Online Appendix for "Primary-Auctions for Event Tickets: Eliminating the Rents of 'Bob the Broker'?"

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A Selection Concerns

Our eBay data are not a complete census of secondary-market activity, and for this reason one might worry that our results in Section V.A. are affected by selection. We have three specific concerns.

The first potential concern is that our analysis only uses *successful* eBay listings – listings where either the seller's Buy-it-Now price was accepted, or where the seller's auction elicited bids of at least their reserve price. For any particular listing, our estimate of aftermarket value conditional on success would be higher than that conditional on failure, so we might worry about a positive bias entering our analysis of the secondary market. This would cause us to *over*-estimate the prevalence of profitable resale opportunities. Since our main results in Section V.A. suggest that arbitrage profits are small, we do not need to worry about this type of selection driving our main results.[1]

A second potential concern is specific to eBay Buy-it-Now listings. Suppose that eBay sellers set their Buy-it-Now price equal to their tickets' true average aftermarket value plus a noise term that represents seller error. We then will observe more sales when the seller error term is negative than when it is positive. This will cause a negative bias in our estimate of secondary-market values, and hence cause us to *under*-estimate the returns to speculation. However, arbitrage profits are actually higher for pure Buy-it-Now listings (+\$23.86) than for all other listings (-\$13.46).[2] Thus, our results do not appear to be driven by eBay sellers who set their Buy-it-Now prices too low.

[1] Our main results are also not driven by selection stemming from the fact that seat numbers are not typically observed by eBay buyers. This institutional feature would allow professional resellers to make profits by acquiring low-quality seats within rows in the primary market, as long as such behavior is not fully accounted for by buyers in the secondary market. Such selection, though, would again cause us to over-estimate the returns to speculation.

[2] This finding is consistent with results in Einav et al. (2015), which suggest that Buy-it-Now transaction prices are consistently higher than non-BIN transaction prices, across a wide variety of eBay categories.

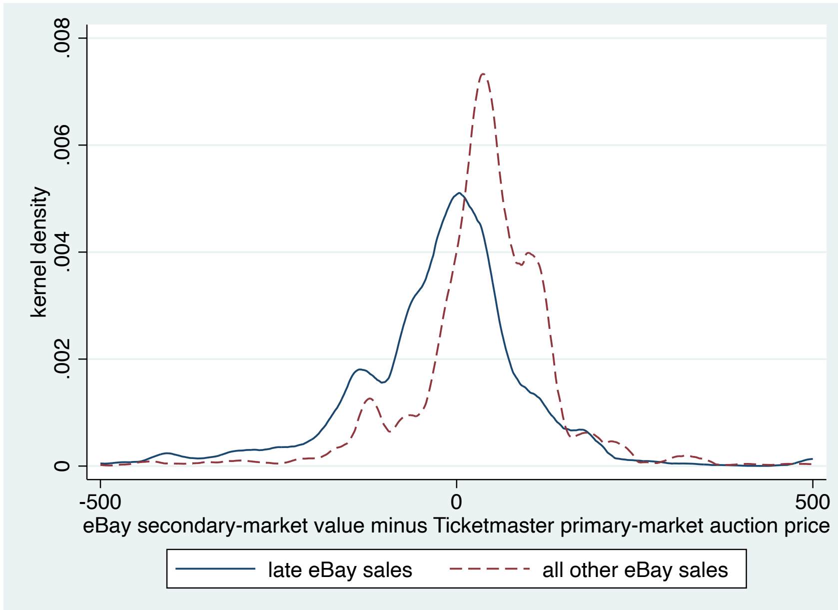
A third potential concern relates to our only seeing the portion of the aftermarket that occurs on eBay, as opposed to other venues such as StubHub. If the eBay component of the aftermarket is a random sample of the total aftermarket, then our use of just eBay data will cause us to have less power than if we had the full aftermarket, but will not cause bias. If the eBay component is non-random, however, this could cause bias. No arbitrage logic partly mitigates this concern: if eBay prices are systematically lower than StubHub prices (each net of fees), then arbitrageurs can buy on eBay to resell on StubHub (cf. Sweeting, 2012). However, we worry about the following: what if a seller's strategy is to initially post their tickets at a high fixed price on a venue such as StubHub, and only if that fixed-price posting is unsuccessful, to run an auction on eBay. That is, what if sellers use eBay as a last-minute "salvage market" to ensure that their tickets are sold.[3] In this case eBay prices will be lower than average, which would cause us to under-estimate the returns to speculation. Given the direction of our results in Section V.A., this is an important concern.

To address this third concern, we compare the distribution of potential resale profits (as defined in the main text) associated with resales on eBay that occur close to the event date, when we should worry about salvage-market effects, with the potential resale profits associated with all other eBay resales. See Figure A1. eBay sales in the last 30 days before the event are associated with mean potential resale losses of -\$24.16 per ticket, whereas the potential resale profits associated with all other eBay sales are +\$40.93 per ticket. Notice as well that the early distribution has a higher mode than does the late distribution (small positive profits), and that the late distribution has a fatter left tail (large losses). While these findings are consistent with the declining-price phenomenon documented by Sweeting (2012), they also suggest that we should be worried about salvage-market effects being present in our data.[4]

[3]The institutional reason why one might worry that sellers initially post on StubHub and then salvage on eBay, as opposed to the other way around, is the difference between the two venues' fee structures. StubHub does not charge listing fees and allows sellers to maintain their fixed price listing for as long as they like; eBay does charge listing fees, and depending on the type of seller most listings last for 7-10 days. Thus, posting a ticket at a high fixed price for a long period of time is free on StubHub (not counting opportunity costs), but costly on eBay. Pushing in the other direction, StubHub provides stronger buyer protection and transaction support than does eBay (e.g., ensuring that the buyer successfully receives the tickets from the seller), and these services may be especially valuable when the amount of time before an event is limited.

[4]Another interesting feature of the comparison of early to late eBay resales is that the primary-market auction prices are substantially more informative of early resale values: the R^2 of early eBay prices on TM primary-market auction prices is 0.77, versus 0.55 for late, and 0.66 for the full sample. This is consistent with results in Sweeting (2012), which show that the variance of secondary-market prices is much higher in the final days before an event.

Figure A1. Potential resale profits, late eBay sales versus all other eBay sales



Notes: Late eBay sales are defined as eBay sales that occur within the last 30 days before the event. For more details, see the text.

