Dynamic Matching in Overloaded Waiting Lists

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Online Appendix

B Nonlinear Waiting Costs

This appendix extends the model to allow for discounting. Agents pay a linear waiting cost c per period until they are assigned, and discount waiting costs and the value of the assigned item by $\delta < 1$ per period. Formally, the period t_0 utility of an agent of type θ who is assigned item x_t item in period $t \geq t_0$ is

$$u_{\theta}^{t_0}(X,t) = -\sum_{\tau=t_0}^{t-1} \delta^{\tau-t_0} c + \delta^{t-t_0} v_{\theta}(x).$$

where the valuation $v_{\theta}(x)$ is $v > 0$ for a matching item and 0 for mismatched items:

$$v_{\theta}(x) = \begin{cases} v & (\theta, X) \in \{(\alpha, A), (\beta, B)\} \\ 0 & (\theta, X) \in \{(\alpha, B), (\beta, A)\} \end{cases}.$$

Note that past waiting costs are sunk costs, and that $u_{\theta}^{t_0}$ is independent of the time when the agent joined the waiting-list.

To analyze the misallocation rate under the FCFS buffer-queue policy, denote by $U_{\alpha}(A^{(k)})$ the utility of an α agent who receives the k-th future A item to arrive. The probability that the k-th item will arrive in exactly t periods is $(p_A)^k(1-p_A)^{t-k}\binom{t-1}{k-1}$, and we have that

$$\begin{aligned} U_{\alpha}(A^{(k)}) &= \sum_{t=k}^{\infty} (p_A)^k (1-p_A)^{t-k} \binom{t-1}{k-1} u_{\alpha}^{t_0}(A, t_0+t) \\ &= \frac{1}{1-\delta}\left((c+v(1-\delta)) \left(\frac{\delta p_A}{1-\delta(1-p_A)} \right)^k - c \right). \end{aligned}$$

Thus, an α agent will be willing to decline a B item if

$$\left(\frac{\delta p_A}{1-\delta\,(1-p_A)} \right)^k \geq \frac{c}{c+v(1-\delta)},$$

or, equivalently,

$$k \leq \frac{\log\left(\frac{c}{c+v(1-\delta)}\right)}{\log\left(\frac{\delta p_A}{1-\delta(1-p_A)}\right)}.$$

These derivations allow us to extend our results from Section 3 to settings in which agents discount future periods.

Theorem 5. *Let agents have a linear cost of waiting of $c \geq 0$ per period and discount the value of assigned items by $\delta < 1$ per period. Assume $p_A = p_\alpha$. The misallocation rate under the FCFS buffer-queue mechanism is*

$$\xi^{FCFS} = \frac{2p_A p_B}{p_B K^A + p_A K^B + 1},$$

where $K^A = \left\lfloor \log\left(\frac{c}{c+v(1-\delta)}\right) / \log\left(\frac{\delta p_A}{1-\delta(1-p_A)}\right) \right\rfloor$ *and* $K^B = \left\lfloor \log\left(\frac{c}{c+v(1-\delta)}\right) / \log\left(\frac{\delta p_B}{1-\delta(1-p_B)}\right) \right\rfloor$.

C The Disjoint-Queues Mechanism

This section considers the disjoint-queues (DQ) mechanism, which asks agents to report their preferences as soon as they join the waiting-list. The mechanism holds two separate queues, one for A items and one for B items, and asks agents to select and join a single queue. Agents observe the length of both queues when they make their choice. Both queues follow an FCFS policy. Once agents join a queue, they wait in that queue until they are assigned to the item of that queue.[49]

Misallocation happens under the DQ mechanism for similar reasons that misallocation happens under the FCFS buffer-queue mechanism: the random arrivals of agents and items may result in a temporary imbalance between the demand from agents and the available supply of items. Under the DQ mechanism, this imbalance is realized in the form of a difference between the expected wait in the two queues. When this difference grows too large, agents will join the queue with a shorter wait regardless of their type, possibly resulting in misallocation.

In contrast to buffer-queue mechanisms, the analysis of the DQ mechanism requires a specification of the agent arrival process. For simplicity, we assume that in each period, one new agent joins the waiting-list and then one item arrives. In addition, assume that

[49]This mechanism is a simplified version of mechanisms commonly used by public housing authorities, where applicants are asked to select a single project-specific waiting-list they would like to join (e.g., the New York City Housing Authority (New York City Public Housing Authority, 2015)).

initially both queues are of equal length.

We capture the dynamics of the DQ mechanism using techniques similar to those used in section 5 and Appendix A. Denote by ℓ_A the number of agents in the A queue and by ℓ_B the number of agents in the B queue. By assumption, each period one agent joins one of the queues and one agent is assigned, and therefore the total number of agents remains a constant we denote by $2M$. We assume $2M$ is sufficiently large so that neither queue is ever empty. The Markov chain used to capture the dynamics of the DQ mechanism is described within the proof of the following Lemma.

Lemma 11. *Assume $p_A = p_\alpha = p$. For all sufficiently large M, the misallocation rate under the disjoint-queues mechanism is*

$$\xi^{DQ} = \frac{2p(1-p)}{\lfloor p\,(1-p)\,\bar{w} - (1-2p)M \rfloor + \lfloor p\,(1-p)\,\bar{w} + (1-2p)\,M \rfloor + 2}.$$

Notice $\xi^{DQ} \approx \frac{2p(1-p)}{2p(1-p)\bar{w}+2}$ (by ignoring the integer constraints), which is similar to the misallocation rate under an FCFS buffer-queue mechanism. The difference is due to the assumption that in each period, one item and one agent arrive. A different arrival process can lead to a different misallocation probability.

Proof. When an agent joins the waiting-list, he observes the length of both queues. The number of misallocated items is equal to the number of agents who join their mismatched queue. To calculate the latter, we establish a Markov chain that captures the dynamics of this system.

Let $\Delta = \ell_A/p - \ell_B/\,(1-p)$ be the difference between the expected wait of the two queues. An α agent will prefer to join the A queue if

$$v - c \cdot \frac{\ell_A}{p} \geq -c \cdot \frac{\ell_B}{1-p},$$

or

$$\Delta \leq v/c = \bar{w}.$$

Similarly, a β agent will prefer to join the B queue if $-\Delta \leq \bar{w}$. Thus, for sufficiently large M, neither queue is ever empty.

