Online Appendix

Macroeconomic Dynamics with Rigid Wage Contracts

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April 18, 2022

A Solution to the general contracting problem

Consider the contracting problem (2). The total wage income can be reformulated in terms of the hours-productivity relation $N(\cdot)$ by a change of variable:

$$\int_0^N W(n)dn + W_{min} = \int_0^Z W(N(z))N'(z)dz + W_{min} = \int_0^Z zF'(N(z))N'(z)dz + W_{min}$$

The firm's problem can thus be expressed in terms of $N(\cdot)$.

$$\max_{N(\cdot),W_{min}} \quad \mathbb{E}\left[ZF(N(Z)) - \int_0^Z zF'(N(z))N'(z)dz - W_{min}\right]$$
$$s.t. \quad \mathbb{E}\left(V\left(\int_0^Z zF'(N(z))N'(z)dz + W_{min}\right) - v(N(Z))\right) = \underline{\text{U}}$$

We have now transformed the problem from choosing a relationship between wage payments and hours (implying a relationship between hours and productivity) to choosing a relationship between hours and productivity (implying a relationship between wage payments and hours).

We reformulate the problem so that we can directly apply Euler's equation from calculus of variations. First, note that $ZF(N(Z)) - \int_0^Z aF'(N(a))N'(a)dz = \int_0^Z F(N(a))dz$ by integration by parts. Therefore, the problem can be reformulated as

$$\max_{N(\cdot),W_{min}} \mathbb{E}\left[-W_{min} + \int_0^Z F(N(z))dz\right] \quad s.t. \quad \mathbb{E}\left(V\left(ZF(N(Z)) - \int_0^Z F(N(z))dz + W_{min}\right) - v(N(Z))\right) = \underline{U}.$$

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Write $I(Z) = \int_0^Z F(N(z))dz$. Note that $I'(Z) = F(N(Z))$. The problem can then be written as

$$\max_{I(\cdot),W_{min}} \mathbb{E}\left[-W_{min} + I(Z)\right] \qquad s.t. \qquad \mathbb{E}\left(V\left(ZI'(Z) - I(Z) + W_{min}\right) - v(F^{-1}(I'(Z)))\right) = \underline{\text{U}}$$

We will use tools from calculus of variations to solve this problem. First, we rewrite the problem with a Lagrange multiplier on the constraint,

$$\max_{I(\cdot),W_{min},\lambda} \mathbb{E}\left[-W_{min} + I(Z)\right] + \lambda(\mathbb{E}\left(V\left(ZI'(Z) - I(Z) + W_{min}\right) - v(F^{-1}(I'(Z)))\right) - \underline{U})$$

By taking the first-order conditions with respect to W_{min} and λ, we get the optimality conditions

$$\frac{1}{\mathbb{E}(U'(ZI'(Z) - I(Z) + W_{min}))} = \lambda,$$
$$\mathbb{E}\left(V\left(ZI'(Z) - I(Z) + W_{min}\right) - v(F^{-1}(I'(Z)))\right) = \underline{\text{U}}.$$

The problem of choosing $I(\cdot)$ is on a form where Euler's equation for an extremum applies. Let

$$F(Z, I, I') = \left[-W_{min} + I + \lambda\left(V(ZI' + W_{min} - I) - v(F^{-1}(I')) - \underline{\text{U}}\right)\right] f_Z(Z).$$

Euler's equation states that optimality requires $F_I = \frac{d}{dZ} F_{I'}$.