A conservative response is to discard secondary-market data from the final days before the concert occurs as possibly tainted, and interpret the +\$40.93 mean potential resale profits prior to the last 30 days as a conservative upper bound on potential resale profits.[5] A second response is that failing to resell early is a real risk in this market (cf. Board and Skrzypacz, 2016),[6] and that the returns to speculation should be calculated based on the full sample of early and late eBay resales. A piece of evidence in support of this latter interpretation comes from looking at the difference between early eBay resales conducted using eBay's pure fixed-price selling format and eBay's pure auction format. Early pure BIN listings are associated with large, statistically significant profits of \$48.24 per ticket. Early pure auctions, by contrast, are associated with negative profits of -\$20.00 per ticket.[7] Early BIN listings are where we should be most worried about our first selection concern,

[5]The 95% confidence interval of this pre-last-30-days estimate is [\$26.64, \$54.28]. If we discard the final 15 days, rather than 30, the estimate is \$34.82 (95% CI: [\$21.63, \$47.87]).

[6]Board and Skrzypacz (2016) characterize the optimal dynamic mechanism for sellers of perishable goods when buyers are forward looking. The optimal mechanism involves declining posted prices, followed by an auction in the final period. The auction can be interpreted as a salvage market, since its purpose is to ensure sale (modulo an optimally set reserve price) in the last period before the good expires. See Sweeting (2012) for further discussion of this paper and related dynamic pricing literature.

[7]Following Einav et al. (2015), we classify an eBay sale as pure fixed price if the listing uses a Buy-it-Now price and does not allow bidders to bid less than the BIN amount, and classify an eBay sale as a pure auction if it does

namely, that we only observe arbitrage profits for successful eBay listings. Using a high BIN price early also is consistent with optimal dynamic pricing behavior (Board and Skrzypacz, 2016). Early auction listings, on the other hand, should represent an unbiased estimate of the aftermarket value of the tickets at that particular moment in time. The fact that these profits are negative suggests that the TM primary-market auctions are not leaving large positive secondary-market profits on the table.[8]

B Additional Robustness Tests

B.A. Robustness of Main Results to Alternative Matching Specifications

As discussed in Section IV.C., there are four potential ways to match primary- and secondary-market observations within the same concert-section-row tuple. In our main specification, we match each Ticketmaster primary-market transaction with the average price of the eBay secondary-market transactions for the c-s-r in question. This approach allows us to exploit all of the variation in the winning bids in Ticketmaster's high-quality dataset. In this section, we consider the robustness of our results to three alternative matching specifications. First, we match each secondary-market transaction with the average price of the primary-market transactions for the c-s-r in question. Next, for each c-s-r, we match the average price of the primary-market transactions with the average price of the secondary-market transactions. Last, for each c-s-r, we match the minimum primary-market auction price with the average secondary-market price. We consider this specification to address the possibility that, within a c-s-r, bidders who paid higher prices in the primary-market auctions were less likely to resell their tickets in the secondary market.

Table B1 lists the average profits associated with buying tickets in the TM primary-market auctions and then reselling in the eBay secondary market, for each of four above-discussed matching specifications. 95% confidence intervals are calculated using the bootstrap, with the data clustered at the concert level (cf. footnote 27 in the main text).

not use a BIN and uses a low starting bid. The -\$20.00 figure in the text defines a low starting bid as $<50\%$ of the ticket's face value. If we use $<10\%$ instead, the figure is -\$15.75.

[8]The early pure auctions exercise can also be interpreted as a response to the first selection concern described above. Since pure auctions nearly always result in a sale, one need not worry about bias from the use of only successful eBay listings. Under this interpretation, the -\$20.00 can be viewed as a conservative lower bound on potential resale profits.

Over all of our specifications, the 95% confidence intervals admit estimates of potential resale profits, net of eBay transaction fees, ranging from -\$38.52 to +\$18.59.

Table B1 – Summary statistics for potential resale profits, and price discovery regression results, under different approaches to data aggregation.

						Price-Discovery Regression		
Unit of Analysis	N	Avg. Profit ($)	Clustered CI ($)	Mode ($)	Skewness	Slope	Constant ($)	R^2
TM Transaction	8,425	6.07	[-7.57, 18.59]	28.46	-0.69	0.85	47.38	0.66
(main specification)						(0.07)	(16.90)	
eBay Transaction	3,532	-20.45	[-38.52, -3.44]	22.89	-0.61	0.76	40.03	0.57
						(0.06)	(14.98)	
CSR Combination	1,646	-18.17	[-26.91, -9.42]	22.50	-1.75	0.77	43.64	0.65
						(0.07)	(15.82)	
CSR Combination	1,646	-7.41	[-16.27, 1.35]	25.00	-1.09	0.79	46.82	0.63
(min. TM price)						(0.07)	(15.92)	

Notes: For details on the different approaches to data aggregation, see Section IV.C. Confidence intervals are calculated at the 95% level. Standard errors are clustered at the concert level and are calculated via bootstrap. Modes are calculated from kernel density estimates that apply the Epanechnikov kernel function to a grid of 1,500 points on the interval [-$500, +$500]. The price-discovery regression results pertain to the regression of eBay secondary-market value on TM primary-market auction price.

B.B. Robustness of Main Results to Assumptions on eBay Transaction Fees

While all eBay sellers pay eBay fees (i.e., final-value fees and insertion fees), we do not observe whether a given eBay seller transacted using PayPal, or some other method such as cash or check. Therefore, in our main specification, we subtract eBay fees from the eBay transaction price, but do not consider PayPal fees, which were roughly a bit less than 3% of the eBay sale price at the time of our data.[9]

Table B2 reports gross profits, profits net of eBay fees, and profits net of both eBay fees and PayPal fees. Over all of our these specifications, the 95% confidence intervals admit estimates of profits ranging from -$16.10 to +$30.31.

[9] See footnote 22 in the main text for further details on PayPal fees.

Table B2 – Summary statistics for arbitrage profits, and price discovery regression results, under different assumptions on eBay transaction fees.

						Price-Discovery Regression		
	N	Avg. Profit ($)	Clustered CI ($)	Mode ($)	Skewness	Slope	Constant ($)	R^2
Resale Profits Gross of eBay Fees	8,425	17.03	[2.81, 30.31]	37.20	-0.48	0.87	52.98	0.66
						(0.07)	(17.05)	
Resale Profits Net of eBay Fees (main specification)	8,425	6.07	[-7.57, 18.59]	28.46	-0.69	0.85	47.38	0.66
						(0.07)	(16.90)	
Resale Profits Net of eBay Fees and PayPal Fees	8,425	-1.94	[-16.10, 10.86]	24.23	-0.97	0.83	45.80	0.66
						(0.07)	(16.44)	

Notes: For details on eBay fees, see Section IV.B. For all other aspects of the table see the notes to Table B1.

