The Trickling Up of Excess Savings

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Online Appendix

A Deriving the main model and its extensions

A.1 Characterizing household with bonds-in-utility preferences.

The objective of each household i is to choose consumption to maximize discounted flow utility

$$\int_0^\infty e^{-\rho t} \left(u(c_{it}) + v_i(a_{it})\right) dt \tag{3}$$

where u is flow utility from consumption and v_i is type-specific utility from assets, subject to the flow budget constraint

$$\dot{a}_{it} = r_t a_{it} + \theta_i Y_t - c_{it} \tag{4}$$

where θ_i is the type's share of aggregate income.

This problem delivers the standard intertemporal Euler equation

$$\frac{\dot{c}_{it}}{c_{it}} = -\frac{u'(c_{it})}{u''(c_{it})c_{it}} \left(\frac{v_i'(a_{it})}{u'(c_{it})} + r_t - \rho\right) \tag{5}$$

Now, multiplying both sides by c_{it} and taking a first-order approximation of (5) around the steady state, we have

$$d\dot{c}_{it} = \sigma_i^{-1} c_i \left(\frac{v_i''(a_i)}{u'(c_i)} da_{it} + \sigma_i(\rho - r)\frac{dc_{it}}{c_i} + dr_t\right) \tag{6}$$

where we define $\sigma_i \equiv -\frac{u''(c_i)c_i}{u'(c_i)}$. Linearizing (4) gives

$$d\dot{a}_{it} = r da_{it} + a_i dr_t - dc_{it} + \theta_i dY_t \tag{7}$$

Characterizing policy function around the steady state: relating to flow MPCs. First, we want to characterize the consumption policy function for this agent around the steady state in the absence of shocks to future r_t or Y_t. Suppose that it is given locally by $dc_{it} = m_i da_{it}$. Plugging this into (6) gives

$$m_i d\dot{a}_{it} = \sigma_i^{-1} c_i \left(\frac{v_i''(a_i)}{u'(c_i)} da_{it} + \sigma_i(\rho - r)\frac{m_i da_{it}}{c_i}\right)$$

and then plugging in $d\dot{a}_{it} = (r - m_i) da_{it}$ from (7) and dividing the above by da_{it} gives the relation

$$m_i(r - m_i) = \sigma_i^{-1} c_i \frac{v_i''(a_i)}{u'(c_i)} + (\rho - r) m_i \tag{8}$$

Under the assumption of $r = 0$ in the steady state, simplifies to just

$$m_i^2 + \rho m_i + \sigma_i^{-1} c_i \frac{v_i''(a_i)}{u'(c_i)} = 0 \tag{9}$$

(9) gives a quadratic equation that we can solve for the flow MPC m_i in terms of primitives around the steady state.[5] Additionally, if we plug (8) into (5), and again enforce $r = 0$ in steady state, we obtain a linearized Euler equation where the curvature of the bond-in-utility function shows up entirely through m_i:

$$d\dot{c}_{it} = -(m_i^2 + \rho m_i) da_{it} + \rho dc_{it} + \sigma_i^{-1} c_i dr_t \tag{10}$$

In general, we will consider the $\rho \to 0$ limit, as well as a case where u is CRRA, at which point (10) simplifies further to just

$$d\dot{c}_{it} = -m_i^2 da_{it} + \sigma^{-1} c_i dr_t \tag{11}$$

Similarly, with the assumption of $r = 0$, (7) simplifies to

$$d\dot{a}_{it} = a_i dr_t - dc_{it} + \theta_i dY_t \tag{12}$$

Characterizing household policy functions. Define $dc_{it}^P \equiv dc_{it} - m_i da_{it}$ to be the first-order change in a household's consumption policy function, which equals the first-order change in consumption relative to steady state, minus the effect of assets. Note that then plugging into (7) and assuming $r = 0$ gives

$$\begin{aligned} d\dot{c}_{it}^P &= d\dot{c}_{it} - m_i d\dot{a}_{it} \\ &= d\dot{c}_{it} - m_i a_i dr_t + m_i dc_{it} - m_i \theta_i dY_t \\ &= d\dot{c}_{it} - m_i a_i dr_t + m_i dc_{it}^P + m_i^2 da_{it} - m_i \theta_i dY_t \end{aligned} \tag{13}$$

