On-line Appendix

Property Taxation, Zoning, and Efficiency in a Dynamic Tiebout Model

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1 Proofs

1.1 Proof of Proposition 1

Suppose first that $O^*_{Hi} > 0$ for all Hi. Since O^* is an equilibrium steady state, then, $N(O^*) = dO^*$. Thus, since $O^*_{Hi} > 0$ for all Hi, it must be the case that there is new construction of both types of houses in both communities. Accordingly, housing prices must equal construction costs so that $P(O^*) = (C_L, C_S, C_L, C_S)$. It must also be the case that the fraction of large houses in each community is the same; that is, $\lambda_1(O^*) = \lambda_2(O^*) = \lambda^*$. For if one community had a greater fraction of large houses, the public service surplus enjoyed by large house owners in that community would be higher than in the other which would violate (12). Since both house prices and the fraction of large houses are the same across the two communities, it follows from (9) and (10) that service levels and taxes are also the same. If a majority of households own large houses ($\lambda^* \geq 1/2$), then, from (8) and (9), the public service level will be g^*_L and, from (13), households live in large houses only if their preference exceeds the expression in (17). If a majority of households own small houses ($\lambda^* < 1/2$), the public service level is g^*_S and households live in large houses only if their preference exceeds the expression in (18).

It remains to consider the possibility that $O^*_{Hi} = 0$ for some house type Hi. Since the stock of housing must be sufficient to accommodate those households who need to reside in the area, we know that $\sum_i \sum_H O^*_{Hi} = 1$. It cannot be the case that there are no small houses ($O^*_{S1} = O^*_{S2} = 0$) because this would imply that the equilibrium prices of small houses must exceed C_S. This reflects the assumption that the distribution of household types has full support on the interval $[0, \overline{\theta}]$. This means that there are household types who are close to indifferent between large and small houses and they would be willing to pay close to C_L for a small house. By assumption, C_L is larger than C_S.

Suppose there are no large houses ($O^*_{L1} = O^*_{L2} = 0$). Then both communities would consist of small houses implying that $\lambda_1(O^*) = \lambda_2(O^*) = 0$. The steady state price of small houses would be C_S and communities would choose public good service levels level equal to $g^*(c)$ and a tax rate of $cg^*(c)/C_S$. The steady state price of large houses would have to be less than or equal to C_L so that construction firms had no incentive to supply large homes. In order for this to be satisfied, all households must weakly prefer to live in a small house given that small and large houses are priced at replacement cost and the tax rate is $cg^*(c)/C_S$. This requires that

$$\overline{\theta} \leq (1 - \delta(1-d))(C_L - C_S) + \frac{c(C_L - C_S)}{C_S} g^*(c).$$

All we have assumed is that $\overline{\theta}$ exceeds $(1-\delta(1-d))(C_L - C_S)$, so this is possible given our assumptions. However, the existence of such a steady state is perfectly consistent with the Proposition which does not require that $\lambda_i(O^*) > 0$.

We are left with two possibilities: there are no small houses in just one community ($O^*_{Si} = 0$ for some i) and there are no large houses in just one community ($O^*_{Li} = 0$ for some i). The first possibility is easy to rule out. Suppose that community 1 were the community in which there were no small houses (i.e., $O^*_{S1} = 0$). It cannot be the case that $O^*_{L2} > 0$, because then $(P_{L1}(O^*), P_{L2}(O^*), P_{S2}(O^*))$ would equal (C_L, C_L, C_S) and large home owners would strictly prefer to buy large homes in community 1. It cannot be the case that $O^*_{L2} = 0$, because then $(P_{L1}(O^*), P_{S1}(O^*), P_{S2}(O^*))$ would equal (C_L, P_{S1}, C_S) for some $P_{S1} \leq C_S$, the public service levels in both communities would equal $g^*(c)$, and the tax rates would be $cg^*(c)/C_L$ and $cg^*(c)/C_S$ respectively. Small home owners would then strictly prefer to buy small homes in community 1.

The second possibility is harder to rule out and this is where Assumption 1 comes into play. Suppose that community 1 is the community in which there are no large houses (i.e., $O^*_{L1} = 0$). Then the steady state prices $(P_{S1}(O^*), P_{L2}(O^*), P_{S2}(O^*))$ must equal (C_S, C_L, C_S). Moreover, the public service level in community 1 must equal $g^*(c)$, and the tax rate would be $cg^*(c)/C_S$. In community 2, the tax price for small home owners would equal $cC_S/\left(C_L\lambda_2(O^*) + C_S(1-\lambda_2(O^*))\right)$ which is lower than the community 1 tax price of c. It must be the case that large home owners are in the majority in community 2 (i.e., $\lambda_2(O^*) \geq 1/2$) because otherwise small house owners would strictly prefer to buy in community 2 to benefit from the lower tax price. Given that home prices are the same, in order for small home owners to be indifferent between buying in the two communities, it must be the case that their public service surplus is the same. This requires that the fraction of large houses in community 2 $\lambda_2(O^*)$ satisfies the equation:

$$B(g^*(c)) - cg^*(c) = \left[\begin{array}{c} B(g^*(\frac{cC_L}{C_L\lambda_2(O^*)+C_S(1-\lambda_2(O^*))})) \\ -\frac{cC_S}{C_L\lambda_2(O^*)+C_S(1-\lambda_2(O^*))} g^*(\frac{cC_L}{C_L\lambda_2(O^*)+C_S(1-\lambda_2(O^*))}) \end{array} \right].$$

It follows from this discussion that a necessary condition for the existence of a steady state in which one community is all small houses and the other has a mix of houses is that there exists $\lambda \geq 1/2$ such that

$$B(g^*(c)) - cg^*(c) = B(g^*(\frac{cC_L}{C_L\lambda + C_S(1-\lambda)})) - \frac{cC_S}{C_L\lambda + C_S(1-\lambda)} g^*(\frac{cC_L}{C_L\lambda + C_S(1-\lambda)})$$

Since the right hand side of this equation is increasing in λ there will not exist an asymmetric equilibrium if

$$B(g^*(c)) - cg^*(c) < B(g^*(\frac{2cC_L}{C_L + C_S})) - \frac{2cC_S}{C_L + C_S} g^*(\frac{2cC_L}{C_L + C_S}). \tag{30}$$

Intuitively, this is saying that the surplus a small homeowner would obtain living in a community in which 1/2 of the homes are large and the public service level is chosen by large home owners, is strictly larger than that they would enjoy living in a community in which all homes are small. We will show that inequality (30) is true under Assumption 1.

