The Limits of Multi-Product Price Discrimination

Online Appendix

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A Online Appendix

A.1 A Mussa-Rosen Setting

In this section we show that our main results extend to a setting with linear valuations and non-linear costs, as in Mussa and Rosen (1978).

The set of products is $A \subseteq R^+$, and A includes the outside option 0. Consumers' types are $T = \{1, \ldots, n\}$, and their values are linear, $v^i_a = v^i \cdot a$ for $v^i \in R^+$, where $v^1 < \ldots < v^n$. Let $v^0 = 0$. Producing product a costs the seller $C(a) \in R^+$, where $C(0) = 0$. Each type i has a unique efficient product $\bar{a}(i)$, which maximizes $va - C(a)$ across products a, and different types can have different efficient products. Notice that a higher type has a weakly higher efficient product. The seller's optimal mechanism (x, p) for market f is an IC and IR mechanism that maximizes the expected profit $E_{i\sim f, a\sim x(i)}[p(i) - C(a)]$ over all IC and IR mechanisms.

A mechanism (x, p) is a non-screening mechanism for a set of types $T' = \{i_1, \ldots, i_m\}$, $i_1 < \ldots < i_m$, if it sells to each of these types his efficient product at the highest possible price. That is, the allocation of type i_j is efficient and this type pays

$$p(i_j) = \sum_{j'=1}^{j} v^{i_{j'}} (\bar{a}(i_{j'}) - \bar{a}(i_{j'-1})),$$

where $i_0 = 0$. Types $i' < i_1$ are excluded and types i', $i_j \leq i' < i_{j+1}$ obtain the allocation and payment of type i_j (where we define $i_{m+1} = n + 1$). We denote this mechanism by $N^{T'}$. In the special case where all types have the same efficient alternative, mechanism $N^{T'}$ sells that efficient alternative at the value of type i_1 to all types i_1 and higher, generalizing our formulation in the main body of the paper.

A market f is efficient if $N^{supp(f)}$ is optimal for f, where $supp(f)$ is the support of f. A market f is a non-screening market if $N^{T'}$ is optimal for f for some T', that is, any product that is sold is efficient for some type that buys that product. A market is a screening market if it is not a non-screening market. Any optimal mechanism for a screening market sells some product that is not efficient for any type that buys that product. If all types have the same efficient product, any optimal mechanism of a screening mechanism sells an inefficient product (other than 0) to some type. These definitions coincide with those in the main body of the paper when all types have the same efficient product. All other notions are defined as before.

In the standard Mussa-Rosen setting, the set of products is $A = R^+$ and the cost function is assumed to be twice continuously differentiable, increasing, and strictly convex. In such a setting, each market f has a unique optimal mechanism M (the allocation of each type is the unique product that maximizes the virtual surplus). Consider some type i with an inefficient allocation in M. By Lemma 3, for the first-best consumer surplus to be achievable, mechanism M must be optimal for any segment f'. But some such segment must contain i, and because the optimal mechanism for the segment is unique and M is optimal for the segment, the segment is inefficient. This means that first-best consumer surplus and the surplus triangle are unachievable for any inefficient market f.

With a finite set A of products, several mechanisms may be optimal for a market. We turn to this case and assume that $C(a) > 0$ for all $a \neq 0$ but we do not assume that the cost function is increasing nor that it is convex.

We first consider the case of two types, 1 and 2. We show that, similarly to Proposition 1 and Proposition 2, first-best consumer surplus and the surplus triangle are achievable for an inefficient market if and only if all markets are non-screening markets. We start with the following lemma, which is similar to Lemma 4. We identify each market by its fraction $q \in [0, 1]$ of type 2 consumers. Recall that $\mathcal{F}(M)$ denotes the sets of markets for which mechanism M is optimal.

Lemma A.1 *There exist q_1 and q_2, $0 \leq q_1 \leq q_2 \leq 1$, such that $\mathcal{F}(N^{\{1,2\}}) = [0, q_1], \mathcal{F}(N^{\{2\}}) = [q_2, 1]$, and $\mathcal{F}(M) \subseteq [q_1, q_2]$ for any mechanism $M \neq N^{\{1,2\}}, N^{\{2\}}$.*