Now, substituting (10) for $d\dot{c}_{it}$ into this, and writing $dc_{it} = dc_{it}^P + m_i da_{it}$, we get

$$\begin{aligned} d\dot{c}_{it}^P &= -(m_i^2 + \rho m_i) da_{it} + \rho dc_{it}^P + \rho m_i da_{it} + \sigma_i^{-1} c_i dr_t - m_i a_i dr_t + m_i dc_{it}^P + m_i^2 da_{it} - m_i \theta_i dY_t \\ &= (\rho + m_i) dc_{it}^P + \sigma_i^{-1} c_i dr_t - m_i a_i dr_t - m_i \theta_i dY_t \end{aligned} \tag{14}$$

where we see that the $(m_i^2 + \rho m_i) da_{it}$ cancel. (14) implies that

$$dc_{it}^P = \int_0^\infty e^{-(\rho + m_i)s} (-\sigma_i^{-1} c_i dr_{t+s} + m_i a_i dr_{t+s} + m_i \theta_i dY_{t+s}) ds \tag{15}$$

i.e. that the change in consumption policy is the discounted forward-looking average of substitution effects of interest rates $-\sigma_i^{-1} c_i dr_{t+s}$, income effects of interest rates $m_i a_i dr_{t+s}$, and changes in aggregate income $m_i \theta_i dY_{t+s}$. The discount factor is $\rho + m_i$.

[5] See Auclert, Rognlie and Straub (2018) for the equivalent quadratic equation in discrete time.

A.2 Description of benchmark general equilibrium environment

We suppose that each type $i = 1, \ldots, N$ supplies n_{it} hours of effective labor to the market in steady state, leading to aggregate labor supply of $N_t = \sum_i n_{it}$. Each unit of labor produces one unit of goods, $Y_t = N_t$, and the goods market is competitive, so that the real wage is always 1. We assume that nominal wages are sticky, and that for any level of aggregate labor demand N_t that deviates from the steady state, the rationing rule increases the effective labor of each type proportionately: $n_{it} = \frac{N_t}{N} n_i$. We define $\theta_i = n_i / N$ to be the share of effective labor supplied by type i; differences in θ_i across groups can reflect differences in population or differences in productivity. Then labor income of each group is $n_{it} = \theta_i Y_t$.

Since nominal wage inflation will not matter for real outcomes under our assumptions, we leave the Phillips curve for wages (and the underlying disutility function from labor) unspecified. See Auclert, Rognlie and Straub (2018) for more details.

We assume that we are in the neighborhood of the steady state, that r_t is held constant by monetary policy, and that agents assume future aggregate income will be at its steady state level. Then because all forward-looking inputs to their problems are fixed, agents follow the their steady-state consumption policy function $dc_{it} = m_i da_{it}$, and the budget constraint will be given by (12) with $dr_t = 0$, i.e. $d\dot{a}_{it} = -dc_{it} + \theta_i dY_t$. Further, goods market clearing implies that $dY_t = \sum_i dc_{it}$.

These three equations describe our benchmark model; for economy of notation, in the paper, we replace dc_{it} with c_{it}, da_{it} with a_{it}, and dY_t with Y_t, with all variables implicitly denoting first-order deviations from steady state.