Defining the function $\varphi : [C_S, C_L] \to \Re$ as follows:

$$\varphi(x) = B(g^*(\frac{2cx}{x + C_S})) - \frac{2cC_S}{x + C_S} g^*(\frac{2cx}{x + C_S}),$$

it suffices to show that $\varphi(C_L)$ exceeds $B(g^*(c)) - cg^*(c)$. Note that $\varphi(C_S)$ equals $B(g^*(c)) - cg^*(c)$, and so inequality (30) must be true if $\varphi'(x) \geq 0$ over the relevant range. Differentiating, we have that

$$\varphi'(x) = \left(B'(g^*(\frac{2cx}{x + C_S})) - \frac{2cC_S}{x + C_S} \right) \frac{dg^*}{dx} + \frac{2cC_S}{(x + C_S)^2} g^*(\frac{2cx}{x + C_S}).$$

Using the fact that for any tax price ρ, we have that $B'(g^*(\rho)) = \rho$, we can write this as

$$\varphi'(x) = \left(\frac{2c(x - C_S)}{x + C_S} \right) \frac{dg^*}{dx} + \frac{2cC_S}{(x + C_S)^2} g^*(\frac{2cx}{x + C_S}).$$

Noting that

$$\frac{dg^*}{dx} = \frac{dg^*(\frac{2cx}{x+C_S})}{d\rho} \frac{d\rho}{dx} = \frac{dg^*(\frac{2cx}{x+C_S})}{d\rho} \left(\frac{2cC_S}{(x + C_S)^2} \right),$$

we can write

$$\varphi'(x) = \frac{2cC_S}{(x + C_S)^2} \left[\left(\frac{2c(x - C_S)}{x + C_S} \right) \frac{dg^*(\frac{2cx}{x+C_S})}{d\rho} + g^*(\frac{2cx}{x + C_S}) \right].$$

It follows that inequality (30) must be true if for all $x \in [C_S, C_L]$

$$g^*(\frac{2cx}{x + C_S}) \geq -\left(\frac{2c(x - C_S)}{x + C_S} \right) \frac{dg^*(\frac{2cx}{x+C_S})}{d\rho}.$$

Recalling that $\varepsilon_d(\rho)$ denotes the elasticity of demand for public services at tax price ρ, this is equivalent to

$$\varepsilon_d(\frac{2cx}{x+C_S}) \leq \frac{1}{1-\frac{C_S}{x}}.$$

This follows immediately from Assumption 1. ■

2 Existence of a steady state with no zoning

In Section IIID, we claim that there will exist a steady state with no zoning. Given the discussion in the text, this will be true if either there exists λ^* greater than 1/2 satisfying the equation

$$\lambda^* = 1 - F((1-\delta(1-d) + \frac{cg_L^*}{\lambda^* C_L + (1-\lambda^*)C_S})(C_L - C_S)),$$

or there exists λ^* less than 1/2 satisfying the equation

$$\lambda^* = 1 - F((1-\delta(1-d) + \frac{cg_S^*}{\lambda^* C_L + (1-\lambda^*)C_S})(C_L - C_S)).$$

In the former case, there exists a steady state in which the fraction of large houses in each community is λ^* and the public service level is g_L^* and in the latter there exists a steady state in which the fraction of large houses in each community is λ^* and the public service level is g_S^*.

Define the function $\varphi(\lambda)$ on the interval $[0,1]$ as follows:

$$\varphi(\lambda) = \begin{cases} 1 - F((1-\delta(1-d) + \frac{cg^*(\frac{cC_S}{C_L\lambda + C_S(1-\lambda)})}{\lambda C_L + (1-\lambda)C_S})(C_L - C_S)) & \text{if } \lambda \in [0,1/2) \\ 1 - F((1-\delta(1-d) + \frac{cg^*(\frac{cC_L}{C_L\lambda + C_S(1-\lambda)})}{\lambda C_L + (1-\lambda)C_S})(C_L - C_S)) & \text{if } \lambda \in [1/2,1] \end{cases}.$$

Then it is enough to show that there exists a solution to the equation $\lambda^* = \varphi(\lambda^*)$. Consider the behavior of the function $\varphi(\lambda)$ on the interval $[0,1]$. First, note that because $cg^*(c)/C_S$ exceeds $cg^*(c)/C_L$, we have that $\varphi(0) \leq \varphi(1)$. Second, note that we can assume without loss of generality that

$$(1-\delta(1-d) + \frac{cg^*(c)}{C_S})(C_L - C_S) < \overline{\theta},$$

for if this inequality is not satisfied then $\lambda^* = 0$ is a solution. This assumption implies that $\varphi(0) > 0$. Third, since $(1-\delta(1-d) + \frac{cg^*(c)}{C_L})(C_L - C_S)$ is positive, we have that $\varphi(1) < 1$. Fourth, given the properties of the public service benefit function $B(g)$, the optimal public service level $g^*(\rho)$ is a continuous function of the tax price. This implies that the function $\varphi(\lambda)$ is continuous on $[0, 1/2)$ and continuous on $[1/2, 1]$. Finally, given that $g^*(\rho)$ is decreasing in ρ, we have that

$$\begin{aligned}\lim_{\lambda \nearrow 1/2} \varphi(\lambda) &= 1 - F((1 - \delta(1-d) + \frac{2cg^*(\frac{2cC_S}{C_L - C_S})}{C_L - C_S})(C_L - C_S)) \\ &\leq 1 - F((1 - \delta(1-d) + \frac{2cg^*(\frac{2cC_L}{C_L - C_S})}{C_L - C_S})(C_L - C_S)) = \varphi(1/2)\end{aligned}$$

With this understanding of the behavior of the function $\varphi(\lambda)$, we can now prove that there must exist a solution to the equation $\lambda^* = \varphi(\lambda^*)$. Suppose there does not exist a solution on the interval $[0, 1/2)$. Then we know that since $\varphi(0) > 0$ and $\varphi(\lambda)$ is continuous on $[0, 1/2)$, it must be the case that $\varphi(\lambda) > \lambda$ for all λ on the interval $[0, 1/2)$. It follows that $1/2 < \lim_{\lambda \nearrow 1/2} \varphi(\lambda) \leq \varphi(1/2)$. But we know that $\varphi(1)$ is less than 1. Given that $\varphi(\lambda)$ is continuous on $[1/2, 1]$, there must exist a λ^* between 1/2 and 1 such that $\lambda^* = \varphi(\lambda^*)$. ■

2.1 Proof of Proposition 2

Suppose that community 1 is the zoned community. If O^* is a steady state, then, under zoning, it must be the case that $O^*_{S1} = 0$ and hence $\lambda_1(O^*) = 1$. It must also be the case that $O^*_{L2} = 0$ and hence that $\lambda_2(O^*) = 0$. To see why, suppose, to the contrary, that $O^*_{L2} > 0$. Then it must be the case that the steady state price of large houses in both communities is C_L. Since the price of small houses in community 2 is C_S, the tax price of public services is lower for large house owners in community 1. But this means public service surplus enjoyed by large house owners in community 1 is higher than in community 2 which would violate (12). Since $P_{L1}(O^*) = C_L$ and $\lambda_1(O^*) = 1$ and $P_{S2}(O^*) = C_S$ and $\lambda_2(O^*) = 0$, it follows from (9) and (10) that

$$(g_1(O^*), \tau_1(O^*)) = (g^*(c), \frac{cg^*(c)}{C_L})$$

and that

$$(g_2(O^*), \tau_2(O^*)) = (g^*(c), \frac{cg^*(c)}{C_S}).$$

Households living in community 1 pay property taxes equal to $\frac{cg^*(c)}{C_L} C_L = cg^*(c)$ and households living in community 2 pay property taxes equal to $\frac{cg^*(c)}{C_S} C_S = cg^*(c)$. From (4), it follows that a household of type θ will prefer living in a large house in community 1 to a small house in community 2 if

