Online Appendix: Optimal Insurance: Dual Utility, Random Losses and Adverse Selection

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Online Appendix A: Stochastic Mechanisms

We provide an example showing that a stochastic mechanism may be more profitable than the optimal deterministic mechanism. Consider an agent with risk preference represented by $g(x) = x^2$ and assume that $\theta \sim U[0, 1]$. Suppose that there is a single deterministic loss level $l$, and let the agent’s type $\theta$ be the probability that a loss occurs. The optimal deterministic mechanism consists of full insurance to types $\theta \geq \frac{2}{3}$ at a price $\frac{8}{9}l$, and no insurance for lower types. Using this mechanism, the insurer’s profit is $\frac{1}{54}l \approx 0.185l$.

Consider now a stochastic direct mechanism of the form $(t(\theta), p(\theta), l)$ such that: type $\theta$ pays a premium $t(\theta)$; in exchange, when a loss occurs, the insurer fully reimburses the agent’s loss with (conditional) probability $1 - p(\theta)$. Note that the above class of mechanisms includes the optimal deterministic mechanism.

If type $\theta$ reports to be type $\theta'$ he receives $-t(\theta') - l$ with probability $p(\theta')\theta$ and receives $-t(\theta')$ otherwise. Thus, in the proposed mechanism, this type of agent has a payoff of

$$\tilde{U}(\theta, \theta') = -l - t(\theta') + g(1 - p(\theta')\theta)l$$

One can verify that $(t(\theta), p(\theta), l)$ is incentive compatible if and only if $p$ is non-increasing and

$$\tilde{U}(\theta) = \tilde{U}(\bar{\theta}) - l \int_{\theta}^1 p(z)g(1 - p(z)z)dz,$$

where we write $\tilde{U}(\theta) = \tilde{U}(\theta, \theta)$ for short. The above conditions imply that

$$t(\theta) = -l - \tilde{U}(\bar{\theta}) + g(1 - p(\theta)\theta)l + l \int_{\theta}^1 p(z)g'(1 - p(z)z)dz.$$

By using similar arguments to that of Lemma 1, it can be shown that the individual
rationality constraint holds if and only if

\[ \bar{U}(\theta) \geq -l(1 - g(1 - \theta)) = 0. \]

From now onward, we only consider mechanism for which \( \bar{U}(\theta) = 0 \). The insurer’s profit is

\[
\pi(p, t) = \int_{\theta}^{\bar{\theta}} [t(\theta) - (1 - p(\theta))\theta l] f(\theta) d\theta
\]

\[
= -l + \int_{\theta}^{\bar{\theta}} \left[ g(1 - p(\theta))l + l \int_{\theta}^{\bar{\theta}} p(z)g'(1 - p(z)z) dz - (1 - p(\theta))\theta l \right] f(\theta) d\theta
\]

\[
= l \int_{\theta}^{\bar{\theta}} \left[ g(1 - p(\theta)) - (1 - p(\theta))\theta + \frac{1 - F(\theta)}{f(\theta)} p(\theta)g'(1 - p(\theta)) \right] f(\theta) d\theta - l
\]

\[
= l \int_{\theta}^{\bar{\theta}} \left[ \theta (3\theta - 2)p^2 - (3\theta - 2)p + 1 - \theta \right] d\theta - l.
\]

To obtain this equality, we used integration by parts:

\[
\int_{\theta}^{\bar{\theta}} [1 - F(\theta)] p(\theta)g'(1 - p(\theta)) d\theta - \int_{\theta}^{\bar{\theta}} f(\theta) \left[ \int_{\theta}^{\bar{\theta}} p(z)g'(1 - p(z)z) dz \right] d\theta = 0.
\]

The optimal \( p \) is then given by

\[
p^*(\theta) = \begin{cases} 
1 & \text{if } \theta \leq \frac{1}{2} \\
\frac{1}{2\theta} & \text{if } \frac{1}{2} < \theta < \frac{2}{3} \\
0 & \text{if } \theta \geq \frac{2}{3}.
\end{cases}
\]

That is, within the above described class of potentially stochastic mechanisms, it is optimal to offer no insurance to agents with type below \( \frac{1}{2} \), to offer unconditional full insurance to those with type above \( \frac{2}{3} \), and to offer to reimburse the loss with (conditional) probability \( 1 - \frac{1}{2\theta} \) to intermediate types in \( (\frac{1}{2}, \frac{2}{3}) \). This mechanism yields, approximately, an expected profit of \( 0.188l > 0.185l \), and is thus superior to the optimal deterministic mechanism.
Online Appendix B: Finite Number of Losses

