Online Appendix to "Robust Predictions for DSGE Models with Incomplete Information"

Ryan Chahrour Robert Ulbricht

August 25, 2021

A Additional Proofs and Results

A.1 Market Clearing in the Primal Economy

Here we supplement our discussion in the main body with a parametric example that demonstrates how imposing market clearing in the primal economy precisely rules out the case where agents have *exactly* no information about prices.

Consider again the example from Section 2.2. Concretely, suppose that market clearing in the incomplete information economy requires

$$\bar{\mathbb{E}}_t^h[r_t] \equiv \int \mathbb{E}[r_t|\mathcal{I}_{i,t}^h] \,\mathrm{d}i = -y_t. \tag{A.1}$$

The analog condition in the primal representation is given by

$$r_t + \tau_t = -y_t, \tag{A.2}$$

where $\tau_t = \int \tau_{i,t} \,\mathrm{d}i$ is the average expectation error across housholds. Suppose that the joint process for y_t and τ_t is parametrized by

$$\tau_t = \alpha y_t + u_t,$$

where $\alpha > 0$ parametrizes the correlation of τ_t with y_t, and u_t is orthogonal to y_t. Here, the restriction that $\alpha > 0$ is an immediate implication of Theorem 1.[1] Substituting τ_t into

[1] Specifically, using the notation of the theorem, we have $\mu_{i,t} = r_t + \tau_{i,t}$. Hence, for $\tau_{i,t}$ to be implementable, it must satisfy $\mathrm{Cov}[\tau_{i,t}, r_t + \tau_{i,t}] = 0$, or equivalently, $\mathrm{Cov}[\tau_t, r_t + \tau_t] = -\mathrm{Var}[\Delta\tau_{i,t}]$. Using (A.2) to substitute for r_t, we obtain $\mathrm{Cov}[\tau_t, y_t] = \mathrm{Var}[\Delta\tau_{i,t}] > 0$ and, hence, $\alpha > 0$.

(A.2), the market clearing rate is pinned down by $r_t = -(\alpha+1)y_t - u_t$, which entails an "equilibrium expectation" of

$$\bar{\mathbb{E}}_t^h[r_t] = r_t + \tau_t = \frac{1}{\alpha+1}\left(r_t + u_t\right). \tag{A.3}$$

We conclude that simply assuming that the exogenous wedge process is finite ($|\alpha| < \infty$), as required under the conditions of Theorem 1, and imposing market clearing in the primal representation implies that $\bar{\mathbb{E}}_t^h[r_t]$ is strictly increasing in r_t.

Intuitively, for $\bar{\mathbb{E}}_t^h[r_t]$ to be strictly increasing in r in the underlying incomplete information economy, households must observe some (possibly noisy) signal about r_t. To demonstrate this, suppose that each household forms their expectation about r_t based on a single signal,

$$s_{i,t} = r_t + u_{i,t},$$

with $u_{i,t} \sim \mathcal{N}(0, \sigma_u^2)$. For simplicity, further suppose that in equilibrium $r_t \sim \mathcal{N}(0, \sigma_r^2)$ for some endogenously determined σ_r^2. Then

$$\mathbb{E}[r_t|\mathcal{I}_{it}] = \gamma\left(r_t + u_{i,t}\right), \qquad \bar{\mathbb{E}}_t^h[r_t] = \gamma\left(r_t + \bar{u}_t\right), \tag{A.4}$$

for $\gamma \equiv \frac{\sigma_r^2}{\sigma_r^2+\sigma_u^2} \in [0,1]$ and $\bar{u}_t \equiv \int u_{it}\,\mathrm{d}i$. Here, γ parametrizes the informativeness of the signal, and the stochastic term $\bar{u}_t$ captures the possibility that $\{u_{i,t}\}$ are correlated in the cross-section. Substituting $\bar{\mathbb{E}}_t^h[r_t]$ into (A.1), we obtain the market-clearing interest rate,

$$r_t = -y_t/\gamma - \bar{u}_t.$$

Clearly, for a market-clearing rate to exists, it must be that the price signal is informative; i.e., $\gamma > 0$ (or, equivalently, $\sigma_u < \infty$). This corresponds precisely to the case where the average expectation $\bar{\mathbb{E}}_t^h[r_t]$ is strictly increasing in r_t.

A.2 Proof of Lemma 1

The characterization for $\hat{y}_t$ is immediate. To solve for π_t, let $\pi_t = \pi(L)u_t$, define

$$\tilde{A}(L)u_t \equiv \begin{bmatrix} -1 & \xi \end{bmatrix} A(L)u_t = \xi\tau_t^x - \tau_t^c,$$

and substitute in (36) to obtain

$$\pi(L)u_t = \phi^{-1}\left[(L^{-1}-1)\tilde{A}(L) + L^{-1}\pi(L)\right]_+ u_t$$

where $[\cdot]_+$ sends negative powers of L to zero. Applying the z-transform, we obtain the following functional equation

$$(z^{-1}-\phi)\pi(z) = (1-z^{-1})\tilde{A}(z) + z^{-1}\tilde{A}_0 + z^{-1}\pi_0. \tag{A.5}$$

Stationarity requires π to be analytic on the unit disk (Whiteman, 1983). Evaluating (A.5) at $z=\phi^{-1}\in(-1,1)$, therefore, pins down

$$\pi_0 = (1-\phi^{-1})\tilde{A}(\phi^{-1}) - \tilde{A}_0,$$

so that

$$\pi(z) = \frac{(1-z)\tilde{A}(z) - (1-\phi^{-1})\tilde{A}(\phi^{-1})}{\phi z - 1}.$$

A.3 Proof of Proposition 6

To begin, combine Proposition 5 with equation (39) to obtain the following lemma.

