{a}, \theta)$ with probability one, and subject to the incentive constraints discussed. Let V_{FR} (full-random) be the value of this program.

Note that among the incentive constraints is that for each announced type, and for each $\hat{v}$, the agent should not want to vary his action from the recommended one. Hence, for each θ, and with σ-probability one,

$$\int \varphi(\hat{v}(x)) f(x|\hat{a})dx \geq C(a, \hat{s}, \theta).$$

Also, an agent should not want to locally lie about their type and then follow the recommended action for the announced type. Hence, letting $\hat{S}(\theta) = \int \hat{s} d\sigma(\hat{a}, \hat{s}, \hat{v}|\theta)$ be the equilibrium surplus of type θ in the randomized mechanism, and recalling that θ enters $\hat{s}$ only through c, we have by the envelope theorem that

$$\hat{S}'(\theta) = \int (-c_\theta(\hat{a}, \theta)) d\sigma(\hat{a}, \hat{s}, \hat{v}|\theta).$$

But then, V_{FR} is at most equal to V_{RR} (relaxed-random) where V_{RR} is defined by

$$V_{RR} = \max_{\mu} \int \left(\int \left(B(\hat{a}) - C(\hat{a}, \hat{s}, \theta) \right) d\sigma(\hat{a}, \hat{s}, \hat{v}|\theta) \right) h(\theta) d\theta$$
$$s.t. \int \hat{s} d\sigma(\hat{a}, \hat{s}, \hat{v}|\theta) = \bar{u} + \int_{\underline{\theta}}^{\theta} \left(\int (-c_\theta(\hat{a}, \tau)) d\sigma(\hat{a}, \hat{s}, \hat{v}|\theta) \right) d\tau.$$

Let V_{RD} (relaxed-deterministic) be the value of $\mathcal{P}_D$, in which menus are restricted to be deterministic. We claim $V_{RD} = V_{RR}$. To see this, let σ^* be optimal in the relaxed-random program. Let $a^*(\theta) = \int \hat{a} d\sigma^*(\hat{a}, \hat{s}, \hat{v}|\theta)$, $S^*(\theta) = \int \hat{s} d\sigma^*(\hat{a}, \hat{s}, \hat{v}|\theta)$, and $S^{**}(\theta) = \bar{u} + \int_{\underline{\theta}}^{\theta} (-c_\theta(a^*(\tau), \tau)) d\tau$. Since $-c_\theta$ is convex in a (recall $c_{aa\theta} \leq 0$), we have

$$S_\theta^{**} = -c_\theta(a^*(\theta), \theta) \leq \int (-c_\theta(\hat{a}, \theta)) d\sigma^*(\hat{a}, \hat{s}, \hat{v}|\theta) = S_\theta^*,$$

and so, since $S^{**}(\underline{\theta}) = S^*(\underline{\theta}) = \bar{u}$, we have $S^{**} \leq S^*$. But then, since $B - C$ is concave in (a, u_0), and decreasing in u_0,

$$\begin{aligned}
V_{RR} &= \int \left(\int (B(\hat{a}) - C(\hat{a}, \hat{s}, \theta)) d\sigma^*(\hat{a}, \hat{s}, \hat{v}|\theta) \right) h(\theta) d\theta \\
&\leq \int (B(a^*(\theta)) - C(a^*(\theta), S^*(\theta), \theta)) h(\theta) d\theta \\
&\leq \int (B(a^*(\theta)) - C(a^*(\theta), S^{**}(\theta), \theta)) h(\theta) d\theta \\
&\leq V_{RD},
\end{aligned}$$

where the last inequality follows since by construction (a^*, S^{**}) is feasible in the relaxed deterministic problem. So, $V_{FR} \leq V_{RD}$, and thus if the solution to the relaxed deterministic program is feasible, then it is in fact optimal even if randomization is allowed. □

6.6 Analysis of Social Planner's Problem

Our techniques apply beyond our profit-maximizing principal. As an example, consider a social planner who cares about both the firm and the members of society.[12] For example, the planner may be designing a tax code that raises and redistributes income, and also affects effort incentives.

