Online Appendix

ACE – Analytic Climate Economy
by Christian Traeger

A. Further Results, Model Transformation, and Climate Parametrization

A1. CES-Preferences for Consumption, Investment Goods, and Production Example

This section introduces a variety of final goods. The representative agent has CES preferences over the final consumption goods and investment in equation (4) becomes a composite.

Let $I_c \in \{1, ..., \bar{I}_c\}$ and $I_I = \{\underline{I}_I, ..., I\}$ denote the sets of final consumption goods and investment components, respectively, where $\bar{I}_c, \underline{I}_I, I \in \mathbb{N}$. These sets can coincide, overlap, or be disjoint. The representative agent consumes the share $x_{l,t}$ of good $c_{l,t}$.[35] As in the base model, a representative consumer maximizes the discounted sum of the log of a consumption aggregate C_t (objective 8). But now, the consumption aggregate is a CES-aggregator over a variety of different final goods

$$C_t = \Big(\sum_{l \in I_C} a_{l,t} (x_{l,t} c_{l,t})^{s_t} \Big)^{\frac{1}{s_t}} \tag{A.1}$$

with good-specific weights $a_{l,t}$ and substitutability index $s_t \leq 1$ for all t and $l \in I_C$. The final goods follow a production process of the form

$$c_{l,t} = A_{l,t} K_{l,t}^{\alpha} N_{l,t}^{1-\alpha-\nu} d_{l,t}^{\nu} \, [1 - D(T_{1,t})] \quad \text{for } l = 1, ..., I \tag{A.2}$$

with good-specific technology, capital, and labor as well as an energy intermediate $d_{l,t}$. The energy intermediate represents the different substitutabilities across energy sources in different sectors. In each sector, the energy intermediate

$$d_{l,t} = \Big(\sum_{i \in \Theta_l} e_{i,t}^{\tilde{s}_{l,t}} \Big)^{\frac{1}{\tilde{s}_{l,t}}} \tag{A.3}$$

is a CES-combination of different energy sources, where $\Theta_l \subset \{1, ..., I_E\}$ specifies the subset of primary energy sources used in the production of good l. To convert

[35] For goods that are only used in consumption, $l \in I_c \backslash I_I$, it is $x_{l,t} = 1$. For goods that are used in both the consumption and the investment process, $l \in I_c \cap I_I$, the consumption share $x_{l,t}$ is endogenously chosen and the remaining share $1 - x_{l,t}$ enters investment. For goods that are only used in the production-investment process, $l \in I_I \backslash I_c$, it is $x_{l,t} = 0$.

a fossil fuel or renewable resource into energy we need capital and labor and I assume

$$e_{i,t} = g_{i,t}(A_{I+i,t}, K_{I+i,t}, N_{I+i,t}, E_{i,t})$$
$$\text{satisfying } \; g_{i,t}(A_{I+i,t}, \gamma K_{I+i,t}, N_{I+i,t}, E_{i,t}) = \gamma^{\tilde{\alpha}} g_{i,t}(A_{I+i,t}, K_{I+i,t}, N_{I+i,t}, E_{i,t}),$$

for all $i \in \{1, ..., I_E\}$. Some fossil fuels $E_{i,t}$ will be subject to the intertemporal scarcity constraint (2). I use a general function g to include Cobb-Douglas as well as formulations with a "bliss point", i.e., a finite emission level beyond which more coal or more oil no longer increases production – a feature satisfied in DICE. I refer to an accompanying paper for a more detailed discussion of these features and their implications for emissions (Traeger 2021*a*). Here, I analyze the implications on the optimal carbon tax. Finally, the investment composite

$$I_t = \Big(\sum_{l \in I_I} b_{l,t} c_{l,t}^{\zeta_t} \Big)^{\frac{1}{\zeta_t}} \tag{A.4}$$

with investment weights $b_{l,t}$ and substitutability index $\zeta_t \leq 1$ for all t and $l \in I_I$ replaces equation (3) of the base model. This economy features $I_A = I_K = I_N = I + I_E$ different sectors.

PROPOSITION 4: *The preference and production extensions of this section replace equations* (1) *and* (3) *of the base ACE by equations (A.1-A.4). The SCC, expressed in terms of aggregate consumption equivalents, becomes*

$$SCC_t = \frac{C_t}{1-\beta\kappa} \frac{\beta}{M_{pre}} \xi_0 \left[(\mathbf{1} - \beta \boldsymbol{\sigma})^{-1} \right]_{1,1} \sigma^{forc} \left[(\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \right]_{1,1} \tag{A.5}$$

where $\kappa \equiv \alpha + \tilde{\alpha}\nu$*.*

Assuming coinciding parameter values, the value of the SCC in equation (A.5) coincides with that of Proposition 2. Also equation (11) of the base model implicitly contains the expression $\frac{C_t}{1-\beta\kappa}$. However, the base model assumes a single consumption-investment good. This assumption results in the simple and constant consumption rate $1 - \beta\kappa$, implying that $\frac{C_t}{1-\beta\kappa} = \frac{(1-\beta\kappa)Y_t^{net}}{1-\beta\kappa} = Y_t^{net}$. More generally, the SCC is proportional to consumption, which plays the role of inverse marginal utility in the present period, translating the welfare loss from emitting a unity of CO_2 into today's consumption equivalents and, thus, USD.

A2. Capital Depreciation

This section derives the capital equation of motion (4), quantifies the correction factor, and discusses the model's implication that the consumption *rate* (but not level) is unaffected by climate states. The usual capital accumulation equation, enriched by climate damages, is

$$K_{t+1} = Y_t[1 - D(T_{1,t})] - C_t + (1 - \delta_k)K_t \ .$$

Defining the consumption rate $x_t = \frac{C_t}{Y_t[1-D(T_{1,t})]}$ and recognizing that $Y_t[1 - D(T_{1,t})] - C_t = K_{t+1} - (1 - \delta_k)K_t$ implies[36]

$$K_{t+1} = Y_t[1 - D(T_{1,t})](1 - x_t)\left[1 + \frac{1-\delta_k}{\frac{K_{t+1}}{K_t} - (1-\delta_k)}\right] \ .$$

Defining the capital growth rate $g_{k,t} = \frac{K_{t+1}}{K_t} - 1$, I obtain the stated equation of motion (4) for capital.

I evaluate the correction factor based on the Penn World Tables (Feenstra, Inklaar and Timmer 2015). For 2019, currently the latest year of the Penn Word Tables 10.0, the global depreciation parameter is $\delta_k^{annual} = 0.0439$, so $\delta_k \approx 0.44$. Averaging capital growth over the past 10 years (using 2017USD values) delivers $g_k^{annual} = 0.02949$ or $g_k \approx 0.29$. The resulting correction factor is $\left[\frac{1+g_{k,t}}{\delta_k + g_{k,t}}\right] \approx 1.8$. (for a 5 year time step the correction factor would be 3.1, and for an annual time step 14). For the US, $\delta_k^{annual} = 0.046$ and $g_k^{annual} = 0.013$, resulting in a decadal correction factor of 1.9 (or a correction factor of 3.6 for a five year time step).

Treating the growth and depreciation correction in squared brackets as exogenous remains an approximation. The extension shows that the model is robust against the immediate criticism of not being able to represent the correct capital evolution and capital output ratio, and against the agent's neglect of capital value beyond immediate next period usage. Yet, the crucial implications of the assumptions underlying equation (4) is that the investment rate is independent of the climate states. It is the price to pay for an analytic solution. The remainder of this section shows that this price seems small.

Figure A1 tests ACE's result (and implicit assumption) that the optimal consumption rate is independent of the climate states. The figure depicts the optimal consumption rate generated by a recursive DICE implementation with an annual time step and, thus, an annual capital decay structure of the usual form (Traeger 2012*b*).[37] It also abandons the assumption of logarithmic utility, further stacking the cards against ACE's assumptions. The first two graphs in the figure depict the control rules for DICE-2013's $\eta = 1.45$ (inverse of the intertemporal elasticity of substitution). These two graphs state the optimal consumption rate

[36]The step uses $K_{t+1} = Y_t[1 - D(T_{1,t})] - Y_t[1 - D(T_{1,t})]x_t + (Y_t[1 - D(T_{1,t})] - C_t) \times \frac{(1-\delta_k)K_t}{Y_t[1-D(T_{1,t})]-C_t}$.

[37]The recursive implementation based on the Bellman equation solves for the optimal control rule as a function of the states derlivering the full control surface depicted here. This recursive implementation has a slightly simplified climate change model compared to the original DICE model, but matches the Maggic6.0 model, used also as the DICE benchmark, similarly well.

for the years 2025 and 2205. The third graph in the figure depicts the optimal consumption rate for the lower value $\eta = 0.66$ calibrated by the long-run risk literature (Vissing-Jørgensen and Attanasio 2003, Bansal and Yaron 2004, Bansal, Kiku and Yaron 2010, Chen, Favilukis and Ludvigson 2013, Bansal, Kiku and Yaron 2012, Bansal et al. 2014, Collin-Dufresne, Johannes and Lochstoer 2016, Nakamura, Sergeyev and Steinsson 2017)

The qualitative behavior is the same for all graphs in Figure A1. Overall, the figure shows that the optimal consumption rate is largely independent of the climate states (if the vertical axis started at zero the variation of the control rule would be invisible). At current temperature levels, the optimal consumption rate does not depend on the CO_2 concentrations. This result is in accordance with the theoretical result under ACE's assumption set. However, the optimal consumption rate increases slightly for higher temperatures. It increases by less than a percentage point from no warming to a 3C warming at low CO_2 concentrations. The increase is lower at higher CO_2 concentrations.