$$\theta + B(g^*(c)) - cg^*(c) - C_L + \delta(1-d)C_L \geq B(g^*(c)) - cg^*(c) - C_S + \delta(1-d)C_S$$

or, equivalently, if their preference θ exceeds θ^e as defined in (22). ∎

2.2 Proof of Proposition 3

Let

$$\left\langle \begin{array}{c} Z(O), P(O,Z), N(O,Z), (g_1(O,Z), g_2(O,Z)), \\ (\tau_1(O,Z), \tau_2(O,Z)), (V_\theta(O,Z), V_\theta(O), \alpha_\theta(O,Z))_\theta, \xi(\theta,O,Z) \end{array} \right\rangle$$

be an equilibrium with endogenous zoning and let O^* be a steady state. Suppose, contrary to the Proposition, that O^* is both efficient and strongly locally stable. By definition, this implies that either $Z(O^*) = (1,0)$ and O^* equals $(1 - F(\theta^e), 0, 0, F(\theta^e))$ or $Z(O^*) = (0,1)$ and O^* equals $(0, F(\theta^e), 1 - F(\theta^e), 0)$. For concreteness, assume the former, so that the large houses are in community 1. By the definition of a steady state, this implies that $N(O^*) = d(1 - F(\theta^e), 0, 0, F(\theta^e))$.

We will show that all residents of community 2 would be better off imposing zoning if the state were O^*, which is inconsistent with the fact that $Z_2(O^*) = 0$. Recall that community 2 consists of all households of type $\theta \le \theta^e$ and that at the time of voting they all own small houses. From (24), the equilibrium continuation payoff for a resident of type θ in community 2 is

$$(1-d)C_S + \mu V_\theta(O^*) + (1-\mu)\frac{y}{1-\delta}.$$

Since O^* is a steady state, we have that for all $\theta \le \theta^e$

$$V_\theta(O^*) = y + B(g^e) - cg^e - C_S + \delta[(1-d)C_S + \mu V_\theta(O^*) + (1-\mu)\frac{y}{1-\delta}],$$

which implies that

$$V_\theta(O^*) = V^* \equiv \frac{y + B(g^e) - cg^e - C_S + \delta[(1-d)C_S + (1-\mu)\frac{y}{1-\delta}]}{1-\delta\mu}.$$

Note that this continuation payoff is independent of θ.

The continuation payoff for a resident of type θ in community 2 if community 2 deviates from equilibrium play and implements zoning is

$$(1-d)P_{S2}(O^*,(1,1)) + \mu V_\theta(O^*,(1,1)) + (1-\mu)\frac{y}{1-\delta}.$$

To evaluate this, we need to know what happens following community 2's deviation to impose zoning. In the period of the deviation, the initial stock of housing will be $O_0 = O^*$, new construction will be $N_0 = N(O^*,(1,1))$, housing prices will be $P_0 = P(O^*,(1,1))$, public service levels will be $(g_{10}, g_{20}) = (g_1(O^*,(1,1)), g_2(O^*,(1,1)))$, and tax rates will be $(\tau_{10}, \tau_{20}) = (\tau_1(O^*, (1,1)), \tau_2(O^*, (1,1)))$. The housing stock at the beginning of the period following the deviation will be $O_1 = (1-d)O_0 + N_0$. Define the sequence of housing stocks $\langle O_t(O_1)\rangle_{t=1}^{\infty}$ inductively as follows: $O_1(O_1)$ equals O_1 and $O_{t+1}(O_1)$ equals $(1-d)O_t(O_1) + N(O_t(O_1))$ where $N(O_t(O_1))$ is the equilibrium level of new construction associated with housing stocks $O_t(O_1)$; that is, $N(O_t(O_1))$ equals $N(O_t(O_1), Z(O_t(O_1)))$. Then, $O_t = O_t(O_1)$ is the housing stock that will prevail at the beginning of the period t periods after the deviation. In that period, zoning rules will be $Z_t = Z(O_t)$, new construction will be $N_t = N(O_t)$, housing prices will be $P_t = P(O_t)$, public service levels will be $(g_{1t}, g_{2t}) = (g_1(O_t), g_2(O_t))$, and tax rates will be $(\tau_{1t}, \tau_{2t}) = (\tau_1(O_t), \tau_2(O_t))$.

By assumption the steady state O^* is strongly locally stable. Since $\|\mathbf{O}^* - \mathbf{O}^*\| = 0$ and $N_{L10} + N_{S10} + N_{L20} + N_{S20} = d$, this implies that $\lim_{t\to\infty} O_t = O^*$ and that $Z_t = (1,0)$ for all $t \geq 1$. With this information, we can now establish three properties of the sequence of policies that follow community 2's deviation.

Property 1: For sufficiently large t, $P_{L1t} = C_L$ and $P_{S2t} = C_S$.

Proof of Property 1: To prove this it is enough to show that for sufficiently large t, $N_{L1t} > 0$ and $N_{S2t} > 0$. But this follows from the fact that $\lim_{t\to\infty} O_t = O^*$ and that $O^* = (1 - F(\theta^e), 0, 0, F(\theta^e))$. ■

Property 2: For all $t = 0, ..., \infty$, $(\lambda_{1t}, \lambda_{2t}) = (1, 0)$, where λ_{it} is the fraction of post-construction houses that are large in community i, t periods after the deviation.

Proof of Property 2: We know that $\lambda_{10} = 1$ because $(O_{L10}, O_{S10}) = (1 - F(\theta^e), 0)$ and, since zoning is in place in community 1 in the period of deviation, we have that $N_{S10} = 0$. Moreover, by strong local stability, $Z_{1t} = 1$ for all $t \geq 1$ and thus it must be that $\lambda_{1t} = 1$ for all t.

We know that $(O_{L20}, O_{S20}) = (0, F(\theta^e))$. We also claim that for all $t \geq 0$, $N_{L2t} = 0$. To see this, suppose, to the contrary, that $N_{L2t} > 0$. Then, in order for households to want to buy these houses it must be that

$$[B(g_{2t}) - \tau_{2t}C_L] - C_L + \delta(1-d)P_{L2t+1} \geq [B(g^e) - cg^e] - P_{L1t} + \delta(1-d)P_{L1t+1}.$$

But because community 2 has small houses and community 1 does not, it must be that

$$B(g_{2t}) - \tau_{2t}C_L < B(g^e) - cg^e.$$

Thus, for the above inequality to hold, we must have that $P_{L2t+1} > P_{L1t+1}$. But we know that community arbitrage implies that

$$[B(g_{2t+1}) - \tau_{2t+1}P_{L2t+1}] - P_{L2t+1} + \delta(1-d)P_{L2t+2} = [B(g^e) - cg^e] - P_{L1t+1} + \delta(1-d)P_{L1t+2}.$$

But again because community 2 has small houses, it must be that

$$[B(g_{2t+1}) - \tau_{2t+1}P_{L2t+1}] < [B(g^e) - cg^e]\,.$$

Thus, we require $P_{L2t+2} > P_{L1t+2}$. Continuing this line of argument, we conclude that $P_{L2t} > P_{L1t}$ for all $t = 1, ..., \infty$. But we know from Property 1 that for sufficiently large t, it must be that $P_{L1t} = C_L$. It follows that for all $t \geq 0$, $\lambda_{2t} = 0$. ■

Property 3: $P_{S20} > C_S$ and for all $t \geq 1$, $P_{S2t} = C_S$.