To model such a situation, reinterpret the agent as a continuum of agents of different types with density h, and assume that $B - C$ reflects the profits of a firm on any given agent. The social planner cares about the total surplus, $\int Sh$, of the members of society with weight $1-\eta$, and on the total profits of the firm, $\int (B-C)h$, with weight η.[13] The planner faces participation constraint for the firm that $\int (B - C)h \geq K$ for some exogenously given K. Agents have outside option $\bar{u}$, and their types and actions remain hidden. For simplicity, we work in the linear probability setting. The case $\eta = 1$ is our original monopolist's problem. When $\eta = 0$, the planner has production technology B and utilitarian preferences over the utility of the agents, with K reflecting the other spending needs of the planner net of her outside resources.

The critical realization is that the difficult part of this problem–an agent can both misrepresent their ability, and then choose any effort level–is unaffected by the change in the objective function. Hence, in the linear setting, *IMC* remains necessary and sufficient for a solution to the relaxed program to induce a feasible solution in the full problem. Because of this, the planner will optimally choose to use a menu of Holmström-Mirrlees contracts, and we can characterize the

[12]Many other objective functions are also amenable to what follows.

[13]The form of this integral embeds a separability assumption on the firm's profits across agents.

problem of the planner as

$$\max_{\alpha,S} \eta \int (B-C)h + (1-\eta)\int Sh \qquad (P_S)$$
$$s.t. \int (B-C)h \geq K,\ S(\underline{\theta}) \geq \bar{u},\ S' = -c_\theta\ \forall\theta,\ \text{and } IMC,$$

where the second two constraints weaken IC_S to recognize that IR may not bind at $\underline{\theta}$.

Signing the distortions to effort in our society will depend on the condition that $C_{u_0}(\alpha(\cdot), S(\cdot), \cdot)$ is strictly increasing, so that it is more expensive to give extra utility to those who are better off.[14] Recall from Theorem 5 that for a given interval $[\theta_1, \theta_2]$, $\phi(\theta) = 0$ for $\theta < \theta_1$, $\phi(\theta) = \int_{\theta_1}^{\theta} (-c_{a\theta}/c_{aa})\, d\tau$ for $\theta \in [\theta_1, \theta_2]$, and $\phi(\theta) = \int_{\theta_1}^{\theta_2} (-c_{a\theta}/c_{aa})\, d\tau$ for $\theta > \theta_2$.

Theorem 8 (Social Planner) *Let FOP hold, let $F_{aa} = 0$, let $C(\cdot,\cdot,\theta)$ be strictly convex, let $C_{u_0}(\alpha(\cdot), S(\cdot), \cdot)$ be strictly increasing, and let $\theta_1 < \theta_2$ be such that IMC is slack immediately to the left of θ_1 and right of θ_2. Then,*

$$\int_{\theta_1}^{\theta_2} (B_a - C_a)\frac{1}{c_{aa}} h \geq \int \left(C_{u_0} - \int C_{u_0} h\right) \phi h > 0. \qquad (24)$$

If $S(\underline{\theta}) > \bar{u}$ at the optimum, then the weak inequality is an equality, and anywhere that IMC is slack,

$$B_a - C_a = \frac{-c_{a\theta}}{h} \int_{\theta}^{\bar{\theta}} \left(C_{u_0} - \int C_{u_0} h\right) h. \qquad (25)$$