Note that, in the absence of a reaction of monetary policy ($r_t = 0$ for all t), we have that the cumulative output response is

$$\frac{\int_0^\infty Y_t dt}{B} = \left(1 - \frac{a_{N0}}{B}\right) \frac{1}{\theta_N} = \frac{\text{Share of initial transfer not given to super-rich}}{\text{Income share of super-rich}} \tag{16}$$

This follows from the fact that $\dot{a}_{Nt} = \theta_N Y_t$, so applying Proposition 1, $\theta_N \int_0^\infty Y_t dt = \int_0^\infty \dot{a}_{Nt} dt = B - a_{N0}$. Equation (16) expresses the cumulative multiplier from the deficit-financed transfer as a simple ratio of two sufficient statistics, the share of the transfer not initially given to the super-rich to their income share.

A.3 Extensions

With rational expectations. When agents have rational expectations and do perceive future dY_t, in the limit $\rho \to 0$, then their consumption is simply characterized by (11). We further assume that monetary policy keeps the real interest rate constant, $dr_t = 0$, so this gives

$$d\dot{c}_{it} = -m_i^2 da_i$$

Finally, we assume no steady state assets $a_i = 0$, so that equation (12) is

$$d\dot{a}_{it} = \theta_i dY_t - dc_{it}$$

Redefining $c_{it} \equiv dc_{it}$, $a_{it} \equiv da_{it}$, $Y_t \equiv dY_t$ for simplicity, the model is now:

$$\dot{c_{it}} = -m_i^2 a_{it}; \qquad \dot{a}_{it} = \theta_i Y_t - c_{it}; \qquad Y_t = \sum_{i=1}^{N} c_{it} \tag{17}$$

With monetary response. Now suppose that r_t does vary over time according to some monetary rule $dr_t = \phi dY_t$ that increases the real interest rate to offset a boom in demand. Assume that this path of real interest rates is perfectly anticipated by households, but that households still do not anticipate changes in aggregate income. (For instance, households might see the term structure of borrowing rates directly from financial markets, but not have similar exposure to their own incomes; these are level-1 households in Farhi and Werning 2019)

To avoid large instantaneous income effects (since in reality assets will have longer duration and their returns will be insulated from interest rate changes), and to avoid needing to specify a taxation rule for the government, we assume here that steady-state assets of all types are zero. Also, changes in future incomes do not appear in (14), since the household does not perceive them when choosing policy. Hence, together with our other simplifications, (14) becomes simply $d\dot{c}_{it}^P = m_i dc_{it}^P + \sigma^{-1} c_i dr_t$, and $dc_{it} = m_i da_{it} + dc_{it}^P$. This modification to consumption is the only first-order departure from the benchmark framework.

Note that since the effects of monetary policy are discounted by m_i, high i types with lower m_i will have a larger consumption response to interest rates. Therefore, a rise in real interest rates in response to excess savings will cause high i to spend relatively less, leaving them with more wealth and speeding the process of trickling up.

To summarize, the equations are:

$$\begin{aligned}
d\dot{c}_{it}^P &= m_i dc_{it}^P + \sigma^{-1} c_i dr_t \\
dc_{it} &= m_i da_{it} + dc_{it}^P \\
d\dot{a}_{it} &= -dc_{it} + \theta_i dY_t \\
dY_t &= \sum_{i=1}^{N} dc_{it}
\end{aligned}$$

Plugging in the monetary response $dr_t = \phi dY_t$, assuming further that steady state $c_i = \theta_i$, and switching notation back to levels, we obtain:

$$\begin{aligned}
\dot{c}_{it}^P &= m_i c_{it}^P + \sigma^{-1}\phi\theta_i Y_t \\
c_{it} &= m_i a_{it} + c_{it}^P \\
\dot{a}_{it} &= -c_{it} + \theta_i Y_t \\
Y_t &= \sum_{i=1}^N c_{it}
\end{aligned}$$

Note that in particular type N agent is Ricardian, with Euler equation $dc_{Nt} = \sigma^{-1} c_N dr_t$. In condensed form, these equations read:

$$\dot{c}_{it}^P = m_i c_{it}^P + \sigma^{-1}\theta_i\phi Y_t; \qquad \dot{a}_{it} = \theta_i Y_t - m_i a_{it} - c_{it}^P; \qquad Y_t = \sum_{i=1}^N \left(m_i a_{it} + c_{it}^P\right) \tag{18}$$