The graphs confirm that also in DICE, and in a model with regular annual capital decay structure and not exactly log-utility, the investment rate is not used as a primary measure of climate change policy. The rate does not respond to the CO_2 concentration, which is a measure of expected warming. Only once the temperature dependent damages set in, the consumption rate slightly increases and the investment rate goes down. Instead of reflecting climate policy, this (minor) climate dependence of the consumption rate reflects a response to the damages incurred: these damages reduce the cake to be split into investment and consumption, then, a slightly higher fraction goes to consumption. This response is lower when CO_2 concentrations are high: then the social planner expects high temperatures and damages also in the future and is more hesitant to reduce investment.

A3. Transformation to linearity in states

For notational convenience, I introduce the normalized vector $\boldsymbol{\mathcal{K}}_t \equiv \frac{\boldsymbol{K}_t}{K_t}$ characterizing the distribution of capital over sectors whose components satisfy $\sum_{i=1}^{I_K} \mathcal{K}_{i,t} = 1$. To obtain the equivalent linear-in-state-system, I replace aggregate capital $K_t = \sum_{i=1}^{I_K} K_{i,t}$ by logarithmic capital $k_t \equiv \log K_t$. I replace temperature levels in the atmosphere and the different ocean layers by the transformed exponential temperature states $\tau_{i,t} \equiv \exp(\xi_i T_{i,t})$, $i \in \{1, ..., l\}$. I collect these transformed temperature states in the vector $\boldsymbol{\tau}_t \in \mathbb{R}^l$. Finally, I use the consumption rate $x_t = \frac{C_t}{Y_t[1-D(T_{1,t})]}$, rather than absolute consumption, as the consumption-investment control. Only the rate will be separable from the system's states. Homogeneity of the production function implies that

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = K_t^\kappa F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t).$$

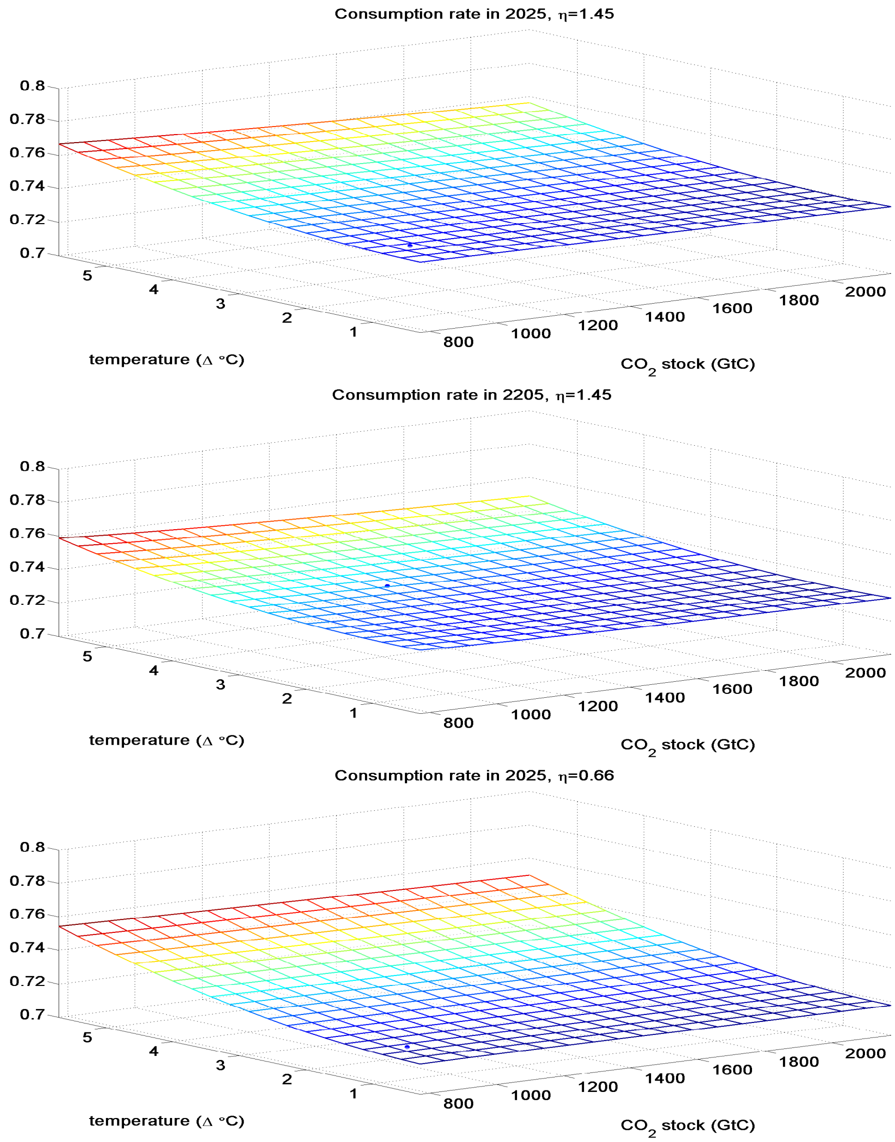

Figure A1. : The graphs analyze the climate (in-)dependence of the optimal consumption rate x^* in the wide-spread DICE model, relying on the control rules of the recursive implementation by Traeger (2012*b*). The first two graphs assume the DICE-2013 value $\eta = 1.45$, the third graph follows the long-run risk literature with $\eta = \frac{2}{3}$. The blue dot in each graph indicates the expected optimal control and prevailing temperature-CO_2 combination along the optimal policy path in the given year.

Then, welfare as a function of the consumption rate is

$$\begin{aligned}
u(x_t) &\equiv \log C_t = \log x_t + \log Y_t + \log[1 - D(T_{1,t})] \\
&= \log x_t + \kappa \log K_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \xi_0 \tau_{1,t} + \xi_0 .
\end{aligned}$$

The Bellman equation in terms of the transformed state variables is

$$\begin{aligned}
V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}} & \log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \\
& -\xi_0 \tau_{1,t} + \xi_0 + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, t+1) \ , \qquad \text{(A.6)}
\end{aligned}$$

and is subject to the following linear equations of motion and constraints. The equations of motion for the effective capital stock and the carbon cycle are

$$\begin{aligned}
k_{t+1} &= \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \xi_0 \tau_{1,t} + \xi_0 + \log(1-x_t) \\
&\quad + \log[1 + g_{k,t}] - \log[\delta_k + g_{k,t}] \qquad \text{(A.7)} \\
\boldsymbol{M_{t+1}} &= \boldsymbol{\Phi} \boldsymbol{M}_t + \boldsymbol{b} \left(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo} \right) \ . \qquad \text{(A.8)}
\end{aligned}$$

where $\boldsymbol{b} = \boldsymbol{e}_1$ in ACE-DICE and $\boldsymbol{b} = \boldsymbol{a}$ in ACE-Joos. Using equation (10), I transform the temperature's equation of motion (7) for layer $i \in \{1, ..., l\}$ to

$$\begin{aligned}
T_{i,t+1} = \frac{1}{\xi_1} \log \Big(& (1-\sigma_{i,i-1}-\sigma_{i,i+1}) \exp[\xi_1 T_{i,t}] \\
& + \sigma_{i,i-1} \exp[\xi_1 T_{i-1,t}] + \sigma_{i,i+1} \exp[\xi_1 T_{i+1,t}] \Big) \ .
\end{aligned}$$

Using the definitions $\sigma_{ii} = 1-\sigma_{i,i-1}-\sigma_{i,i+1}$ and $\tau_{i,t} = \exp(\xi_1 T_{i,t})$ I find

$$\begin{aligned}
\exp(\xi_1 T_{i,t+1}) &= \sigma_{i,i} \exp[\xi_1 T_{i,t}] + \sigma_{i,i-1} \exp[\xi_1 T_{i-1,t}] + \sigma_{i,i+1} \exp[\xi_1 T_{i+1,t}] \\
\Rightarrow \tau_{i,t+1} &= \sigma_{i,i} \tau_{i,t} + \sigma_{i,i-1} \tau_{i-1,t} + \sigma_{i,i+1} \tau_{i+1,t}, i \in \{2, ..., l\},
\end{aligned}$$

still using $\sigma_{l,l+1} = 0$ for notational convenience (see footnote 12). Noting that

$$\exp[\xi_1 T_{0,t}] = \exp\left[\xi_1 \frac{s}{\eta} F_t\right] = \exp\left[\xi_1 \frac{s}{\log 2} \log \frac{\boldsymbol{dM}_t + G_t}{M_{pre}}\right] = \frac{\boldsymbol{dM}_t + G_t}{M_{pre}},$$

the equation for atmospheric temperature ($i = 1$) becomes

$$\tau_{1,t+1} = \sigma_{1,1} \tau_{1,t} + \sigma_{1,0} \frac{\boldsymbol{dM}_t + G_t}{M_{pre}} + \sigma_{1,2} \tau_{2,t} \ .$$