Proof of Property 3: From Property 1 we know that for sufficiently large t it must be that $(P_{L1t}, P_{S2t}) = (C_L, C_S)$. Let $\widehat{t}$ be the largest period in which $(P_{L1t}, P_{S2t}) \neq (C_L, C_S)$. Suppose first that $\widehat{t} = 0$. Then all we need to show is that $P_{S20} > C_S$. We know that $O_0 = (1 - F(\theta^e), 0, 0, F(\theta^e))$. From Property 2, we know that $N_0 = (d, 0, 0, 0)$ and hence that $P_{L10} = C_L$. We also know that $P_{L11} = C_L$ and that $P_{S21} = C_S$. Suppose to the contrary that $P_{S20} \leq C_S$. Then, it must be that all types with preferences less than θ^e strictly prefer small houses in community 2 implying that demand is at least equal to $F(\theta^e)$. Supply, however, is equal to $(1-d)F(\theta^e)$.

Now suppose that $\widehat{t} \geq 1$. Since $Z_{\widehat{t}} = (1, 0)$, there are two possibilities in period $\widehat{t}$: (i) $P_{L1\widehat{t}} = C_L$, $P_{S2\widehat{t}} < C_S$, and $N_{L1\widehat{t}} = d$, and (ii) $P_{L1\widehat{t}} < C_L$, $P_{S2\widehat{t}} = C_S$, and $N_{S2\widehat{t}} = d$. We now show that possibility (i) cannot arise. Suppose, to the contrary, that it does arise. Then, given that $(P_{L1\widehat{t}+1}, P_{S2\widehat{t}+1}) = (C_L, C_S)$, we know that it must be that $(1-d)O_{L1\widehat{t}} + d \leq 1 - F(\theta^e)$ and that $(1-d)O_{S2\widehat{t}} \geq F(\theta^e)$. This is because all households with types θ less than θ^e will strictly prefer to purchase a small house in period $\widehat{t}$ at these prices. Thus, we must have that $O_{S2\widehat{t}} \geq F(\theta^e)/(1-d)$ in order for the housing market to clear. Now consider period $\widehat{t} - 1$. Suppose that $(1-d)O_{S2\widehat{t}-1} < F(\theta^e)$. Then, since $O_{S2\widehat{t}} \geq F(\theta^e)/(1-d)$, there must be new construction of small houses in community 2 in period $\widehat{t} - 1$. In that case, $P_{S2\widehat{t}-1} = C_S$, but since the price of small houses falls in period $\widehat{t}$, no households with types greater than θ^e will want to purchase a small house in community 2. Accordingly, we have that $(1-d)O_{S2\widehat{t}-1} + N_{S2\widehat{t}-1} \leq F(\theta^e)$. But then we have that

$$O_{S2\widehat{t}} = (1-d)O_{S2\widehat{t}-1} + N_{S2\widehat{t}-1} < F(\theta^e)/(1-d),$$

which is a contradiction. Thus, $O_{S2\widehat{t}-1} \geq F(\theta^e)/(1-d)$. This in turn implies that $P_{L1\widehat{t}-1} = C_L$, $P_{S2\widehat{t}-1} < C_S$, and that $N_{L1\widehat{t}-1} = d$. Again, there can be no new construction of small houses, because all households of type greater than θ^e will want large houses. Continuing this line of argument, we conclude that for all $t = 1, ..., \widehat{t}$, we must have that $O_{S2t} \geq F(\theta^e)/(1-d)$. But since $O_{S20} = F(\theta^e)$ and $N_{S20} = 0$, we have that

$$O_{S21} = (1-d)O_{S20} + N_{S20} = (1-d)F(\theta^e) < F(\theta^e)/(1-d)$$

which is a contradiction. We conclude therefore that it cannot be that $P_{L1\widehat{t}} = C_L$, $P_{S2\widehat{t}} < C_S$, and $N_{L1\widehat{t}} = d$.

We have therefore established that in period $\widehat{t}$, $P_{L1\widehat{t}} < C_L$, $P_{S2\widehat{t}} = C_S$, and $N_{S2\widehat{t}} = d$. If $\widehat{t} \geq 2$, consider period $\widehat{t}-1$. Again, there are two possibilities: (i) $P_{L1\widehat{t}-1} = C_L$, $P_{S2\widehat{t}-1} < C_S$, and $N_{L1\widehat{t}-1} = d$, and (ii) $P_{L1\widehat{t}-1} < C_L$, $P_{S2\widehat{t}-1} = C_S$, and $N_{S2\widehat{t}-1} = d$. Using similar logic, we can again show that possibility (i) cannot arise.

Continuing on in this way, we conclude that for all $t = 1, ..., \widehat{t}$, we have that $P_{L1\widehat{t}} < C_L$, $P_{S2\widehat{t}} = C_S$, and $N_{S2\widehat{t}} = d$. Now consider period 0, the period the deviation becomes effective. We know that $O_0 = (1 - F(\theta^e), 0, 0, F(\theta^e))$, that $N_0 = (d, 0, 0, 0)$ and that $P_{L10} = C_L$. We also know that $P_{L11} \leq C_L$ and that $P_{S21} = C_S$. We now argue that $P_{S20} > C_S$. Suppose to the contrary that $P_{S20} \leq C_S$. Then, it must be the case that in period 0 all types with preferences less than θ^e strictly prefer small houses in community 2 implying that demand is at least equal to $F(\theta^e)$. Supply, however, is equal to $(1-d)F(\theta^e)$. ■

We can now complete the proof of the Proposition. Consider the payoff of a household of type 0 under the deviation. As the household with the lowest preference for large houses, this household can expect to remain in small houses in community 2 for as long as it remains in the area. Thus, given that for all $t \geq 1$, $P_{S2t} = C_S$ and $\lambda_{2t} = 0$, we have that

$$V_0(O^*, (1,1)) = y + B(g^e) - cg^e - P_{S20} + \delta[(1-d)C_S + \mu V^* + (1-\mu)\frac{y}{1-\delta}].$$

This household will favor imposing zoning if

$$(1-d)P_{S20} + \mu V_0(O^*, (1,1)) + (1-\mu)\frac{y}{1-\delta} - \left[(1-d)C_S + \mu V^* + (1-\mu)\frac{y}{1-\delta}\right] > 0.$$

This difference equals

$$(1-\mu-d)\left[P_{S20} - C_S\right],$$

which is positive given that $1 - \mu > d$. It follows that households of type 0 are in favor of imposing zoning.