Thus, in the same average sense as in Theorem 5, effort is distorted downward, and it is distorted downward pointwise where both $S(\underline{\theta}) \geq \bar{u}$ and IMC is slack. To see the intuition for (24), consider the perturbation in the proof of Theorem 5, in which one lowers the effort of types in an interval so as to lower their marginal cost of effort by a constant. In addition, move utility between the firm and all agents uniformly so as to return the firm to its original profit level, hence assuring that $\int (B-C)h \geq K$ remains satisfied. When the action is lowered on $[\theta_1, \theta_2]$, society becomes more equal. This raises the utility of the lowest type and hence IR continues to hold. The resulting redistribution of surplus from the well-off to the less well-off saves money, since in particular, $C_{u_0} - \int C_{u_0} h$ is the cost of moving a util from society in general to θ, and ϕ is increasing and thus primarily increases surplus for the already well-off. So, for optimality, lowering effort on this interval must lower total output, and we have (24). If $S(\underline{\theta}) \geq \bar{u}$ is not binding at the optimum, as will hold if either $\bar{u}$ is low (people simply cannot leave the society) or if society is rich enough, then the perturbation is feasible in both directions, and (24) holds with equality.[15] Equation (25) follows where IMC is slack.

[14] The complexity is that while higher types are paid more on average, for sufficiently low output levels they are paid less than if they had announced a lower type.

[15] The claim about a rich society follows because c_θ is bounded, and thus so is $S(\bar{\theta}) - S(\underline{\theta})$. Hence, if the average

If (25) holds globally, then the planner optimally penalizes effort for all types except the extremes. But, unlike the monopolist, she does so to achieve a more egalitarian outcome subject to the "resource constraint" represented by K, and so one with higher average utility. The planner also utilizes the (moral-hazard constrained) efficient effort level at both extremes of types.[16] This is intuitive, since the reallocation of income from those below θ to those above θ is vacuous at each extreme, and more generally, when θ is close to either extreme, changing the relative utility of types below and above θ involves moving less and less money around. This generalizes a result in the optimal taxation literature (see Seade (1977), and Salanie (2011) for a good summary).

Before proving the theorem, let us discuss the monotonicity assumption on C_{u_0}. Consider first the two-outcome case, where at any given θ, $C_{u_0} = \alpha\varphi_h' + (1-\alpha)\varphi_l'$, with $v_h = S + c + (1-\alpha)\, c_a$, and $v_l = S + c - \alpha c_a$, and so, since $S_\theta - c_\theta = 0$, $(v_h)_\theta = (1-\alpha)\,(c_a)_\theta$ and $v_l = -\alpha\,(c_a)_\theta$. But then,

$$(C_{u_0})_\theta = \alpha' \left(\varphi_h' - \varphi_l'\right) + \alpha\,(1-\alpha)\,(c_a)_\theta \left(\varphi_h'' - \varphi_l''\right),$$

where the first term is strictly positive since φ is strictly convex, and the second term is positive, since ρ is concave, which one can show is equivalent to $\varphi''' \geq 0$. So, for the two outcome case, C_{u_0} is indeed strictly increasing in θ.

More generally, since $C_{u_0}(\alpha(\theta), S(\theta), \theta) = \int \varphi'(v(\theta, x)) f(x|\alpha(\theta))dx$ is an identity, we have $(C_{u_0})_\theta = \alpha' \int \varphi' f_a + \int \varphi'' v_\theta f$. The first term reflects that higher types exert higher effort and so are more likely to attain higher outcomes. It is bounded strictly above zero, using *IMC,* that φ is strictly convex, and strict $MLRP$. For the square-root case, φ'' is constant, and so, since $\int v_\theta f = 0$, the second term disappears, $\int v_\theta f = 0$, and so we are again done.

More informally, when u is "close" enough to square root, the second term will be small, and we will have C_{u_0} strictly increasing. Finally, in our society, utility gaps are bounded uniformly from top to bottom, since c_θ is bounded. So, if society is rich enough–either because K is small or B is large–then all members of society will be quite well-off. But then, as formalized in Section VII.B, under mild conditions, φ'' will again be close to constant over the relevant ranges, the relevant integral will be small, and we again have C_{u_0} strictly increasing.[17]

Proof of Theorem 8 Modify the perturbation from the proof of Theorem 5 so that any change in profit to the firm is redistributed in utility-equivalent terms to the agents. That is, $S(\underline{\theta}; \varepsilon)$ equals $s(\varepsilon)$, where

$$\int \left(B(\alpha(\theta;\varepsilon)) - C\left(\alpha(\theta;\varepsilon), s(\varepsilon) - \int_{\underline{\theta}}^{\theta} c_\theta\left(\alpha\left(\tau;\varepsilon\right), \tau\right) d\tau, \theta\right)\right) h = \int (B - C) h,$$

member of society is well off, so is the least well-off.