We compute the derivatives in turn. Write $\mathcal{V}(\cdot) = v(F^{-1}(\cdot))$. We suppress the argument of $V(\cdot)$. The density function of Z is given by f_Z.

$$\begin{aligned}
F_I &= [1 - \lambda V'(\ldots)] f_Z(Z) \\
F_{I'} &= \lambda(ZV'(\ldots) - \mathcal{V}'(\ldots)) f_Z(Z) \\
\frac{d}{dZ} F_{I'} &= \lambda(V'(\ldots) + Z^2 V''(\ldots) I'' - \mathcal{V}''(\ldots) I'') f_Z + \\
&\quad \lambda(ZV'(\ldots) - \mathcal{V}'(\ldots)) f'_Z(Z)
\end{aligned}$$

Therefore, optimality requires

$$[1 - \lambda V'(\ldots)] f_Z(Z) = \lambda[V'(\ldots) + Z^2 V''(\ldots) I''(Z) - \mathcal{V}''(\ldots) I''(Z)] f_Z + \lambda[ZV'(\ldots) - \mathcal{V}'(\ldots)] f'_Z(Z).$$

or

$$I'' = \frac{\lambda^{-1} - 2V' - \frac{f'_Z}{f_Z}(ZV' - \mathcal{V}')}{Z^2 V'' - \mathcal{V}''}.$$

which is an ordinary differential equations.

Wage payments are $W^s = ZI'(Z) + I(Z) + W_{min}$. Hours are $N = F^{-1}(I'(Z))$. Therefore, expressing wage payments as a function of hours, we arrive at $W^s = (I')^{-1}(F(N))F(N) - I((I')^{-1}(F(N))) + W_{min}$.

B The role of complete markets

In the static model in Section II, we assumed that workers can trade a complete set of financial securities. In this section, we elaborate on how the assumption of complete markets implies that workers' marginal utility of consumption is independent of the idiosyncratic productivity shocks, and, therefore, that the outcome of the contracting problem under complete markets is identical to the contracting problem assuming that all workers belong to a "representative family", for which the marginal utility of consumption only depends on aggregate shocks.

In the main text, aggregate productivity was expected to be constant and we considered the response to an unforeseen "MIT" shock. To clarify the role of complete markets in insuring against idiosyncratic but not aggregate shocks, we here describe the general case with the ex-ante distribution for aggregate productivity being non-degenerate.

After the realization of the shocks, the full state consists of all the idiosyncratic productivities, A_i for $i \in [0,1]$, and aggregate productivity A. The idiosyncratic productivities are i.i.d. and drawn from the distribution f_{A_i}. Aggregate productivity is independent of idiosyncratic productivities, and drawn from the distribution f_A.

Since markets are complete, there is a price $f_{A_i}(a_i)f_A(a)P_{a_i,a}$ of an Arrow-Debreu security delivering one unit of consumption when $A_i = a_i$ and $A = a$. Because idiosyncratic risk is diversifiable, we have $P_{a_i,a} = P_a$ where $f_A(a)P_a$ is the price of an Arrow-Debreu security delivering one unit of consumption when $A = a$.

Worker's consumption problem With complete markets, the worker's consumption problem involves trading Arrow-Debreu claims to consumption (with price $f_A(a)P_a$ for a claim to consumption when $A = a$). Given a wage contract, inducing a relationship between wage payments and productivity given by $W^s(N(A_iA))$, the consumption problem of the worker is

$$\max_{C(\cdot,\cdot)} \; \mathbb{E}[u(C(A_i, A)]$$
$$s.t \quad \int\int f_{A_i}(A_i)f_A(A)P_A(C(A_i,A) - e(A))dA_i dA = \int\int f_{A_i}(A_i)f_A(A)P_A W^s(N(A_iA))dA_i dA.$$

By taking first-order conditions, it is immediate that consumption does not depend on idiosyncratic productivity, so we write consumption only as a function of the aggregate state, $C(A)$. Optimality implies that the ratio $u'(C(A))/P_A$ is independent of A (and equal to the Lagrange multiplier on the constraint).

Further, the right-hand side of the constraint can be rewritten as $\mathbb{E}[P_A W^s(N(A_i A))]$, the time-zero market value of future wage payments, so the consumption problem of the worker defines an indirect utility function $V(X)$ of the market value of future wage payments $X = \mathbb{E}[P_A W^s(N(A_i A))]$. By the envelope theorem, $V'(X) = u'(C(A))/P_A$ for all A.