We can capture the dynamics of the mechanism using a Markov chain whose states are possible values of Δ. The value of Δ changes when either an agent joins the waiting-list or an item arrives and an agent is assigned. If a period begins with $\Delta \in [-\bar{w}, \bar{w}]$ with probability p_α, the new agent joins the A queue, increasing ℓ_A by 1, and with probability p_β, the new agent joins the B queue, increasing ℓ_B by 1. If $\Delta \notin [-\bar{w}, \bar{w}]$, the new agent always joins the shorter queue. Immediately after the agent joins, the item arrives; with probability p_A, an A arrives and ℓ_A decreases by 1, and with probability p_B, a B arrives and ℓ_B decreases by 1. Thus, in every period, the value of Δ can change by $0, -\gamma$, or $+\gamma$ where $\gamma = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$. We assume that initially both queues hold the same number of agents, and therefore the initial value of Δ is given by $\Delta_0 = \frac{M}{p} - \frac{M}{1-p}$.

We have that $\Delta - \Delta_0$ is always a multiple of γ and write $\Delta = \Delta_0 + k \cdot \gamma$. Denote the maximal value of k such that $\Delta \in [-\bar{w}, \bar{w}]$ by

$$\begin{aligned} k^A &= \max \{k \in \mathbb{Z} \mid \ \Delta_0 + k \cdot \gamma \leq \bar{w}\} \\ &= \max \{k \in \mathbb{Z} \mid \ k \leq (\bar{w} - \Delta_0) / \gamma\} \\ &= \lfloor (\bar{w} - \Delta_0) / \gamma \rfloor , \end{aligned}$$

and denote the minimal value of k such that $\Delta \in [-\bar{w}, \bar{w}]$ by

$$\begin{aligned} k^B &= \min \left\{ k \in \mathbb{Z} \mid v - c \cdot \frac{\ell_B}{p_B} \geq -c \cdot \frac{\ell_A}{p_A} \right\} \\ &= \min \{k \in \mathbb{Z} \mid \ -\Delta_0 - k \cdot \gamma \leq \bar{w}\} \\ &= \min \{k \in \mathbb{Z} \mid \ k \geq (-\bar{w} - \Delta_0) / \gamma\} \\ &= \lceil (-\bar{w} - \Delta_0) / \gamma \rceil \\ &= - \lfloor (\bar{w} + \Delta_0) / \gamma \rfloor . \end{aligned}$$

We capture the dynamics of the system by a Markov chain whose states S are possible values of k at the beginning of a period

$$S = \{k^B - 1, k^B, \ldots, \ldots, k^A, k^A + 1\},$$

and transition probabilities are given by

$$P\left(s_t | s_{t-1}\right) = \begin{cases} p_\alpha p_B & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} + 1 \\ p_\beta p_A & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} - 1 \\ p_\alpha p_A + p_\beta p_B & s_{t-1} \in \left[k^B, k^A\right], \ s_t = s_{t-1} \\ p_A & s_{t-1} = k^B - 1, \ \ s_t = s_{t-1} \\ p_B & s_{t-1} = k^B - 1, \ \ s_t = s_{t-1} + 1 \\ p_B & s_{t-1} = k^A + 1, \quad s_t = s_{t-1} \\ p_A & s_{t-1} = k^A + 1, \quad s_t = s_{t-1} - 1 \ , \end{cases}$$

We next solve for the stationary distribution π. Equating the flow between $s = k$ and

$s' = (k+1)$ for $k, (k+1) \in \left[k^B, k^A\right]$, we get that

$$\pi(k) p_\alpha p_B = \pi\left(k+1\right) p_\beta p_A,$$

and given that $p_\alpha = p_A$, we get

$$\pi(k) = \pi\left(k+1\right).$$

Denote $\tau = \pi(k)$ for any $k \in \left[k^B, k^A\right]$. For $s = k^A + 1$ we get the flow equation

$$\pi(k^A+1) p_A = \pi(k^A) p_\alpha p_B,$$

giving

$$\begin{aligned} \pi(k^A+1) &= (1-p)\tau \\ \pi(k^B-1) &= p\tau\,. \end{aligned}$$

Equating the sum of probabilities to 1, we get that

$$\begin{aligned} 1 &= \sum_{k \in [k^B, k^A]} \pi(k) + \pi(k^A+1) + \pi(k^B-1) \\ &= \left(k^A - k^B + 1\right)\tau + (1-p)\tau + p\tau \\ &= \left(k^A - k^B + 2\right)\tau, \end{aligned}$$

and therefore

$$\tau = \frac{1}{k^A - k^B + 2}.$$

Misallocation happens when the state is either $k^A + 1$ or $k^B - 1$ and the new agent is mismatched; therefore, the misallocation rate is

$$\begin{aligned} \xi^{DQ} &= p_\alpha \pi(k^A+1) + p_\beta \pi(k^B-1) \\ &= \frac{2p(1-p)}{k^A - k^B + 2} \\ &= \frac{2p(1-p)}{\lfloor(\bar{w} - \Delta_0)/\gamma\rfloor + \lfloor(\bar{w} + \Delta_0)/\gamma\rfloor + 2} \\ &= \frac{2p(1-p)}{\lfloor p\,(1-p)\,\bar{w} - (1-2p)M\rfloor + \lfloor p\,(1-p)\,\bar{w} + (1-2p)\,M\rfloor + 2}. \end{aligned}$$

□

D Omitted Proofs

This appendix includes omitted proofs in the order of their appearance in the main text. Sections 5 and Appendix A develop the technical tools that are used throughout the paper, and proofs of results from Section 3 and Section 6 rely on results from these sections. The proof of Lemma 2 from Section 2 uses Corollary 1. The proof of Theorem 2 is a generalization of the proofs of Lemma 7 and Corollary 3. The proof of Theorem 1 from Section 3 relies on results from Section 5.