Note that the linearity in $M_{1,t}$ requires $\xi_1 = \frac{\log 2}{s}$ as stated in the proposition. Then, using the definition $\sigma^{forc} = \sigma_{1,0}$, the individual equations of motion for generalized temperature can be collected into the vector equation

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{\boldsymbol{dM}_{t+j} + G_t}{M_{pre}}\boldsymbol{e}_1 \ . \tag{A.9}$$

Finally, the equation of motion for the resource stock is

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d. \tag{A.10}$$

The underlying constraints within periods are

$\sum_{i=0}^{I} N_{i,t} = 1,\ N_{i,t} \geq 0,\ \sum_{i=1}^{I_K} \mathcal{K}_{i,t} = 1,\ \mathcal{K}_{i,t} \geq 0,\ \boldsymbol{R}_t \geq 0,$

and the initial states are given. The present paper assumes that the optimal labor and capital allocation across sectors has an interior solution and that scarce resources are stretched over the infinite time horizon along the optimal path, avoiding boundary value complications.

A4. Illustrating a Two Layer Carbon Cycle

In the simple and insightful case of two carbon reservoirs the carbon cycle's transition matrix is $\boldsymbol{\Phi} = \begin{pmatrix} 1-\delta_{\text{Atm}\to\text{Ocean}} & \delta_{\text{Ocean}\to\text{Atm}} \\ \delta_{\text{Atm}\to\text{Ocean}} & 1-\delta_{\text{Ocean}\to\text{Atm}} \end{pmatrix}$, where e.g. $\delta_{\text{Atm}\to\text{Ocean}}$ characterizes the fraction of carbon in the atmosphere transitioning into the ocean in a given time step. The conservation of carbon implies that both columns add to unity: carbon that does not leave a layer ($\delta_{\cdot\to\cdot}$) stays $(1-\delta_{\cdot\to\cdot})$. The shadow value becomes

$$\varphi_{M,1} = \beta\varphi_{\tau,1}\sigma^{forc}M_{pre}^{-1}(1-\beta)^{-1}\left[1+\beta\frac{\delta_{\text{Atm}\to\text{Ocean}}}{1-\beta(1-\delta_{\text{Ocean}\to\text{Atm}})}\right]^{-1}.$$

The shadow value becomes less negative if more carbon flows from the atmosphere into the ocean (higher $\delta_{\text{Atm}\to\text{Ocean}}$). It becomes more negative for a higher persistence of carbon in the ocean (higher $1-\delta_{\text{Ocean}\to\text{Atm}}$). These impacts on the SCC are straight forward: the carbon in the ocean is the "good carbon" that does not contribute to the greenhouse effect. In round brackets, the root $(1-\beta)^{-1}$ noted in Proposition 2 makes the expression so sensitive to a low rate of pure time preference.

A common approximation of atmospheric carbon dynamics is the equation of motion of the early DICE 1994 model. Here, carbon in excess of preindustrial levels decays as in $M_{1,t+1} = M_{pre} + (1-\delta_{decay})(M_{1,t} - M_{pre})$. The shadow value formula becomes

$$\varphi_{M,1} = \beta\varphi_{\tau,1}\sigma^{forc}M_{pre}^{-1}\big(1-\beta(1-\delta_{decay})\big)^{-1},$$

which misses the long-run carbon impact and the SCC's sensitivity to pure time preference.

A5. Illustrating a Two Layer Atmosphere-Ocean Temperature System

The two layer example of atmosphere-ocean temperature dynamics has the transition matrix $\boldsymbol{\sigma} = \begin{pmatrix} 1-\sigma_1^{up}-\sigma_1^{down} & \sigma_1^{down} \\ \sigma_2^{up} & 1-\sigma_2^{up} \end{pmatrix}$. The corresponding term of the SCC (equation 11) takes the form

$$\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{11} = \left(1 - \beta \underbrace{(1-\sigma_1^{down}-\sigma_1^{up})}_{\text{persistence in atmosphere}} - \frac{\beta^2 \sigma_1^{down}\sigma_1^{up}}{1-\beta \underbrace{(1-\sigma_2^{up})}_{\text{pers. in ocean}}}\right)^{-1}.$$

Persistence of the warming in the atmosphere or in the oceans increases the shadow cost. Persistence of warming in the oceans increases the SCC proportional to the terms σ_1^{down} routing the warming into the oceans and σ_1^{up} routing the warming back from the oceans into the atmosphere. The discount factor β accompanies each of these transition coefficients as each of them causes a one period delay. Taking the limit of $\beta \to 1$ confirms that an analogue to Proposition 2's part (3) does not hold for the temperature system

$$\lim_{\beta\to 1} \varphi_{\tau,1} = -\xi_0(1+\varphi_k)\left[(1-\boldsymbol{\sigma})^{-1}\right]_{11} = -\frac{\xi_0(1+\varphi_k)}{\sigma_1^{up}} \neq \infty. \tag{A.11}$$

As the discount rate approaches zero, the transient temperature dynamics characterized by σ_1^{down} and σ_2^{up} becomes irrelevant for evaluation, and only the weight σ_1^{up} reducing the warming persistence below unity contributes.[38]

Extending on the "missing time preference sensitivity" in the general case, note that temperature is an intensive variable: it does not scale proportional to mass or volume (as is the case for the extensive variable carbon). The columns of the matrix $\boldsymbol{\sigma}$ do not sum to unity. As a consequence of the mean structure in equation (7), however, the rows in the ocean layers' transition matrix sum to unity. The first row determining next period's atmospheric temperature sums to a value smaller than unity: it "misses" the weight that the mean places on radiative forcing. The "missing weight" is a consequence of the permanent energy exchange with outer space. Radiative forcing characterizes a flow equilibrium of incoming and outgoing radiation.

[38] I note that the carbon cycle lacks the reduction in persistence deriving from the forcing weight σ_1^{up}. With this observation, equation (A.11) gives another illustration of the impact of mass conservation in the case of carbon: "$\sigma_1^{up} \to 0 \Rightarrow \varphi_{\tau,1} \to \infty$". Note that in the SCC formula σ_1^{up} cancels, as it simultaneously increases the impact of a carbon change on temperature. This exact cancellation (in the limit $\beta \to 1$) is a consequence of the weights σ_1^{up} on forcing and $1-\sigma_1^{up}$ on atmospheric temperature summing to unity. The result that a warming pulse has a transitional impact and an emission pulse has a permanent impact on the system is independent of the fact that these weights sum to unity.

A6. Climate Parametrization

ACE-DICE uses Nordhaus and Sztorc's (2013) carbon cycle for the DICE 2013 model. For a 10 year time step, the transition matrix is

$$\Phi = \begin{bmatrix} 0.8240 & 0.0767 & 0 \\ 0.1760 & 0.9183 & 0.0007 \\ 0 & 0.0050 & 0.9993 \end{bmatrix} \tag{A.12}$$

This matrix is obtained by rescaling the original coefficients of the DICE 2013 model as follows

$$\Xi^* = (0.088, 0.0025, 0.03832888889, 0.00033750); \qquad \Xi = \Xi^* \frac{step}{5}; \tag{A.13}$$

$$\Phi = \begin{bmatrix} [1-\Xi_1 & \Xi_3 & 0 \\ \Xi_1 & 1-\Xi_2-\Xi_3 & \Xi_4 \\ 0 & \Xi_2 & 1-\Xi_4 \end{bmatrix} \tag{A.14}$$

where $step = 10$ is the time step. I refer to the working paper version for a graph demonstrating that the rescaled 10 year carbon cycle dynamics is virtually indistinguishable from the 5 year dynamics using DICE 2013 BAU emissions (Traeger 2018).

ACE-Joos uses Joos et al.'s (2013) impulse response parametrization, which is

$$\boldsymbol{a} = (a_0, a_1, a_2, a_3)^\top = (0.2240, 0.2824, 0.2763)^\top \quad \text{and} \quad \boldsymbol{\tau} = (394.4, 36.54, 4.304)$$
$$\text{in } \Delta M_{1,t} = a_0 + \sum_{i=1}^{3} a_i \exp\left(-\frac{t}{\tau_i}\right).$$

For ACE-Joos' box representation and a 10 year time step, these model parameters translate into the transition matrix

$$\Phi = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0.9750 & 0 & 0 \\ 0 & 0 & 0.7606 & 0 \\ 0 & 0 & 0 & 0.0979 \end{bmatrix} \tag{A.15}$$

where the diagonal entries are $\gamma_i = \exp\left(-\frac{step}{\tau_i}\right)$ and $\boldsymbol{a} = (a_0, a_1, a_2, a_3)^\top$ characterizes the fraction of carbon going into a particular box as in the original Joos et al. (2013) formulation.