B.C. Robustness of Results on Bidder Experience

In Section VI, we show that experienced bidders, defined as those who win at least 10 TM auctions, earn modest positive arbitrage profits from buying in the TM auctions and reselling on eBay. The potential resale profits of experienced bidders are significantly higher than those of inexperienced bidders, namely, those who win less than 10 TM auctions. Since our classification of experienced and inexperienced bidders is somewhat arbitrary, we consider the robustness of our results on bidder experience to an alternative classification.

In particular, we define an experienced bidder as one who wins TM auctions for concerts in at least two different cities, performed by at least two different artists. Therefore inexperienced bidders are those who win tickets to just one event, those who are avid followers of a given artist (who follow just one artist across several cities), and those who simply enjoy live musical events (i.e., those who attend several concerts in a given city). The reasoning behind this alternative specification is that bidders who are not professional resellers are likely to fall into one of the three aforementioned categories of inexperienced bidders. Table B3 shows that the results on bidder experience are qualitatively unchanged when we employ our alternative classification. In both the main and the alternative specifications, (i) experienced bidders have small positive potential resale profits (significant at 1% and 10%, respectively), (ii) inexperienced bidders have essentially zero potential resale profits, and (iii) the profits of experienced bidders are significantly larger than those of inexperienced bidders (significant at 1%). Our results also remain qualitatively unchanged if we define either measure of bidder experience based on the number of auctions the bidder participated in rather than the number of auctions the bidder won.

Table B3 – Summary statistics for potential resale profits, and price discovery regression results, for different definitions of bidder experience.

						Price-Discovery Regression		
	N	Avg. Profit ($)	95% CI ($)	Mode ($)	Skewness	Slope	Constant ($)	R^2
Experienced bidders								
$\geq$ 10 transactions	1,785	19.49	[5.32, 33.04]	29.94	0.75	0.95	32.03	0.67
(main specification)						(0.09)	(19.14)	
$\geq$ 2 cities and artists	2,597	15.02	[-1.96, 29.87]	30.17	-0.50	0.90	41.88	0.68
						(0.09)	(21.12)	
Inexperienced bidders								
< 10 transactions	6,640	2.47	[-12.26, 15.65]	27.83	-0.89	0.83	49.17	0.65
(main specification)						(0.07)	(17.18)	
< 2 cities or artists	5,828	2.09	[-12.66, 15.28]	27.16	-0.79	0.82	50.45	0.64
						(0.06)	(16.10)	

Notes: For details on bidder experience measures, see Section VI. For all other aspects of the table see the notes to Table B1.

C Proofs

Proof of Proposition 1

Proof. Let us look for a symmetric equilibrium in which all bidders use the same bidding function $b(\cdot)$. We initially assume and later prove that $b(\cdot)$ is strictly increasing, so that there is a one-to-one relationship between bids and valuations. Therefore we can think of a bid $\hat{b}$ as the submission of a valuation $\hat{v}$, such that $\hat{b} = b(\hat{v})$. The bidder can then be thought of as choosing the submitted valuation $\hat{v}$ optimally, given the bidding function $b(\cdot)$. The bidder thus maximizes her expected value from the auction by solving the following program:

$$\max_{\hat{v}} \sum_{k=1}^{K} [v\alpha_k - b(\hat{v})] P_k(\hat{v}),$$

where

$$P_k(x) = \binom{n-1}{k-1} F(x)^{n-k} (1 - F(x))^{k-1} \tag{C.1}$$

is the probability that a bidder with valuation x wins the k^{th} object. In order for $b(\cdot)$ to define a symmetric equilibrium, the first-order condition requires that the bidder's expected value must be maximized at her true valuation v. That is,

$$\sum_{k=1}^{K} \Big([v\alpha_k - b(v)] P_k'(v) - P_k(v) b'(v) \Big) = 0. \tag{C.2}$$

We can use (C.2) to solve for the equilibrium bidding function as follows. Rearranging terms,

$$\begin{aligned}
\sum_{k=1}^{K} v\alpha_k P_k'(v) &= \sum_{k=1}^{K} \left(b(v) P_k'(v) + P_k(v) b'(v) \right) \\
&= \frac{d}{dv}\left(b(v) \sum_{k=1}^{K} P_k(v) \right) \\
\implies b(v) \sum_{k=1}^{K} P_k(v) - b(0) \sum_{k=1}^{K} P_k(0) &= \int_0^v x \sum_{k=1}^{K} \alpha_k P_k'(x)\, dx.
\end{aligned}$$

But $P_k(0) = 0, \forall k \in \{1, \cdots, K\}$, and so

$$b(v) = \frac{1}{\sum_{k=1}^{K} P_k(v)} \int_0^v x \sum_{k=1}^{K} \alpha_k P_k'(x)\, dx. \tag{C.3}$$

We can then use integration by parts to derive (1) in the main text:

$$\begin{aligned}
b(v) &= \frac{1}{\sum_{k=1}^{K} P_k(v)} \sum_{k=1}^{K} \int_0^v x\alpha_k P_k'(x)\, dx \\
&= \frac{1}{\sum_{k=1}^{K} P_k(v)} \left(\sum_{k=1}^{K} \alpha_k \left[x P_k(x) \right] \Big|_{x=0}^{x=v} - \sum_{k=1}^{K} \int_0^v \alpha_k P_k(x)\, dx \right) \\
&= \frac{1}{\sum_{k=1}^{K} P_k(v)} \left(\sum_{k=1}^{K} P_k(v)(v\alpha_k) - \sum_{k=1}^{K} \int_0^v \alpha_k P_k(x)\, dx \right).
\end{aligned}$$

To finish the proof we must confirm, as we had assumed above, that the bidding function (1) in the main text is strictly increasing. Using (C.3), notice that

$$
\begin{aligned}
b'(v) &= \frac{\left(\sum_{k=1}^{K} P_k(v)\right)\left(v\sum_{k=1}^{K}\alpha_k P_k'(v)\right) - \left(\sum_{k=1}^{K} P_k'(v)\right)\left(\int_0^v x\sum_{k=1}^{K}\alpha_k P_k'(x)\,dx\right)}{\left(\sum_{k=1}^{K} P_k(v)\right)^2} \\
\implies Sign\left\{b'(v)\right\} &= Sign\left\{\left(\sum_{k=1}^{K} P_k(v)\right)\left(v\sum_{k=1}^{K}\alpha_k P_k'(v)\right) - \left(\sum_{k=1}^{K} P_k'(v)\right)\left(\int_0^v x\sum_{k=1}^{K}\alpha_k P_k'(x)\,dx\right)\right\}.
\end{aligned}
$$

Lemma 1. For any $x \in (0, \bar{v})$, $\sum_{k=1}^{K}\alpha_k P_k'(x) > 0$.