[16]Indeed, note that the integral in (25) is single-peaked and zero at the extremes.

[17]What stands in the way of a fully general result is that beyond the two outcome case, v_θ need not have tidy crossing properties, especially over over ironed regions, since c_a is unchanging, but l is changing with a.

with the rhs evaluated at the candidate solution, and where we thus have

$$s'(0) = \frac{1}{\int C_{u_0} h} \left(\int_{\theta_1}^{\theta_2} (B_a - C_a) \frac{1}{c_{aa}} h - \int C_{u_0} \phi h \right). \tag{26}$$

Since the firm's profit is unaffected, the rate of change in the objective function with respect to ε is thus, for $\eta < 1$,

$$\begin{aligned} (1-\eta) \frac{d}{d\varepsilon} \left(\int Sh \right) \Big|_{\varepsilon=0} &\underset{s}{=} s'(0) + \int \phi h \\ &\underset{s}{=} \int_{\theta_1}^{\theta_2} (B_a - C_a) \frac{1}{c_{aa}} h - \int \left(C_{u_0} - \int C_{u_0} h \right) \phi h, \end{aligned} \tag{27}$$

where $\int \left(C_{u_0} - \int C_{u_0} h \right) \phi h > 0$ by Lemma 12, since C_{u_0} is strictly increasing, ϕ is increasing and not everywhere constant, and $\int \left(C_{u_0} - \int C_{u_0} h \right) h = 0$.

Assume by contradiction that $\int_{\theta_1}^{\theta_2} (B_a - C_a) \frac{1}{c_{aa}} h < \int (C_{u_0} - \int C_{u_0} h) \phi h$. Then, since $\int C_{u_0} h$ and ϕ are both positive, and from (26), we have $s'(0) < 0$, so that for ε small but negative, IR holds and the perturbation is feasible. But, from (27), this deviation is strictly profitable, a contradiction.[18] If IR does not bind, then the perturbation is also feasible for ε small and positive, and so the weak inequality in (24) must be an equality.

The proof of (25) follows as before, by noting that the inequality in (24) is an equality on a neighborhood, differentiating on both sides with respect to θ_2, using that for $\theta < \theta_2$, $\phi_{\theta_2} = 0$, while for for $\theta > \theta_2$, ϕ_{θ_2} is $-c_{a\theta}/c_{aa}$ evaluated at θ_2, and rearranging. □

As for the profit maximizing principal case (Footnote 9), our societal optimum *cannot* in general be implemented without the announcement or menu phase of the mechanism. Hence, for example, in an optimal tax code, people of different abilities will be selected into different tax schemes mapping gross into net incomes. It is intriguing to think about how such a tax code would be implemented, since the announcement of type must occur prior to the choice of effort, and so, for example, at the beginning of one's career.

7 Existence and Differentiability in the Moral Hazard Problem

Let W be the domain of the utility function, an interval with infimum $\underline{w}$ and supremum $\bar{w}$. Let $\underline{v} = \lim_{w \to \underline{w}} u(w)$, and let $\bar{v} = \lim_{w \to \bar{w}} u(w)$. Let $\mathcal{E}$ be the set of (a, u_0, θ) such that the relaxed moral hazard problem in Section $\mathcal{P}_{MH}$ admits a solution $\hat{v}$ where $\hat{v}(\underline{x}) > \underline{v}$ and $\hat{v}(\bar{x}) < \bar{v}$. If we let $\underline{\tau} = \lim_{w \to \underline{w}} (1/u'(w))$, and $\bar{\tau} = \lim_{w \to \bar{w}} (1/u'(w))$, then it is easy to show that $\hat{v}(\underline{x}) > \underline{v}$ if and only if $\lambda + \mu l(\underline{x}|a) > \underline{\tau}$ for the associated Lagrange multipliers, and similarly, that $\hat{v}(\bar{x}) < \bar{v}$ if and