My calibration of the non-linear temperature dynamic system to MAGICC6.0

delivers the transition matrix

$$\sigma = \begin{bmatrix} 0.000000027 & 0.461806 & 0 \\ 0.076520 & 0.895994 & 0.0274865 \\ 0 & 0.003993 & 0.996007 \end{bmatrix} \tag{A.16}$$

for a 10 year time step, where $\sigma^{forc} = 0.538194$. The thin red-dashed line in Figure 4 calibrates to MAGICC6.0 using an annual time step and finds

$$\sigma_{ann} = \begin{bmatrix} 0.711637 & 0.142193 & 0 \\ 0.007035 & 0.991129 & 0.001836 \\ 0 & 0.000379 & 0.999621 \end{bmatrix} \tag{A.17}$$

going along with $\sigma^{forc} = 0.146170$. Comparing the two matrices shows that atmospheric temperature's autoregression hardly plays a role on the decadal scale, but does play a role in a model with an annual time step. I use this calibration only to point out the similarity between a one year and a 10 year time step impulse response. I do not recommend ACE for an annual time step given the approximation of capital depreciation. For a 5 year time step, the calibration delivers the transition matrix

$$\sigma_{5y} = \begin{bmatrix} 0.0000104 & 0.486022 & 0 \\ 0.036187 & 0.953354 & 0.01045 \\ 0 & 0.001918 & 0.998081 \end{bmatrix} \tag{A.18}$$

going along with $\sigma^{forc} = 0.513966$.[39]

In all scenarios I am using a climate sensitivity of $cs = 3$ and, thus, $\xi_1 = \frac{\log 2}{3} = 0.231049$. The base scenario's damage coefficient is $\xi_0 = 0.02219$. HSP damages are characterized by the damage semi-elasticity $\xi_0 = 0.09808$ and under HSP* by $\xi_0 = 0.10876$. I use the IMF's October 2020 forecast for global purchasing power parity output in 2020, which is 130.187 trillion USD. The IMF's corresponding investment rate forecast is 0.26108 (IMF 2020).

[39] For a 5 year time step, simulating capital evolution with the corresponding correction factor works fine, but it would not seem reasonable to endogenously calibrate time preference from the investment rate. For this purpose, it seems better to follow Anderson and Brock's (2021) suggestion to endow ACE with a logarithmic capital depreciation model instead; van der Ploeg and Rezai (2021) follow this suggestion, but use the TCRE model for their climate component.

B. Proofs and Calculations

B1. Proof of Proposition 1

1) Sufficiency: I show that the affine value function

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t \tag{B.1}$$

solves the linear-in-state system corresponding to the equations of sections I.A and I.B with the functional form assumptions presented in Proposition 1. Appendix A.A3 transformed these assumptions into the linear-in-state-system summarized by equations (A.6-A.10), which I take as point of departure. Note that the coefficient on the resource stock has to be time-dependent: the shadow value of the exhaustible resource increases (endogenously) over time following the Hotelling rule.

The controls in the equations of motion (A.7)-(A.10) are additively separated from the states. Therefore, maximizing the right hand side of the resulting Bellman equation delivers optimal control rules that are independent of the state variables. These controls are functions of the shadow values, but independent of the states. Solving the Bellman equation then amounts to a set of coefficient matching conditions determining the shadow values.

I provide the details simultaneously for the different model versions where $T_{0,t} = \frac{s}{\eta}F_t$ (original "delayed" timing) or $T_{0,t} = \frac{s}{\eta}F_{t+1}$ (advanced timing) and where radiative forcing derives from the carbon cycle model (with forcing equation 6) or the box model based on Joos et al. (2013) (with forcing equation 6') and for the model where M_{t+1} is governed by equation (5) or equation (5'). For this purpose, I define $j \equiv 0$ for the delayed system and $j \equiv 1$ for the advanced system so that $T_{0,t} = \frac{s}{\eta}F_{t+j}$ in both cases. To cover the carbon cycle model of ACE-DICE simultaneously with the impulse response model of ACE-Joos (see Section I.B), I define $\boldsymbol{b} \equiv \boldsymbol{e}_1$ and $\boldsymbol{d} \equiv \boldsymbol{e}_1^\top$ for the case of a carbon cycle under equation (5). I define $\boldsymbol{b} \equiv \boldsymbol{\Phi}\boldsymbol{e}_1$ and $\boldsymbol{d} \equiv \boldsymbol{e}_1^\top$ for the case of a carbon cycle under equation (5'). I define $\boldsymbol{b} = \boldsymbol{a}$ (the model's weight vector) and $\boldsymbol{d} \equiv (1, 1, 1, 1)$ in the case of the impulse response model under equation (5). I define $\boldsymbol{b} = \boldsymbol{\Phi}\boldsymbol{a}$ and $\boldsymbol{d} \equiv (1, 1, 1, 1)$ in the case of the impulse response model under equation (5').[40]

Inserting the value function's trial solution (equation B.1) and the next period states (equations A.7-A.10) into the (deterministic) Bellmann equation (A.6) de-

[40] The definition of $\boldsymbol{d}$ implies that $\boldsymbol{d}\boldsymbol{M}_t$ is atmospheric carbon $M_{1,t}$ in the carbon cycle model and the sum of the different carbon boxes $\sum_{i=0}^{3} M_{i,t}$ in the impulse response model.

livers

$$\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}} \log x_t + \beta\varphi_k \log(1-x_t) \tag{B.2}$$
$$+ (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)$$
$$- (1+\beta\varphi_k)\xi_0\tau_{1,t} + (1+\beta\varphi_k)\xi_0 + \lambda_t^N \big(1 - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big)$$
$$+ \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big)$$
$$+ \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^d\right) + \beta\varphi_{t+1}$$
$$+ \beta\boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{b}\big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\right)$$
$$+ \beta\boldsymbol{\varphi}_\tau^\top \left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \boldsymbol{e}_1\sigma^{forc}\frac{\boldsymbol{dM}_{t+j} + G_t}{M_{pre}}\right).$$

In the case of advanced timing ($j = 1$), it is $\boldsymbol{dM}_{t+1} = \boldsymbol{d}\left(\boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{b}\big(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\right)$. In the general case I can therefore write $\boldsymbol{dM}_{t+j} = \boldsymbol{d}\left(\boldsymbol{\Phi}^j\boldsymbol{M}_t + j\boldsymbol{b}\big(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\right)$, where the dummy j eliminates additional contributions in the case of the original "delayed" timing.

Maximizing the right hand side of the Bellman equation over the consumption rate yields

$$\frac{1}{x} - \beta\varphi_k\frac{1}{1-x} = 0 \quad \Rightarrow \quad x^* = \frac{1}{1+\beta\varphi_k}\ . \tag{B.3}$$

The optimal labor, capital, and resource inputs depend on the precise assumptions governing production and energy sector, i.e., the specification of $F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)$. For a well-defined energy system, I obtain unique solutions for these optimal inputs as functions of the technology levels, shadow values, and current states. In detail, the first order conditions for the capital shares deliver

$$(1+\beta\varphi_k)\frac{\frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial\mathcal{K}_{i,t}}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} = \lambda_t^{\mathcal{K}}$$
$$\Leftrightarrow \mathcal{K}_{i,t} = \frac{1}{\lambda_t^{\mathcal{K}}}(1+\beta\varphi_k)\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)$$
$$\Rightarrow \lambda_t^{\mathcal{K}} = \sum_{i=1}^{I_K}(1+\beta\varphi_k)\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)$$
$$\Rightarrow \mathcal{K}_{i,t} = \frac{\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\sum_{i=1}^{I_K}\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)},$$

which is an explicit equation only in the case of constant elasticities $\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial \mathcal{K}_{i,t}} \frac{\mathcal{K}_{i,t}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}$, and an implicit equation that has to be solved together with the other first order conditions otherwise. Analogously, the first order conditions for the labor input deliver

$$\begin{aligned}
&(1+\beta\varphi_k)\frac{\frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial N_{i,t}}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)} = \lambda_t^N \\
&\Rightarrow N_{i,t} = \frac{\sigma_{Y,N_i}(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\sum_{i=1}^{I_N}\sigma_{Y,N_i}(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)},
\end{aligned}$$

with elasticities $\sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial N_{i,t}} \frac{N_{i,t}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}$. The first order conditions for a scarce (fossil) resource input are

$$\begin{aligned}
&(1+\beta\varphi_k)\frac{\frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial E_{i,t}}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)} = \beta\Big(\varphi_{R,i,t+1} - \boldsymbol{\varphi}_M^\top \boldsymbol{b} - j\varphi_{1,\tau}\sigma^{forc}\frac{\boldsymbol{db}}{M_{pre}}\Big) \\
&\Leftrightarrow E_{i,t} = \frac{(1+\beta\varphi_k)\sigma_{Y,E_i}(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\beta\big(\varphi_{R,i,t+1} - \boldsymbol{\varphi}_M^\top \boldsymbol{b} - j\varphi_{1,\tau}\sigma^{forc}\frac{\boldsymbol{db}}{M_{pre}}\big)}
\end{aligned}$$

with elasticities $\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial E_{i,t}} \frac{E_{i,t}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}$. Recall that j is a dummy eliminating the final contribution to the denominator in the case of "delayed" timing. The first order conditions for a non-scarce resource input are analogous but without the shadow cost term $\beta\varphi_{R,i,t+1}$.