Proof. Let $x \in (0,\bar{v})$. Define $\alpha_{K+1} \equiv 0$. We have

$$
\begin{aligned}
\sum_{k=1}^{K}\alpha_k P_k'(x) &= \sum_{k=1}^{K}\alpha_k\binom{n-1}{k-1}F(x)^{n-k-1}(1-F(x))^{k-2}\left[(n-k)(1-F(x))-(k-1)F(x)\right]f(x) \\
&= (n-1)f(x)\left(\sum_{k=1}^{K}\alpha_k\binom{n-2}{k-1}F(x)^{n-k-1}(1-F(x))^{k-1} - \sum_{k=2}^{K}\alpha_k\binom{n-2}{k-2}F(x)^{n-k}(1-F(x))^{k-2}\right) \\
&= (n-1)f(x)\left(\sum_{k=1}^{K}\alpha_k\binom{n-2}{k-1}F(x)^{n-k-1}(1-F(x))^{k-1} - \sum_{k=1}^{K}\alpha_{k+1}\binom{n-2}{k-1}F(x)^{n-k-1}(1-F(x))^{k-1}\right) \\
&= \sum_{k=1}^{K}\left[\alpha_k-\alpha_{k+1}\right](n-1)\binom{n-2}{k-1}F(x)^{n-k-1}(1-F(x))^{k-1}f(x) \\
&> 0.
\end{aligned}
$$

The first equality follows from the definition of $P_k'(\cdot)$, and the second results from algebraic manipulation. The third equality is obtained by shifting the index of the second sum and using $\alpha_{K+1} = 0$. The last equality results from grouping terms, and the inequality is due to the fact that $\{\alpha_k\}_{k=1}^{K+1}$ is a strictly decreasing sequence. This completes the proof of the Lemma.

□

From Lemma 1,

$$
\begin{aligned}
\int_0^v x\sum_{k=1}^{K}\alpha_k P_k'(x)\,dx &< v\int_0^v \sum_{k=1}^{K}\alpha_k P_k'(x)\,dx \\
&= v\sum_{k=1}^{K}\alpha_k P_k(v).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\sum_{k=1}^{K} P_k(v)\right)\left(v\sum_{k=1}^{K}\alpha_k P_k'(v)\right) - \left(\sum_{k=1}^{K} P_k'(v)\right)\left(\int_0^v x\sum_{k=1}^{K}\alpha_k P_k'(x)\,dx\right) \\
&> v\left[\left(\sum_{k=1}^{K} P_k(v)\right)\left(\sum_{k=1}^{K}\alpha_k P_k'(v)\right) - \left(\sum_{k=1}^{K} P_k'(v)\right)\left(\sum_{k=1}^{K}\alpha_k P_k(v)\right)\right]
\end{aligned}
$$

Therefore it suffices to show that

$$\begin{aligned}
\left(\sum_{k=1}^{K} P_k(v)\right)\left(\sum_{k=1}^{K} \alpha_k P_k'(v)\right) - \left(\sum_{k=1}^{K} P_k'(v)\right)\left(\sum_{k=1}^{K} \alpha_k P_k(v)\right) &\geq 0 \\
\iff \sum_{j=1}^{K}\sum_{k=1}^{K} P_k(v)\,\alpha_j P_j'(v) - \sum_{j=1}^{K}\sum_{k=1}^{K} P_j'(v)\,\alpha_k P_k(v) &\geq 0 \\
\iff \sum_{j=1}^{K}\sum_{k=1}^{K} (\alpha_j - \alpha_k)\, P_k(v)\, P_j'(v) &\geq 0 \\
\iff \sum_{j=1}^{K-1}\sum_{k=j+1}^{K} \underbrace{(\alpha_j - \alpha_k)}_{>0,\text{ since } j<k} \left[P_k(v)\,P_j'(v) - P_j(v)\,P_k'(v)\right] &\geq 0.
\end{aligned}$$

where the second, third and fourth inequalities each follow from rearranging terms. It is thus sufficient to show that, for any $v \in [0, \bar{v}]$ and any $j < k$,

$$P_k(v)\,P_j'(v) - P_j(v)\,P_k'(v) \geq 0.$$

Note that

$$\begin{aligned}
&P_k(v)\,P_j'(v) \\
&= \left[\binom{n-1}{k-1} F(v)^{n-k}(1-F(v))^{k-1}\right]\left[\binom{n-1}{j-1} F(v)^{n-j-1}(1-F(v))^{j-2}\left[(n-j)(1-F(v)) - (j-1)F(v)\right] f(v)\right] \\
&= \binom{n-1}{k-1}\binom{n-1}{j-1} F(v)^{2n-j-k-1}(1-F(v))^{j+k-3}\left[(n-j) - (n-1)F(v)\right] f(v).
\end{aligned}$$

The first equality is due to the definitions of $P_k(\cdot)$ and $P_j'(\cdot)$ and the second is obtained from algebraic simplification. It follows that

$$\begin{aligned}
P_k(v)\,P_j'(v) - P_j(v)\,P_k'(v) &= \binom{n-1}{k-1}\binom{n-1}{j-1} F(v)^{2n-j-k-1}(1-F(v))^{j+k-3}\left[(n-j)-(n-k)\right] f(v) \\
&= (k-j)\binom{n-1}{k-1}\binom{n-1}{j-1} F(v)^{2n-j-k-1}(1-F(v))^{j+k-3} f(v) \\
&\geq 0,
\end{aligned}$$

as required. Therefore, the bidding function (1) in the main text is indeed strictly increasing. This in turn implies that the resulting allocation is efficient.

□

Proof of Proposition 2

Proof. The proof will proceed by induction. Bidders draw their private valuations from $F(\cdot)$, and the mechanics of the auction are as described previously. We will construct K auctions that, if held sequentially, yield the GEA.