Lemma 2. *Fix a (zero mean) MA($\bar{h}$) process τ for (τ_t^c, τ_t^p) and set $\Theta_{i,t}^{\text{sym}}$ as in (22). Then there exists an information structure consistent with Assumptions 1–3 that implements τ in the incomplete-information economy, if and only if there exists a (zero mean) MA($\bar{h}$) process $\Delta\tau$ such that*

$$\Gamma_s(\tau,\epsilon) = -\Lambda_s(\Delta\tau,\Delta f) \quad \textit{for all } s\geq 0, \tag{A.6}$$

where

$$\begin{aligned}\Gamma_s(\tau,\epsilon) &\equiv \text{Cov}[\tau_t, (\text{d}y_{t-s}, \text{d}y_{t-s}, \epsilon_{t-s})]\\ \Lambda_s(\Delta\tau,\Delta f) &\equiv \text{Cov}[\Delta\tau_{i,t}, (\Delta\text{d}c_{i,t-s}, \Delta\text{d}y_{i,t-s}, \Delta\text{d}a_{i,t-s})].\end{aligned}$$

Equipped with Lemma 2, our proof proceeds in two steps. First, we derive the mappings $(\tau,\epsilon)\mapsto\Gamma_s$ and $(\Delta\tau,\Delta f)\mapsto\Lambda_s$ in closed form. Second, with this explicit characterization at hand, we complete the proof by constructing processes for $\Delta\tau$ and Δf that for any given (τ,ϵ) satisfy the conditions of Lemma 2.

Characterization of Γ_s The mapping Γ_s is immediate from (37),

$$\Gamma_s(\tau,\epsilon) = \xi \mathrm{Cov}[\tau_t, \mathrm{d}\tau^x_{t-s}] \times [1,1,0] + \mathrm{Cov}[\tau_t, \epsilon_{t-s}] \times [1,1,1]. \tag{A.7}$$

Characterization of Λ_s We now solve the "Delta-economy" for the endogenous law of motions for $\Delta \mathrm{d}c_{i,t}$ and $\Delta \mathrm{d}y_{i,t}$. The equilibrium of the Delta-economy is defined by (27), (28), (40), (41), which can be written as follows:

$$\begin{aligned}
\Delta p_{i,t} &= -\eta^{-1}\Delta y_{i,t} + z_{i,t} \\
\beta b_{i,t} &= b_{i,t-1} + \Delta y_{i,t} - \Delta c_{i,t} + \Delta p_{i,t} \\
\Delta c_{i,t} &= \mathbb{E}_t[\Delta c_{i,t+1} - \Delta\tau^c_{i,t+1}] + \Delta\tau^c_{i,t} \\
\Delta y_{i,t} &= \xi(\Delta y_{i,t} - \Delta c_{i,t} + \Delta p_{i,t} + \Delta\tau^p_{i,t}) + \Delta a_{i,t}
\end{aligned}$$

The system can be written more compactly as

$$\mathbb{E}_t[\mathrm{d}\Delta y_{i,t+1}] = \delta\mathbb{E}_t[\xi^{-1}\mathrm{d}\Delta a_{i,t+1} + \mathrm{d}z_{i,t+1} + \mathrm{d}\Delta\mathrm{d}\tau^p_{i,t+1} - \mathrm{d}\Delta\tau^c_{i,t+1}] \tag{A.8}$$
$$\beta b_{i,t} = b_{i,t-1} + \xi^{-1}(\Delta y_{i,t} - \Delta a_{i,t}) - \Delta\tau^p_{i,t} \tag{A.9}$$

where $\delta \equiv (\eta^{-1} + \xi^{-1} - 1)^{-1}$, and consumption is determined by

$$\Delta c_{i,t} = -\delta^{-1}\Delta y_{i,t} + z_{i,t} + \Delta\tau^p_{i,t} + \xi^{-1}\Delta a_{i,t}. \tag{A.10}$$

Fix some process $(\Delta\tau^c_{i,t}, \Delta\tau^p_{i,t}, \Delta a_{i,t}, z_{i,t})' = B(L)\upsilon_{i,t}$, where $B(L)$ is a square-summable matrix-polynomial in non-negative powers of the lag operator L and the vector $\upsilon_{i,t}$ are white noise shocks. Conjecture

$$\Delta y_{i,t} = \xi(\beta-1)b_{i,t-1} + \Phi(L)\upsilon_{i,t}. \tag{A.11}$$

Substituting (A.11) in (A.9), it must be that

$$\Phi(L)\upsilon_{i,t} = \xi\beta\mathrm{d}b_{i,t} + \xi\Delta\tau^x_{i,t} + \Delta a_{i,t}. \tag{A.12}$$

Using (A.11) to eliminate $\Delta\mathrm{d}y_{i,t+1}$ in (A.8), we have

$$(\beta-1)\xi\mathrm{d}b_{i,t} + \left[(L^{-1}-1)\Phi(L)\right]_+ \upsilon_{i,t} = \begin{bmatrix} -\delta & \delta & \delta\xi^{-1} & \delta \end{bmatrix} \left[(L^{-1}-1)B(L)\right]_+ \upsilon_{i,t} \tag{A.13}$$

where $[\cdot]_+$ sends the negative powers of L to zero. Further using (A.13) to eliminate $\mathrm{d}b_{i,t}$ in