Solving the (potentially simultaneous) system of first order conditions, I obtain the optimal controls $\boldsymbol{N}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, $\boldsymbol{\mathcal{K}}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, and $\boldsymbol{E}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$. I will suppress the detailed dependencies below for notational convenience. Knowing these solutions is crucial for determining the precise output path and energy transition under a given policy regime. However, the SCC and, thus, the carbon tax depend only on the structure and optimization of the controls but not on their quantification.

Inserting the (general) control rules into the maximized Bellman equation and collecting terms that depend on state variables on the left hand side delivers

$$
\begin{aligned}
&\big(\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j\big)\boldsymbol{M}_t + \big(\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + (1+\beta\varphi_k)\xi_0\boldsymbol{e}_1^\top\big)\boldsymbol{\tau}_t \\
&\big(\varphi_k - (1+\beta\varphi_k)\kappa\big)k_t + \big(\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top\big)\boldsymbol{R}_t \\
&\;+\varphi_t \;=\; \beta\varphi_{t+1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.4)}\\
&\qquad\qquad \left.\begin{aligned}
&+\;\log x_t^*(\varphi_k) + \beta\varphi_k\log(1-x_t^*(\varphi_k)) + (1+\beta\varphi_k)\xi_0 \\
&+(1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*) \\
&+\beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) - \beta\boldsymbol{\varphi}_{R,t+1}^\top\boldsymbol{E}_t^{d*} \\
&+\beta\boldsymbol{\varphi}_M^\top\boldsymbol{b}\big(\textstyle\sum_{i=1}^{I^d}E_{i,t} + E_t^{exo}\big) + \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M^{pre}}G_t \\
&+j\beta\varphi_{1,\tau}\sigma^{forc}\boldsymbol{db}\big(\textstyle\sum_{i=1}^{I^d}E_{i,t} + E_t^{exo}\big).
\end{aligned}\right\} \equiv B(\cdot)
\end{aligned}
$$

The equality holds for all levels of the state variables if and only if the coefficients in front of the state variables vanish, and the evolution of φ_t satisfies the state independent part of the equation. Setting the states' coefficients to zero yields

$$\varphi_k - (1+\beta\varphi_k)\kappa = 0 \qquad \Rightarrow \varphi_k = \frac{\kappa}{1-\beta\kappa} \tag{B.5}$$

$$\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j = 0 \Rightarrow \boldsymbol{\varphi}_M^\top = \frac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1} \tag{B.6}$$

$$\boldsymbol{\varphi}_\tau^\top + (1+\beta\varphi_k)\xi_0\boldsymbol{e}_1^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} = 0 \qquad \Rightarrow \boldsymbol{\varphi}_\tau = -\xi_0(1+\beta\varphi_k)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1} \tag{B.7}$$

$$\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top = 0 \qquad \Rightarrow \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ . \tag{B.8}$$

The initial values $\boldsymbol{\varphi}_{R,0}^\top$ of the scarce resources depend on the precise evolution of the economy and, thus, depends on assumptions about production and the energy sector. Using the shadow value of log capital in equation (B.3) results in the optimal consumption rate $x^* = 1 - \beta\kappa$. Then equation equation (B.4) turns into the condition

$$\varphi_t - \beta\varphi_{t+1} = B(\cdot). \tag{B.9}$$

This condition will be satisfied by picking the sequence $\varphi_0, \varphi_1, \varphi_2, ...$ Equation (B.9) does not pin down the initial value φ_0. The additional condition $\lim_{t\to\infty}\beta^t V(\cdot) = 0 \Rightarrow \lim_{t\to\infty}\beta^t\varphi_t = 0$ pins down this initial value φ_0 ensuring that the value function is normalized just as the infinite sum of optimized utility (Stokey and Lucas 1989, chapter 4.1). Yet, optimal policy does not dependent on the sequence $\varphi_0, \varphi_1, \varphi_2, \varphi_3, ...$.

2) Necessity: The affine value function solves the system if and only if it is linear-in-state. I have to show that no other transformation of capital or temperature, no other damage function, and no other non-linear mean can achieve the linear-in-state transformation of the equations in sections I.A and I.B. I take as common knowledge that only the log-transformation of capital will solve the system with an affine value function.

To obtain a linear-in-state structure, transformed atmospheric temperature has to be linear in atmospheric carbon. By assumption, temperature evolves as a generalized mean:

$$\mathfrak{M}_i(T_{i-1,t}\ ,\ T_{i,t}\ ,\ T_{i+1,t}) = f^{-1}[\sigma_{i,i-1}f(T_{i-1,t}) + \sigma_{i,i}f(T_{i,t}) + \sigma_{i,i+1}f(T_{i+1,t})]$$

and atmospheric equilibrium temperature for a given forcing is

$$T_{0,t} = \frac{s}{\eta}F_t = \frac{s}{\log 2}\log\frac{M_{1,t+j}+G_t}{M_{pre}}\ ,$$

which is logarithmic in the atmospheric carbon stock. The equation of motion of atmospheric temperature $T_{1,t}$ is therefore

$$T_{1,t+1} = \mathfrak{M}_1(T_{0,t}\ ,\ T_{1,t}\ ,\ T_{2,t}) = f^{-1}[\sigma_{1,0}f(T_{0,t}) + \sigma_{1,1}f(T_{1,t}) + \sigma_{1,2}f(T_{2,t})]$$
$$\Leftrightarrow f(T_{1,t+1}) = \sigma_{1,0}f\left(\frac{s}{\log 2}\log\frac{M_{1,t+j}+G_t}{M_{pre}}\right) + \sigma_{1,1}f(T_{1,t}) + \sigma_{1,2}f(T_{2,t}). \quad \text{(B.10)}$$

First, equation (B.10) implies that $f(T_{1,t})$ and $f(T_{2,t})$ have to be linear to permit a linear-in-state interaction between generalized atmospheric and upper ocean temperature (atmospheric temperature appears on both left and right side of the equality). Second, equation (B.10) implies that $f\left(\frac{s}{\log 2}\log\frac{M_{1,t+j}+G_t}{M_{pre}}\right)$ has to be linear in $M_{1,t+j}$ to permit a linear-in-state interaction between generalized atmospheric temperature and atmospheric carbon. Thus, $f(z) = \exp\left(\frac{\log 2}{s}z\right)$ up to positive affine transformation. Yet, positive affine transformations of f leave the generalized mean unchanged as they simply cancel with the inverse (Hardy, Littlewood and Polya 1964). Note that this step fixes both the functional form of f and the parameter $\xi_1 = \log\frac{2}{s}$.[41] Consequently, the generalized temperature state delivering a linear-in-state dynamics and a linear contribution to the value function has to be $\tau_{i,t} = \exp(\xi_1 T_{i,t})$ for $i \in \{1,2\}$. It follows inductively from

$$f(T_{i,t+1}) = \sigma_{i,i-1}f(T_{i-1,t}) + \sigma_{i,i}f(T_{i,t}) + \sigma_{i,i+1}f(T_{i+1,t})$$

for $i = 2,...,l-1$ that $\tau_{i,t} = \exp(\xi_1 T_{i,t})$ has to hold for all $i \in \{1,...,l\}$, up to affine transformations with a joint multiplicative constant.

[41] The earlier working paper version uses a slightly generalized version of the generalized mean $\mathfrak{M}_1(\cdot)$ permitting additional degrees of freedom (Traeger 2015). However, additional quality of the fit achieved with these additional weight did not warrant the complications in the presentation.

Finally, I show that damages have to be of the form stated in equation (9). Taking the logarithm of the capital's equation of motion (4) delivers

$$\log K_{t+1} = \log Y_t + \log[1 - D(T_{1,t})] + \log(1 - x_t) + \log\left[\frac{1 + g_{k,t}}{\delta_k + g_{k,t}}\right] ,$$

where $\log Y_t$ is linear in the state $k_t = \log K_t$. To render the system linear in the states, at any time t, there have to exist two constants $c_1, c_2 \in \mathbb{R}$ such that

$$\log[1 - D(T_{1,t})] = c_1 \tau_{1,t} + c_2 = c_1 \exp(\xi_1 T_{1,t}) + c_2$$
$$\Rightarrow D(T_{1,t}) = 1 - \exp(c_1 \exp(\xi_1 T_{1,t}) + c_2).$$

Moreover, $c_1 = -c_2 \equiv \xi_0 \in \mathbb{R}$ follows from the requirement that damages are zero at $T_{1,t} = 0$.