D.1 Proofs from Section 2

Proof of Lemma 1. Consider an arbitrary assignment μ and an arrival process specified by $\chi : \mathcal{I} \to \{t \geq 0\}$ where $\chi(i)$ is the arrival time of agent $i \in \mathcal{I}$. Assume the world ends after period T and let $\mathcal{I}_T = \{i \in \mathcal{I} | \chi(i) \leq T\}$ denote the set of agents that arrive before period T, and let $\mathcal{I}_T(\mu) = \{\mu(t) \mid t \leq T\}$ denote the set of agents that were assigned under μ before period T. Let $\xi_t \in \{0, 1\}$ be an indicator equal to 1 if the item x_t is misallocated under μ. The sum of agents utilities up to time T under μ is

$$\mathrm{WF}_T = \sum_{t=0}^{T} \left((1 - \xi_t) \cdot v + \xi_t \cdot 0 - c \cdot (t - \chi(\mu(t))) \right) - \sum_{i \in \mathcal{I}_T \setminus \mathcal{I}_T(\mu)} c \cdot (T - \chi(i))$$

where the first summation gives the total utility of agents in $\mathcal{I}_T(\mu)$ and the second summation gives the total utility of the remaining unassigned agents. Rewriting, we have that

$$\mathrm{WF}_T = v \cdot \left(T - \sum_{t=0}^{T} \xi_t \right) + \sum_{t=0}^{T} c \cdot (T - t) - \sum_{i \in \mathcal{I}_T} c \cdot (T - \chi(i))$$

Since the last two arguments do not depend on μ, they will cancel out when we take the difference between the welfare under the two assignments μ, μ'. Therefore the relative welfare of an assignment depends only on the number of misallocations $\sum_{t=0}^{T} \xi_t$. □

Proof of Lemma 2. The lemma follows from the observation that the full information mechanism is equivalent to a buffer-queue mechanism with $K_\alpha = K_\beta = M - 1$ together with Corollary 1. □

D.2 Proofs from Section 3

Proof of Lemma 3. Consider an α agent in position k. The α agent always accepts an A item. If the α agent is offered a B item, it must be that the agents in positions $1, \dots, k-1$ declined the B item and are waiting for an A.[50] Thus, the agent's expected wait for an A is the expected number of periods until k copies of A arrive, which is k/p_A. Thus, the α agent will decline a B if and only if $v - c \cdot k/p_A \geq 0$, or $k \leq p_A v/c = p_A \bar{w}$. Symmetrically, a β agent in position k declines an A item if and only if $k \leq p_B \bar{w}$. □

Proof of Theorem 1. Theorem 1 is a direct corollary of Theorem 3 and Lemma 5.

The expressions for the WFL are obtained by substituting in the expression for the misallocation rate and the formulas for the maximal BQ sizes. When the system is balanced, the WFL is

$$\begin{aligned}
\mathrm{WFL}^{\mathrm{WLWD}} &= v\xi^{\mathrm{WLWD}} \\
&= v\frac{2p(1-p)}{(1-p)K^A + pK^B + 1} \\
&= v\frac{2p(1-p)}{(1-p)\lfloor p\bar{w}\rfloor + p\lfloor (1-p)\bar{w}\rfloor + 1}.
\end{aligned}$$

When the system is unbalanced, the WFL is

$$\begin{aligned}
\mathrm{WFL}^{\mathrm{WLWD}} &= v\left(\xi^{\mathrm{WLWD}} - |p_A - p_\alpha|\right) \\
&= v \cdot (p_A - p_\alpha)\frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} - v \cdot |p_A - p_\alpha|.
\end{aligned}$$

If $p_\alpha < p_A$,

$$\begin{aligned}
&v \cdot (p_A - p_\alpha)\frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} - v \cdot (p_A - p_\alpha) \\
=&v \cdot (p_A - p_\alpha)\frac{2\left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} \\
=&\frac{2v \cdot |p_A - p_\alpha|}{\left(p_\alpha/p_A\right)^{-(K^A+1)}\left(p_\beta/p_B\right)^{K^B+1} - 1}.
\end{aligned}$$

[50] To see this claim is true, observe that the problem of the agent in the first position is stationary, and he will either immediately take a mismatched B or wait for an A (by assumption, the agent waits for an A when both options give the same utility). This argument implies the problem of the agent in the second position is stationary, and the claim follows by induction.

If $p_\alpha > p_A$, we perform similar operations and obtain

$$\begin{aligned}
& v \cdot (p_A - p_\alpha) \frac{\left(p_\beta/p_B\right)^{K^B+1} + \left(p_\alpha/p_A\right)^{K^A+1}}{\left(p_\beta/p_B\right)^{K^B+1} - \left(p_\alpha/p_A\right)^{K^A+1}} + v \cdot (p_A - p_\alpha) \\
= & \frac{2v \cdot |p_A - p_\alpha|}{\left(p_\alpha/p_A\right)^{K^A+1} \left(p_\beta/p_B\right)^{-(K^B+1)} - 1} \,.
\end{aligned}$$

We can combine the results from these two cases into one expression:

$$\mathrm{WFL}^{\mathrm{WLWD}} = \frac{2v \, |p_A - p_\alpha|}{\left(\left(p_\alpha/p_A\right)^{K^A+1} \left(p_\beta/p_B\right)^{-(K^B+1)}\right)^{sgn(p_\alpha - p_A)} - 1} \,.$$

□

Proof of Corollary 1. The maximal sizes in the FCFS BQ system are $K^A = \lfloor p_A \bar{w} \rfloor = \lfloor p_A v/c \rfloor$ and $K^B = \lfloor p_B \bar{w} \rfloor = \lfloor p_B v/c \rfloor$. As $c \to 0$, we have that $K^A, K^B \to \infty$. The waiting cost c affects ξ and WFL only through K^A and K^B. Hence, by Theorem 3 we have that

$$\lim_{c \to 0} \xi^{\mathrm{WLWD}} = \lim_{K^A, K^B \to \infty} \xi^{\mathrm{WLWD}} = |p_A - p_\alpha| \,,$$

and

$$\lim_{c \to 0} \mathrm{WFL}^{\mathrm{WLWD}} = \lim_{K^A, K^B \to \infty} v \left(\xi^{\mathrm{WLWD}} - |p_A - p_\alpha|\right) = 0 \,.$$

□

Proof of Corollary 2. If the system is balanced, we have that

$$\lim_{v \to \infty} \mathrm{WFL}^{\mathrm{WLWD}} = \lim_{v \to \infty} v \frac{2p(1-p)}{(1-p) \lfloor pv/c \rfloor + p \lfloor (1-p)v/c \rfloor + 1} \,.$$