First, a second-price English auction is conducted among the n bidders for K units of object K, each worth v_i to bidder i (recall that $\alpha_K = 1$). The last K surviving bidders win the object, and pay the price at which the last drop-out occurred.

Once the above auction has concluded, the K winners of the auction immediately enter an all-pay English auction in which K bidders compete for $K-1$ upgrades from object K to object $K-1$, each worth $(\alpha_{K-1} - \alpha_K) v_i$ to bidder i. After one drop-out, the remaining bidders win the object, and all bidders (including the losing bidder) pay the price at which the drop-out occurred. Notice that at the end of this auction, the $K-1$ winners have object $K-1$, the loser keeps object K, and the total payment across the two auctions of all K participants is the drop-out price of the losing bidder in the auction for the upgrade to object $K-1$.

Another all-pay auction is then held among the $K-1$ winners of the previous auction, offering $K-2$ upgrades from object $K-1$ to object $K-2$. Proceeding in this manner, a total of $K-1$ all-pay auctions are conducted sequentially. In each such auction, the bidders are the winners of the previous auction, and an upgrade of one quality level is awarded to all bidders but one. The process naturally stops after the quality upgrade from object 2 to object 1 has been sold.

In aggregate, bidders who do not win any object pay nothing, and bidders who win an object pay the price at which they drop out (except for the winner of object 1, who pays the price at which the second-to-last survivor drops out). Therefore when the above K auctions are conducted sequentially, the resulting composite auction is the GEA.

Now let us analyze any of the above-described all-pay auctions in isolation: $k+1$ bidders compete for k objects, each worth $(\alpha_k - \alpha_{k+1}) v_i$ to bidder i. The following Lemma will be useful in our proof.

Lemma 2. *The unique symmetric perfect Bayesian equilibrium of the above all-pay English auction is characterized by:*

$$T(v; \underline{v}, k) = k(\alpha_k - \alpha_{k+1}) \int_{\underline{v}}^{v} x h(x)\, dx.$$

Proof. This result follows immediately from Lemma 3 of Bulow and Klemperer (1999).

□

Consider also the initial second-price auction for object K in isolation.

Lemma 3. *The unique symmetric perfect Bayesian equilibrium of the initial second-price English auction is characterized by:*

$$T(v; \underline{v}, K) = v - \underline{v}.$$

Proof. Since losing bidders do not pay, a bidder v would strictly prefer to remain in the auction if the total time that has elapsed is less than v, since winning object K would give her positive surplus in this event. Conversely, she would strictly prefer to drop out if the total time that has elapsed is greater than v, because she would get negative surplus if she were to win the object. Therefore she will drop out when the total time that has elapsed is v. When all bidders follow this strategy, her waiting time at any instant can be written as $T(v; \underline{v}, k) = v - \underline{v}$.

$\square$

In what follows, we will refer to the subgame that begins with the auction for the upgrade to object k as subgame k. Now analyze subgame 1, from the perspective of a bidder with valuation v. Recall that this is the last subgame of the GEA. There is one other bidder remaining, and the bidders are competing for one upgrade from object 2 to object 1. Both bidders have already won object 2, but importantly the benefit from this object is sunk. Likewise their waiting costs from surviving to this point in the game are sunk. Therefore we can conclude from Lemma 2 that the unique symmetric perfect Bayesian equilibrium of this subgame is defined by (2) in the main text, with $k = 1$.

Next, consider subgame k, $1 < k < K$. Consider a bidder with value v, and suppose that all other players bid according to (2) in the main text. Suppose additionally that the bidder in question knows that she will follow the proposed strategies in each subgame $j < k$, conditional on surviving until subgame j is reached. We know from Lemma 2 that (2) gives the myopic best response of the bidder in the auction for the upgrade to object k. Hence it could not be optimal to deviate by dropping out earlier than she would under the proposed equilibrium - the bidder would be giving up positive expected utility in the present auction and possibly in future auctions.

Suppose instead that the bidder drops out later than she would under (2), i.e., she plays as if she has valuation $v^* > v$. There are three possibilities, each of which occur with positive probability:

1. Both types v^* and v would not win object k. Then the bidder strictly prefers to bid as type v, since she would have a lower drop-out price by doing so.

2. Both types v^* and v would win object k. Then since the bidder follows $T(v; \cdot, \cdot)$ in all future auctions, her expected payoff is the same from playing as either type.

3. Type v would drop out in the auction for object k, but type v^* would win object k. Then at the start of subgame $k-1$, v is less than the lowest possible type of the other bidders who have not dropped out, $\underline{v}_{k-1}$. Since the bidder follows $T\left(v; \underline{v}_{k-1}, k-1\right)$ in this auction, she must drop out immediately. Therefore her expected utility from misrepresenting her type differs from that under truthful play only in terms of her payoff from the auction for object k. But from Lemma 2, her expected surplus in this auction is maximized by playing truthfully.

Thus the bidder strictly prefers playing truthfully to bidding as type $v^* > v$. It follows that when all other bidders follow the proposed equilibrium strategies and the bidder knows that she will follow (2) in all future subgames, her optimal waiting time in the auction for object k is also given by $T(v; \cdot, k)$. By induction, then, (2) defines the unique symmetric perfect Bayesian equilibrium of subgame k, $k < K$.

Moving to the first auction, consider again a bidder with value v, who knows that all other bidders are bidding according to (2), and that she will also follow the proposed equilibrium strategies in future subgames $k < K$, if she is still active in those subgames. When all other players follow the proposed strategies, they drop out precisely when the price equals their valuation of object K. Lemma 3 tells us that this strategy is the myopic best response in the auction for object K. Using an argument analogous to the one above, the bidder cannot profitably deviate from (2) by playing as though her valuation is higher or lower than v. By an additional step of induction, we can thus conclude that (2) defines the unique symmetric perfect Bayesian equilibrium of the GEA.

Finally, the fact that the equilibrium is efficient follows immediately from (2); $\forall k \in \{1, 2, \cdots, K\}$, $T(\cdot; \underline{v}, k)$ is clearly strictly increasing.