B2. Proof of Proposition 2

Proof of Parts (1&2): A. Original "delayed" timing & atmospheric carbon approach:
Here, $j \equiv 0$, $\boldsymbol{b} \equiv \boldsymbol{e}_1$ and $\boldsymbol{d} \equiv \boldsymbol{e}_1^\top$. Then, $\boldsymbol{d}\boldsymbol{\Phi}^j = \boldsymbol{e}_1^\top$. The SCC is the negative of the shadow value of atmospheric carbon expressed in money-measured consumption units. Inserting equation (B.5) for the shadow value of log-capital and (B.7) for the shadow value of atmospheric temperature (first entry of the vector) into equation (B.6) characterizing the shadow value of carbon in the different reservoirs delivers

$$\boldsymbol{\varphi}_M^\top = -\xi_0 \left(1 + \beta \frac{\kappa}{1 - \beta\kappa}\right) \left[(1 - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta \sigma^{forc}}{M_{pre}} \boldsymbol{e}_1^\top (\mathbf{1} - \beta\boldsymbol{\Phi})^{-1} .$$

The expression characterizes the social cost in terms of welfare units. This marginal welfare cost translates into a consumption change as follows: $du_t = \frac{1}{C_t} dC_t = \frac{1}{x^* Y_t^{net}} dC_t \Rightarrow dC_t = (1 - \beta\kappa) Y_t^{net} du_t$. Thus, observing that $\left(1 + \beta \frac{\kappa}{1-\beta\kappa}\right) = \frac{1}{1-\beta\kappa}$, the SCC in consumption units is

$$SCC = -(1 - \beta\kappa) Y_t^{net} \varphi_{M,1} = Y_t^{net} \; \xi_0 \left[(1 - \beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta \sigma^{forc}}{M_{pre}} \left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1} .$$

B. Advanced timing & emission release approach:
Here, $j \equiv 1$, $\boldsymbol{b} \equiv \boldsymbol{\Phi}\boldsymbol{e}_1$ and $\boldsymbol{d} \equiv \boldsymbol{e}_1^\top$. The SCC now follows from the impact of an emission unit on future welfare. The right side of equation (B.B1) shows that the impact of period t emissions on next period's value function is

$$\beta \boldsymbol{\varphi}_M^\top \boldsymbol{b} \left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo} \right) + j \beta \boldsymbol{\varphi}_{1,\tau} \sigma^{forc} \frac{\boldsymbol{d}\boldsymbol{b}\left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo} \right),}{M_{pre}}.$$

In the present case, $j=1$, $\boldsymbol{b} \equiv \boldsymbol{\Phi}\boldsymbol{e}_1$, $\boldsymbol{d} \equiv \boldsymbol{e}_1^\top$ and, thus, $\boldsymbol{db} = \boldsymbol{e}_1^\top \boldsymbol{\Phi}\boldsymbol{e}_1 = \Phi_{1,1}$. By equations (B.6) and (B.7) the SCC in utils is

$$
\begin{aligned}
&-\beta\boldsymbol{\varphi}_M^\top \boldsymbol{b} - \beta\boldsymbol{\varphi}_{1,\tau}\sigma^{forc}\tfrac{\boldsymbol{db}}{M_{pre}} = \beta\tfrac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}}\boldsymbol{d}\boldsymbol{\Phi}(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1}\boldsymbol{b} - \beta\boldsymbol{\varphi}_{1,\tau}\sigma^{forc}\tfrac{\boldsymbol{db}}{M_{pre}} \\
&= -\tfrac{\beta\sigma^{forc}}{M_{pre}}\varphi_{\tau,1}\left(\beta\boldsymbol{d}\boldsymbol{\Phi}(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1}\boldsymbol{b} + \boldsymbol{db}\right) \\
&= -\tfrac{\beta\sigma^{forc}}{M_{pre}}\varphi_{\tau,1}\left(\boldsymbol{db} + \beta\boldsymbol{d}\boldsymbol{\Phi}\left(\textstyle\sum_{i=0}^\infty \beta^i\boldsymbol{\Phi}^i\right)\boldsymbol{b}\right) = -\tfrac{\beta\sigma^{forc}}{M_{pre}}\varphi_{\tau,1}\left(\boldsymbol{db} + \boldsymbol{d}\left(\textstyle\sum_{i=0}^\infty \beta^{i+1}\boldsymbol{\Phi}^{i+1}\right)\boldsymbol{b}\right) \\
&= -\tfrac{\beta\sigma^{forc}}{M_{pre}}\varphi_{\tau,1}\left(\boldsymbol{d}\left(\textstyle\sum_{i=0}^\infty \beta^i\boldsymbol{\Phi}^i\right)\boldsymbol{b}\right) = -\tfrac{\beta\sigma^{forc}}{M_{pre}}\varphi_{\tau,1}\left(\boldsymbol{d}(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1}\boldsymbol{b}\right) \\
&= -\tfrac{\beta\sigma^{forc}}{M_{pre}}\xi_0(1+\beta\varphi_k)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\left(\boldsymbol{d}(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1}\boldsymbol{b}\right)
\end{aligned}
$$

where I used the Neuman series $(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1} = \sum_{i=0}^\infty \beta^i\boldsymbol{\Phi}^i$.

Proof of Part (3): Mass conservation of carbon implies that the columns of Φ add to unity. In consequence, the vector with unit entry in all dimensions is a left and, thus, right eigenvector. The corresponding eigenvalue is one and the determinant of $\boldsymbol{1}-\beta\boldsymbol{\Phi}$ has the root $1-\beta$. It follows from Cramer's rule (or as an application of the Cayley-Hamilton theorem) that the entries of the matrix $(\boldsymbol{1}-\beta\boldsymbol{\Phi})^{-1}$ are proportional to $(1-\beta)^{-1}$.

B3. General Remarks on Population Change (Section III.E)

Population change is a special case of Proposition 3 that I prove below. The Proposition states the analytic closed-form result obtained under the assumption of a constant population growth rate. The quantitative results in Table 1 rely on non-constant population growth. They use the UN population growth scenario delivering decadal growth factors 1.0967, 1.0761, 1.0583, 1.0428, 1.0303, 1.0205, 1.0127, 1.0061. I assume population to be stationary after 2100, which is in line with the corresponding UN data that almost converges by its end year 2100. I first calculate the shadow value of atmospheric carbon using the constant population growth solution (with zero growth) for 2100 (see proposition). Then, I recursively calculate the present shadow value using the equations (B.15-B.17) derived below towards the end of the proof.

No temperature lag under population change. The scenarios omitting temperature lag under a non-stationary population change adjust equations (B.16) and (B.17) as follows. The absence of temperature dynamics or delay eliminates the term containing the generalized heat transition matrix $\boldsymbol{\sigma}$ in equation (B.16). As a result the temperature's shadow value is directly determined by the damage coefficient and the shadow value of log-capital. It also eliminates the parameter σ^{forc} from equation (B.17). Again, the quantitative solution for the UN growth scenario first calculates the stationary shadow values post 2100 (merely omitting σ^{forc} and $\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}$) and then recursively calculates the present shadow values using equation (B.15) and the modified versions of equations (B.16) and (B.17) discussed above.

B4. Proof of Proposition 3

The welfare objective (14) replaces the term $\log C_t$ on the r.h.s. of the (untransformed) Bellman equation by the term

$$\textstyle\sum_{i\in P} \alpha_{i,t} \log c_{i,t} + \lambda_{p,t}\Big(C_t - \sum_{i\in P} p_{i,t} c_{i,t}\Big), \tag{B.11}$$

using the Lagrange multiplier $\lambda_{p,t}$. ACE now maximizes over all $c_{i,t}$, $i \in P$, as well as aggregate consumption C_t. Group i's consumption levels $c_{i,t}$ only appears in the Bellman equation in the terms stated in equation (B.11). Thus, the first order condition for group i's consumption is

$$\frac{\alpha_{i,t}}{c_{i,t}} = \lambda_{p,t} p_{i,t} \quad \Rightarrow \quad c_{i,t} = \frac{\alpha_{i,t}}{p_{i,t}} \lambda_{p,t}^{-1}.$$

Thus, the consumption constraint yields

$$C_t = \sum_{i\in P} p_{i,t} c_{i,t} = \sum_{i\in P} p_{i,t} \frac{\alpha_{i,t}}{p_{i,t}} \lambda_{p,t}^{-1} = \lambda_{p,t}^{-1} \sum_{i\in P} \alpha_{i,t} = \lambda_{p,t}^{-1} \alpha_t \quad \Rightarrow \quad \lambda_{p,t} = C_t^{-1} \alpha_t.$$

The FOC for aggregate consumption replaces the earlier marginal utility C_t^{-1} with the shadow value of the consumption constraint $\lambda_{p,t} = C_t^{-1}\alpha_t$. Thus, in part i), where $\alpha_t = 1$ for all t, the Bellman equation remains unaltered and so does the solution for the SCC.