Because $x - 1 \leq \lfloor x \rfloor \leq x + 1$, we have that

$$(1-p) \lfloor pv/c \rfloor + p \lfloor (1-p)v/c \rfloor + 1 = 2p(1-p)v/c + r(v)$$

for $r(v)$ satisfying $-10 \leq r(v) \leq 10$ for all $v > 0$. Therefore,

$$\begin{aligned}\lim_{v\to\infty} \mathrm{WFL}^{\mathrm{WLWD}} &= \lim_{v\to\infty} v\frac{2p(1-p)}{2p(1-p)v/c + r\,(v)} \\ &= \lim_{v\to\infty} c \cdot \frac{2p(1-p)}{2p(1-p) + r(v)\cdot c/v} \\ &= c\,.\end{aligned}$$

If $p_\alpha > p_A$, we have that

$$\begin{aligned}\mathrm{WFL}^{\mathrm{WLWD}} &= \frac{2v\cdot(p_\alpha - p_A)}{\left({}^{p_\alpha}\!/\!{}_{p_A}\right)^{K^A+1}\left({}^{p_\beta}\!/\!{}_{p_B}\right)^{-K^B-1} - 1} \\ &= 2\cdot(p_\alpha - p_A)\,\frac{v\cdot b^{K^B+1}}{a^{K^A+1} - b^{K^B+1}} \\ &= 2\cdot(p_\alpha - p_A)\,\frac{v\cdot b^{\lfloor p_B v/c\rfloor+1}}{a^{\lfloor p_A v/c\rfloor+1} - b^{\lfloor p_B v/c\rfloor+1}}\,,\end{aligned}$$

where $a = p_\alpha/p_A > 1$ and $b = p_\beta/p_B < 1$. Because $\lim_{v\to\infty} v \cdot b^{\lfloor p_B v/c\rfloor+1} = 0$ and $\lim_{v\to\infty} a^{\lfloor p_A v/c\rfloor+1} - b^{\lfloor p_B v/c\rfloor+1} = \infty$, we have that

$$\begin{aligned}\lim_{v\to\infty} \mathrm{WFL}^{\mathrm{WLWD}} &= \lim_{v\to\infty} 2\cdot(p_\alpha - p_A)\,\frac{v\cdot b^{\lfloor p_B v/c\rfloor+1}}{a^{\lfloor p_A v/c\rfloor+1} - b^{\lfloor p_B v/c\rfloor+1}} \\ &= 0.\end{aligned}$$

A symmetric argument shows that if $p_\alpha < p_A$, we have $\lim_{v\to\infty} \mathrm{WFL}^{\mathrm{WLWD}} = 0$. □

D.3 Proofs from Section 4

Proof of Theorem 2. In the unique equilibrium under the information disclosure Υ^*, an α agent joins the buffer-queue if he is offered a B and informed that $k \in \{1, \ldots, 2p_A\bar{w} - 1\}$, and takes the current item otherwise. To see that this is an equilibrium, observe that $k \geq 2p_A\bar{w}$ implies the expected wait is above $\bar{w}$, and therefore any α agent best responds by taking the current item. By Lemma 10 in Appendix A, in equilibrium an α agent who is informed that $k < 2p_A\bar{w}$ is equally likely to be in either of the positions $1, \ldots, \lfloor 2p_A\bar{w}\rfloor - 1$. Thus, the agent believes his expected wait is $\lfloor 2p_A\bar{w}\rfloor/2p_A \leq \bar{w}$ and best responds by waiting for an A. By Little's Law, the expected wait conditional on being informed that $k < 2p_A\bar{w}$ is lower than $\bar{w}$ if some agents do not join the buffer-queue after being informed that $k < 2p_A\bar{w}$. Therefore, it is a dominant strategy for an α agent to join the buffer-queue if offered a B and informed that $k < 2p_A\bar{w}$.

Next, we show that information disclosure Υ^* gives the minimal welfare loss.[51] With-

[51] This part of the proof follows from the proof of Theorem 7 in online appendix D of the working paper version which extends Lemma 7 and Corollary 3. For completeness, we provide a proof here.

out loss we can restrict attention to information disclosures which send a recommendation whether mismatched agents should take the item or join the buffer-queue, and equilibria in which the agents follow the recommended action. To constitute an equilibrium, the expected wait conditional on receiving a recommendation to join the buffer-queue must be less than $\bar{w}$. To formally specify this IC constraint, we calculate the stationary distribution of the system.

As in Lemma 4, because agents join the buffer-queue only if they are offered a mismatched item, the possible states are $k \in \mathbb{Z}$ where $k > 0$ corresponds to k agents of type α waiting in the A buffer-queue and $k \leq 0$ corresponds to $|k|$ agents of type β waiting in the B buffer-queue. As in Appendix A, we used an extended state space $\mathbb{Z} \times \{\phi, A, B\}$ that includes within-period states that indicate whether an item is currently being offered.

Let $f(k)$ denote the probability that a mismatched agent is given the recommendation to join the buffer-queue in state k. It will be helpful to denote $F^B(k) = \prod_{i=0}^{k-1} f(i)$ for $k > 0$, $F^A(k) = \prod_{i=0}^{|k|-1} f(-i)$ for $k < 0$, and $F^B(0) = F^A(0) = 1$.

Denote the stationary distribution by π, where $\pi(k) = \pi^{\phi}(k)$ is the stationary probability of state (k, ϕ), and $\pi^B(k)$ is the stationary probability of (k, B) (and likewise for π^A). We follow the same steps as in Lemma 10 to calculate the stationary distribution π. For any $k > 0$, the mechanism must visit state (k, B) between every two visits to (k, ϕ), and vice versa. Therefore, we have that

$$\pi^B(k) = \pi^{\phi}(k).$$

For $k = 0$, we have that the state $(0, B)$ can only be reached from the state $(0, \phi)$ by an arrival of a B item. Therefore,

$$\pi^B(0) = p_B \pi^{\phi}(0) = (1 - p)\pi^{\phi}(0).$$

Because the flow through the cut between $s \leq k$ and $s \geq k + 1$ must be zero (see Figure 11) we have that for $k \geq 0$

$$p_A \pi^{\phi}(k + 1) = p_{\alpha} f(k) \pi^B(k)\,,$$

or

$$\pi^{\phi}(k + 1) = f(k)\pi^B(k)\,.$$

For $k = 0$ we have

$$\begin{aligned} \pi^{\phi}(1) &= f(0)\pi^B(0) \\ &= f(0)(1 - p)\pi(0). \end{aligned}$$