[18] In the case $\eta = 1$, the perturbed solution has IR strictly slack, and so the firm can lower surplus to all types for a strict increase in profits.

only if $\lambda + \mu l(\bar{x}|a) < \bar{\tau}$.

Lemma 14 *The set $\mathcal{E}$ is open. The multipliers λ and μ are twice continuously differentiable functions of (a, u_0, θ) on $\mathcal{E}$. Hence, so are the functions $\tilde{v}$ and C.*

Proof Let

$$G(\lambda, \mu, a, u_0, \theta) = \begin{pmatrix} g_1(\lambda, \mu, a, u_0, \theta) \\ g_2(\lambda, \mu, a, u_0, \theta) \end{pmatrix},$$

where

$$g_1(\lambda, \mu, a, u_0, \theta) = \int \rho(\lambda + \mu l(x|a)) f(x|a) dx - c(a, \theta) - u_0,$$
$$g_2(\lambda, \mu, a, u_0, \theta) = \int \rho(\lambda + \mu l(x|a)) f_a(x|a) dx - c_a(a, \theta).$$

Let $(a^0, u_0^0, \theta^0) \in \mathcal{E}$, let λ^0 and μ^0 be the associated Lagrange multipliers, and let $\kappa^0 = (\lambda^0, \mu^0, a^0, u_0^0, \theta^0)$. Then, $G(\kappa^0) = 0$, and by definition of $\mathcal{E}$, $\lambda^0 + \mu^0 l(\underline{x}|a^0) > \underline{\tau}$, and $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\tau}$. We need to show that λ and μ are implicitly defined as C^1 functions of (a, u_0, θ) on a neighborhood of (a^0, u_0^0, θ^0). Since $\lambda + \mu l(\underline{x}|a)$ and $\lambda + \mu l(\bar{x}|a)$ are continuous in (λ, μ, a), it would follow from this that $\mathcal{E}$ is open. We proceed in several steps. Let ψ map $1/u'$ into money, that is, ψ solves $1/u'(\psi(\tau)) = \tau$. Then we can write ρ as $\rho(\tau) = u(\psi(\tau))$.

Step 1. We first show that $g_{1\lambda}$ exists at κ^0, and is equal to $\int \rho'(\lambda^0 + \mu^0 l(x|a^0)) f(x|a^0) dx$. To show this, we must first show that it is valid to differentiate under the integral. This requires that $\rho(\lambda + \mu l(x|a)) f(x|a)$ be integrable. Since f is continuous on the compact interval $[\underline{x}, \bar{x}]$, it is bounded, and so it is enough to show that $|\rho(\lambda + \mu l(x|a))|$ is bounded. But,

$$\rho(\lambda + \mu l(x|a)) \leq \rho(\lambda^0 + \mu^0 l(\bar{x}|a^0)) < \infty,$$