$\square$

Proof of Proposition 3

Proof. Since the sealed-bid and ascending TM auctions are efficient, and the lowest type gets zero surplus in both auctions, Myerson's Lemma implies that all bidders have the same expected surplus in both auctions. Moreover, all bidders have the same expected surplus in both auctions conditional on losing (namely, zero). It follows that all bidders must also have the same expected surplus in both auctions conditional on winning some object. Hence any bidder's expected surplus conditional on winning some object, in either auction, can be expressed as

$$\begin{aligned} s(v; n_{pro}, n_{fan}) &= \frac{\sum_{k=1}^{K}[v\alpha_k - b(v; n_{pro}, n_{fan})]P_k(v; n_{fan}, n_{pro})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})} \\ &= \sum_{k=1}^{K}[v\alpha_k - b(v; n_{pro}, n_{fan})]\left(\frac{P_k(v; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})}\right) \end{aligned}$$

where

$$P_k(v; n_{pro}, n_{fan}) = \binom{n_{pro} + n_{fan} - 1}{k-1} F(v; n_{pro}, n_{fan})^{n_{pro}+n_{fan}-k}\big(1 - F(v; n_{pro}, n_{fan})\big)^{k-1}$$

where we make the dependence of $F(\cdot)$ on n_{pro} and n_{fan} explicit in equation (3) in the main text, and $b(\cdot)$ is the sealed-bid auction equilibrium bidding function (1) in the main text.

Plugging in the bidding function from (1) and doing some algebraic manipulation, we find that

$$s(v; n_{pro}, n_{fan}) = \int_0^v \frac{\sum_{k=1}^{K} \alpha_k P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})} dx. \tag{C.4}$$

Now we split up the analysis into two cases, $v \in [w - \epsilon, w)$ and $v = w$, and analyze the behavior of the conditional surplus function $s(\cdot)$ as $n_{pro} \to \infty$.

(i) $\underline{v \in [w - \epsilon, w)}$:

First, fix $v \in (w - \epsilon, w)$. We will discuss the case $v = w - \epsilon$ separately at the end. Define $\pi_{n_{pro}}(x) \equiv \frac{\sum_{k=1}^{K} \alpha_k P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^{K} P_k(v;n_{pro},n_{fan})}$, so that (C.4) becomes

$$s(v; n_{pro}, n_{fan}) = \int_0^v \pi_{n_{pro}}(x)dx. \tag{C.5}$$

Now we aim to show that $\pi_{n_{pro}}(x) \to 0$ for all $x \in (0, v)$. To do this, we will first show that the conditional probability terms $\frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^{K} P_k(v;n_{pro},n_{fan})}$ go to 0 for all $x \in (0, v)$ and all $k \in \{1, \ldots, K\}$.

Fix $x \in (0, v)$. For all $k \in \{1, \ldots, K\}$ we get that

$$\begin{aligned} \lim_{n_{pro}\to\infty} \frac{P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})} &\le \lim_{n_{pro}\to\infty} \frac{P_k(x; n_{pro}, n_{fan})}{P_k(v; n_{pro}, n_{fan})} \\ &= \lim_{n_{pro}\to\infty} \left[\left(\frac{F(x; n_{pro}, n_{fan})}{F(v; n_{pro}, n_{fan})}\right)^{n_{pro}+n_{fan}-k} \left(\frac{1 - F(x; n_{pro}, n_{fan})}{1 - F(v; n_{pro}, n_{fan})}\right)^{k-1}\right] \end{aligned}$$

Since we now have a limit of the form $\lim_n f(n)^{g(n)}$ we use a trick to evaluate the limit properly by reexpressing the quantity using exponents and logarithms.

$$
\begin{aligned}
&\lim_{n_{pro}\to\infty}\left[\Big(\frac{F(x;n_{pro},n_{fan})}{F(v;n_{pro},n_{fan})}\Big)^{n_{pro}+n_{fan}-k}\Big(\frac{1-F(x;n_{pro},n_{fan})}{1-F(v;n_{pro},n_{fan})}\Big)^{k-1}\right]\\
&=\left[\lim_{n_{pro}\to\infty}\exp\left\{\frac{\log(\frac{F(x;n_{pro},n_{fan})}{F(v;n_{pro},n_{fan})})}{\frac{1}{n_{pro}+n_{fan}-k}}\right\}\right]\left[\lim_{n_{pro}\to\infty}\Big(\frac{1-F(x;n_{pro},n_{fan})}{1-F(v;n_{pro},n_{fan})}\Big)^{k-1}\right]\\
&=\exp\left\{\lim_{n_{pro}\to\infty}\frac{\log(\frac{F(x;n_{pro},n_{fan})}{F(v;n_{pro},n_{fan})})}{\frac{1}{n_{pro}+n_{fan}-k}}\right\}\Big(\lim_{n_{pro}\to\infty}\frac{1-F(x;n_{pro},n_{fan})}{1-F(v;n_{pro},n_{fan})}\Big)^{k-1}\\
&\quad\text{passing limits through both continuous functions}\\
&=\exp\left\{\frac{\log(\lim_{n_{pro}\to\infty}\frac{F(x;n_{pro},n_{fan})}{F(v;n_{pro},n_{fan})})}{\lim_{n_{pro}\to\infty}\frac{1}{n_{pro}+n_{fan}-k}}\right\}\Big(\lim_{n_{pro}\to\infty}\frac{1-F(x;n_{pro},n_{fan})}{1-F(v;n_{pro},n_{fan})}\Big)^{k-1}\\
&\quad\text{distributing limits in the first term and passing further through the continuous}\log(\cdot)\\
&=\exp\left\{\frac{\log(\frac{F_{pro}(x)}{F_{pro}(v)})}{\lim_{n_{pro}\to\infty}\frac{1}{n_{pro}+n_{fan}-k}}\right\}\Big(\frac{1-F_{pro}(x)}{1-F_{pro}(v)}\Big)^{k-1}\\
&\quad\text{finding the well-defined limits in both terms}\\
&=\exp\{-\infty\}\Big(\frac{1-F_{pro}(x)}{1-F_{pro}(v)}\Big)^{k-1}\\
&\quad\text{since}\log(\frac{F_{pro}(x)}{F_{pro}(v)})<0\text{ because }x<v\\
&=0,
\end{aligned}
$$

and so $\lim_{n_{pro}\to\infty}\frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(v;n_{pro},n_{fan})}\le 0$.