Now consider households of type $\theta \in (0, \theta^e]$. As noted, the continuation payoff for these residents if zoning is not implemented is exactly the same as for a type 0 household. On the other hand, since a type θ household can always make the same choices as a type 0 household, it must be the case that $V_\theta(O^*, (1,1)) \geq V_0(O^*, (1,1))$. It therefore follows that

$$\begin{aligned}
& (1-d)P_{S20} + \mu V_\theta(O^*, (1,1)) + (1-\mu)\frac{y}{1-\delta} - \left[(1-d)C_S + \mu V^* + (1-\mu)\frac{y}{1-\delta}\right] \\
\geq \;& (1-\mu-d)\left[P_{S20} - C_S\right] > 0.
\end{aligned}$$

Thus, households of type $\theta \in (0, \theta^e]$ also favor imposing zoning. ■

2.3 Extension of Proposition 3 to three communities

Note first there is a sense in which Proposition 3 is trivially true when we have three communities. Take an efficient steady state in which, say, communities 2 and 3 are unzoned and the small houses are allocated in community 2 so that $O^* = (1-F(\theta^e), 0, 0, F(\theta^e), 0, 0)$ and $Z(O^*) = (1,0,0)$. Consider for ε small and positive the stock $O = (1-F(\theta^e), 0, 0, F(\theta^e) - \varepsilon, \varepsilon, 0)$. Thus, compared with O^*, O features a small number of large houses in community 3. Define $\langle O_t(O)\rangle_{t=0}^{\infty}$ in the usual way and assume that $Z(O_t(O)) = Z(O^*)$ for all t. Then it cannot be the case that $\lim_{t\to\infty} O_t(O) = O^*$. This is because, by virtue of having a small fraction of large houses, community 3 is now a more attractive place to build new small houses than community 2. This means the fraction of small houses in community 3 will grow relative to that in community 2 and $\lim_{t\to\infty} O_t(O) = (1-F(\theta^e), 0, 0, 0, 0, F(\theta^e))$. It follows that O^* is not strongly locally stable in the sense defined earlier.

Nonetheless, from an efficiency perspective there is nothing troubling about this example. Whether in the long run all the small housing is located in community 2, community 3, or both, is immaterial. The difficulty is that the division of construction across the two unzoned communities is arbitrary and small differences between the two communities will force it in one or the other direction. To reflect this, we modify our definition of strong local stability. For a given equilibrium with endogenous zoning and any given zoning rules Z, let $\Phi(Z)$ be the set of housing stocks O that are steady states and have the property that $Z(O) = Z$. Then, we say that a steady state O^* is strongly locally stable if there exists $\varepsilon > 0$ such that for any initial stock O with the property that $\|O - O^*\| < \varepsilon$, the sequence of housing stocks $\langle O_t((1-d)O + N)\rangle_{t=0}^{\infty}$ converges to some steady state in $\Phi(Z(O^*))$ for any arbitrary vector of new construction N such that $N_{L1} + N_{S1} + N_{L2} + N_{S2} = d$. Moreover, the associated zoning rules $\langle Z(O_t((1-d)O + N))\rangle_{t=0}^{\infty}$ equal $Z(O^*)$ in all periods. This definition reduces to our earlier one in the case in which there is a unique steady state associated with a particular set of zoning rules.

We now demonstrate that Proposition 3 holds with this new more general definition of strong local stability. Suppose, to the contrary, that there exists a political equilibrium with endogenous zoning with an equilibrium steady state O^* which is both efficient and strongly locally stable. It must be the case that $Z(O^*)$ involves at least one community zoning and one community not zoning. Without loss of generality, assume that $Z_1(O^*) = 1$ and $Z_3(O^*) = 0$. If $Z_2(O^*) = 1$, then community 3 is a monopoly supplier of small homes and we can use the the logic from Proposition 3 to obtain a contradiction. Thus, assume that $Z_2(O^*) = 0$. This implies that $O^* = (1 - F(\theta^e), 0, 0, F(\theta^e) - x, 0, x)$ for some $x \in [0, F(\theta^e)]$.

Now consider for ε small and positive the stock

$$O(\varepsilon) = (1 - F(\theta^e), 0, \varepsilon, F(\theta^e) - x - \varepsilon, 0, x).$$

This differs from the steady state in that community 2 has a small fraction of large homes. We will show that for sufficiently small ε, $Z_2(O(\varepsilon)) \neq 0$ which will contradict the assumption O^* is strongly locally stable. To see this recall that if O^* is strongly locally stable it must be the case that for sufficiently small $\varepsilon > 0$ we have that for any arbitrary vector of new construction N such that $N_{L1} + N_{S1} + N_{L2} + N_{S2} = d$, $\langle O_t((1-d)O(\varepsilon) + N)\rangle_{t=0}^{\infty}$ converges to some steady state in $\Phi((1,0,0))$ and the associated zoning rules $\langle Z(O_t((1-d)O(\varepsilon) + N))\rangle_{t=0}^{\infty}$ equal $(1, 0, 0)$ in all periods. In particular, this must be true for $N = dO(\varepsilon)$, which implies that $Z(O(\varepsilon))$ must equal $(1, 0, 0)$.

Suppose first that the residents of community 2 follow the postulated equilibrium and do not impose zoning. As the efficient steady state is strongly locally stable, the future play of the equilibrium will have community 1 implementing zoning and communities 2 and 3 not. All new construction of small homes will occur in community 2 and the price of new small homes will be less than C_S in community 3. All new construction of large homes will occur in community 1 and the price of large homes in community 2 will be less than C_L. Let λ_{2t}^0 be the fraction of large homes (post-construction) in community 2 after $t = 0,....,\infty$ periods and let P_{L2t}^0 be the price of such houses.

Now suppose that the residents of community 2 were to deviate from the postulated equilibrium behavior by imposing zoning. By strong local stability, in all subsequent periods, households anticipate that zoning rules will return to steady state levels. In the period of the deviation, new construction of small homes will occur in community 3. However, the price of small homes in community 2 must be higher than those in community 3 because of the beneficial fiscal externality created by the presence of large homes. Let P_{S20}^1 be the price of small homes in community 2 in the period of the deviation. The value of large homes in community 2 will also be higher than on the equilibrium path as will the fraction of large homes. Let P_{L20}^1 be the price of large homes in community 2 and λ_{20}^1 the fraction. Following the period of deviation, there will be a lower fraction of small homes in community 2 which will increase the price of large homes relative to the equilibrium. Let λ^1_{2t} denote the fraction (post-construction) after t periods and let P^1_{L2t} denote the price. The price of small homes in community 2 will return to C_S and the price of small homes in community 3 will be less than the construction cost.