By induction, for any $k > 0$ we have

$$\pi^{\phi}(k) = \pi^{B}(k) = (1-p)\pi(0)F^{B}(k).$$

We solve for $\pi(0)$ using that the total probability is equal to 1 to get that

$$\pi^{\phi}(0) \quad = \quad \frac{1}{2}\cdot\frac{1}{1+(1-p)\sum_{k=1}^{\infty}F^{B}(k)+p\sum_{k=1}^{\infty}F^{A}(-k)}.$$

We can now formulate the IC constraint, which requires that the expected wait conditional on the current item and on receiving a recommendation to join the buffer-queue $\mathsf{w}^A, \mathsf{w}^B$ must be less than $\bar{w}$. We calculate w^A by using Little's Law (which states that the average wait is equal to the average number of agents in the system divided by the arrival rate). Observe that if all agents follow the mechanism's recommendation, then w^A is the expected wait for a randomly approached α agent who joins the buffer-queue. Consider states in which α agents are waiting. The arrival rate of α approached agents is equal to the rate at which α approached agents are assigned (otherwise, w^A is infinite), which is p_A. We calculate the average number of agents in the buffer-queue by taking the stationary distribution restricted to the states in which α agents are waiting. Therefore, the expected wait of a random α agent is

$$\begin{aligned}\mathsf{w}^A \quad &= \quad \frac{1}{p_A}\frac{\sum_{k=1}^{\infty}\pi^{B}(k)\cdot k}{\sum_{k=1}^{\infty}\pi^{B}(k)} \\ &= \quad \frac{1}{p}\frac{\sum_{k=1}^{\infty}F^{B}(k)\cdot k}{\sum_{k=1}^{\infty}F^{B}(k)},\end{aligned}$$

and that the IC constraint can be written as:

$$\sum_{k=1}^{\infty}F^{B}(k)\cdot k \leq \bar{w}\cdot p\sum_{k=1}^{\infty}F^{B}(k).$$

Given the stationary distribution, we can calculate the misallocation rate. Agents are approached in states (k, A) or (k, B). Transitions from these states result in misallocation if the current item is offered to a mismatched agent and the agent does not join the buffer-queue. Therefore,

$$\xi \quad = \quad \frac{\sum_{k=0}^{\infty}p_{\alpha}\left(1-f^{B}(k)\right)\pi^{B}(k)+\sum_{k=0}^{\infty}p_{\beta}\left(1-f^{A}(-k)\right)\pi^{A}(-k)}{\sum_{k=0}^{\infty}\pi^{B}(k)+\sum_{k=0}^{\infty}\pi^{A}(k)}.$$

To simplify this expression, note

$$\begin{aligned}\sum_{k=0}^{\infty} p_\alpha \left(1 - f^B(k)\right) \pi^B(k) &= \sum_{k=0}^{\infty} p \left(1 - f^B(k)\right) (1-p)\pi(0) \prod_{i=0}^{k-1} f^B(i) \\ &= p(1-p)\pi(0) \cdot \sum_{k=0}^{\infty} \left(\prod_{i=0}^{k-1} f^B(i) - \prod_{i=0}^{k} f^B(i) \right) \\ &= p(1-p)\pi(0) \cdot \left(1 - \lim_{k\to\infty} F^B(k)\right) .\end{aligned}$$

The denominator, which is the probability that the next transition will approach a new agent, is equal to 1/2 by the previous derivation. The IC constraint together with the monotonicity of F^B imply that $\lim_{k\to\infty} F^B(k) = 0$. Thus, for any IC information disclosure we have

$$\xi = \frac{2p(1-p)}{1 + (1-p)\sum_{k=1}^{\infty} F^B(k) + p\sum_{k=1}^{\infty} F^A(-k)}.$$

Taking these results together, we find the minimal misallocation rate out of all IC information disclosure policies is bounded by the solution to the following optimization problem:

$$\begin{aligned} &\text{Minimize}_{F^A,F^B} && \tfrac{2p(1-p)}{1+(1-p)\sum_{k=1}^{\infty} F^B(k)+p\sum_{k=1}^{\infty} F^A(-k)} \\ &s.t. && \textstyle\sum_{k=1}^{\infty} (k - \bar{w}p) \cdot F^B(k) \le 0 \\ & && \textstyle\sum_{k=1}^{\infty} (k - \bar{w}(1-p)) \cdot F^A(-k) \le 0 \\ & && 1 \ge F^B(k) \ge F^B(k+1) \ge 0 \qquad \forall k > 0 \\ & && 1 \ge F^A(-k) \ge F^A(-k-1) \ge 0 \quad \forall k > 0 \,. \end{aligned}$$

This optimization problem can be decomposed to two separate optimization problems, one for the domain where α approached agents are present:

$$\begin{aligned} &\underset{F^B}{\text{Maximize}} && \textstyle\sum_{k=1}^{K^A} F^B(k) \\ &s.t. && \textstyle\sum_{k=1}^{\infty} (k - \bar{w}p) \cdot F^B(k) \le 0 \\ & && 1 \ge F^B(k) \ge F^B(k+1) \ge 0 \quad \forall k > 0 \end{aligned} \tag{1}$$

and the analogous problem for the domain in which β approached agents are present.

The optimal solution to (1) is given by

$$F^*(k) = \begin{cases} 1 & k < L^* \\ x^* & k = L^* \end{cases}$$

for $L^* \in \mathbb{N}$ and $x^* \in [0,1)$, and the first constraint must bind. Thus, the optimal solution corresponds to $L^* = \lfloor 2p\bar{w} \rfloor - 1$ and $x^* = [2p\bar{w}] = 2p\bar{w} - \lfloor 2p\bar{w} \rfloor$. In particular, when $2p\bar{w} - 1$ is an integer we have that $L^* = K^A$ and $x^* = 0$, which is equivalent to information disclosure Υ^*. Therefore, we find that any information disclosure policy achieves a weakly worse misallocation rate than information disclosure Υ^*, proving the result. □

D.4 Proofs from Section 5

Proof of Lemma 4. Follows from Appendix A. □

Proof of Theorem 3. Note that every agent is approached exactly once, and every agent who declines an item will eventually receive his preferred item (after waiting in the BQ). Therefore, the misallocation rate is equal to the fraction of offers in which a mismatched agents takes the current item.