For part ii), the FOC for aggregate consumption picks up another constant and I have to recalculate the FOC for the aggregate consumption rate. After optimizing the individual consumption levels $c_{i,t}$, the equations above imply that period t's wefare contribution is

$$\begin{aligned}\sum_{i\in P} \alpha_{i,t} \log c_{i,t} &= \sum_{i\in P} \alpha_{i,t} \log\left(\frac{\alpha_{i,t}}{p_{i,t}} \lambda_{p,t}^{-1}\right) = \sum_{i\in P} \alpha_{i,t} \log\left(\frac{\alpha_{i,t}}{\alpha_t p_{i,t}} C_t\right) \\ &= \alpha_t \log C_t + \underbrace{\sum_{i\in P} \alpha_{i,t} \left[\log \frac{\alpha_{i,t}}{\alpha_t} - \log p_{i,t}\right]}_{\equiv \bar{\alpha}_t},\end{aligned}$$

where $\bar{\alpha}_t$ is an exogenous additive constant. The crucial change to the Bellman equation derives from the constant α_t multiplying aggregate consumption. After expressing the objective in terms of the aggregate consumption rate, Bellman equation (A.6) turns into

$$V(k_t, \boldsymbol{\tau_t}, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}} \alpha_\mathrm{t}(\log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t))$$
$$+ \alpha_\mathrm{t}(-\xi_0 \tau_{1,t} + \xi_0) + \bar{\alpha}_\mathrm{t} + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, t{+}1) \ . \tag{B.12}$$

From here, there are two ways forward. One can use the original trial solution for the value function permitting the shadow values to change over time. More elegantly, at least for the case of a constant growth rate, take the trial solution

$$V(k_t, \boldsymbol{\tau_t}, \boldsymbol{M}_t, \boldsymbol{R}_t, t) = \alpha_t \varphi_k k_t + \alpha_t \boldsymbol{\varphi}_M^\top \boldsymbol{M_t} + \alpha_t \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \alpha_t \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t. \tag{B.13}$$

Plugging this trial-solution into (B.12) and dividing by α_t delivers

$$\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t$$
$$= \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}} \log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) + -\xi_0 \tau_{1,t} + \xi_0 + \frac{\bar{\alpha}_t}{\alpha_t}$$
$$+ \beta \frac{\alpha_{t+1}}{\alpha_t} \left(\varphi_k k_{t+1} + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_{t+1} + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_{t+1} + \boldsymbol{\varphi}_{R,t+1}^\top \boldsymbol{R}_{t+1} + \varphi_{t+1} \right) \ . \tag{B.14}$$

Denoting the, by assumption, constant growth factor of the intergenerational weight α_t by $g = \frac{\alpha_{t+1}}{\alpha_t}$, equation (B.14) corresponds to the original dynamic programming problem with discount factor βg replacing the original discount factor β. The constant $\frac{\bar{\alpha}_t}{\alpha_t}$ affects the absolute welfare level in utils, however, it has no impact on the shadow values (see derivation of the original shadow values). Thus, we obtain the same shadow value formula for $\varphi_{M,1}$ in utils as before. Yet, our new value function (B.13) multiplies this $\varphi_{M,1}$ with the intergenerational weight α_t. At the same time, we have derived the shadow value of aggregate consumption as $\lambda_{p,t} = C_t^{-1} \alpha_t$, and the conversion factor from utils to consumption changes to $dC_t = C_t \alpha_t^{-1} dW_t$. The novel α_t^{-1} in the conversion factor cancels the multiplier α_t in front of the shadow value $\varphi_{M,1}$ and, therefore, the only change in the SCC formula remains $\beta \to \beta g$.

In the *general case with non-constant populations growth*, the shadow values in equation (B.14) have to pick up time indices

$$\varphi_{k,t} k_t + \boldsymbol{\varphi}_{M,t}^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_{\tau,t}^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_t$$
$$= \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}} \log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) + -\xi_0 \tau_{1,t} + \xi_0 + \frac{\bar{\alpha}_t}{\alpha_t}$$
$$+ \beta \frac{\alpha_{t+1}}{\alpha_t} \left(\varphi_{k,t+1} k_{t+1} + \boldsymbol{\varphi}_{M,t+1}^\top \boldsymbol{M}_{t+1} + \boldsymbol{\varphi}_{\tau,t+1}^\top \boldsymbol{\tau}_{t+1} + \boldsymbol{\varphi}_{R,t+1}^\top \boldsymbol{R}_{t+1} + \varphi_{t+1} \right) \ .$$

Letting $g_t = \frac{\alpha_{t+1}}{\alpha_t}$, and plugging in the expressions for the next period states as in equation (B.2) delivers the first order condition for the consumption rate

$$\frac{1}{x_t} = \beta\varphi_{k,t+1}\frac{g_t}{1-x_t} \quad \Rightarrow \quad x_t^* = \frac{1}{1+\beta g_t \varphi_{k,t+1}} \ .$$

Also the other controls remain independent of the states. Next, I have to insert the controls into the maximized Bellman equation and collect terms that depend on the state variables on the left as in equation (B.4). All that changes w.r.t. the earlier derivation is that we distinguish present from next period shadow values, which are multiplied by the growth factor g_t

$$\begin{aligned}
&\big(\boldsymbol{\varphi}_{M,t}^\top - \beta g_t \boldsymbol{\varphi}_{M,t+1}^\top \boldsymbol{\Phi} - \beta g_t \varphi_{\tau,1,t+1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j\big)\boldsymbol{M}_t \\
&+\big(\boldsymbol{\varphi}_{\tau,t}^\top - \beta g_t \boldsymbol{\varphi}_{\tau,t+1}^\top \boldsymbol{\sigma} + (1+\beta g_t \varphi_{k,t+1})\xi_0 \boldsymbol{e}_1^\top\big)\boldsymbol{\tau}_t \\
&+\big(\varphi_{k,t} - (1+\beta g_t \varphi_{k,t+1}\kappa\big)k_t + \big(\boldsymbol{\varphi}_{R,t}^\top - \beta g_t \boldsymbol{\varphi}_{R,t+1}^\top)\big)\boldsymbol{R}_t + \varphi_t = \beta\varphi_{t+1} + B(\cdot)
\end{aligned}$$

where $B(\cdot)$ is defined in equation (B.4). The resulting recursion equations for the shadow values that eliminate the state coefficients are

$$\varphi_{k,t} - (1+\beta g_t \varphi_{k,t+1})\kappa = 0 \qquad \Rightarrow \varphi_{k,t} = \kappa + \beta g_t \kappa \varphi_{k,t+1} \tag{B.15}$$

$$\begin{aligned}
&\boldsymbol{\varphi}_{\tau,t}^\top + (1+\beta g_t \varphi_{k,t+1})\xi_0 \boldsymbol{e}_1^\top - \beta g_t \boldsymbol{\varphi}_{\tau,t+1}^\top \boldsymbol{\sigma} = 0 \\
&\qquad \Rightarrow \boldsymbol{\varphi}_{\tau,t}^\top = \beta g_t \boldsymbol{\varphi}_{\tau,t+1}^\top \boldsymbol{\sigma} - (1+\beta g_t \varphi_{k,t+1})\xi_0 \boldsymbol{e}_1^\top
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
&\boldsymbol{\varphi}_{M,t}^\top - \beta g_t \boldsymbol{\varphi}_{M,t+1}^\top \boldsymbol{\Phi} - \beta g_t \varphi_{\tau,1,t+1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j = 0 \\
&\qquad \Rightarrow \boldsymbol{\varphi}_{M,t}^\top = \frac{\beta g_t \varphi_{\tau,1,t+1}\sigma^{forc}}{M_{pre}}\boldsymbol{d\Phi}^j + \beta g_t \boldsymbol{\varphi}_{M,t+1}^\top \boldsymbol{\Phi}.
\end{aligned} \tag{B.17}$$

or, e.g., with ACE-DICE standard timing

$$\Rightarrow \boldsymbol{\varphi}_{M,t}^\top = \frac{\beta g_t \varphi_{\tau,1,t+1}\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top + \beta g_t \boldsymbol{\varphi}_{M,t+1}^\top \boldsymbol{\Phi}.$$

Once population stabilizes we are in a stationary state where our original solution holds. From that stationary state, equations (B.15-B.17) deliver the recursion to calculate the present shadow values, first solving (B.15), then (B.16), and then (B.17).

As earlier, the SCC converts the shadow value of atmospheric carbon into consumption units using the relation derived above, $dC_t = C_t\alpha_t^{-1}dW_t$. The shadow value of atmospheric carbon is now $\alpha_t\varphi_{M,1,t}$ because of the differing trial solution. Thus, once again the α_t cancels and the conversion of the shadow value into consumption units works equivalently to the cases without population weighting.