Now consider the incentives to deviate for the types who own small homes in community 2. The payoff on the equilibrium path for a type θ who owns a small house in community 2 and will continue to live in a small house is

$$(1-d)C_S + \mu V^0_0 + (1-\mu)\frac{y}{1-\delta}.$$

The continuation value V^0_0 is the first element of the sequence $\langle V^0_t \rangle^\infty_{t=0}$ defined inductively by

$$\begin{aligned} V^0_t &= B(g^*(\frac{cC_S}{\lambda^0_{2t}P^0_{L2t} + (1-\lambda^0_{2t})C_S})) - \frac{cC_S}{\lambda^0_{2t}P^0_{L2t} + (1-\lambda^0_{2t})C_S} g^*(\frac{cC_S}{\lambda^0_{2t}P^0_{L2t} + (1-\lambda^0_{2t})C_S}) - C_S \\ &\quad + \delta[(1-d)C_S + \mu V^0_{t+1} + (1-\mu)\frac{y}{1-\delta}] \end{aligned}$$

with end point condition

$$V^0_\infty = \frac{B(g^*(c)) - cg^*(c) - C_S + \delta[(1-d)C_S + (1-\mu)\frac{y}{1-\delta}]}{1-\delta\mu}.$$

The payoff under the deviation for a type θ who owns a small house in community 2 and will continue to live in a small house is

$$(1-d)P^1_{S20} + \mu V^1_0 + (1-\mu)\frac{y}{1-\delta},$$

where

$$\begin{aligned} V^1_0 &= B(g^*(\frac{cP^1_{S20}}{\lambda^1_{20}P^1_{L20} + (1-\lambda^1_{20})P^1_{S20}})) - \frac{cP^1_{S20}}{\lambda^1_{20}P^1_{L20} + (1-\lambda^1_{20})P^1_{S20}} g^*(\frac{cP^1_{S20}}{\lambda^1_{20}P^1_{L20} + (1-\lambda^1_{20})P^1_{S20}}) - P^1_{S20} \\ &\quad + \delta[(1-d)C_S + \mu V^1_1 + (1-\mu)\frac{y}{1-\delta}]. \end{aligned}$$

The continuation value V^1_1 is the first element of the sequence $\langle V^1_t \rangle^\infty_{t=1}$ defined inductively by

$$V_t^1 = B(g^*(\frac{cC_S}{\lambda_{2t}^1 P_{L2t}^1 + (1-\lambda_{2t}^1)C_S})) - \frac{cC_S}{\lambda_{2t}^1 P_{L2t}^1 + (1-\lambda_{2t}^1)C_S} g^*(\frac{cC_S}{\lambda_{2t}^1 P_{L2t}^1 + (1-\lambda_{2t}^1)C_S}) - C_S$$
$$+\delta[(1-d)C_S + \mu V_{t+1}^1 + (1-\mu)\frac{y}{1-\delta}]$$

with end point condition

$$V_\infty^1 = \frac{B(g^*(c)) - cg^*(c) - C_S + \delta[(1-d)C_S + (1-\mu)\frac{y}{1-\delta}]}{1-\delta\mu}.$$

The gain from deviating is

$$\Delta = (1-d)[P_{S20}^1 - C_S] + \mu\left[V_0^1 - V_0^0\right]$$

But we have that

$$V_0^1 - V_0^0 = S_0^1 - P_{S20}^1 + \delta\mu V_1^1 - \left[S_0^0 - C_S + \delta\mu V_1^0\right]$$

where S_t^0 and S_t^1 denote public service surplus on the equilibrium path and with the deviation. Thus, we have that

$$\Delta = (1-\mu-d)[P_{S20}^1 - C_S] + \mu\left[S_0^1 - S_0^0\right] + \delta\mu^2\left[V_1^1 - V_1^0\right]. \tag{31}$$

We now claim that

$$\mu\left[S_0^1 - S_0^0\right] \geq -\mu c g^*(\frac{cC_S}{\lambda_{20}^0 P_{L20}^0 + (1-\lambda_{20}^0)C_S})\left(\frac{\left[P_{S20}^1 - C_S\right]\lambda_{20}^1 C_L}{C_S^2}\right). \tag{32}$$

To prove this, note that

$$g^*(\frac{cP_{S20}^1}{\lambda_{20}^1 P_{L20}^1 + (1-\lambda_{20}^1)P_{S20}^1}) = \arg\max_g\left\{B(g) - \frac{cP_{S20}^1}{\lambda_{20}^1 P_{L20}^1 + (1-\lambda_{20}^1)P_{S20}^1} g\right\}$$

and so

$$S_0^1 \geq B(g^*(\frac{cC_S}{\lambda_{20}^0 P_{L20}^0 + (1-\lambda_{20}^0)C_S})) - \frac{cP_{S20}^1}{\lambda_{20}^1 P_{L20}^1 + (1-\lambda_{20}^1)P_{S20}^1} g^*(\frac{cC_S}{\lambda_{20}^0 P_{L20}^0 + (1-\lambda_{20}^0)C_S}).$$

It follows that

$$\mu\left[S_0^1 - S_0^0\right] \geq \mu c g^*(\frac{cC_S}{\lambda_{20}^0 P_{L20}^0 + (1-\lambda_{20}^0)C_S})\left[\frac{cC_S}{\lambda_{20}^0 P_{L20}^0 + (1-\lambda_{20}^0)C_S} - \frac{cP_{S20}^1}{\lambda_{20}^1 P_{L20}^1 + (1-\lambda_{20}^1)P_{S20}^1}\right]$$

Moreover, since $P^1_{L20} \geq P^0_{L20}$, $P^1_{L20} \geq C_S$ and $\lambda^1_{20} > \lambda^0_{20}$, we have that

$$\begin{aligned} & \mu c g^*(\cdot)\left[\frac{C_S}{\lambda^0_{20}P^0_{L20}+(1-\lambda^0_{20})C_S}-\frac{P^1_{S20}}{\lambda^1_{20}P^1_{L20}+(1-\lambda^1_{20})P^1_{S20}}\right] \\ \geq\; & \mu c g^*(\cdot)\left[\frac{C_S}{\lambda^1_{20}P^1_{L20}+(1-\lambda^1_{20})C_S}-\frac{P^1_{S20}}{\lambda^1_{20}P^1_{L20}+(1-\lambda^1_{20})P^1_{S20}}\right] \end{aligned}$$

In addition,

$$\mu c g^*(\cdot)\left[\frac{C_S}{\lambda^1_{20}P^1_{L20}+(1-\lambda^1_{20})C_S}-\frac{P^1_{S20}}{\lambda^1_{20}P^1_{L20}+(1-\lambda^1_{20})P^1_{S20}}\right] \geq -\mu c g^*(\cdot)\left[\frac{(P^1_{S20}-C_S)\,\lambda^1_{20}P^1_{L20}}{C_S^2}\right].$$

Combining (31) and (32), we have

$$\Delta \geq (1-\mu-d)[P^1_{S20}-C_S] - \mu c g^*(\cdot)\left[\frac{(P^1_{S20}-C_S)\,\lambda^1_{20}P^1_{L20}}{C_S^2}\right] + \delta\mu^2\left[V^1_1 - V^0_1\right]. \tag{33}$$

We also claim that $V^1_1 \geq V^0_1$. For this, it is enough to show that for all $t = 1, ..., \infty$, $S^1_t \geq S^0_t$. We have that

$$S^1_t \geq B(g^*(\frac{cC_S}{\lambda^0_{2t}P^0_{L2t}+(1-\lambda^0_{2t})C_S})) - \frac{cC_S}{\lambda^1_{2t}P^1_{L2t}+(1-\lambda^1_{2t})C_S} g^*(\frac{cC_S}{\lambda^0_{2t}P^0_{L2t}+(1-\lambda^0_{2t})C_S}).$$

Thus,

$$S^1_t - S^0_t \geq \left[\frac{1}{\lambda^0_{2t}P^0_{L2t}+(1-\lambda^0_{2t})C_S}-\frac{1}{\lambda^1_{2t}P^1_{L2t}+(1-\lambda^1_{2t})C_S}\right] cC_S g^*(\frac{cC_S}{\lambda^0_{2t}P^0_{L2t}+(1-\lambda^0_{2t})C_S})$$

But we know that $\lambda^1_{2t} \geq \lambda^0_{2t}$, $P^1_{L2t} \geq P^0_{L2t}$ and $P^0_{L2t} \geq C_S$. Thus, we have that

$$\frac{1}{\lambda^0_{2t}P^0_{L2t}+(1-\lambda^0_{2t})C_S} \geq \frac{1}{\lambda^1_{2t}P^1_{L2t}+(1-\lambda^1_{2t})C_S},$$

which implies the result.