An α agent takes a B item if the system is in state (K^A, B) and the approached agent is of type α. Symmetrically, a β agent takes an A item if the system is in state $(-K^B, A)$ and the approached agent is of type β. To obtain the probability of these events conditional on the system making an offer to an agent, we use the stationary distribution calculated in Lemma 10 in Appendix A. Denote the conditional probability of a state where the system makes an offer to an agent by

$$\begin{aligned} \hat{\pi}^X(k) &= \frac{\pi^X(k)}{\sum_k \pi^B(k) + \sum_k \pi^A(k)} \\ &= 2\pi^X(k) \end{aligned}$$

for $X \in \{A, B\}$. From Lemma 10, we have that if the system is balanced, the misallocation rate is

$$\begin{aligned} \xi &= p \cdot \hat{\pi}^B(K^A) + (1-p) \cdot \hat{\pi}^A(-K^B) \\ &= \frac{2p(1-p)}{(1-p)K^A + pK^B + 1}. \end{aligned}$$

If $p_\alpha \neq p_A$, the misallocation rate is

$$
\begin{aligned}
\xi &= p_\alpha \cdot \hat{\pi}^B(K^A) + p_\beta \cdot \hat{\pi}^A(-K^B) \\
&= p_\alpha \cdot p_B \left(\frac{p_\alpha}{p_A}\right)^{K^A} \cdot \frac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}} \\
&+ p_\beta \cdot p_A \left(\frac{p_\beta}{p_B}\right)^{K^B} \cdot \frac{p_A - p_\alpha}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}} \\
&= (p_A - p_\alpha) \frac{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} + p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}}{p_A p_\beta \left(\frac{p_\beta}{p_B}\right)^{K^B} - p_B p_\alpha \left(\frac{p_\alpha}{p_A}\right)^{K^A}} \\
&= (p_A - p_\alpha) \frac{\left(\frac{p_\beta}{p_B}\right)^{K^B+1} + \left(\frac{p_\alpha}{p_A}\right)^{K^A+1}}{\left(\frac{p_\beta}{p_B}\right)^{K^B+1} - \left(\frac{p_\alpha}{p_A}\right)^{K^A+1}} .
\end{aligned}
$$

We now show that the misallocation rate ξ is monotonically decreasing in K^A, K^B. This is immediate if the system is balanced. If $p_\alpha \neq p_A$, consider the derivatives $\frac{d\xi}{dK^A}$ and $\frac{d\xi}{dK^B}$. To simplify notation, we substitute $a = p_\alpha / p_A$ and $b = p_\beta / p_B$:

$$
\begin{aligned}
\frac{d\xi}{dK^A} &= \frac{d}{dK^A} \left[(p_A - p_\alpha) \frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}} \right] \\
&= \frac{2\,(p_A - p_\alpha) \cdot (\log a) \cdot a^{K^A+1} b^{K^B+1}}{\left(b^{K^B+1} - a^{K^A+1}\right)^2} \\
\frac{d\xi}{dK^B} &= \frac{-2\,(p_A - p_\alpha) \cdot (\log b) \cdot a^{K^A+1} b^{K^B+1}}{\left(b^{K^B+1} - a^{K^A+1}\right)^2} .
\end{aligned}
$$

If $p_A > p_\alpha$ (or $a < 1$), we have that $p_A - p_\alpha > 0$ and $\log a < 0$, which implies $\frac{d\xi}{dK^A} < 0$. If $p_A < p_\alpha$ (or $a > 1$), $p_A - p_\alpha < 0$ and $\log a > 0$, which again implies $\frac{d\xi}{dK^A} < 0$. A symmetric argument shows $\frac{d\xi}{dK^B} < 0$.

We next show the misallocation rate ξ converges to $|p_A - p_\alpha|$ as $K^A \to \infty$ or $K^B \to \infty$. If the system is balanced, this is immediate. For an unbalanced system, first consider the case in which $p_A > p_\alpha$, $a < 1$, and therefore $p_B < p_\beta$, $b > 1$:

$$\begin{aligned}
\lim_{K^A\to\infty} \xi &= \lim_{K^A\to\infty} (p_A - p_\alpha) \frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}} \\
&= (p_A - p_\alpha) \frac{b^{K^B+1} + 0}{b^{K^B+1} - 0} \\
&= p_A - p_\alpha = |p_A - p_\alpha| \\
\lim_{K^B\to\infty} \xi &= \lim_{K^B\to\infty} (p_A - p_\alpha) \frac{b^{K^B+1} + a^{K^A+1}}{b^{K^B+1} - a^{K^A+1}} \\
&= (p_A - p_\alpha) \cdot 1 = |p_A - p_\alpha| .
\end{aligned}$$

If $p_A < p_\alpha, a > 1$ and $p_B > p_\beta, b > 1$, we have that

$$\begin{aligned}
\lim_{K^A\to\infty} \xi &= (p_A - p_\alpha) \cdot -1 \\
&= p_\alpha - p_A = |p_A - p_\alpha| \\
\lim_{K^B\to\infty} \xi &= (p_A - p_\alpha) \cdot \frac{0 + a^{K^A+1}}{0 - a^{K^A+1}} \\
&= p_\alpha - p_A = |p_A - p_\alpha| .
\end{aligned}$$

Hence,

$$\lim_{K^A\to\infty} \xi = \lim_{K^B\to\infty} \xi = |p_A - p_\alpha| .$$

□

Proof of Lemma 5. To characterize equilibrium behavior, consider an α agent in position k. The α agent always accepts an A item. If the α agent is offered a B item, it must be that the agents in positions $1, \ldots, k-1$ declined the B item and are waiting for an A.[52] Thus, the agent faces expected wait for an A equal to $w_k = k/p_A$, which is the expected number of periods until k copies of A arrive. Thus, the α agent will decline a B if and only if $w_k \leq \bar{w}$, or $k \leq p_A v/c = p_A \bar{w}$. Symmetrically, a β agent in position k declines an A item if and only if $k \leq p_B \bar{w}$.