where we use that $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\tau}$ by hypothesis, and similarly, $\rho(\lambda + \mu l(x|a)) \geq \rho(\lambda^0 + \mu^0 l(\underline{x}|a^0)) > \infty$, and we are done. Another requirement for passing the derivative through the integral is that $\rho'(\lambda^0 + \mu^0 l(x|a^0)) f(x|a^0)$ is bounded above by an integrable function on some neighborhood of (λ^0, μ^0, a^0). To see this, choose $\underline{\delta}$ and $\bar{\delta}$ such that $\underline{\tau} < \underline{\delta} < \lambda^0 + \mu^0 l(\underline{x}|a^0)$ and $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\delta} < \bar{\tau}$. Then, since $\lambda + \mu l(\underline{x}|a)$ and $\lambda + \mu l(\bar{x}|a)$ are continuous in (λ, μ, a), there is a neighborhood N of (λ^0, μ^0, a^0) such that $\underline{\delta} < \rho(\lambda + \mu l(\underline{x}|a)) < \rho(\lambda + \mu l(\bar{x}|a)) < \bar{\delta}$ on N. But then, for all x, and everywhere on N, $\rho'(\lambda + \mu l(x|a)) \leq \max_{\tau \in [\underline{\delta}, \bar{\delta}]} \rho'(\tau) < \infty$, where the second inequality follows since ρ is continuously differentiable (with $\rho'(\tau) = ((u')^3 / -u'')(\psi(\tau)))$ and $[\underline{\delta}, \bar{\delta}]$ is compact. By Corollary 5.9 in Bartle (1966) (and Billingsley (1995), problem 16.5), we can pass the derivative through the integral and this provides an expression for $g_{1\lambda}$.

Step 2. $g_{1\lambda} = \int \rho'(\lambda + \mu l(x|a)) f(x|a) dx$ is itself continuous in (λ, μ, a) at (λ^0, μ^0, a^0). This follows since $\lambda + \mu l(x|a)$ is, under our conditions, uniformly continuous in (λ, μ, a), and ρ' is uniformly continuous in its argument on $[\underline{\delta}, \bar{\delta}]$.

STEP 3. By similar arguments, $g_{1\mu}$, g_{1a}, $g_{2\lambda}$, $g_{2\mu}$, and g_{2a} are defined as the integral of the relevant derivative, and are continuous. Finally, $g_{i\theta}$ and g_{iu_0} are trivially continuous. Hence, G is continuously differentiable on a neighborhood of κ^0. Then G is twice continuously differentiable, noting in specific that

$$\rho''(\tau) = \frac{(u')^3}{-u''}\left[3\frac{u''}{u'} - \frac{u'''}{u''}\right](\psi(\tau)),$$

and so since u is C^3, ρ'' is continuous on the compact interval $[\underline{\delta}, \bar{\delta}]$, and hence it is bounded.

STEP 4. By Jewitt, Kadan, and Swinkels (2008), $\nabla G(\kappa^0) \neq 0$. Hence, by the Implicit Function Theorem for C^k functions (Fiacco (1983), Theorem 2.4.1), λ and μ are twice continuously differentiable functions of (a, u_0, θ) in a neighborhood of (a^0, u_0^0, θ^0).

STEP 5. Since $\tilde{v}(x, a, u_0, \theta) = \rho(\lambda + \mu l(x|a))$ for all (x, a, u_0, θ), it follows that $\tilde{v}$ is twice-continuously differentiable, and thus so is C, since $C(a, u_0, \theta) = \int \varphi(\tilde{v}(x, a, u_0, \theta))f(x|a)dx$. □

The reader may wonder at the level of detail displayed in this proof. To see that there is something to prove, consider $u = \log w$. Then (see Moroni and Swinkels (2014) for details), it is easy to exhibit first, combinations of c_a, c, and u_0 for which no optimal contract exists, and second, combinations of c_a, c, and u_0 for which the optimal contract has $v(\underline{x}) = -\infty$, and at which the relevant integrals cease to be continuous (let alone differentiable) in the relevant parameters.