Firm's contracting problem The firm is owned by the workers. The firm's problem is to maximize the time-zero market value of the profits of the firm,

$$\begin{aligned} \max_{W(\cdot), W_{min}, N(\cdot)} \; & \mathbb{E}\left[P_A A A_i F(N(AA_i)) - P_A \int_0^{N(AA_i)} W(n)dn - P_A W_{min}\right] \\ s.t. \; & W(N(A)) = AF'(N(A)) \\ & V\left(\mathbb{E}\left[P_A \int_0^{N(AA_i)} W(n)dn + P_A W_{min}\right]\right) - \mathbb{E}[v(N(AA_i))] = \underline{U} \end{aligned}$$

where $V(\cdot)$ is the indirect utility function derived from the worker's consumption problem. With the substitution $X = \mathbb{E}[P_A W^s(N(A_i A))]$, we can rewrite the problem as:

$$\begin{aligned} \max_{W(\cdot), X, N(\cdot)} \; & \mathbb{E}\left[P_A A A_i F(N(AA_i))\right] - X \\ s.t. \; & W(N(A)) = AF'(N(A)), \\ & V(X) - \mathbb{E}[v(N(AA_i))] = \underline{U}. \end{aligned}$$

The first constraint can always be satisfied by adjusting $W(\cdot)$. Optimality for $N(\cdot)$ gives $\mathbb{E}[P_A \tilde{A} F'(N(\tilde{A}))|\tilde{A}] = \lambda v'(N(\tilde{A}))$, where λ is the Lagrange multiplier with respect to the second constraint. Optimality for X gives $1 = \lambda V'(X)$. Substituting out the Lagrange multiplier, we arrive at

$$\mathbb{E}[P_A \tilde{A} F'(N(\tilde{A}))|\tilde{A}] = \frac{1}{V'(X)} v'(N(\tilde{A})).$$

Recall from the worker's problem that $V'(X) = u'(C(A))/P_A$ for all A. Setting the numeraire of prices such that $P_A = u'(C(A))$, we arrive at the wage-hours schedule satisfying

$$\mathbb{E}[u'(C(A)) \tilde{A} F'(N(\tilde{A}))|\tilde{A}] = v'(N(\tilde{A})).$$

B.1 Representative family

We now compare this with the contracting problem when the worker and firm both use the representative family's marginal utility of consumption $u'(C(A))$ as discount factor. The firm's contracting problem is

$$\begin{aligned}\max_{W(\cdot),N(\cdot)}\ & \mathbb{E}[u'(C(A))AA_iF(N(AA_i)) - u'(C(A))W^s(N(AA_i)]\\ s.t.\ \ & AF'(N(A)) = W(N(A)),\\ & \mathbb{E}[u'(C(A))W^s(N(A_iA)) - v(N(A_iA))] = \underline{U}.\end{aligned}$$

Substituting out expected discounted wage payments, observing that $W(\cdot)$ can be adjusted so that the first constraint holds, we arrive at the optimality condition

$$\mathbb{E}[u'(C(A))\tilde{A}F'(N(\tilde{A}))|\tilde{A}] = v'(N(\tilde{A})).$$

Thus hours are identical under complete markets and with a representative family. Absent aggregate shocks, the hours response in both cases reduces to $u'(C)A_iF'(N(A_i)) = v'(N(A_i))$, corresponding to Proposition 1 with $\xi = 1/u'(C)$. From the first constraint, the marginal wage schedule is also identical under complete markets and with a representative household. Finally, the general-equilibrium zero-profit condition implies the same level of the base pay.

C The contracting model with only aggregate shocks

With *only* aggregate shocks, the optimal contract takes into account that high productivity states are also states with low marginal utility of consumption and vice versa. So income effects offset the substitution effect.

In particular, with balanced-growth preferences, the efficient contract implements constant hours (and constant wage payments). The efficient contract implements constant hours by having a zero marginal wage up until the optimal number of hours and infinite marginal wage thereafter.