Proof of Proposition 3. It holds that

\[ H_\theta(z) = \begin{cases} 
1 - \theta & \text{if } z < l_1 \\
1 - \theta + \theta \sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} \leq z < l_k \text{ and } k \in \{2, \ldots, n\} \\
1 & \text{if } z \geq l_n 
\end{cases} \]

and \( \frac{\partial H_\theta(z)}{\partial \theta} = \begin{cases} 
-1 & \text{if } z < l_1 \\
-1 + \sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} < z < l_k \text{ and } k \in \{2, \ldots, n\} \\
0 & \text{if } z > l_n 
\end{cases} \]

In any incentive compatible mechanism, the menu of deductibles \( D(\theta) \) is non-increasing in the probability of accident \( \theta \). In particular, \( D(\theta) \) is continuous almost everywhere.

Fix such a non-increasing menu, and let \( \theta_0 = \bar{\theta} \). Denote by \( \theta_1 = \inf \{ \theta : D(\theta) \leq l_1 \} \).

If this set is empty, define \( \theta_1 = \theta_0 = \bar{\theta} \). Similarly, for \( i \in \{2, \ldots, n\} \) define \( \theta_i = \inf \{ \theta : D(\theta) \leq l_i \} \) with \( \theta_i := \theta_{i-1} \) if the set is empty.

By the monotonicity of \( D(\theta) \), it holds that \( \theta = \theta_n \leq \theta_{n-1} \leq \ldots \leq \theta_1 \leq \theta_0 = \bar{\theta} \). The insurer’s profit becomes then

\[
\pi = \int_{\theta}^{\bar{\theta}} \left[ -\mathbb{E}[L(\theta)] + \int_0^{D(\theta)} \left[ g(H_\theta(z)) - H_\theta(z) - \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} \right] dz \right] f(\theta) d\theta - U(\bar{\theta})
\]

\[
= -\int_{\theta}^{\bar{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta + \int_{\theta}^{\bar{\theta}} \left[ \int_0^{D(\theta)} \left[ g(H_\theta(z)) - H_\theta(z) - \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} \right] dz \right] f(\theta) d\theta
\]

\[
+ \sum_{k=2}^{n} \int_{\theta_{k-1}}^{\theta_k} \left[ \int_0^{D(\theta)} \left[ g(H_\theta(z)) - H_\theta(z) - \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} \right] dz \right] f(\theta) d\theta
\]

\[
\pi = \int_{\theta}^{\bar{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta - U(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} \left[ \int_0^{D(\theta)} \left[ g(1 - \theta) - (1 - \theta) + \frac{1 - F(\theta)}{f(\theta)} g'(1 - \theta) \right] dz \right] f(\theta) d\theta
\]

\[
+ \sum_{k=2}^{n} \int_{\theta_{k-1}}^{\theta_k} \int_0^{D(\theta)} \left[ g(1 - \theta + \theta \sum_{i=1}^{k-1} p_i) - \left( 1 - \theta + \theta \sum_{i=1}^{k-1} p_i \right) \right] dz f(\theta) d\theta
\]

\[
\pi = -\int_{\theta}^{\bar{\theta}} \mathbb{E}[L(\theta)] f(\theta) d\theta - U(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} D(\theta) [g(1 - \theta) - (1 - \theta) + \frac{1 - F(\theta)}{f(\theta)} g'(1 - \theta)] f(\theta) d\theta
\]

\[
+ \sum_{k=2}^{n} \int_{\theta_{k-1}}^{\theta_k} D(\theta) \left[ g(1 - \theta + \theta \sum_{i=1}^{k-1} p_i) - \left( 1 - \theta + \theta \sum_{i=1}^{k-1} p_i \right) \right] f(\theta) d\theta.
\]
By definition, in each interval \([\theta_k, \theta_{k-1}]\), the given deductible \(D(\theta)\) belongs to the interval \([l_{k-1}, l_k]\), where we denote \(l_0 = 0\). Note that, on each interval \([\theta_k, \theta_{k-1}]\), the obtained expression for profit is linear in \(D\):

\[
\int_{\theta_k}^{\theta_{k-1}} D(\theta) \left[ g(1 - \theta + \theta \sum_{i=1}^{k-1} p_i) - \left(1 - \theta + \theta \sum_{i=1}^{k-1} p_i\right) \left(1 - \sum_{i=1}^{k-1} p_i\right) \right] f(\theta) d\theta.
\]