Since $\lim_{n_{pro}\to\infty}\frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(v;n_{pro},n_{fan})}\ge\lim_{n_{pro}\to\infty}0=0$ we have that $\lim_{n_{pro}\to\infty}\frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(v;n_{pro},n_{fan})}=0$ for all $x\in(0,v)$ and all $k\in\{1,\dots,K\}$. Thus we have that for all $x\in(0,v)$

$$
\begin{aligned}
\lim_{n_{pro}\to\infty}\pi_{n_{pro}}(x)&=\lim_{n_{pro}\to\infty}\frac{\sum_{k=1}^K\alpha_kP_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(v;n_{pro},n_{fan})}\\
&=\sum_{k=1}^K\alpha_k\lim_{n_{pro}\to\infty}\Big(\frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(v;n_{pro},n_{fan})}\Big)\\
&=0.
\end{aligned}
$$

By (C.5)

$$\lim_{n_{pro}\to\infty}s(v;n_{pro},n_{fan})=\lim_{n_{pro}\to\infty}\int_0^v\pi_{n_{pro}}(x)dx$$

and so we now want to argue that we can pass the limit into the integral. To do this we will use the Lebesgue Dominated Convergence Theorem.[10]

[10] The Lebesgue Dominated Convergence Theorem (Royden and Fitzpatrick, 2010, p. 88) is the following: *Let $\{f_n\}$ be a sequence of measurable functions on E, a measurable set of the real numbers. Suppose there is a function g that is integrable over E (meaning that $\int_E|g|<\infty$) and dominates $\{f_n\}$ on E in the sense that $|f_n|\le g$ on E for all n. Then, if $\{f_n\}\to f$ pointwise almost everywhere on E (meaning that it converges to f pointwise everywhere except possibly on a set of measure 0), then f is integrable over E and*

Note that the function $\pi_{n_{pro}}(\cdot)$ is defined on $[0, v]$, and we have shown that $\pi_{n_{pro}}(\cdot)$ converges to 0 almost everywhere on $[0, v]$ (it converges at all points except possibly at the measure 0 set $\{0\} \cup \{v\}$). We now need a dominating function to apply the theorem.

By Lemma 1, $\pi_{n_{pro}}(\cdot)$ is increasing on $(0, v)$. It is also continuous and so it must be that $\pi_{n_{pro}}(x) \leq \pi_{n_{pro}}(v)$. Moreover, we have that

$$\pi_{n_{pro}}(v) = \frac{\sum_{k=1}^{K} \alpha_k P_k(v; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})} \leq \alpha_1$$

and so if we define $g(x) \equiv \alpha_1$ then we have our dominating function since $|\pi_{n_{pro}}(x)| \leq g(x)$ for all $x \in [0, v]$ and $\int_0^v |g(x)|dx = v\alpha_1 < \infty$. Thus by applying the Dominated Convergence Theorem, we have that

$$\lim_{n_{pro}\to\infty} s(v; n_{pro}, n_{fan}) = \lim_{n_{pro}\to\infty} \int_0^v \pi_{n_{pro}}(x)dx = \int_0^v \lim_{n_{pro}\to\infty} \pi_{n_{pro}}(x)dx = 0.$$

Since we arbitrarily fixed $v \in (w - \epsilon, w)$ at the beginning, we are done with this case as required.

The proof for $v = w - \epsilon$ uses a similar argument but requires an algebraic simplification before we convert the expression that has the form $\lim_n f(n)^{g(n)}$. For all $x \in (0, w - \epsilon]$ we have $F(x; n_{pro}, n_{fan}) = \frac{n_{fan} F_{fan}(x)}{n_{pro} + n_{fan}}$ and so

$$\begin{aligned}
\lim_{n_{pro}\to\infty} \frac{P_k(x; n_{pro}, n_{fan})}{P_k(v; n_{pro}, n_{fan})} &= \lim_{n_{pro}\to\infty} \left[\left(\frac{F(x; n_{pro}, n_{fan})}{F(v; n_{pro}, n_{fan})} \right)^{n_{pro}+n_{fan}-k} \left(\frac{1 - F(x; n_{pro}, n_{fan})}{1 - F(v; n_{pro}, n_{fan})} \right)^{k-1} \right] \\
&= \lim_{n_{pro}\to\infty} \left[\left(\frac{F_{fan}(x)}{F_{fan}(w - \epsilon)} \right)^{n_{pro}+n_{fan}-k} \left(\frac{1 - \frac{n_{fan}}{n_{fan}+n_{pro}} F_{fan}(x)}{1 - \frac{n_{fan}}{n_{fan}+n_{pro}} F_{fan}(w - \epsilon)} \right)^{k-1} \right]
\end{aligned}$$

and this goes to 0 for all $x \in (0, w - \epsilon)$ and all $k \in \{1, \ldots, K\}$ since the left term goes to 0 and the right term is finite in the limit. The argument for passing the limit through the integral is identical.

(ii) $\underline{v = w}$:

Define $\pi_{n_{pro}}(x) \equiv \frac{\sum_{k=1}^{K} \alpha_k P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(w; n_{pro}, n_{fan})}$, so that (C.4) becomes

$$s(w; n_{pro}, n_{fan}) = \int_0^w \pi_{n_{pro}}(x)dx. \tag{C.6}$$

In the previous case we showed that $\frac{P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(v; n_{pro}, n_{fan})}$ converged to 0 by bounding this ratio explicitly and showing the bound goes to 0. We now show that $\frac{P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(w; n_{pro}, n_{fan})}$ converges to 0 by showing that the denominator converges to something positive and finite and then showing that the numerator converges to 0.

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Since all the functions we deal with are continuous and defined on intervals, they are measurable, and intervals of the real numbers are measurable sets.

We have that

$$\begin{aligned}
P_k(w;n_{pro},n_{fan}) &= \binom{n_{pro}+n_{fan}-1}{k-1} F(w;n_{pro},n_{fan})^{n_{pro}+n_{fan}-k}\Big(1-F(w;n_{pro},n_{fan})\Big)^{k-1} \\
&= \binom{n_{pro}+n_{fan}-1}{k-1} \Big(\frac{n_{pro}}{n_{pro}+n_{fan}} + \frac{n_{fan}}{n_{pro}+n_{fan}}F_{fan}(w)\Big)^{n_{pro}+n_{fan}-k}\Big(\frac{n_{fan}}{n_{pro}+n_{fan}}\Big(1-F_{fan}(w)\Big)\Big)^{k-1} \\
&= \binom{n_{pro}+n_{fan}-1}{k-1} \Big(1-\frac{n_{fan}}{n_{pro}+n_{fan}}\Big(1-F_{fan}(w)\Big)\Big)^{n_{pro}+n_{fan}-k}\Big(\frac{n_{fan}}{n_{pro}+n_{fan}}\Big(1-F_{fan}(w)\Big)\Big)^{k-1}
\end{aligned}$$

where the second equality follows from $F_{pro}(w)=1$ and the third equality follows from algebraic manipulation. Define $\lambda \equiv n_{fan}(1-F_{fan}(w)) > 0$ and $n \equiv n_{pro}+n_{fan}$. Note that $n \to \infty$ as $n_{pro} \to \infty$. Then we have that

$$P_k(w;n_{pro},n_{fan}) = \binom{n-1}{k-1}\Big(\frac{\lambda}{n}\Big)^{k-1}\Big(1-\frac{\lambda}{n}\Big)^{n-k}.$$