Observe that we can equivalently describe the waiting list with declines as an IC BQ mechanism with a FCFS BQ policy. The choice of an α agent in position k to decline a B item is equivalent to that agent choosing to join the k-th position in the A FCFS BQ,

[52]To see this claim is true, observe that the problem of the agent in the first position is stationary, and he will either immediately take a mismatched B or wait for an A (by assumption, the agent waits for an A when both options give the same utility). This argument implies the problem of the agent in the second position is stationary, and the claim follows by induction.

because under both the agent faces an expected wait of $w_k = k/p_A$, as for an A item they need to wait until k copies of A items arrive. The maximal size of the A BQ is given by the maximal K^A satisfying the IC constraint $w_K^A = K^A/p_A \le \bar{w}$. □

D.5 Proofs from Section 6

Proof of Lemma 6 and Lemma 6'. Observe that α agents must all be assigned before any agent joins the B BQ, and therefore their expected wait cannot be affected by the B BQ policy or the decisions of β agents to join the B BQ. □

Proof of Lemma 7. Consider the dynamics of the BQ, restricting attention to periods when the BQ is not empty. Little's Law (Little 1961) states that if L is the long-term average number of agents in the BQ (conditional on the BQ being non-empty), λ is the long-term average rate at which agents join the BQ, and w is the average time that an agent waits in the BQ, then it holds that

$$L = \lambda \mathsf{w}.$$

Because the number of agents in the BQ is independent of φ (i.e., independent of which agent is selected to receive an item and leave the BQ), we have that L is independent of φ. Using Lemma 10 of Appendix A, we can calculate that

$$\begin{aligned} L &= \sum_{k=1}^{K} k \cdot \frac{\pi(k,\phi)}{\sum_{k=1}^{K} \pi(k,\phi)} \\ &= \begin{cases} \frac{K+1}{2} & p_\alpha = p_A \\ K + \frac{p_A}{p_A - p_\alpha} + \frac{K}{(p_\alpha/p_A)^K - 1} & p_\alpha \neq p_A\,, \end{cases} \end{aligned}$$

where $\pi(k,\phi) / \sum_{k=1}^{K} \pi(k,\phi)$ is the conditional probability that the BQ holds k agents.

We have that $\lambda = p_A$, because the average number of agents that leave the BQ in a period is equal to the probability that an A item arrives (and one agent leaves). Because the number of agents in the BQ is bounded by a constant, the long run average rate at which agents join the BQ is equal to the long-run average rate at which agents leave the BQ. Thus, we have that

$$W(K) = \mathbb{E}[w_{\tilde{k}}] = \frac{L}{p_A} = \begin{cases} \frac{K+1}{2p} & \text{if } p_\alpha = p_A = p \\ \frac{K}{p_A} + \frac{1}{p_A - p_\alpha} + \frac{1}{p_A} \frac{K}{(p_\alpha/p_A)^K - 1} & \text{if } p_\alpha \neq p_A\,. \end{cases}$$

Last, we show that if (K,φ) is a IC buffer-queue policy, then it must be $W(K) \le \bar{w}$. To do so, we calculate $\mathbb{E}[w_{\tilde{k}}]$ from $\{w_k\}_{k=1}^{K}$. Let $\tilde{k}$ be a random variable whose support is

$1, \ldots, K$ and $P(\tilde{k} = k) > 0$ is equal to the probability that an agent in the BQ initially joined at position k. Because each agent who joins the BQ does so in some state (k, B), we have that

$$P(\tilde{k} = k) = \frac{\pi(k, B)}{\sum_{k=1}^{K} \pi(k, B)}.$$

By the law of iterated expectation, we have that

$$\mathbb{E}\left[w_{\tilde{k}}\right] = \sum_{k=1}^{K} w_k \cdot P(\tilde{k} = k).$$

If a BQ policy $\langle K, \varphi \rangle$ is IC, we have that $w_k \leq \bar{w}$ for all $k \leq K$, and therefore $W(K) = \mathbb{E}\left[w_{\tilde{k}}\right] \leq \bar{w}$. □

Proof of Corollary 3. By Lemma 7, if $\mathcal{M} = \left(K^A, \varphi^A, K^B, \varphi^B\right)$ is IC, then

$$W(K^A) = \frac{K^A + 1}{2p_A} \leq \bar{w},$$

or

$$K^A \leq \lfloor 2p_A \bar{w} \rfloor - 1.$$

Similarly, we have that

$$K^B \leq \lfloor 2p_B \bar{w} \rfloor - 1.$$

By Theorem 3, the misallocation rate is decreasing in K^A, K^B, and the bound follows. □

Proof of Theorem 4. By Theorem 3, the misallocation rate is decreasing in K^A, K^B. Therefore, welfare is increasing in K^A, K^B. The sizes K^A, K^B are constrained by the BF-IC requirement, which can be decomposed by Lemma 6' to requiring separately that $\left(K^A, \varphi^A\right)$ and $\left(K^B, \varphi^B\right)$ are each BF-IC. This argument reduces the problem to characterizing the maximal K' for which some φ' exists such that (K', φ') is BF-IC.

For ease of notation, we consider the policy for the A BQ and use p_A for the item arrival probability. Let (K, φ) be the policy of the BQ, and let $w_{k,\sigma}$ be the implied expected waits under the belief σ (note these expected waits are independent of p_α).

We establish an upper bound on K for any BF-IC (K, φ) by looking at the expected wait $w_{k,\hat{\sigma}}$ under the belief $\hat{\sigma} \equiv 1$. That is, $w_{k,\hat{\sigma}}$ is the expected wait when every agent who is offered a B item declines the B item and joins the A BQ. Equivalently, $w_{k,\hat{\sigma}}$ is the expected wait when $p_\alpha = 1$ and all agents are truthful. As in the proof of Lemma 7, we have that the average number of agents in the BQ is given by

$$\begin{aligned} L &= \sum_{k=1}^{K} k \cdot \frac{\pi(k,\phi)}{\sum_{k=1}^{K} \pi(k,\phi)} \\ &= \frac{K}{1-p_A^K} - \frac{p_A}{1-p_A}, \end{aligned}$$

which is monotonically increasing in K. Let $\mathsf{w}_{\hat{\sigma}}$ denote the average time an agent waits in the BQ. Using Little's Law, we have that

$$\begin{aligned} \mathsf{w}_{\hat{\sigma}} &= L/p_A. \\ &= \frac{K}{p_A\left(1-p_A^K\right)} - \frac{1}{1-p_A}. \end{aligned}$$

Let K^* be defined by

$$\begin{aligned} K^* &= \max\left\{K \mid \mathsf{w}_{\hat{\sigma}} \leq \bar{w}\right\} \\ &= \kappa^*\left(\bar{w}, 1, p_A\right). \end{aligned}$$

By the law of iterated expectation, we have

$$\mathsf{w}_{\hat{\sigma}} = \mathbb{E}\left[w_{\tilde{k},\hat{\sigma}}\right] = \sum_{k=1}^{K} w_{k,\hat{\sigma}} \cdot P(\tilde{k}=k).$$

Therefore, if (K,φ) is a BF-IC policy, we have that $\mathsf{w}_{\hat{\sigma}} \leq \bar{w}$ and $K \leq K^*$.