B Proofs of propositions 2 and 3

B.1 Proof of proposition 2

We prove the following claim:

Claim (N)**:** Let $\{\theta_j\}$ be positive and sum to 1, let $m_1 > \ldots > m_N = 0$, let $a_{j0} \geq 0$, and let a_{jt} solve the system of differential equations

$$\dot{a}_{jt} = -m_j a_{jt} + \theta_j \left(\sum_{i=1}^N m_i a_{it} + x_t\right)$$

where $x_t \geq 0$ is an exogenous inflow. Assume $m_j a_{j0}/\theta_j$ strictly falls in j. Then: For any $J \geq 1$ and t

$$\sum_{j=J}^N \dot{a}_{jt} \geq \left(\sum_{j=J}^N \theta_j\right) x_t \tag{19}$$

and for any t

$$\sum_{j=1}^N m_j \dot{a}_{jt} \leq x_t \sum_{j=1}^N m_j \theta_j \tag{20}$$

Claim (N) is strictly more general than proposition 2. Indeed, setting $x_t = 0$, the claim implies $\sum_{j=J}^N \dot{a}_{jt} \geq 0$ for any $J \geq 1$, from which it follows that

$$\sum_{j=J}^N a_{jt'} \geq \sum_{j=J}^N a_{jt}$$

for any dates $t' > t$. The flip-side is $\sum_{j=1}^{J-1} a_{jt'} \leq \sum_{j=1}^{J-1} a_{jt}$.

We proceed to prove claim (N) by induction over N. The induction start with $N = 1$ is trivial.

Next, suppose claim (N) holds. We intend to prove claim $(N+1)$. For that, take $\{\theta_j\}_{j=0}^{N}$ with $\sum_{j=0}^{N} \theta_j = 1$, $m_0 > \ldots > m_N = 0$, $a_{j0} \geq 0$ and a_{jt} described by

$$\dot{a}_{jt} = -m_j a_{jt} + \theta_j \left(\sum_{i=0}^{N} m_i a_{it} + x_t \right)$$

Assume $m_j a_{j0} / \theta_j$ decreases monotonically in j.

Lemma 1. *There is always a positive net flow from type 0 to everyone else. In math,*

$$m_0 a_{0t} > \theta_0 \sum_{i=0}^{N} m_i a_{it}$$

which we can rewrite as

$$m_0 a_{0t} > \frac{\theta_0}{1-\theta_0} \sum_{i=1}^{N} m_i a_{it}$$

Proof. We show this by contradiction. Let τ be the first time at which the are equal,

$$m_0 a_{0\tau} = \frac{\theta_0}{1-\theta_0} \sum_{i=1}^{N} m_i a_{i\tau} \tag{21}$$

This means, up until date $t = \tau$, we can write the evolution of wealth of the types $j > 0$ as

$$\dot{a}_{jt} = -m_j a_{jt} + \theta_j \left(\frac{1}{1-\theta_0} \sum_{i=1}^{N} m_i a_{it} + x_t + m_0 a_{0t} - \frac{\theta_0}{1-\theta_0} \sum_{i=1}^{N} m_i a_{it} \right)$$

where before date τ, $m_0 a_{0t} \geq \frac{\theta_0}{1-\theta_0} \sum_{i=1}^{N} m_i a_{it}$.