(A.12) and applying the z-transform, we obtain the following functional equation

$$\begin{aligned}&(1-\beta^{-1}z)\Phi(z) = \\ &\qquad \begin{bmatrix}-\delta & \delta & \delta\xi^{-1} & \delta\end{bmatrix}[(1-z)B(z)-B_0] + \Phi_0 + (1-\beta^{-1})\begin{bmatrix}0 & \xi & 1 & 0\end{bmatrix}B(z)z. \quad \text{(A.14)}\end{aligned}$$

Evaluating (A.14) at $z = \beta \in (-1, 1)$, pins down Φ_0 and $\Phi(z)$, from which we obtain the following equilibrium process for $\mathrm{d}\Delta y_{i,t} \equiv \mathrm{d}y(L)\upsilon_{i,t}$ and $\mathrm{d}\Delta c_{i,t} \equiv \mathrm{d}c(L)\upsilon_{i,t}$:

$$\mathrm{d}y(z) = \begin{bmatrix}-\delta & \delta & \delta\xi^{-1} & \delta\end{bmatrix}(1-z)B(z) + \begin{bmatrix}\delta & \xi-\delta & 1-\delta\xi^{-1} & -\delta\end{bmatrix}(1-\beta)B(\beta) \quad \text{(A.15)}$$

and

$$\mathrm{d}c_{i,t} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}(1-z)B(z) + \begin{bmatrix}-1 & 1-\delta^{-1}\xi & \xi^{-1}-\delta^{-1} & 1\end{bmatrix}(1-\beta)B(\beta). \quad \text{(A.16)}$$

Collecting equations, we obtain

$$\begin{aligned}\Lambda_s(\Delta\tau, f) = \mathrm{Cov}&\left[\Delta\tau_{i,t}, \begin{bmatrix}1 & 0 & 0 & 0\\ -\delta & \delta & \delta\xi^{-1} & \delta \\ 0 & 0 & 1 & 0\end{bmatrix}(1-L)B(L)\upsilon_{i,t-s}\right] \\ &+ \mathrm{Cov}\left[\Delta\tau_{i,t}, \begin{bmatrix}-1 & 1-\delta^{-1}\xi & \xi^{-1}-\delta^{-1} & 1 \\ \delta & \xi-\delta & 1-\delta\xi^{-1} & -\delta \\ 0 & 0 & 0 & 0\end{bmatrix}(1-\beta)B(\beta)\upsilon_{i,t-s}\right] \quad \text{(A.17)}\end{aligned}$$

for

$$\Delta\tau_{i,t} = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{bmatrix}B(L)\upsilon_{i,t}.$$

Construction of process $\Delta\tau$ and Δf that implement (τ, ϵ) We complete the proof by construction. In particular, we provide an algorithm that for arbitrary $\{\Gamma_s\}_{s=0}^{\bar{h}}$ constructs processes $\Delta\tau$ and Δf that satisfy (A.6).

To begin, substitute (A.17) to (A.6), post-multiply both sides by

$$M \equiv \begin{bmatrix}1 & 1 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & -\xi^{-1} & 1\end{bmatrix},$$

and apply the z-transform, to obtain the equivalent functional equation

$$\tilde{\Gamma}(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left\{ \left[B(z)(1-z^{-1})B(z^{-1})'\right]_+ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}' + \right.$$

$$\left. + B(z)(1-\beta)B(\beta)' \begin{bmatrix} -1 & 1-\delta^{-1}\xi & \xi^{-1}-\delta^{-1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}' \right\} \tag{A.18}$$

where $\tilde{\Gamma}(z) \equiv \mathcal{Z}\{-\Gamma_s M\}_{s\geq 0}$ is the (one-sided) z-transform of $\{-\Gamma_s M\}$, and where B parametrizes the joint process $(\Delta\tau_{i,t}, \Delta f_{i,t})$ as in the characterization of Γ above. In particular, let

$$B(L) = \begin{bmatrix} B_\tau(L) \\ B_a(L) \\ B_z(L) \end{bmatrix}$$

where $B_\tau(z)$ is a lag-polynomial of size $2 \times n$, $B_a(z)$ and $B_z(z)$ are each lag-polynomials of size $1 \times n$, and n is an arbitrary number of innovations. Then (A.18) can be further rewritten as

$$\tilde{\Gamma}_1(z) + \Omega(z) = \left\{(1-z^{-1})B_\tau(z)B_\tau(z^{-1})'\right\}_+ + \Psi(z) + B_\tau(z)B_\tau(\beta)'\Lambda \tag{A.19}$$

and

$$\tilde{\Gamma}_2(z) = \left\{(1-z^{-1})B_\tau(z)B_a(z^{-1})'\right\}_+, \tag{A.20}$$

where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ correspond to the first two and third column of $\tilde{\Gamma}$, respectively, and where

$$\Psi(z) \equiv \left\{B_\tau(z)\left[(1-\beta)B_z(\beta)' \quad (1-z^{-1})B_z(z^{-1})'\right]\right\}_+$$

and

$$\Omega(z) \equiv -(1-\beta)(\xi^{-1}-\delta^{-1})\left[B_\tau(z)B_a(\beta)' \quad 0\right]$$

and

$$\Lambda \equiv \begin{bmatrix} -(1-\beta) & 0 \\ (1-\beta)(1-\delta^{-1}\xi) & 0 \end{bmatrix}.$$

Fix $N \leq \bar{h}$ as the largest non-zero power of z in $\tilde{\Gamma}$. Consider the following parametric structure for B_τ, B_a, and B_z:

$$\begin{bmatrix} B_\tau(z) \\ B_a(z) \\ B_z(z) \end{bmatrix} = \begin{bmatrix} \lambda_\tau(z) & I \\ \lambda_a(z) & (1-z)^{-1}\lambda_{a,0} \\ 0 & \lambda_{z,0} + \lambda_{z,1} z \end{bmatrix}$$

with

$$\lambda_\tau(z) = \left[\lambda_{\tau,1} + \rho z \quad \cdots \quad \lambda_{\tau,N} + \rho^N z^N\right]$$

and

$$\lambda_a(z) = \left[(1-z)^{-1}\lambda_{a,1} \quad \cdots \quad (1-z)^{-1}\lambda_{a,N}\right],$$

and where $\{\lambda_{a,j}, \lambda_{z,j}\}$ are of size 1×2 and $\{\lambda_{\tau,j}\}$ are of size 2×2. Observe that B_τ is at most of order $\bar{h}$ in line with the requirements of Lemma 2.