We proceed to show that $\left(K^*, \varphi^{\text{SIRO}}\right)$ is BF-IC. First, observe that under the belief $\hat{\sigma} \equiv 1$ for any positions $k, k' \leq K^*$, we have that $w_{k,\hat{\sigma}} = w_{k',\hat{\sigma}} = \mathsf{w}_{\hat{\sigma}}$. Because SIRO treats agents at all positions equally, the expected wait at the end of the period under SIRO is independent of the agent's position in the BQ. Under $\hat{\sigma}$, any agent joining the BQ believes the BQ will hold K^* agents at the end of the period. Therefore, all agents who join the BQ receive the same expected wait, and because $w_{k,\hat{\sigma}} = w_{k',\hat{\sigma}}$ for any $k, k' \leq K^*$, we have that $w_{k,\hat{\sigma}} = \mathsf{w}_{\hat{\sigma}} \leq \bar{w}$.

Second, we show that under the BQ policy $\left(K, \varphi^{\text{SIRO}}\right)$ for any belief σ and position k, we have that $w_{k,\sigma} \leq w_{k,\hat{\sigma}}$. To see that this inequality holds, consider an agent joining the BQ at position k, and fix the realized sequence of items arriving in future periods. Given the sequence, let $\ell(n)$ be the number of agents in the BQ when the n-th A item in the sequence arrives. The agent waits until an assignment, and conditional on not being assigned earlier, he is assigned the n-th item with probability $1/\ell(n)$. The agent's expected wait increases as $1/\ell(n)$ decreases, and therefore as $\ell(n)$ increases. The belief $\hat{\sigma}$ implies the maximal possible $\ell(n)$ for each n, regardless of the item process, as the number of agents $\ell(n)$ is maximized when the BQ reaches its maximal size each time a

B item arrives. Averaging over all item arrival sequences, we thus have that $w_{k,\sigma} \leq w_{k,\hat{\sigma}}$.

In summary, we have that $\left(K^A, \varphi^{\text{SIRO}}\right)$ with $K^A = \kappa^* (\bar{w}, 1, p_A)$ is BF-IC, and for any BF-IC BQ policy (K', φ'), it holds that $K' \leq \kappa^* (\bar{w}, 1, p_A)$. □

Proof of Lemma 8. Consider a symmetric equilibrium in which an α agent who is offered position k in the A BQ joins with probability $s(k) \in [0, 1]$. Let $\sigma(k) = p_\alpha \cdot s(k)$ denote the correct equilibrium beliefs. We first show that for any s, the expected wait $w_{k,\sigma}$ is increasing in k. Fix the sequence of items arriving in future periods and compare two trajectories starting in period t, the first starting with k agents and the second with $k' > k$ agents. We couple the two trajectories and consider the number of agents in the BQ in each period until the BQ empties under the first trajectory. Let $\hat{t} > t+1$ be the first period in which the two trajectories have an equal number of agents, or the BQ empties under the first trajectory. Note that in every period from $t+1$ to $\hat{t}$, the BQ has strictly fewer agents under the first trajectory. The probability that a given agent in the BQ is assigned the n-th arriving A item is $1/\ell(n)$ if $\ell(n)$ agents are in the BQ at the beginning of period when the n-th item arrives, which is strictly decreasing in $\ell(n)$. If an A item arrives between $t+1$ and $\hat{t}$, a given agent in the BQ has higher assignment probability under the first trajectory. If any agents remain in the BQ under the first trajectory in period $\hat{t}$, their future assignment probabilities are equal under the two trajectories. Averaging over all sequences of item arrivals, we find that under the first trajectory, a given agent in the BQ has a weakly higher probability of getting assigned in all periods under the first trajectory, and a strictly higher probability in some periods. Therefore, agents face a higher expected wait under the second trajectory, and the expected wait $w_{k,\sigma}$ is strictly monotonically decreasing in k.

The agent's best response implies $s(k) > 0$ only if $w_{k,\sigma} \leq \bar{w}$ and $s(k) < 1$ only if $w_{k,\sigma} \geq \bar{w}$. Together with the monotonicity of $w_{k,\sigma}$, the best response implies the existence of x^* such that[53] $s(k) = \mathbf{1}_{\{k \leq x^*\}} + \mathbf{1}_{\{\lfloor x^* \rfloor < k < \lfloor x^* \rfloor + 1\}} [x^*]$, that is, $s(k) = 1$ for $k \leq \lfloor x^* \rfloor$, $s(k) = 0$ for $k > \lfloor x^* \rfloor + 1$ and $s(k) = [x^*] = x^* - \lfloor x^* \rfloor$ for $k = \lfloor x^* \rfloor + 1$.

The same coupling argument shows $w_{k,\sigma}$ is strictly increasing in x^* (and therefore, a unique equilibrium exists). Let $\left(K^*, \varphi^{\text{SIRO}}\right)$ be the BF-IC SIRO BQ policy with the maximal $K^* = \kappa^* (\bar{w}, 1, p_A)$, and let $w_{k,\hat{\sigma}}$ be the implied wait under the belief $\hat{\sigma} \equiv 1$. Let $\sigma'(k) = p_\alpha \cdot \mathbf{1}_{\{k \leq K^*\}}$ be the belief that agents join position 1 to K^*. From the proof of Theorem 4, we have that

$$w_{k,\sigma'} \leq w_{k,\hat{\sigma}} \leq \bar{w},$$

[53] The fractional part of x^* is denoted by $[x^*] = x^* - \lfloor x^* \rfloor$.

and therefore we must have that $x^* \geq K^*$. □

Proof of Lemma 9. At the end of a period, all agents in the BQ have the same expected utility. The monotonicity arguments used in the proof of Lemma 8 show an agent is weakly better off if no agents join after him. Thus, the last agent would have chosen to join the BQ if he knew no agents would join after him. Because the last agent prefers joining the BQ to being assigned immediately to a mismatched item, all other agents prefer to stay in the BQ as well. □