It now follows from (33) that

$$\Delta \geq [P^1_{S20}-C_S]\left((1-\mu-d) - \mu c g^*(\cdot)\left[\frac{\lambda^1_{20}P^1_{L20}}{C_S^2}\right]\right)$$

Since $\lambda^1_{20} \leq \varepsilon / \left(F(\theta^e) - x\right)$, this must be positive for sufficiently small ε given that $1-\mu > d$.
■

3 Computing equilibrium

3.1 Equilibrium with no zoning

To compute the equilibrium with no zoning, we rely on a "guess and verify" approach. For any initial stock, we first conjecture how the housing stock will evolve and compute the limiting housing prices. Working backwards, we then use the market equilibrium conditions developed in Section IIIB to construct the implied sequence of housing prices. Knowing these and the evolution of the housing stock, we then recover taxes and public service levels. The exact details of the procedure depend upon the particular initial stock we are working with. To illustrate, we explain the procedure for the two scenarios illustrated in Figures 1 and 2.

Scenario 1 In this scenario, the two communities begin with the same housing stocks and, in our conjectured equilibrium, they remain symmetric. The evolution of the housing stock is as follows. Starting from any symmetric allocation, housing stocks will jump to the steady state, if such a jump is feasible given the constraint that new construction must sum to d. If this is not feasible, there will be construction of large homes if the share of large homes is below the steady state level and construction of small homes otherwise. Given this law of motion, the housing stock arrives at the steady state in a finite number of periods. The exact number depends upon the fraction of large houses in the initial state. The housing prices in the "jump" period in which the stock first reaches the steady state will equal their respective construction costs.

Given this knowlege of how the housing stock will evolve and the limiting prices, the prices in the preceding periods are constructed by backwards induction. Given the symmetry of the allocation, the prices will be equal across comunities and hence we just have to solve for two sequences $\langle P_{St} \rangle_{t=1}^{T}$ and $\langle P_{Lt} \rangle_{t=1}^{T}$ where T denotes the period in which the stock arrives at the steady state. For each period, from our knowledge of the evolution of the housing stock, we can compute θ_c from equation (15). Moreover, one of the current prices is known since P_{St} is equal to C_S if small homes are being built, and P_{Lt} is equal to C_L otherwise. Equation (14) gives us an equation in the unknown price because public service levels and tax prices can be expressed as functions of prices (and the housing stock) using (7), (8), and (9). Using this equation, we can solve for the sequence of unknown prices. We do this, using a backward shooting algorithm. This requires solving iteratively (T times) one equation in one unknown. Once we have done this, we can recover the public service levels and taxes from equations (7), (8), and (9).

Scenario 2 In this scenario, one community (community 1) has a larger fraction of large homes than the other, but with both still less than the steady state. In our conjectured equilibrium, at first all new construction is in the form of large homes and occurs in community 1. After a finite number of periods, small home construction begins and also occurs in community 1. Eventually, community 2 will become so small that one period's new construction of large homes will be sufficient to equate the fraction of large homes in the two communities. At this point, the housing stock jumps to the steady state.

The precise timing of this evolution is constructed using a backwards shooting algorithm. First, the period in which the stock jumps to the steady state is conjectured. Expression (14) is used to track backwards the evolution of the stock under the assumption that both types of homes are built in community 1 and hence prices in that community are equal to construction costs. Going backwards, when the housing stock arrives to a point at which the size of community 1 and the number of large homes are such that it could be reached from the initial state in a finite number of periods by building only large homes, the construction of small homes ceases. From then on, again going backwards, all construction is of large homes, and hence the evolution of the stock is known. Equation (14) is used to construct the prices for small homes in community 1 during these periods.

This procedure reveals the evolution of the stock and the price path in community 1. From this, we can compute the public service levels and taxes for community 1. In community 2, there is no new construction until the steady state is reached, so the prices of both home types need to be determined. We employ equation (12) for both home types. Given period $t + 1$ home prices in community 2, these two equalities provide two equations in the two unknown period t prices because period t public service levels and tax prices can be expressed as functions of period t home prices in community 2. Hence community 2's prices can be obtained by solving iteratively these two equations in two unknowns. Again, once we have community 2's prices, we also have the public service levels and taxes for community 2 from equations (7), (8), and (9).

3.2 Equilibrium with exogenous zoning

A different procedure is required to compute the equilibrium with exogenous zoning. This is because guessing the precise evolution of the housing stock is too difficult. While we know that new construction will involve only large homes in community 1 and small homes in community 2, we do not know the precise mix.

The first step in computing the equilibrium is to find the limiting housing prices. We know that the zoned community (community 1) converges asymptotically to one with large homes only and size $1-\theta^e$, and the unzoned community (community 2) to one with small homes only and size θ^e. The limiting prices of large homes in community 1 and of small homes in community 2 are just the construction costs. Since we know θ^c (it equals θ^e), the prices of small homes in community 1 and large homes in community 2 can be obtained from (14).

The second step is to compute the evolution of the housing stock. For this problem, a backwards shooting algorithm cannot be implemented as the stock in period $t+1$ does not necessarily pin down the stock in period t. Instead, we use a variant of a forward shooting algorithm, as described below:

- We set the number of periods for convergence to the steady state to occur. Given the slow decay of the housing stock, a large number of periods are required. The number we use is 400. With this number of periods, the difference between the community sizes in the final period and their steady state values is less than 10^{-4}.
- Since there are only two types of homes that are built, the evolution of the stock can be described by the sequence of new large homes built in community 1 $\langle N_{L1t}\rangle_{t=1}^{400}$. The associated small home construction is given by $N_{S2t} = d - N_{L2t}$.
- We assume first that all N_{L1t} are interior: that is $0 < N_{L1t} < d$, for all t. This implies there is always construction of both home types. This assumption ties down the prices of large homes in community 1 and of small homes in community 2. Given the limit prices, the remaining prices can be computed using equation (14) by backwards induction starting in period 400 in which, by assumption, the stock has converged to the steady state. We then note that condition (12) does not hold for any such sequence – it holds only for the equilibrium one. This suggests the following iterative updating.