Condition (A.20) then simplifies to

$$\tilde{\Gamma}_2(z) = \lambda_\tau(z)\lambda_a' + \lambda_{a,0}'.$$

So for any λ_τ, it suffices to set

$$\lambda_{a,s} = \rho^{-s}\tilde{\Gamma}_{2,s}' \quad \forall s \geq 1, \quad \text{and}$$
$$\lambda_{a,0} = \tilde{\Gamma}_{2,0}' - \sum_{j=1}^{N} \lambda_{\tau,j}'\lambda_{a,j}$$

in order to satisfy orthogonality with respect to $a_{i,t}$.

Regarding condition (A.19), we have that

$$\Pi(z) \equiv \tilde{\Gamma}_1(z) + \Omega(z) - \Lambda - I = \left\{(1-z^{-1})\tau_\tau(z)\tau_\tau(z^{-1})'\right\}_+ + \Psi_0 + \lambda_\tau(z)\lambda_\tau(\beta)'\Lambda$$

where

$$\Omega(z) = -\tilde{\Gamma}_2(z)\left[(\xi^{-1} - \delta^{-1}) \quad 0\right]$$

and

$$\Psi_0 \equiv \Psi(z) = \left[(1-\beta)\left(\lambda_{z,0}' + \beta\lambda_{z,1}'\right) \quad \lambda_{z,0}'\right].$$

Notice that (i) the left-hand side, $\Pi(z)$, is exogenously determined by the aggregate economy that we are trying to implement, and (ii) we have Ψ_0 as a degree of freedom to induce an arbitrary unconditional covariance on the right-hand side. Writing out the right-hand side in the time-domain, we have

$$\Pi_0 = \Psi_0 - \rho\lambda'_{\tau,1} + \frac{\rho^2}{1-\rho^2} + \sum_{j=1}^{N} \lambda_{\tau,j}\lambda'_{\tau,j}(I+\Lambda) + \sum_{j=1}^{N} \rho^j \beta^j \lambda_{\tau,j}\Lambda \tag{A.21}$$

$$\Pi_s = \rho^s \lambda'_{\tau,s}(I+\Lambda) - \rho^{s+1}\lambda'_{\tau,s+1} + \rho^{2s}\beta^s\Lambda. \tag{A.22}$$

Initialized at $\lambda_{N+1} = 0$, (A.22) can be solved recursively backwards for a sequence $\{\lambda_{\tau,s}\}$ that ensures orthogonality with respect to $(c_{i,t-s}, y_{i,t-s})_{s\geq 1}$. Finally, orthogonality with respect to $(c_{i,t}, y_{i,t})$ is achieved by setting Ψ_0 to satisfy (A.21), completing the proof.

A.4 Cyclicality of Inflation in the Quantitative Model

Here we prove a variant of Proposition 2 in the context of our quantitative model, showing that if firms know the location of their demand curve (i.e., $\Theta^f_{i,t}$ contains both $p_{i,t}$ and $y_{i,t}$), then inflation must be procyclical for any expectation-driven business cycles. This holds regardless of whether firms are price or quantity setters.[2]

The result is derived for the more general case where households and firms do not necessarily share the same information. The case of symmetric information follows as corollary.

Proposition 7. *Suppose $\{W_{i,t-s}\}_{s\geq 0} \subseteq \Theta^h_{i,t}$ and $\{Y_{i,t-s}, P_{i,t-s}, W_{i,t-s}\}_{s\geq 0} \subseteq \Theta^f_{i,t}$. Then inflation must be weakly procyclical. Specifically, the correlation with the output gap is bounded below as follows:*

$$\sqrt{\mathrm{Var}[\hat{y}_t]} \leq \xi \frac{\mathrm{Corr}[\hat{y}_t, \pi_t]}{1 - \mathrm{Corr}[\hat{y}_t, \hat{y}_{t-1}]} \sqrt{\mathrm{Var}[\pi_t]}.$$

Proof. The proof proceeds in analog to the one of Proposition 2. Substituting for $w_{i,t}$ using the household's labor supply and taking first differences, orthogonality of the household wedge with respect to $\mathrm{d}w_{i,t}$ requires

$$\mathrm{Cov}[\tau^{p,h}_{i,t}, \zeta \mathrm{d}n_{i,t} + \mathrm{d}c_{i,t} + \pi_t + \mathrm{d}\tau^h_{i,t}] = \mathrm{Cov}[\tau^{p,h}_{i,t}, \pi_t + \mathrm{d}\tau^h_{i,t}] = 0, \tag{A.23}$$

where the first equality exploits that by Theorem 1 $\tau^{p,h}_{i,t} \perp \mu^h_{i,t-s}$ and thus $\tau^{p,h}_{i,t} \perp (n_{i,t-s}, c_{i,t-s})$ for all $s \geq 0$.