B5. Proof of Proposition 4

First, observe that the production processes of the final goods $c_{l,t}$ are all homogenous of degree $\kappa \equiv \alpha + \tilde{\alpha}\nu$ in capital. As a result, both C_t and I_t are also homogenous of degree κ in capital. Moreover, I can pull the production damage factor $[1-D(T_{1,t})]$ from equation (A.2) through the CES aggregators in equations (A.1) and (A.4). Thus,

$$
\begin{aligned}
C_t =& \left(\sum_{l\in I_c} a_{l,t}\left(x_{l,t}A_{l,t}K_{l,t}^{\alpha}N_{l,t}^{1-\alpha-\nu}\left(\left(\sum_{i\in\Theta_l}\left(g_{i,t}(A_{i,t},K_{i,t},N_{i,t},E_{i,t})\right)^{\tilde{s}_{l,t}}\right)^{\frac{1}{\tilde{s}_{l,t}}}\right)^{\nu}\right)^{s_t}\right)^{\frac{1}{s_t}} \\
&\times[1-D(T_{1,t})] \\
=& \underbrace{\left(\sum_{l\in I_c} a_{l,t}\left(x_{l,t}A_{l,t}\mathcal{K}_{l,t}^{\alpha}N_{l,t}^{1-\alpha-\nu}\left(\left(\sum_{i\in\Theta_l}\left(g_{i,t}(A_{i,t},\mathcal{K}_{i,t},N_{i,t},E_{i,t})\right)^{\tilde{s}_{l,t}}\right)^{\frac{1}{\tilde{s}_{l,t}}}\right)^{\nu}\right)^{s_t}\right)^{\frac{1}{s_t}}}_{\equiv\Omega_{C,t}(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)} \\
&\times[1-D(T_{1,t})]K_t^{\alpha+\tilde{\alpha}\nu} = \Omega_{C,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)[1-D(T_{1,t})]K_t^{\alpha+\tilde{\alpha}\nu}
\end{aligned}
$$

and analogously $I_t = \Omega_{I,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)[1-D(T_{1,t})]K_t^{\alpha+\tilde{\alpha}\nu}$. I find

$$
\begin{aligned}
\log C_t &= \log\left(\Omega_{C,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right) + \log[1-D(T_{1,t})] + (\alpha+\tilde{\alpha}\nu)\log K_t \\
&= \log\left(\Omega_{C,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right) - \xi_0\tau_{1,t} + \xi_0 + \kappa k_t,
\end{aligned}
$$

replacing the terms $\log x_t + \log F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)$ in equation (A.6) by the term $\log\left(\Omega_{C,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right)$. Similarly, the equation of motion for log-capital (A.7) changes into

$$
\begin{aligned}
k_{t+1} =& \kappa k_t + \log\left(\Omega_{I,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right) - \xi_0\tau_{1,t} + \xi_0 + \log[1+g_{k,t}] \\
&- \log[\delta_k + g_{k,t}]
\end{aligned}
$$

replacing the terms $\log(1-x_t) + \log F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)$ in equation (A.6) by the term $\log\left(\Omega_{I,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right)$. The maximization on the r.h.s. Bellman equation (A.6) now entails the labor, energy, and capital distribution within and across consumption and investment production processes. Instead of optimizing w.r.t. a single consumption rate, the maximization is now over all the consumption rates $x_{l,t}$ of those goods that can be used in consumption and in investment, i.e., where $l \in I_C \cap I_I$. Let $\hat{\boldsymbol{x}}_t$ denote the vector of endogenously chosen consumption rates, i.e., the vector containing all $x_{l,t}$ with $l \in I_C \cap I_I$. The r.h.s. of the Bellman equation (B.2) on page 51 now reads

$$\max_{\hat{\boldsymbol{x}}_t,\boldsymbol{N_t},\boldsymbol{\mathcal{K}}_t,\boldsymbol{E_t}} \log\left(\Omega_{C,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right)+\beta\varphi_k\log\left(\Omega_{I,t}(\boldsymbol{x}_t,\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)\right)$$
$$+\lambda_t^N\big(\alpha_\mathrm{t}-\textstyle\sum_{i=1}^{I_N}N_{i,t}\big)+\lambda_t^{\mathcal{K}}\big(1-\sum_{i=1}^{I_K}\mathcal{K}_{i,t}\big)-\beta\boldsymbol{\varphi}_{R,t+1}^\top\boldsymbol{E}_t^d$$
$$+\beta\varphi_{M,1}\big(\textstyle\sum_{i=1}^{I^d}E_{i,t}+E_t^{exo}\big)+\ldots$$

where I omit terms that are independent of the controls.

As in the original problem, the solutions to the maximization problems of the r.h.s. Bellman equation are independent of the states because the equation additively separates the terms containing the controls from the terms containing the states. As a result, the FOCs and their solutions change, but the shadow values remain the same as in the original problem (equations B.5-B.8). The change to the SCC formula results from dealing with a consumption variety in the consumption-investment trade-off and the conversion of the shadow price of atmospheric carbon into consumption equivalents.

Expressing the shadow price of atmospheric carbon in equivalents of the consumption aggregate C_t, I still have the conversion factor $du_t=\frac{1}{C_t}dC_t \Rightarrow dC_t = C_t du_t$. In Proposition 2, the simple constant aggregate consumption rate $x^*=(1-\beta\kappa)$ implied $C_t=x^*Y_t^{net}=(1-\beta\kappa)Y_t^{net}$ and the factor $(1-\beta\kappa)$ canceled the shadow value's term $1+\beta\varphi_k=1+\beta\frac{\kappa}{1-\beta\kappa}=\frac{1}{1-\beta\kappa}$. Absent such a simplification, the SCC remains with the term $\frac{C_t}{1-\beta\kappa}$ instead of Y_t^{net}.

r.p.t.p.		Scenario	Parameter changes	Carb Mult	w/o TD USD	SCC in USD	cent per Gallon	Euro-cent per liter
1.4%	1	**base (ACE-DICE)**		4.3	50	**30**	27	6
	2	ACE-Joos (Joos)	Φ	4.3	49	30	27	6
	3	**HSP-damages** (HSP)	ξ_0	4.3	219	**134**	118	27
x 4.3%	4	**productivity recal**	κ, β	2.2	19	**11**	9	2
	5	**productivity eq.** (13)	κ	4.3	56	**34**	30	7
	6	Population growth (pg)	g		63	39	34	8
x 2.3%	7	Pop growth recalib (pg-re)	g, β		40	24	21	5
x 4.3%	8	**HSP & prod recal**	ξ_0, κ, β	2.2	85	**48**	42	10
x 4.3%	9	Joos & HSP & prod recal	$\Phi, \xi_0, \kappa, \beta$	2.1	80	45	39	9
	10	**HSP & prod** (13)	ξ_0, κ	4.3	248	**152**	134	31
x 4.3%	11	prod recal & pg	κ, β, g		23	13	11	3
x 5.2%	12	prod recal & pg-re	κ, β, g		18	10	9	2
	13	prod (13) & pg	κ, g		72	44	39	9
x 4.3%	14	HSP & prod recal & pg	ξ_0, κ, β, g		102	57	50	12
x 5.2%	15	HSP & prod recal & pg-re	ξ_0, κ, β, g		81	45	39	9
	16	HSP & prod (13) & pg	ξ_0, κ, g		319	197	173	40
0.5%	17	**expert discounting**		8.4	109	**75**	66	15
	18	ACE-Joos (Joos)	Φ	8.9	115	79	70	16
	19	**HSP-damages** (HSP)	ξ_0	8.4	480	**331**	291	68
	20	**productivity eq.** (13)	κ	8.4	125	**86**	76	18
	21	Population growth (pg)	g		146	100	88	21
	22	Joos & HSP	Φ, ξ_0	8.9	509	351	308	72
	23	Joos & p (13)	Φ, κ	8.9	133	92	80	19
	24	Joos & pg	Φ, g		156	107	94	22
	25	**Joos & HSP & p** (13)	Φ, ξ_0, κ	8.9	587	**405**	356	83
	26	Joos & HSP & pg	Φ, ξ_0, g		688	473	416	97
	27	Joos & p (13) & pg	Φ, κ, g		180	124	109	25
	28	Joos & HSP & p (13) & pg	Φ, ξ_0, κ, g		795	547	480	112
0.1%	29	low discounting		26	361	290	255	59
	30	**ACE-Joos** (Joos)	Φ	30	409	**328**	288	67
	31	HSP-damages (HSP)	ξ_0	26	1600	1280	1130	262
	32	productivity eq. (13)	κ	26	421	338	296	69
	33	Population growth (pg)	g		500	400	351	82
	34	**Joos & HSP**	Φ, ξ_0	30	1810	**1450**	1270	296
	35	**Joos & prod eq** (13)	Φ, κ	30	476	**382**	335	78
	36	Joos & pg	Φ, g		567	454	398	93
	37	**Joos & HSP & p** (13)	Φ, ξ_0, κ	30	2100	**1690**	1480	345
	38	Joos & HSP & pg	Φ, ξ_0, g		2510	2000	1760	410
	39	Joos & p (13) & pg	Φ, κ, g		660	528	464	108
	40	Joos & HSP & p (13) & pg	Φ, ξ_0, κ, g		2920	2330	2050	477

Table 1—: Quantitative results. "r.p.t.p."=rate of pure time preference. "Carb Mult"= Carbon-based multiplier in SCC (non-stationary for population growth). "w/o TD"= without temperature delay (cutting temperature related terms from SCC). "HSP-damages" uses Section II.C's calibration of damages to Howard and Sterner (2017) and Pindyck (2020). For population growth see Section III.E. The "productivity recal" and "population growth recalibrated" (pg-re) scenarios ("x" in the first column) recalibrate time preference. In contrast, "production eq. (13)" (p 13) and "population growth" (pg) use equation (13) taking β and κ as model input, see Proposition 4 in Appendix A.A1 for details.