  1. Make the initial guess for $\langle N_{L1t}\rangle_{t=1}^{400}$ and compute the associated housing prices. (Our guess is that each element in the sequence is equal to its long run level, $(1-\theta^e)d$).
  2. Start updating the guess in period $t = 1$. Taking period 2 prices from step 1, solve the following three equations for the three unknowns $(N_{L1t}, P_{S1t}, P_{L2t})$: equation (14) written for both communities and expression (12) written for large (or small) homes.
  3. Update the guess for N_{L1t} and (P_{S1t}, P_{L2t}).
  4. Go to the next period and repeat steps 2 and 3. Keep going until the period in which by assumption the steady state is reached (i.e., period 400). The new guess for the entire path for $\langle N_{L1t}\rangle_{t=1}^{400}$ and prices is now constructed.

5. Keep repeating the steps of the above procedure until the maximum difference between the elements in the "old" and "new" construction sequences is less than some specified tolerance (10^{-5} in our case).

- The algorithm above converges, but violates the assumption that N_{L1t} is interior in the first four periods. The algorithm assigns it a value of d, which suggests that in equilibrium in these periods there is no construction of small homes in community 2. Hence we modify the algorithm – in these periods, instead of searching for $(N_{L1t}, P_{S1t}, P_{L2t})$ we postulate that N_{L1t} is equal to d and use the expressions (14) written for both communities and expression (12) to compute the triplet of prices $(P_{S1t}, P_{S2t}, P_{L2t})$. With this modification the algorithm converges.

The iterative procedure just described for computing the evolution of the housing stock simultaneously determines the sequence of housing prices. It only remains to recover the public service levels and taxes for the two communities which we do using equations (7), (8), and (9).

3.3 Equilibrium with endogenous zoning

In our always-zone equilibrium, both communities impose zoning except when a community is empty. New construction occurs in the community which has the largest share of large homes. This is determined by the initial stock, since the community with the highest initial stock of large homes will remain the community with the largest fraction for eternity. By relabeling as necessary, we can call the community with the highest initial share community 1. If the initial stock is such that the two communities have an equal fraction of large homes, then all new construction occurs in community 1.

In this equilibrium, the evolution of the housing stock is straightforward: all new construction is in large homes and it occurs in community 1. This community grows in size and the fraction of its homes that are large converges to one. Community 2 converges in size to zero and the fraction of its homes that are large remains constant.

With the evolution of the housing stock determined, the next step is to compute the associated housing prices. The first point to note is that since the distribution of preference types has full support on $[0, 1]$ there is a positive measure of households who are almost indifferent between large and small homes. Since the fraction of large homes converges to one in the limit, the limiting price of all homes will equal the replacement cost of large homes C_L. Knowing this, and the evolution of the housing stock, enables us to compute home prices in both communities. Given the asymptotic convergence to the limit allocation, a large number of periods is required. As in the case with exogenous zoning, we set this number equal to 400.

In community 1, the price of large homes is always C_L since this is where new construction takes place. From our knowledge of the evolution of the housing stock, we can compute θ_c from equation (15). Using this together with (7), (8), and (9), we can solve for the sequence of small home prices in community 1 $\langle P_{S1t} \rangle_{t=1}^{400}$ from equation (14). As in the no zoning case, we do this using a backward shooting algorithm. This requires solving iteratively (400 times) one equation in one unknown. Once we have done this, we can recover the public service levels and taxes for community 1 from equations (7), (8), and (9).

In community 2, there is no new construction, so the prices of both home types need to be determined. Thus, we have to solve for both $\langle P_{L2t} \rangle_{t=1}^{400}$ and $\langle P_{S2t} \rangle_{t=1}^{400}$. We employ equation (12) for both home types. Given period $t+1$ home prices in community 2, these two equalities provide two equations in the two unknown period t prices because period t public service levels and tax prices can be expressed as functions of period t home prices in community 2. Hence community 2's prices can be obtained by solving iteratively these two equations in two unknowns. Again, once we have community 2's prices, we also have the public service levels and taxes for community 2 from equations (7), (8), and (9).

The main additional complication associated with computing this equilibrium is checking that the majority of residents of each community support zoning. To do this, the first step is to calculate each household type's equilibrium value function. With information about the entire sequence of housing prices, public service levels and taxes, we can compute households' equilibrium value functions from equation (4). Again, backward induction is employed. In the limiting allocation, the value functions for any household type θ with a given type of home in either community is straightforward to compute. Going backward, for each θ we first compute its (within) period housing decision and based on that decision we compute the (beginning of the period) value from owning a home of given type in a given community.

The second step is to checking that for all housing stocks equation (25) is satisfied for a majority of the residents living in each community given that the other community is choosing zoning and given that both communities will play according to the equilibrium in the future. Since it is infeasible to check the condition for every conceivable housing stock, it is necessary to limit the housing stocks we consider. Given that the housing stock $(O_{L1}, O_{S1}, O_{L2}, O_{S2})$ must have aggregate size 1, we can represent any stock as a triplet: the share of large homes in community 1, the share of large homes in community 2, and the size of community 1. We consider a grid of housing stocks in the three dimensional simplex with 13824 (24^3) points. The grid is uniform on $[0,1]^3$, except any 0 is substituted by 0.01 and any 1 is substituted by 0.99.

In order to check equation (25), we need to understand what would happen if one community deviated from the equilibrium by removing zoning. The key task is to solve for prices and new construction in the period of the deviation. This new construction determines the housing stock in the following period. From then on we know what will happen because play follows the equilibrium and the steps just described allow us to solve for the equilibrium path for any initial state. The housing market equilibrium in the period of the deviation is computed by searching for an allocation such that equations (12) and (14) hold when evaluated at the future prices implied by the housing stock generated by this construction allocation.

With information about the entire sequence of housing prices, public service levels and taxes following a deviation, we can compute households' payoffs from the deviation and check whether equation (25) is indeed satisfied for a majority of the residents living in each community. In order to know which households are voting in which district, the equilibrium needs to specify how household types are allocated across the two communities. This is done as follows. Suppose the stock is $(O_{L1}, O_{S1}, O_{L2}, O_{S2})$ and recall that households types are uniformly distributed on $[0, 1]$. Then, in equilibrium all households with types θ less than $O_{S1} + O_{S2}$ are living in a small home and all those with θ above $O_{S1} + O_{S2}$ in a large home. Our equilibrium assumes that for any household type θ less than $O_{S1} + O_{S2}$, a fraction $O_{S1}/(O_{S1} + O_{S2})$ of this type live in community 1 and a fraction $O_{S2}/(O_{S1} + O_{S2})$ live in community 2. Similarly, for any household type θ exceeding $O_{S1} + O_{S2}$, a fraction $O_{L1}/(O_{L1} + O_{L2})$ of this type live in community 1 and a fraction $O_{L2}/(O_{L1} + O_{L2})$ live in community 2. As an example, suppose that $(O_{L1}, O_{S1}, O_{L2}, O_{S2}) = (1/3, 1/6, 1/6, 1/3)$. Then $1/3$ of each type θ less than $1/2$ live in community 1 and $2/3$ live in community 2. Similarly, $2/3$ of each type θ greater than $1/2$ live in community 1 and $1/3$ live in community 2. To aggregate preferences in a given community we need to integrate over the distribution of citizens by their θ type and home ownership. We discretize the distribution of θ using an equi-spaced 51 point grid on [0,1].