Similarly, substituting for $w_{i,t}$ using the firm's labor demand and taking first differences,

[2] Absent nominal rigidity, and given that both $p_{i,t}$ and $y_{i,t}$ are in firms' information sets, there is no difference between price and quantity setting.

orthogonality of the firm wedge with respect to $\mathrm{d}w_{i,t}$ requires

$$\mathrm{Cov}[\tau^{p,f}_{i,t}, \mathrm{d}a_{i,t} + \mathrm{d}p_{i,t} + \mathrm{d}\tau^{p,f}_{i,t}] = \mathrm{Cov}[\tau^{p,h}_{i,t}, \mathrm{d}\tau^{p,f}_{i,t}] = 0. \tag{A.24}$$

Here the first equality follows as $\tau^{p,f}_{i,t} \perp \mu^{f}_{i,t-0}$ implies $\tau^{p,f}_{i,t} \perp n_{i,t-0}$ for all $s \geq 0$ and, hence, $\tau^{p,f}_{i,t} \perp (\mathrm{d}y_{i,t} - \mathrm{d}n_{i,t} + \mathrm{d}p_{i,t})$ under the conditions of the proposition.

Subtracting (A.23) from (A.24), we have

$$\mathrm{Cov}[\tau^{p}_{i,t}, \mathrm{d}\tau^{p}_{i,t} - \pi_t] = 0$$

or

$$(1 - \mathrm{Corr}[\hat{y}_t, \hat{y}_{t-1}])\, \xi^{-1}\mathrm{Var}[\hat{y}_t] - \mathrm{Cov}[\hat{y}_t, \pi_t] = -\left(1 - \mathrm{Corr}[\Delta\tau^{p}_{i,t}, \Delta\tau^{p}_{i,t-1}]\right) \mathrm{Var}[\Delta\tau^{p}_{i,t}] \leq 0,$$

which implies the bound given in the statement of the proposition. □

B Estimation of Unrestricted Wedge Process

Here we describe the methodology for estimating the unrestricted wedges $\hat{\tau}_t$ used in Section 4.

B.1 Description of Methodology

We model the unrestricted wedges as a MA(14) process, which loads on two intrinsic innovations, represented by the 2×1 vector u_t, in addition to the productivity shock ϵ_t,

$$\tau_t = \Phi_\epsilon(L)\epsilon_t + \Phi_u(L)u_t,$$

where $\Phi_\epsilon(L)$ and $\Phi_u(L)$ are square-summable lag polynomials in non-negative powers of L, and ϵ_t and u_t are orthogonal white noise. W.l.o.g., we normalize $\mathrm{Var}[u_t] = I_2$, leaving us to estimate $\gamma_{\mathrm{ma}} \equiv (\Phi_\epsilon, \Phi_u, \sigma_\epsilon)$. For this purpose, we use the generalized method of moments (GMM) to minimize the distance between the model's covariance structure and U.S. data on real per-capita output, inflation, nominal interest rates, and per-capita hours.[3] Let

$$\tilde{\Omega}_T = vech\{\mathrm{Var}[(\tilde{q}_t^{\mathrm{data}}, \dots, \tilde{q}_{t-k}^{\mathrm{data}})]\},$$

denote the empirical auto-covariance matrix of frequency-filtered quarterly US data for $q \equiv (y_t, \pi_t, i_t, n_t)$. We target auto-covariances between zero and $k = 8$ quarters. For the filtering, we use the Baxter and King (1999) approximate high-pass filter with a truncation horizon of 32 quarters, which we denote by $\tilde{q}_t \equiv BK_{32}(q_t)$.[4]

To conserve on the 91 parameters that characterize γ_{ma}, we make two observations, documented in Figure 4 below. First, $\tilde{\Omega}_T$ is well-described by a VAR(1) process for τ_t. Second, a MA(14) truncation of the VAR(1) process that best replicates $\tilde{\Omega}_T$ is almost indistinguishable (in terms of second moments) from the VAR(1) process itself. Accordingly, we construct γ_{ma} by first estimating τ_t as a VAR(1) that is driven by u_t and ϵ_t, and then constructing $\hat{\gamma}_{\mathrm{ma}}$ as the MA(14) truncation of the estimated process.[5]

Let γ_{ar} denote the 10 parameters characterizing the VAR(1) and σ_ϵ. Then the estimator is given by

$$\hat{\gamma}_{\mathrm{ar}} = \underset{\gamma_{\mathrm{ar}}}{\mathrm{argmin}} (\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{\mathrm{ar}}))' W^{-1} (\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{\mathrm{ar}})), \tag{B.1}$$

where $\tilde{\Omega}(\gamma_{\mathrm{ar}})$ is the model analogue to $\tilde{\Omega}_T$ and W is a diagonal matrix with the bootstrapped variances of $\tilde{\Omega}_T$ along the main diagonal. To avoid the issues detailed in Gorodnichenko and Ng (2010), our model analogue $\tilde{\Omega}(\gamma_{\mathrm{ar}})$ is computed after applying the same filtering procedure to the model that we have applied to the data.

A final challenge for estimating the model is that filtering the model can be computational expensive. We address this issue by proving the following equivalence results (see Online Appendix B.3 for proof).

Lemma 3. *Estimator* (B.1) *is equivalent to*

$$\hat{\gamma}_{\mathrm{ar}} = \underset{\gamma_{\mathrm{ar}}}{\mathrm{argmin}} (\Omega_T - \Omega(\gamma_{\mathrm{ar}}))' \tilde{W}^{-1} (\Omega_T - \Omega(\gamma_{\mathrm{ar}})), \tag{B.2}$$

[3] Data range from 1960Q1 to 2012Q4. Real output is given by nominal output divided by the GDP deflator. Inflation is defined as the log-difference in the GDP deflator. Interest rates are given by the Federal Funds Effective rate. Hours are given by hours worked in the non-farm sector. Variables are put in per-capita terms using the non-institutional population over age 16. All data are downloaded from the "Economic Data" archive of the Federal Reserve Bank of St. Louis (FRED).