Proof. Following standard arguments, for any market q the allocation of type 2 is efficient and the allocation of type 1 maximizes

$$(v^1 - qv^2)a - (1 - q)C(a) \tag{2}$$

over all products a (including product 0). Because $\bar{a}(1)$ uniquely maximizes (2) when $q = 0$, (2) is continuous in q, and $\mathcal{F}(N^{\{1,2\}})$ is convex, there exists q_1 such that $\mathcal{F}(N^{\{1,2\}}) = [0, q_1]$. Similarly, because $a = 0$ maximizes (2) for all $q \geq v^1/v^2$, there exists q_2 such that $\mathcal{F}(N^{\{2\}}) = [q_2, 1]$. ■

If $q_1 = q_2$, then all markets are non-screening markets, as shown in Figure 1 (a). In this case, any inefficient market, i.e., a market in $(q_1, 1)$, can be segmented into markets q_1 and 1, achieving first-best consumer surplus. If $q_1 < q_2$, as shown in Figure 1 (b), then some markets are screening markets. In this case, first-best consumer surplus is not achievable for any inefficient market, i.e., a market in $(q_1, 1)$. This is because any segmentation of such a market into efficient segments must include a segment with a different optimal mechanism. The proof of the following proposition is identical to that of Proposition 1 and Proposition 2 and is omitted.

Proposition A.1 *The following are equivalent:*

1. *First-best consumer surplus is achievable for any inefficient market q.*

2. *The surplus triangle is achievable for any non-singleton market q.*

3. $q_1 = q_2$.

We next show that our main result, that first-best consumer surplus and the surplus triangle are unachievable for any screening market, continues to hold with any number of types. The key step is to establish a lemma that parallels Lemma 5. We start with some simplifications, then state some known characterizations of optimal mechanisms, and then state and prove the lemma.

Notice that in any optimal mechanism, the allocation of a type must minimize the expected cost over all distributions over alternatives that have the same expectation. Otherwise, we can lower the allocation cost of that type without changing the expected allocation, and thus without changing the incentives. So for the rest of this section, to simplify notation, we let $x(i)$ denote the expected allocation of type i, and we let $C(x(i))$ denote the minimum cost over all allocations with expectation $x(i)$. This extended cost function is convex because it is the convex hull of the original cost function.

Throughout the rest of this section we use two standard properties (Myerson, 1981; see Vohra, 2011, for the formulation used here with finitely many types). First, there exists a payment rule p such that a mechanism (x, p) is IC and IR if and only if the (expected) allocation $x(i)$ is monotone non-decreasing in i. Second, for such a monotone allocation rule x, the optimal profit for a market f among all IC and IR mechanisms (x, p) is achieved with the payment rule

$$p(i) = \sum_{j=1}^{i} v^j (x(j) - x(j-1)), \tag{3}$$

where $x(0) := 0$, and this optimal profit is equal to the virtual surplus

$$\sum_{i=1}^{n} [R^f(i)x(i) - f_i C(x(i))],$$

where $R^f(i)$,

$$R^f(i) = v^i f_i - (v^{i+1} - v^i) \sum_{j>i} f_j,$$

is the marginal revenue of type i.[7]

Lemma A.2 *Consider a market f' that has an optimal mechanism $M = (x, p)$, and let T_1 be the set of types that get a higher allocation than lower types in M, that is, $T_1 = \{i : x(i) > x(i-1)\}$ (and $x(0) := 0$). Suppose that $N^{supp(f')}$ is also optimal for f'. Then, N^{T_1} is also optimal for f'.*

Proof. We first show that $T_1 \subseteq T_2 := \{\bar{i}(f'), \ldots, n\} \cup supp(f')$, where $\bar{i}(f')$ is the highest type in the support of f'. Suppose for contradiction that $x(i) > x(i-1)$ for some i such that $i < \bar{i}(f')$ and $i \notin supp(f')$. Consider setting $x(i) = x(i-1)$ and $p(i) = p(i-1)$. Because i is not in the support of f', this change doesn't affect the profit obtained from i. But by the payment identity (3), this change increases the payment of all higher types, resulting in higher expected profit because types higher than i have a positive probability in f'. We can also assume that all types higher than $\bar{i}(f')$ obtain their efficient allocation, that is, mechanism N^{T_2} is optimal for f'. This is because mechanism $N^{supp(f')}$ is optimal, and the two mechanisms are identical for all types in the support of f'.

[7] If f has full support, then the virtual surplus can be written in a more familiar form

$$\sum_{i=1}^{n} f_i[\phi^f(i)x(i) - C(x(i))],$$

where

$$\phi^f(i) = v^i - \frac{(v^{i+1} - v^i) \sum_{j>i} f_j}{f_i}$$

is the virtual value of type i. But because the virtual value is undefined if $f_i = 0$, and because we want to allow for segments that do not have full support, we use the formulation with marginal revenue.