Since this is a Binomial probability with parameters $(n-1,\lambda/n)$ and index $k-1 \geq 0$, a well known statistics results gives us that it converges to a Poisson probability with parameter λ and index $k-1 \geq 0$ as n grows large for any fixed n_{fan}. Thus we have that for all $k \in \{1,\ldots,K\}$

$$\lim_{n_{pro}\to\infty} P_k(w;n_{pro},n_{fan}) = \frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda} = \frac{\Big(n_{fan}(1-F_{fan}(w)\Big)^{k-1}}{(k-1)!}e^{-n_{fan}\Big(1-F_{fan}(w)\Big)}$$

which is positive and finite for any finite n_{fan}.

Thus, we have that for all $x \in (0,w)$ and all $k \in \{1,\ldots,K\}$ each conditional probability term becomes

$$\lim_{n_{pro}\to\infty} \frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K P_k(w;n_{pro},n_{fan})} = \frac{\lim_{n_{pro}\to\infty} P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^K \frac{\Big(n_{fan}(1-F_{fan}(w)\Big)^{k-1}}{(k-1)!}e^{-n_{fan}\Big(1-F_{fan}(w)\Big)}}.$$

Now we aim to argue that $\lim_{n_{\text{pro}}\to\infty} P_k(x;n_{pro},n_{fan}) = 0$ for all $x \in (0,w)$ and all $k \in \{1,\ldots,K\}$. Note that for all $x \in (0,w)$ and all $k \in \{1,\ldots,K\}$, $(\Big(1-F(x;n_{pro},n_{fan})\Big)^{k-1} \leq 1$ and so

$$\begin{aligned}
P_k(x;n_{pro},n_{fan}) &= \binom{n_{pro}+n_{fan}-1}{k-1} F(x;n_{pro},n_{fan})^{n_{pro}+n_{fan}-k}\Big(1-F(x;n_{pro},n_{fan})\Big)^{k-1} \\
&\leq \binom{n_{pro}+n_{fan}-1}{k-1} F(x;n_{pro},n_{fan})^{n_{pro}+n_{fan}-k} \\
&\leq \frac{(n_{pro}+n_{fan}-1)^{k-1}}{(k-1)!F(x;n_{pro},n_{fan})^{k-n_{pro}-n_{fan}}}
\end{aligned}$$

where we use the fact that $\frac{n!}{(n-k)!} \leq n^k$ in the last inequality and push the $F(\cdot)$ term into the denominator.

Now let $G(x) \equiv \max\{F_{pro}(x), F_{fan}(x)\}$. Then for all $x \in (0,w)$ we have

$$F(x;n_{pro},n_{fan}) \leq G(x) < 1.$$

Hence,

$$P_k(x;n_{pro},n_{fan}) \leq \frac{(n_{pro}+n_{fan}-1)^{k-1}}{(k-1)!F(x;n_{pro},n_{fan})^{k-n_{pro}-n_{fan}}} \leq \frac{(n_{pro}+n_{fan}-1)^{k-1}}{(k-1)!G(x)^{k-n_{pro}-n_{fan}}}.$$

As $n_{pro} \to \infty$ the last expression above has the form $\frac{\infty}{\infty}$, and so we use l'Hopital's Rule. Applying this rule $(k-1)$ times yields

$$\begin{aligned}
\lim_{n_{pro}\to\infty} \frac{(n_{pro}+n_{fan}-1)^{k-1}}{(k-1)!G(x)^{k-n_{pro}-n_{fan}}} &= \lim_{n_{pro}\to\infty} \frac{G(x)^{n_{pro}+n_{fan}-k}}{(-\log[G(x)])^{k-1}} \\
&= 0.
\end{aligned}$$

Thus we have that $\lim_{n_{pro}\to\infty} P_k(x; n_{pro}, n_{fan}) = 0$ for all $x \in (0, w)$ and all $k \in \{1, \ldots, K\}$ since probabilities must be non-negative. This gives us that $\lim_{n_{pro}\to\infty} \frac{P_k(x;n_{pro},n_{fan})}{\sum_{k=1}^{K} P_k(w;n_{pro},n_{fan})} = 0$ for all $x \in (0, w)$ and all $k \in \{1, \ldots, K\}$. So for all $x \in (0, w)$ we have that

$$\begin{aligned}
\lim_{n_{pro}\to\infty} \pi_{n_{pro}}(x) &= \lim_{n_{pro}\to\infty} \frac{\sum_{k=1}^{K} \alpha_k P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(w; n_{pro}, n_{fan})} \\
&= \sum_{k=1}^{K} \alpha_k \lim_{n_{pro}\to\infty} \left(\frac{P_k(x; n_{pro}, n_{fan})}{\sum_{k=1}^{K} P_k(w; n_{pro}, n_{fan})} \right) \\
&= 0.
\end{aligned}$$

By (C.6) we have that

$$\lim_{n_{pro}\to\infty} s(w; n_{pro}, n_{fan}) = \lim_{n_{pro}\to\infty} \int_0^w \pi_{n_{pro}}(x) dx$$

and we now want to pass the limit into the integral. To do this we appeal again to Lebesgue Dominated Convergence. We have shown convergence almost everywhere on $[0, w]$ using the same method used in the previous case. Using Lemma 1 again we can use the same dominating function $g(x) \equiv \alpha_1$. Thus by applying the Dominated Convergence Theorem, we have that

$$\lim_{n_{pro}\to\infty} s(w; n_{pro}, n_{fan}) = \lim_{n_{pro}\to\infty} \int_0^w \pi_{n_{pro}}(x) dx = \int_0^w \lim_{n_{pro}\to\infty} \pi_{n_{pro}}(x) dx = 0$$

as required for this case.

We have shown that $\forall v \in [w - \epsilon, w]$, $\lim_{n_{pro}\to\infty} s(v; n_{pro}, n_{fan}) = 0$ as required.

$\square$