[4] The Baxter and King (1999) filter requires specification of a lag-length $\bar{\tau}$ for the approximation. We set $\bar{\tau}$ to their recommended value of 12.

[5] Our estimator penalizes excessively persistent dynamics beyond the usual business cycle horizon by imposing a numerical penalty on impulse responses beyond 32 quarters.

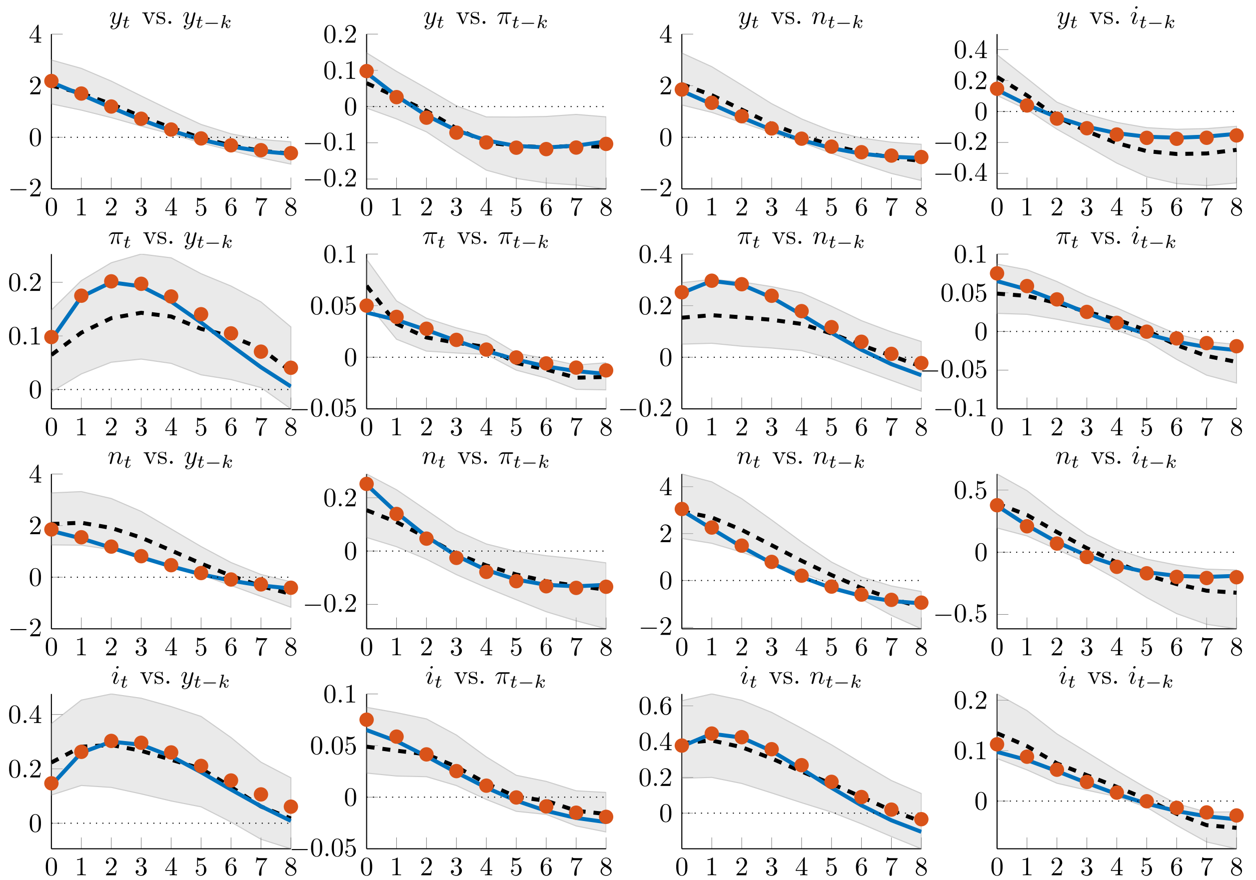

Figure 4: Business cycle comovements in the data and predicted by the estimated model. Note.—All covariances are multiplied by 100 to improve readability. Dashed black lines show the empirical covariance structure $\tilde{\Omega}_T$ together with 90 percent confidence intervals depicted by the shaded areas. Solid blue lines show the corresponding model moments for the VAR(1) case, $\tilde{\Omega}(\hat{\gamma}_{\text{ar}})$. Red dots show the model moments for the truncated MA(14) case, $\tilde{\Omega}(\hat{\gamma}_{\text{ma}})$. Each row i and column j in the table shows the covariances between $\tilde{q}_t^i$ and $\tilde{q}_{t-k}^j$ with lags $k \in \{0, 1, \ldots, 8\}$ depicted on the x-axis.

where $\Omega \equiv vech\{\text{Var}[(\text{d}s_t, \ldots, \text{d}s_{t-K})]\}$ *and* $\tilde{W} \equiv (\Xi' W^{-1} \Xi)^{-1}$ *for* $K = k + 2\bar{\tau}$. *The transformation matrix* Ξ *is defined in* (B.7).