Now let x^{T_1} and x^{T_2} denote the allocation rules of mechanisms N^{T_1} and N^{T_2}, respectively. Let T_3 be the set of types at which the efficient allocation increases, $T_3 = \{i : \bar{a}(i) > \bar{a}(i-1)\}$. Consider set of types $\hat{T} = T_1 \cup T_3 = \{j_1, \ldots, j_r\}$, for some r. The set $\hat{T}$ partitions the set of types into intervals $j_\ell, \ldots, j_{\ell+1} - 1$ for $\ell \in 0, \ldots, r$, where we set $j_0 = 1, j_{r+1} = n+1$. We show that mechanism N^{T_1} is optimal by showing that for any interval $j_\ell, \ldots, j_{\ell+1} - 1$, the virtual surplus of N^{T_1} is weakly higher than the virtual surplus of M

$$\sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i) x^{T_1}(i) - f'_i C(x^{T_1}(i))] \geq \sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i) x(i) - f'_i C(x(i))], \tag{4}$$

and therefore N^{T_1} must also be optimal. We do so inductively, from $\ell = r$ to $\ell = 0$.

Consider an interval $j_\ell, \ldots, j_{\ell+1} - 1$ of types. If $j_\ell \in T_1$, then all types in this interval obtain their efficient product in both N^{T_1} and N^{T_2} and so (4) is trivially satisfied. If $j_\ell \in T_3$, then consider the smallest $\ell' \geq \ell$ such that $j_{\ell'} \in T_1$ ($j_{\ell'} = n+1$ if no such type exists), which means that the allocation in mechanism M is constant for all types in j_ℓ to $j_{\ell'} - 1$ but there is a jump at $j_{\ell'}$, and also that the efficient allocation has a jump at j_ℓ. Now consider two marginal changes to mechanisms M and N^{T_2}.

First, consider a mechanism M' that is identical to M except that for each type i from j_ℓ to $j_{\ell'} - 1$, with probability ϵ the allocation is modified from $x(i)$ to $\bar{a}(i)$. This change increases the expected allocation of types j_ℓ to $j_{\ell'} - 1$, but because there is a jump in the allocation of M at $j_{\ell'}$, for ϵ small enough, the allocation of mechanism M' is monotone. Because mechanism M is optimal, this change must weakly lower virtual surplus,

$$\epsilon \sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) x(i) - f'_i C(x(i))] \geq \epsilon \sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) \bar{a}(i) - f'_i C(\bar{a}(i))]. \tag{5}$$

Similarly, consider a mechanism M' that is identical to N^{T_2} except that for each type i from j_ℓ to $j_{\ell'} - 1$, with probability ϵ the allocation is modified from $\bar{a}(i)$ to $x(i)$. This change lowers the allocation of types j_ℓ to $j_{\ell'} - 1$, but because there is a jump in the allocation of N^{T_2} at j_ℓ, for ϵ small enough, the allocation of mechanism M' is monotone. Because mechanism N^{T_2} is optimal, this change must weakly lower virtual surplus,

$$\epsilon \sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) x(i) - f'_i C(x(i))] \leq \epsilon \sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) \bar{a}(i) - f'_i C(\bar{a}(i))]. \tag{6}$$

Given (5) and (6), we conclude that

$$\sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) x(i) - f'_i C(x(i))] = \sum_{i=j_\ell}^{j_{\ell'}-1} [R^{f'}(i) \bar{a}(i) - f'_i C(\bar{a}(i))]. \tag{7}$$

By induction, we have

$$\sum_{i=j_{\ell+1}}^{j_{\ell'}-1} [R^{f'}(i)x(i) - f'_i C(x(i))] = \sum_{i=j_{\ell+1}}^{j_{\ell'}-1} [R^{f'}(i)\bar{a}(i) - f'_i C(\bar{a}(i))]. \tag{8}$$

Subtracting (8) from (7), we have

$$\sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i)x(i) - f'_i C(x(i))] = \sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i)\bar{a}(i) - f'_i C(\bar{a}(i))].$$

Now notice that allocation x is constant on the interval j_ℓ to $j_{\ell+1}-1$, and the same holds for the efficient allocation. So we have

$$(\sum_{i=j_\ell}^{j_{\ell+1}-1} f'_i)(C(\bar{a}(j_\ell)) - C(x(j_\ell))) = (\sum_{i=j_\ell}^{j_{\ell+1}-1} R^{f'}(i))(\bar{a}(j_\ell) - x(j_\ell)).$$