In what follows, we solve for the optimal contract under general CRRA consumption preferences

$$\frac{C^{1-\gamma} - 1}{1-\gamma}$$

with $0 \leq \gamma < 1$, and we solve for the contract for $\gamma = 1$ by taking the limit from below.

C.1 Model with aggregate shock

We maintain the assumption $e = 0$, implying that $C = Y$. The firm's problem is

$$\max_{W(\cdot),W_{min}N(\cdot)} \mathbb{E}[\Lambda(A)(AF(N(A)) - W^s(N(A)))] \qquad s.t. \qquad \mathbb{E}[\Lambda W^s(N(A)) - v(N(A))] = \underline{U},$$
$$AF'(N(A)) = W(N(A)),$$

where we use the notation $W^s(N) = \int_0^N W(n)dn + W_{min}$. The random variable $\Lambda(A)$, the stochastic discount factor, is determined in general equilibrium by $\Lambda = u'(C) = u'(Y)$.

In analogy with the proof of Proposition 1, we solve for the discounted wage payments $\Lambda(A)W^s(N(A))$ in the constraint and substitute into the objective, arriving at

$$\max_{W^s(\cdot),N(\cdot)} \mathbb{E}[\Lambda AF(N(A)) - v(N(A))] \qquad s.t. \qquad \mathbb{E}[\Lambda(A)W^s(N) - v(N)] = \underline{U},$$
$$AF'(N(A)) = W(N(A)),$$

The first best is obtained if $\Lambda(A)AF'(N(A)) = v'(N(A))$. In general equilibrium, $\Lambda(A) = u'(AF(N(A)))$. Putting those two together, we get that a first-best contract satisfies

$$u'(AF(N(A)))AF'(N(A)) = v'(N(A)).$$

With $u'(C) = C^{-\gamma}$, $F(N) = N^{1-\alpha}$ and $v'(N) = \kappa N^{\psi}$, we get after som algebra,

$$N(A) = \left(\frac{1-\alpha}{\kappa}\right)^{1/((1+\psi)-(1-\gamma)(1-\alpha))} A^{1/((1+\psi)/(1-\gamma)-(1-\alpha))}.$$

The wage implementing the first best must satisfy $(1-\alpha)AN(A)^{-\alpha} = W(N(A))$ or

$$W(N) = (1-\alpha)^{-\gamma/(1-\gamma)}\kappa^{1/(1-\gamma)}N^{(\gamma+\psi)/(1-\gamma)}.$$

This marginal wage schedule is increasing and convex iff $\gamma < 1$. For $\gamma < 1$, the optimality condition for the firm in the second period is thus also a sufficient condition.

Total wage payments are given by

$$\int_0^N W(n)dn + W_{min} = \frac{(1-\gamma)(1-\alpha)^{-\gamma/(1-\gamma)}\kappa^{1/(1-\gamma)}}{1+\psi} N^{(1+\psi)/(1-\gamma)} + W_{min}$$

Substituting in productivity, we get that total wage payments, as a function of productivity, are given by

$$W^s(N(A)) = \frac{1-\gamma}{1+\psi} \frac{(1-\alpha)^{\frac{1}{1-\gamma}\left(\frac{(1+\psi)}{(1+\psi)-(1-\alpha)(1-\gamma)}-\gamma\right)}}{\kappa^{\frac{1}{1-\gamma}\left(\frac{(1+\psi)}{(1+\psi)-(1-\alpha)(1-\gamma)}-1\right)}} A^{\frac{(1+\psi)}{(1+\psi)-(1-\alpha)(1-\gamma)}} + W_{min}$$

C.2 Balanced-growth preferences, $\gamma = 1$

Taking the limits from below,

$$\lim_{\gamma \to 1_-} \frac{1}{1-\gamma} \left(\frac{(1+\psi)}{(1+\psi)-(1-\alpha)(1-\gamma)} - \gamma \right) = 1 + \frac{1-\alpha}{1+\psi},$$

and

$$\lim_{\gamma \to 1_-} \frac{1}{1-\gamma} \left(\frac{(1+\psi)}{(1+\psi)-(1-\alpha)(1-\gamma)} - 1 \right) = \frac{1-\alpha}{1+\psi},$$

we get that total wage payments are constant and equal to W_{min} in the limit when $\gamma \to 1_-$. The marginal wage is 0 when $\kappa N^{1+\psi}/(1-\alpha) < 1$ and infinity when $\kappa N^{1+\psi}/(1-\alpha) > 1$.