The lemma establishes an *exact* equivalence (as opposed to an asymptotic equivalence) between the original GMM estimator (B.1) and an alternative estimator where the unfiltered model is estimated (in first differences) on unfiltered data and the filtering is achieved by replacing W with $\tilde{W}$. Using (B.2) in place of (B.1), estimation becomes straightforward as the mapping from γ_{ar} to $\Omega(\gamma_{\text{ar}})$ is available in closed form.

B.2 Fit

Figure 4 compares the predicted model moments with the targeted data moments. The dashed black lines show the empirical covariance structure $\tilde{\Omega}_T$ along with 90-percent confidence intervals (depicted by the shaded areas). The solid blue and red lines show the corresponding moments for the estimated model for the VAR(1) and MA(14) truncation of the wedges, respectively. Each row i and column j in the table of plots shows the covariances between $\tilde{q}_t^i$ and $\tilde{q}_{t-k}^j$ with lags $k \in \{0, 1, \ldots, 8\}$ depicted on the horizontal axis. Despite the parametric restriction on τ_t and a_t and the fact that we have less shocks than data series, the unrestricted-wedge model does a very good job at capturing the auto-covariance structure of the four time series. In addition, there is no notable difference between the VAR(1) and MA(14) truncation of τ_t.

B.3 Proof of Lemma 3

Let

$$J = \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma)\right)' W^{-1} \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma)\right) \tag{B.3}$$

denote the penalty function in terms of BK-filtered moments, where the filter is applied to both the data and the model. In this appendix, we demonstrate how the penalty can be expressed in terms of the variance over unfiltered first-differenced moments, $\Omega \equiv vech\left\{\mathrm{Var}\left(\mathrm{d}q_{t-K}^t\right)\right\}$, where d is the first-difference operator, and $K \equiv k + 2\bar{\tau}$ with $\bar{\tau}$ denoting the approximation horizon of the BK-filter.[6] Specifically, for any positive-semidefinite W we show that J in (B.3) is equivalent to

$$J = \left(\Omega_T - \Omega(\gamma)\right)' \tilde{W}^{-1} \left(\Omega_T - \Omega(\gamma)\right), \tag{B.4}$$

with $\tilde{W} \equiv \left(\Xi' W^{-1} \Xi\right)^{-1}$ replacing W (a closed-form expression for Ξ is given below).

The Baxter and King (1999) filtered version of s_t takes the form

$$\tilde{q}_t = \sum_{j=-\bar{\tau}}^{\bar{\tau}} a_j q_{t-j}$$

where $\tilde{q}_t$ is stationary by construction. For the high-pass filter used in this paper, the weights $\{a_j\}$ are given by

$$a_j = \tilde{a}_j - \frac{1}{2\bar{\tau}+1} \sum_{j=-\bar{\tau}}^{\bar{\tau}} \tilde{a}_j$$

[6] The first-difference filter is applied to the unfiltered variables to ensure stationarity for variables that have a unit root. Our transformation includes an adjustment term that corrects for the fact that the filtered moments in $\tilde{\Omega}$ are about levels rather than first-differences.

with

$$\tilde{a}_0 = 1 - \bar{\omega}/\pi, \quad \tilde{\alpha}_{j\neq 0} = -\sin(j\bar{\omega})/(j\pi), \quad \bar{\omega} = 2\pi/32.$$

To construct the filter-matrix Ξ, rewrite $\tilde{q}_t$ in terms of growth rates to get

$$\tilde{q}_t = \sum_{j=-\bar{\tau}}^{\bar{\tau}} \sum_{l=0}^{\infty} a_j \mathrm{d}q_{t-j-l}.$$

Noting that $\sum_{j=-\bar{\tau}}^{\bar{\tau}} a_j = 0$, we can simplify to get

$$\tilde{q}_t = B\mathrm{d}q_{t-\bar{\tau}-j}^{t+\bar{\tau}}$$

where

$$B = [b_{-\bar{\tau}}, \ldots, b_{\bar{\tau}}] \otimes I_n, \tag{B.5}$$

$n = 4$ is the number of variables in $\tilde{q}_t$, and $b_s = \sum_{j=-\bar{\tau}}^{s} \alpha_j$.

Letting L_j define the backshift matrix

$$L_j = \left[0_{n(2\bar{\tau}+1),nj}, \quad I_{n(2\bar{\tau}+1)}, \quad 0_{n(2\bar{\tau}+1),n(k-j)}\right], \tag{B.6}$$

we then have that

$$\tilde{\Sigma}_j \equiv \mathrm{Cov}(\tilde{q}, \tilde{q}_{t-j}) = BL_0 \Sigma^K L_j' B',$$

or, equivalently,

$$vec(\tilde{\Sigma}_j) = (BL_j \otimes BL_0)\, vec(\Sigma^K).$$

To complete the construction of Ξ, define selector-matrices P_0 and P_1 such that

$$vech(\tilde{\Sigma}^k) = P_0 \begin{bmatrix} vec(\tilde{\Sigma}_0) \\ \vdots \\ vec(\tilde{\Sigma}_k) \end{bmatrix}$$

and

$$vec(\Sigma^K) = P_1 vech(\Sigma^K).$$

Stacking up $vec(\tilde{\Sigma}_j)$, we then get

$$\tilde{\Omega} = \Xi\Omega$$

where

$$\Xi = P_0 \begin{bmatrix} BL_0 \otimes BL_0 \\ \vdots \\ BL_k \otimes BL_0 \end{bmatrix} P_1 \tag{B.7}$$

with B and L_j as in (B.5) and (B.6). Substitution in (B.3) yields (B.4).