Because C is convex and $x(j_\ell) \leq x^{T_1}(j_\ell) \leq \bar{a}(j_\ell)$,

$$(\sum_{i=j_\ell}^{j_{\ell+1}-1} f'_i)(C(x^{T_1}(j_\ell)) - C(x(j_\ell))) \leq (\sum_{i=j_\ell}^{j_{\ell+1}-1} R^{f'}(i))(x^{T_1}(j_\ell) - x(j_\ell)).$$

Finally, because $x^{T_1}(j_\ell)$ is constant on the interval j_ℓ to $j_{\ell+1}-1$, the above inequality becomes

$$\sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i)x(i) - f'_i C(x(i))] \leq \sum_{i=j_\ell}^{j_{\ell+1}-1} [R^{f'}(i)x^{T_1}(i) - f'_i C(x^{T_1}(i))].$$

Therefore, by induction, mechanism N^{T_1} is optimal. ■

Similar to Lemma 5, the logic behind Lemma A.2 does not rely on a characterization of optimal mechanisms. Such a closed-form characterization is available in the Mussa-Rosen setting but requires ironing to allow for all segmentations. Since the ironed intervals may be different in different segments, the closed-form solution is not convenient to work with. Our approach avoids this issue.

Lemma A.2 implies our result as follows.

Proposition A.2 *First-best consumer surplus is not achievable for any screening market.*

Proof. Consider a market f with an optimal mechanism $M = (x, p)$, and suppose that first-best consumer surplus is achievable for f. Consider a segmentation of market f that achieves first-best consumer surplus, and take any segment f'. By Lemma 3, M is optimal for f' and f' is efficient so $N^{supp(f')}$ is also optimal for f'. Lemma A.2 shows that N^{T_1} is also optimal for f', where $T_1 = \{i : x(i) > x(i-1)\}$ (and $x(0) := 0$). Because N^{T_1} is optimal for every segment, it is also optimal for f and f is a non-screening market. ■

As discussed above, if all types have the same efficient product, a non-screening market is one in which only that efficient product is sold and some types are possibly excluded, so Proposition A.2 generalizes Theorem 1 (Proposition A.2 is not a strict generalization of Theorem 1 because it only concerns linear values). One may wonder if it is possible to strengthen Proposition A.2 using a perhaps more natural definition of a non-screening market. Let us say that a market f is a *strong* non-screening market if $N^{T'}$ is optimal for f, where for some i, T' contains all types i or higher, $T' = \{i, \ldots, n\}$. Compare this with a non-screening market f, which is when $N^{T'}$ is optimal for some T'. So if a market is a strong non-screening market, then it is a non-screening market, but the reverse does not necessarily hold. Our current proof does not show that if first-best consumer surplus is achievable for f, then f is a strong non-screening market. The reason is that a type $i \in supp(f)$ need not be in the support of every segment f'. So even if i is served efficiently in every segment that contains i, this type need not be served efficiently in f. In contrast, Lemma A.2 shows that all the types in T' are in the support of, and are served efficiently in, every segment.

We leave a generalization of Theorem 2 and Theorem 3 to the Mussa-Rosen setting for future work. A conceptual question is what the right definitions of screening and non-screening markets are. The strong definition discussed above does not seem to be the right definition. To see this, consider the third statement of Theorem 2, "every market is a non-screening market." This statement cannot hold with the strong definition for any set of at least three types that have different efficient products. For example, consider three types and a market that contains a large fraction of type 1 consumers and a small fraction of type 3 consumers, so mechanism $N^{1,3}$ is optimal. Then type 2 is served inefficiently so this is not a strong non-screening market (we can also construct a market with full support by adding a small fraction of type 2 such that $N^{1,3}$ is still optimal). This market, however, is a non-screening market, so it is possible that Theorem 2 and Theorem 3 can be generalized using our definition of non-screening. The decomposition algorithm of Bergemann, Brooks, and Morris (2015) does not directly work here because there are seven possible non-screening mechanisms for a market with full support over three types, one for each non-empty subset of $\{1, 2, 3\}$, unlike the special case with the same efficient product where there are three such mechanisms.