D Missing steps to the proof to Proposition 6

Derivation of Equation (13) The solution to (12) is characterized by a first order condition for each level of Z:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \frac{(\beta\theta)^t \Lambda_t}{P_t} \left(P_t A_t f_A(Z/(P_t A_t))\right) \left(Z F'(N(Z)) - \frac{P_t}{\Lambda_t} v'(N(Z)) \right) = 0$$

where $f(\cdot)$ is the density of the distribution of idiosyncratic productivities $A_{i,t}$. The density $P_t A_t f_A(Z/(P_t A_t))$ is the density of nominal productivity Z at time t. Since Z is given, write $N = N(Z)$.

To a first order in aggregate variables, the first-order condition is

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\theta)^t \Lambda_{ss} f_A(Z) \left(Z F'(N) - \frac{(1+\hat{p}_t - \hat{\lambda}_t)}{\Lambda_{ss}} v'(N) \right) = 0$$

or equivalently to a first order in N (with $N = N_{ss}(1+\hat{n})$),

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\theta)^t \Lambda_{ss} f_A(Z) \left(Z F'(N_{ss})(1-\alpha\hat{n}) - \frac{(1+\hat{p}_t - \hat{\lambda}_t)}{\Lambda_{ss}} v'(N_{ss})(1+\psi\hat{n}) \right) = 0.$$

We now remove the outer constants Λ_{ss} and $f_A(\tilde{A})$, and note that $ZF'(N_{ss}) = v'(N_{ss})/\Lambda_{ss}$ to arrive at

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\theta)^t \left(-\alpha\hat{n} - \hat{p}_t + \hat{\lambda}_t - \psi\hat{n} \right) = 0.$$

or, recalling the notation $\hat{n} = \hat{n}(Z)$,

$$\hat{n}(Z) = -\frac{1}{\alpha + \psi}(1 - \beta\theta)\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\theta)^t \left(\hat{p}_t - \hat{\lambda}_t \right).$$

which is Equation (13).

Derivation of Equation (15) We use the following simple lemma, which lies behind new-Keynesian Phillips curves:

Lemma D.1. *Let* $X_t = (1-\theta)\sum_{k=0}^{\infty} \theta^k x_{t-k}$ *and let* $x_t = (1-\beta\theta)\sum_{k=0}^{\infty} (\beta\theta)^k z_{t+k}$. *Then*

$$\Delta X_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}(z_t - X_t) + \beta \Delta X_{t+1}.$$

Proof. We have that $X_t = (1-\theta)x_t + \theta X_{t-1}$. Therefore, $\Delta X_t = (1-\theta)(x_t - X_{t-1})$. Furthermore, $x_t = (1-\beta\theta)z_t + (\beta\theta)x_{t+1}$. Therefore,

$$X_t = (1-\theta)(1-\beta\theta)z_t + (1-\theta)(\beta\theta)x_{t+1} + \theta X_{t-1}.$$

or

$$\theta(X_t - X_{t-1}) = (1-\theta)(1-\beta\theta)(z_t - X_t) + (1-\theta)(\beta\theta)(x_{t+1} - X_t).$$

Finally, noting that $(1-\theta)(x_{t+1} - X_t) = X_{t+1} - X_t$ and rearranging gives the sought after expression. □

Applying the lemma with $X_t = \hat{w}_t^{all}$, $x_t = \hat{\xi}_t$, and $z_t = \hat{p}_t - \hat{\lambda}_t$ gives Equation (15).