Define $\tilde{\theta}_j \equiv \frac{\theta_j}{1-\theta_0}$ for $j = 1, \ldots, N$ and

$$\tilde{x}_t \equiv (1-\theta_0) \left(x_t + m_0 a_{0t} - \frac{\theta_0}{1-\theta_0} \sum_{i=1}^{N} m_i a_{it} \right)$$

Observe that, at date $t = \tau$, $\tilde{x}_\tau = (1-\theta_0)\, x_\tau$. Then, we can apply the induction hypothesis on types $j = 1, \ldots, N$. This establishes that, at date $t = \tau$,

$$\sum_{j=1}^{N} m_j \dot{a}_{j\tau} \leq \left(\sum_{j=1}^{N} \tilde{\theta}_j m_j \right) \tilde{x}_\tau = \left(\sum_{j=1}^{N} \theta_j m_j \right) x_\tau$$

and so

$$
\begin{aligned}
m_0 \dot{a}_{0\tau} &= -m_0^2 a_{0\tau} + m_0\theta_0 \left(\sum_{i=0}^{N} m_i a_{i\tau} + x_\tau \right) = m_0\theta_0 x_\tau \\
&\geq \frac{m_0\theta_0}{\sum_{j=1}^{N} \theta_j m_j} \sum_{j=1}^{N} m_j \dot{a}_{j\tau} \geq \frac{\theta_0}{1-\theta_0} \sum_{j=1}^{N} m_j \dot{a}_{j\tau}
\end{aligned}
$$

where we used the fact that m_j falls monotonically in j. This is a contradiction to τ being the first time for which (21) holds with equality, given that at date 0,

$$
\frac{m_0 a_{00}}{\theta_0} > \sum_{i=0}^{N} \theta_i \frac{m_i a_{i0}}{\theta_i}
$$

which falls from $m_j a_{j0}/\theta_j$ strictly falling in j. □

Lemma 2. *The equations* (19) *and* (20) *hold for the economy with* $N+1$ *types.*

Proof. Now that we established the positive flow from type 0 to the other types, it follows directly that

$$
\sum_{j=J}^{N} \dot{a}_{jt} \geq \left(\sum_{j=J}^{N} \tilde{\theta}_j \right) \tilde{x}_t = \left(\sum_{j=J}^{N} \theta_j \right) x_t
$$

for any $J \geq 1$. Moreover, total wealth grows at rate x_t, so

$$
\sum_{j=0}^{N} \dot{a}_{jt} = x_t
$$

Hence (19) holds. (20) follows from (19), because

$$
\begin{aligned}
\sum_{j=1}^{N} m_j \dot{a}_{jt} &= m_1 \sum_{j=1}^{N} \dot{a}_{jt} - \sum_{k=2}^{N} (m_{k-1} - m_k) \sum_{j=k}^{N} \dot{a}_{jt} \\
&\leq m_1 x_t \sum_{j=1}^{N} \theta_j - \sum_{k=2}^{N} (m_{k-1} - m_k) \left(\sum_{j=k}^{N} \theta_j \right) x_t \\
&\leq x_t \sum_{j=1}^{N} m_j \theta_j
\end{aligned}
$$

□

Lemma 2 establishes claim $(N+1)$ and thus concludes our proof by induction.

B.2 Proof of proposition 3

We can write the law of motion for assets (dropping t subscripts) as

$$\dot{a}_i = -m_i a_i + \theta_i \sum_j m_j a_j \tag{22}$$

or, in stacked form,

$$\dot{a} = (-M + \theta m')a$$

where $M = \text{diag}(m)$. Define $A \equiv -M + \theta m'$. Note that $M^{-1/2} A M^{1/2} = -I + (M^{-1/2}\theta)(M^{1/2}m)'$ should have the same eigenvalues as A. Perron-Frobenius implies that $(M^{-1/2}\theta)(M^{1/2}m)'$ has a unique largest (real) eigenvalue with corresponding positive eigenvector, and then the largest eigenvalue of A is this minus 1.

Since we have already shown in proposition 1 that this system is globally stable, the largest eigenvalue of A must be negative. Call this $-\lambda$. We see that

$$-\lambda v_i = -m_i v_i + \theta_i \sum_j m_j v_j$$
$$v_i = \frac{\theta_i}{m_i - \lambda} \sum_j m_j v_j$$

Note that the eigenvector v would not be everywhere positive if λ was greater than or equal to any m_i. We conclude that $\lambda < m_i$.