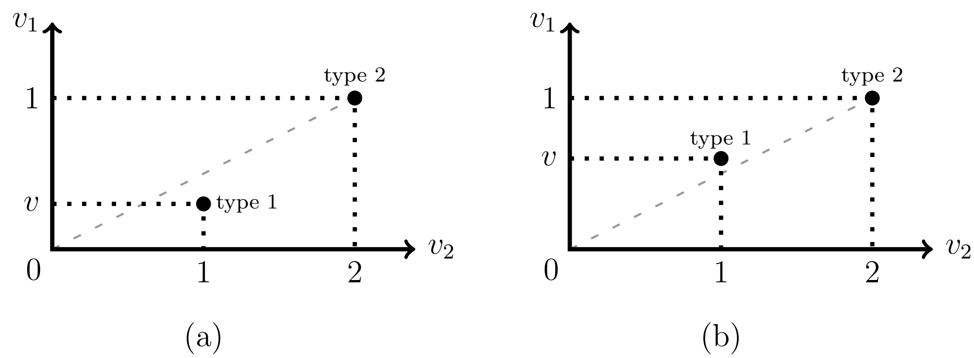

Figure A.1: Type 1 has valuations $(v, 1)$ and type 2 has valuations $(1, 2)$ for product 1 and product 2. Case (a) corresponds to $v \leq 0.5$, and case (b) corresponds to $v \geq 0.5$.

A.2 A Two Type Example

In this section we discuss a parametric example to highlight our results. We directly calculate the closed form expression for the maximum consumer surplus and compare it to first-best consumer surplus. Even though the calculations are straightforward, they are not easily extendable beyond this example.

Suppose that there are two products and two types. A type 1 consumer has valuation $v \in (0, 1)$ for one unit and valuation 1 for two units. A type 2 consumer has valuation 1 for one unit and valuation 2 for two units. The two types are illustrated in Figure A.1, in which case (a) corresponds to $v \leq 0.5$ and case (b) corresponds to $v \geq 0.5$. A market q consists of a fraction $1 - q$ of type 1 consumers and a fraction q of type 2 consumers.

To identify maximum consumer surplus in different markets, it is useful to first identify the optimal mechanism in each market. Consider the following three mechanisms and their revenue in a market q. Mechanism N^1 offers product 2 at price 1. Mechanism N^2 offers product 2 at price 2. Mechanism S screens; it offers each consumer a choice between buying product 1 at price v or product 2 at price $v + 1$. It can be shown that for any market q, one of these three mechanisms is optimal, as illustrated in Figure A.2. If $v \leq 0.5$, then mechanisms N^1 is optimal for markets in $[0, 0.5]$ and mechanism N^2 is optimal for markets in $[0.5, 1]$. If $v \geq 0.5$, then mechanism N^1 is optimal for markets in $[0, 1 - v]$, mechanism S is optimal for markets in $[1 - v, v]$, and mechanism N^2 is optimal for markets in $[v, 1]$.

Next, we compute the (average) consumer surplus in each market q generated by the optimal

Figure A.2: Optimal mechanisms for the two cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.

Figure A.3: Consumer surplus $CS(q)$ for the two cases (a) $v \leq 0.5$ and (b) $v \geq 0.5$.

mechanism for that market.[8] Type 1 does not receive any information rents in any optimal mechanism. Thus, consumer surplus $CS(q)$ in market q is q times the utility of type 2 in the optimal mechanism for that market. Consumer surplus $CS(q)$ is illustrated in Figure A.3.

A segmentation of market q is a distribution μ over markets $[0,1]$ such that $E_{q' \sim \mu}[q'] = q$. The maximum consumer surplus is $MCS(q) = \max_\mu E_{q' \sim \mu}[CS(q')]$, that is, the highest consumer surplus across all segmentations μ. The maximum consumer surplus is obtained by concavifying the function CS. That is, $MCS(q) = \overline{CS}(q)$, where $\overline{CS}$ is the lowest concave function that is point-wise at least as high as CS.

The maximum consumer surplus $MCS(q)$ is at least $CS(q)$ and at most first best consumer surplus $FB(q)$, which is the surplus from the efficient allocation (that is, product 2 for each type) minus the seller's revenue in market q. If the optimal mechanism for market q implements the efficient allocation, then the two bounds are equal, that is, $CS(q) = FB(q)$, so $CS(q) = MCS(q) = FB(q)$. This is the case for a market q for which mechanism N^1 is optimal and for market $q = 1$ which contains only type 2 consumers and for which mechanism N^2 is optimal. We refer to such markets as efficient, and otherwise as inefficient. If a market is efficient, then there is no scope for market segmentation to increase consumer surplus.