Depending on the sign of the integrand, the above expression is maximized with respect to \(D\) at an extreme point of the respective feasible set, i.e., either at \(D^*(\theta) = l_{k-1}\) or at \(D^*(\theta) = l_k\). Thus, the profit from the given mechanism can be increased by changing all deductibles \(D(\theta)\) on the interval \([\theta_k, \theta_{k-1}]\) to the value of \(D^*(\theta)\) that maximizes the above expression. The obtained \(D^*\) is non-increasing by construction, and thus also implementable. Hence, we have shown that the search for an optimal mechanism can be confined to menus consisting of at most \(n + 1\) deductibles, where each deductible equals either zero, or one of the possible losses. 

**Proof of Corollary 1.** Here

\[
H_\theta(z) = \begin{cases} 
1 - \theta & \text{if } z < l \\
1 & \text{if } z \geq l
\end{cases}
\quad \text{and} \quad \frac{\partial H_\theta(z)}{\partial \theta} = \begin{cases} 
-1 & \text{if } z \leq l \\
0 & \text{if } z \geq l
\end{cases}.
\]

The insurer’s profit becomes:

\[
\pi = \int_\theta^\beta \left[ -\mathbb{E}[L(\theta)] + \int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta}] dz \right] f(\theta) d\theta - U(\theta)
\]

\[
= -l + \int_\theta^\beta \left[ \int_0^{D(\theta)} [g(1 - \theta) - (1 - \theta) + \frac{1 - F(\theta)}{f(\theta)} g'(1 - \theta)] dz \right] f(\theta) d\theta - U(\theta)
\]

\[
= -l + \int_\theta^\beta D(\theta) \left[ g(1 - \theta) - (1 - \theta) + \frac{1 - F(\theta)}{f(\theta)} g'(1 - \theta) \right] f(\theta) d\theta - U(\theta).
\]

The above expression is linear in \(D\), and hence the pointwise maximum in the above expression is attained at an extreme point of the feasible set: it can be either at \(D = l\) or at \(D = 0\), depending on the sign of the virtual value. 

**Online Appendix C: Binary Lotteries**

In this Appendix we document several instances of well-known, non-expected utility formulations that coincide with Yaari’s dual utility on the class of binary lotteries (e.g., in an insurance framework with a single, deterministic loss).
1. Gul’s [1991] disappointment-averse preferences with linear utility over outcomes:

\[ U(x) = \frac{\alpha}{1 + (1 - \alpha)\beta} \mathbb{E}[x|x \geq CE(x)] + \frac{(1 - \alpha) (1 + \beta)}{1 + (1 - \alpha)\beta} \mathbb{E}[x|x < CE(x)] \]

where \( CE(x) \) is a certainty equivalent of lottery \( x \in X \), \( \alpha \) is the probability that the outcome of the lottery is above its certainty equivalent, and \( \beta \) is a parameter. For binary lotteries, the above functional form is a special case of Yaari’s dual utility with\(^2\)

\[ g(p) = \frac{p}{1 + (1-p)\beta}. \]

2. Versions of the disappointment aversion theories due to Loomes and Sugden [1986], and Jia et al. [2001] with linear utility over outcomes:

\[ U(x) = \mathbb{E}(x) + (e - d)\mathbb{E} \left[ \max \{ x - \mathbb{E}(x), 0 \} \right], \]

where \( e > 0, d > 0 \). For binary lotteries, this is a special case of Yaari’s dual utility with

\[ g(p) = p(1 + e - d) + (d - e)p^2. \]

Risk aversion (either in the weak or strong sense) is obtained when \( e < d \).

3. The modified Mean-Variance preferences (see Rockafellar et al. [2006]) with linear utility over outcomes are given by\(^3\):

\[ U(x) = \mathbb{E}(x) - \frac{1}{2} r \mathbb{E} \left[ \| x - \mathbb{E}(x) \| \right], \]

where \( r \in [0, 1] \). For binary lotteries this is again a special case of Yaari’s preferences where

\[ g(p) = p - rp(1 - p). \]

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\(^{1}\)This is implicit. See also Cereia-Voglio et al. [2020] for an explicit formulation.

\(^{2}\)(Weak) risk aversion corresponds then to \( \beta > 1/2 \) and aversion to mean-preserving spreads corresponds to \( \beta > 0 \).

\(^{3}\)The modification relative to the standard mean-variance preferences is needed in order to ensure consistency with FOSD.
References