C Comparative Statics With Countercyclical Inflation

In analogue to Figure 2, we explore comparative statics with respect to the parametrization of the micro-shocks, but for the case where inflation is countercyclical with $\gamma_{\hat{y}\pi} = -.3$. The results, shown in Figure 5, display the same qualitative pattern as for the procyclical case explored in the main text. While the maximal volatility is higher, we again see a clear positive relationship between $\sigma_{\hat{y}}^{\max}$ and the volatilities of the micro shocks. As before, the impact of idiosyncratic demands shocks is most relevant, paralleling their key role in the procyclical case.

Here we do not include the cases without demand uncertainty ($p_{i,t} \in \Theta_{i,t}^{f}$), because in line with our discussion in the main text, in these cases inflation is necessarily procyclical (see Online Appendix A.4 for a formal proof). Intuitively, this reflects again the discrepancy in propagation underlying the pro- and countercyclical inflation cases: While procyclical inflation is tied to nominal misperception and expectation errors about aggregate prices, countercylical inflation is tied to expectation errors regarding local demand, and thus is impossible to implement when $p_{i,t}$ is observed by firms. (See also the explanations given in the context of Figure 1.)

References

Baxter, Marianne, and Robert G. King. 1999. "Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series." *Review of Economics and Statistics*, 81(4): 575–593.

Gorodnichenko, Yuriy, and Serena Ng. 2010. "Estimation of DSGE Models When the Data Are Persistent." *Journal of Monetary Economics*, 57(3): 325–340.

Whiteman, Charles. 1983. *Linear Rational Expectations Models: A User's Guide.* University of Minnesota Press.

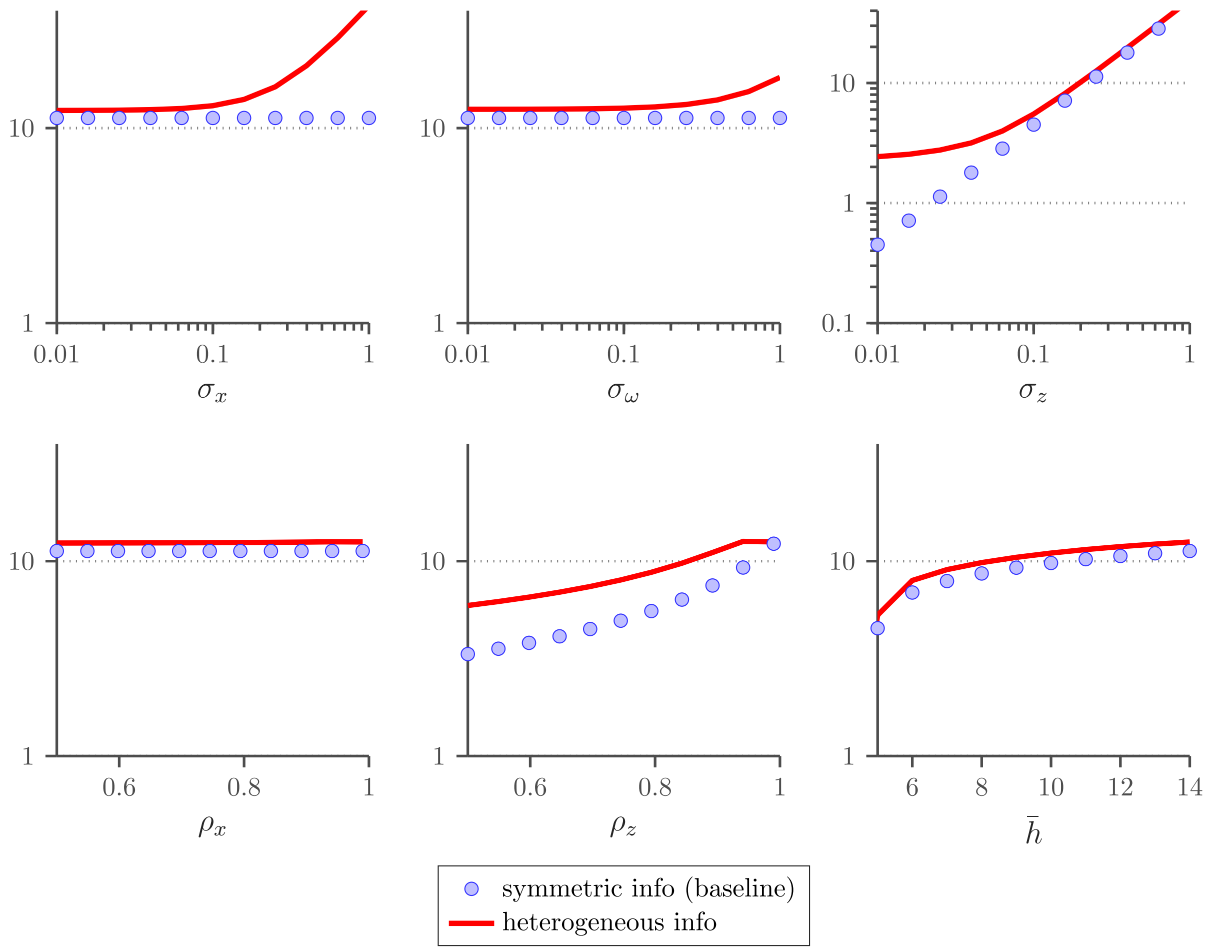

Figure 5: Analogue to Figure 2 with countercyclical inflation. The graphs show the maximal output volatility $\sigma_{\hat{y}}^{\max}$ (denominated in percentage deviations from the balanced growth path) that can be generated by incomplete information for the case where $\rho_{\hat{y}} = .9$ and $\gamma_{\hat{y}\pi} = -.3$.