We can now address the possibility of achieving first-best consumer surplus for all markets $q \in [0,1]$. The relationship between maximum consumer surplus, MCS, and first-best consumer

[8] If there is more than one optimal mechanism we choose the one with higher consumer surplus.

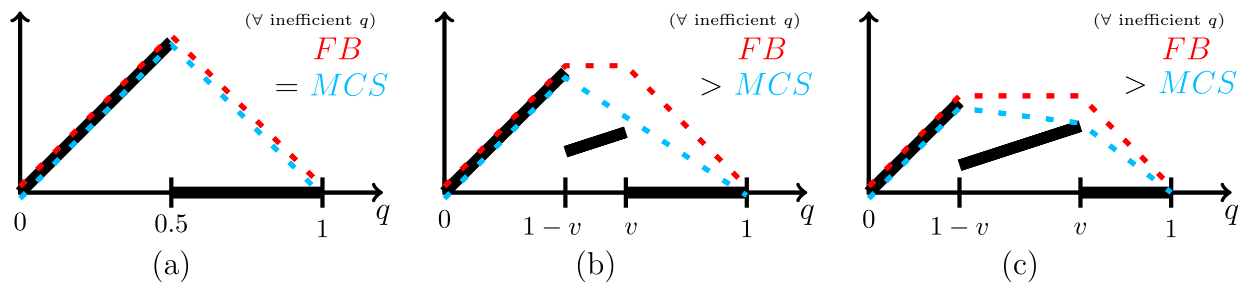

Figure A.4: The relationship between CS, MCS, and FB: (a) $v \in (0, 0.5]$, (b) $v \in (0.5, \frac{\sqrt{5}-1}{2})$, and (c) $v \in [\frac{\sqrt{5}-1}{2}, 1)$.

surplus, FB, is illustrated in Figure A.4, and depends on the value of v. If v is in $(0.5, 1)$, as in Figure A.4 (b) and (c), then first-best consumer surplus is not achievable for any inefficient market. The only difference between cases (b) and (c) in Figure A.4 is that in the former, $MCS(q)$ strictly exceeds $CS(q)$ for every inefficient market q whereas in the latter $MCS(q) = CS(q)$ for market $q = v$.[9] If $v \in (0, 0.5]$, as in Figure A.4 (a), then first-best consumer surplus is achievable for all markets. Equivalently, first-best consumer surplus is achievable for all markets if and only if for every market either mechanism N^1 or N^2 is optimal, that is, the seller does not find it profitable to screen consumers.

This example can also be used to illustrate how close MCS is to FB when FB is not achievable. If v is in $(0.5, 1)$, as in Figure A.4 (b) and (c), then the ratio FB/MCS increases in q in the interval $(1-v, v)$ and decreases in the interval $(v, 1)$. Consider the maximal point $q = v$. At this point, we have $FB = 1 - v$; If $v \in (0.5, \frac{\sqrt{5}-1}{2})$, then $MCS = \frac{(1-v)^2}{v}$ so $FB/MCS = \frac{v}{1-v}$, which increases in v; If $v \in [\frac{\sqrt{5}-1}{2}, 1)$, then $MCS = (1-v)v$, so $FB/MCS = \frac{1}{v}$, which decreases in v.

What is the economic significance of v being greater than or smaller than 0.5? For type 2 consumers, product 2 is twice as valuable as product 1. For type 1 consumers, whether v is greater than or smaller than 0.5 determines whether product 2 is more than or less than twice as valuable as product 1. In other words, when $v \leq 0.5$ the second unit of the product is relatively more complementary to the first unit of the product for type 1 consumers than for type 2 consumers, and vice versa when $v > 0.5$.

Turning to the surplus triangle, it is trivially achievable for markets with a single type of consumer ($q = 0$ and $q = 1$). For all other markets, the same conditions that characterize achiev-

[9]See Haghpanah and Siegel (2021) for a detailed investigation of when MCS strictly exceeds CS.

ability of first-best consumer surplus also characterize when the surplus triangle is achievable for every market or no market (efficient or inefficient). Indeed, whenever first-best consumer surplus is not achievable, the surplus triangle is clearly not achievable. And the results of Bergemann, Brooks, and Morris (2015) show that when the seller does not find it optimal to screen, that is, in every market q only offers two units as a bundle, the entire surplus triangle is achievable.

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