In the text we have assumed that we can exchange differentiation and integration when differentiating $\int vf$ with respect to θ and a. This can be justified as follows:

Lemma 15 *Let $(\alpha(\theta^0), S(\theta^0), \theta^0) \in \mathcal{E}$. Then, for all a, $\int v(x, \theta)f(x|a)dx$ is differentiable in θ at θ^0, with*

$$\frac{\partial}{\partial\theta}\int v(x, \theta^0)f(x|a)dx = \int v_\theta(x, \theta^0)f(x|a)dx,$$

and similarly, $\int v(x, \theta^0)f(x|a)dx$ is differentiable in a at a, with

$$\frac{\partial}{\partial a}\int v(x, \theta^0)f(x|a)dx = \int v(x, \theta^0)f_a(x|a)dx.$$

Proof We will show the result for the case of differentiation by θ since the other case is similar. We must show first that $v(x, \theta^0)f(x|a)$ is integrable. This follows as before since

$$|v(x, \theta^0)| \leq \max\left(|v(\underline{x}, \theta^0)|, |v(\bar{x}, \theta^0)|\right) < \infty.$$

Next we show that, under decoupling, v_θ exists and it is uniformly bounded. To see this, note first that $v(x,\theta) = \rho(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))$ and so

$$v_\theta(x,\theta) = \rho'(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))(\lambda'(\theta) + \mu'(\theta) l(x|\alpha(\theta)) + \mu(\theta) l_a(x|\alpha(\theta))\alpha'(\theta)).$$

As before, let $\underline{\tau} < \underline{\delta} < \lambda^0 + \mu^0 l(\underline{x}|a^0)$, and let $\lambda^0 + \mu^0 l(\bar{x}|a^0) < \bar{\delta} < \bar{\tau}$. Since α is continuous, for all θ sufficiently close to θ^0, $\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)) \in [\underline{\delta}, \bar{\delta}]$, and so, as before, $\rho'(\lambda(\theta) + \mu(\theta) l(x|\alpha(\theta)))$ is uniformly bounded on a neighborhood of θ^0. Also, since α and S are C^1, $\lambda(\theta)$ and $\mu(\theta)$ are continuously differentiable on some neighborhood of θ^0. But then, since l and l_a are uniformly bounded, we can also uniformly bound $(\lambda'(\theta) + \mu'(\theta) l(x|\alpha(\theta)) + \mu(\theta) l_a(x|\alpha(\theta))\alpha'(\theta))$ on the relevant neighborhood. It follows that v_θ is uniformly bounded on the neighborhood, and the lemma follows from Bartle (1966), Corollary 5.9. □

Finally, we need to know that $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ. By Moroni and Swinkels (2014), one set of conditions is given by the following lemma.

Lemma 16 *Assume that $\bar{w} = \bar{v} = \infty$, $\underline{w} = \underline{v} = -\infty$, $\underline{\tau} = 0$, and $\bar{\tau} = \infty$. Then, for all (a, u_0, θ), $(a, u_0, \theta) \in \mathcal{E}$.*

Proof Direct from Moroni and Swinkels (2014). □

This lemma, however, does not cover important cases such as $u = \log w$ or $u = \sqrt{w}$, because in each case, $\underline{w} = 0 > -\infty$. Our next lemma covers $u = \sqrt{w}$, but not $u = \log w$.

Lemma 17 *Let $\bar{w} = \bar{v} = \infty$, $\underline{w} = 0$, and $\bar{\tau} = \infty$. Assume further that $\rho'(\tau)\tau$ is increasing and diverges in τ. Then, there is a threshold $\hat{u}$ such that for all $\bar{u} \geq \hat{u}$, $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ.*

Proof For any given a, and $\mu > 0$, let $i(\mu, a) = \int \rho(\mu(l(x|a) - l(\underline{x}|a))) f_a(x|a) dx$. Note that

$$\begin{aligned} i(\mu, a) &= \int \rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu l_x(x|a)(-F_a(x|a)) dx \\ &= \int \frac{1}{l(x|a) - l(\underline{x}|a)} [\rho'(\mu(l(x|a) - l(\underline{x}|a)))\mu(l(x|a) - l(\underline{x}|a))] l_x(x|a)(-F_a(x|a)) dx, \end{aligned}$$

and so, since $\rho'(\tau)\tau$ is increasing in τ, it follows that the bracketed term, and hence $i(\cdot, a)$, is increasing in μ. Let $m = \min_a l(\bar{x}|a) - l(\underline{x}|a) > 0$, and let

$$\sigma = -\min_{\left\{(x,a) \mid \frac{m}{2} \leq l(x|a) - l(\underline{x}|a) \leq \frac{3m}{4}\right\}} l_x(x|a) F_a(x|a) > 0.$$

Then,

$$
\begin{aligned}
& i(\mu, a) \\
& \geq \int_{\left\{x \mid \frac{m}{2} \leq l(x|a)-l(\underline{x}|a) \leq \frac{3m}{4}\right\}} \frac{\rho'(\mu(l(x|a)-l(\underline{x}|a)))\mu(l(x|a)-l(\underline{x}|a))}{l(x|a)-l(\underline{x}|a)} l_x(x|a)(-F_a(x|a))dx \\
& \geq \frac{4\sigma}{3m} \int_{\left\{x \mid \frac{m}{2} \leq l(x|a)-l(\underline{x}|a) \leq \frac{3m}{4}\right\}} \rho'(\mu(l(x|a)-l(\underline{x}|a)))\mu(l(x|a)-l(\underline{x}|a))dx \\
& \geq \frac{4\sigma}{3m} \rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2} \int_{\left\{x \mid \frac{m}{2} \leq l(x|a)-l(\underline{x}|a) \leq \frac{3m}{4}\right\}} dx \geq \frac{4\sigma}{3m} \frac{m}{4\max_{\{x,a\}} l_x(x|a)} \rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2} \\
& = \frac{\sigma}{3\max_{\{x,a\}} l_x(x|a)} \rho'\left(\mu\frac{m}{2}\right)\mu\frac{m}{2},
\end{aligned}
$$

where the first inequality follows from the fact that the integrand is positive, the second from $l(x|a) - l(\underline{x}|a) \leq 3m/4$, the third from the monotonicity of $\rho'(\tau)\tau$, and the fourth by integration. Note that the lower bound on $i(\mu, a)$ thus obtained diverges in μ. Hence, there exists $\hat{\mu}$ such that $i(\mu, a) > c_a(a, \bar{\theta})$ for all a, and $\mu > \hat{\mu}$. Let

$$
\hat{u} = \max_a \int \rho(\hat{\mu}(l(x|a) - l(\underline{x}|a)))f(x|a)dx \leq \rho\left(\hat{\mu}\max_a(l(\bar{x}|a) - l(\underline{x}|a))\right) < \infty.
$$

It follows from Proposition 1 of Moroni and Swinkels (2014), along with $i(\cdot, a)$ increasing, that $(\alpha(\theta), S(\theta), \theta) \in \mathcal{E}$ for all θ for any $\bar{u} > \hat{u}$. In particular, at any θ, $S(\theta) + c(\alpha(\theta), \theta) > \bar{u} > \hat{u}$. □

Finally, let us consider the case $u = \log w$ (for which $\rho'(\tau)\tau$ is identically 1, so the previous result does not apply). Then, as in the proof of the previous lemma,

$$
\begin{aligned}
i(\mu, a) &\geq \frac{4\sigma}{3m} \int_{\left\{x \mid \frac{m}{2} \leq l(x|a)-l(\underline{x}|a) \leq \frac{3m}{4}\right\}} \rho'(\mu(l(x|a)-l(\underline{x}|a)))\mu(l(x|a)-l(\underline{x}|a))dx \\
&= \frac{4\sigma}{3m} \int_{\left\{x \mid \frac{m}{2} \leq l(x|a)-l(\underline{x}|a) \leq \frac{3m}{4}\right\}} dx \geq \frac{4\sigma}{3m} \frac{m}{4\max_{x,a} l_x(x|a)} \equiv s,
\end{aligned}
$$

and so, if we assume that $c_a(\bar{a}, \bar{\theta}) \leq s$, then Proposition 1 of Moroni and Swinkels (2014) applies.

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