Online Appendix for

A Linear Panel Model with Heterogeneous Coefficients and Variation in Exposure

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This appendix formalizes claims made in the paper.

Claim 1. In the setting of Section “The Possibility of Heterogeneous Coefficients,” the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data $x = \{x_{10}, ..., x_{S0}\}$ for states $s \in \{1, ..., S\}$, is given by

$$E(\hat{\beta}|x) = \frac{\text{Cov}(\beta_s (1-x_{s0}), (1-x_{s0}))}{\text{Var}(1-x_{s0})}$$

where Cov $(\cdot, \cdot)$ and Var $(\cdot)$ denote the sample covariance and variance, respectively, and the expectation $E(\hat{\beta}|x)$ is taken with respect to the distribution of the errors $\varepsilon_{st}$ conditional on the data $x = \{x_{10}, ..., x_{S0}\}$.

Proof. With only two time periods the TWFE estimator of the exposure model is equivalent to an OLS estimator of the first-differenced model

$$y_{s1} - y_{s0} = \delta_1 - \delta_0 + \beta (1-x_{s0}) + \varepsilon_{s1} - \varepsilon_{s0}.$$ 

Therefore the TWFE estimator based on the given sample is

$$\hat{\beta} = \frac{\text{Cov}(y_{s1} - y_{s0}, 1-x_{s0})}{\text{Var}(1-x_{s0})}.$$

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From the heterogeneous model we have that
\[ y_{s1} - y_{s0} = \delta_1 - \delta_0 + \beta_s (1 - x_{s0}) + \varepsilon_{s1} - \varepsilon_{s0} \]
and therefore
\[ \hat{\beta} = \frac{\text{Cov} (\beta_s (1 - x_{s0}) , 1 - x_{s0})}{\text{Var} (1 - x_{s0})} + \frac{\text{Cov} (\varepsilon_{s1} - \varepsilon_{s0} , 1 - x_{s0})}{\text{Var} (1 - x_{s0})}. \]
If \((\varepsilon_{s1} - \varepsilon_{s0})\) is mean zero conditional on \((1 - x_{s0})\) then the expected value of \(\hat{\beta}\) conditional on the data \(x = \{x_{10}, ..., x_{S_0}\}\) is
\[ \mathbb{E} (\hat{\beta} | x) = \frac{\text{Cov} (\beta_s (1 - x_{s0}) , 1 - x_{s0})}{\text{Var} (1 - x_{s0})}. \]

**Corollary 1.** In the setting of Section “The Possibility of Heterogeneous Coefficients,” if \(\beta_s\) is independent of \(x_{s0}\) across states \(s\), then the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data \(x = \{x_{10}, ..., x_{S_0}\}\) for states \(s \in \{1, ..., S\}\), is given by
\[ \mathbb{E} (\hat{\beta} | x) = \mathbb{E} (\beta_s) \]
for \(\mathbb{E} (\beta_s)\) the expected value of \(\beta_s\). Here the expectation \(\mathbb{E} (\hat{\beta} | x)\) is taken with respect to the distribution of the errors \(\varepsilon_{st}\) and coefficients \(\beta_s\) conditional on the data \(x\).

**Proof.** Based on a similar proof for Claim 1, we have that
\[ \mathbb{E} (\hat{\beta} | x) = \frac{\mathbb{E} (\text{Cov} (\beta_s (1 - x_{s0}) , 1 - x_{s0})))}{\text{Var} (1 - x_{s0})} \]
where the expectation is now taken with respect to the distribution of the errors \(\varepsilon_{st}\) as well as \(\beta_s\) conditional on the data \(x = \{x_{10}, ..., x_{S_0}\}\). By the independence of \(\beta_s\) and \(x_{s0}\), we have that
\[ \mathbb{E} (\text{Cov} (\beta_s (1 - x_{s0}) , 1 - x_{s0})) = \text{Cov} (\mathbb{E} (\beta_s) (1 - x_{s0}) , 1 - x_{s0}) = \mathbb{E} (\beta_s) \text{Var} (1 - x_{s0}), \]
and therefore that
\[ E(\hat{\beta}|x) = E(\beta_s). \]

\[ \square \]

**Corollary 2.** In the numerical example of Section “The Possibility of Heterogeneous Coefficients,” the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data \( x = \{x_{10}, \ldots, x_{S0}\} \) for states \( s \in \{1, \ldots, S\} \), lies outside the range of coefficients \([\min_s \beta_s, \max_s \beta_s]\) if and only if \( \lambda \neq 0 \). The same continues to hold when the sample is extended to include a totally unaffected state.

**Proof.** From Claim 1 we have that
\[ E(\hat{\beta}|x) = \frac{\text{Cov}(\beta_s (1 - x_{s0}), 1 - x_{s0})}{\text{Var}(1 - x_{s0})}. \]

Because in the numerical example \( \beta_s = 1 + 0.5\lambda - \lambda x_{s0} \), we have that
\[ E(\hat{\beta}|x) = 1 + 0.5\lambda - \lambda C \]
for
\[ C = \frac{\text{Cov}(x_{s0} (1 - x_{s0}), (1 - x_{s0}))}{\text{Var}(1 - x_{s0})}. \]

In the setting of Section “The Possibility of Heterogeneous Coefficients,” given the data \( x = \{x_{10}, \ldots, x_{S0}\} \) where \( x_{s0} = 0.245 + s/100 \) for \( s = 1, \ldots, 50 \), by direct calculation we have that \( C \approx 0 \), which means that
\[ E(\hat{\beta}|x) = 1 + 0.5\lambda. \]

If we add to the sample a totally unaffected state \( s = 0 \) with \( x_{00} = 1 \), and the remaining states \( s = 1, \ldots, 50 \) continue to follow \( x_{s0} = 0.245 + s/100 \), by direct calculation we have that \( C \approx 0.087 \), which means that
\[ E(\hat{\beta}|x) \approx 1 + 0.413\lambda. \]

Therefore, with or without a totally unaffected state, when \( \lambda > 0 \) we have \( E(\hat{\beta}|x) > \beta_s \) for all \( s \) because \( \max_s \beta_s = 1 + 0.245\lambda \). Similarly, with or without a totally unaffected state, when \( \lambda < 0 \) we have \( E(\hat{\beta}|x) < \beta_s \) for all \( s \) because
\[ \min_s \beta_s = 1 + 0.245\lambda. \] Finally, with or without a totally unaffected state, when \( \lambda = 0 \) we have \( \mathbb{E}\left(\hat{\beta}|x\right) = 1 = \mathbb{E}\left(\beta_s\right) = \max_s \beta_s = \min_s \beta_s. \)

**Claim 2.** In the setting of Section “The Possibility of Heterogeneous Coefficients,” there exists no estimator \( \hat{\beta}' \) that can be expressed as a function of the data \( \{(x_{s0}, y_{s0}, y_{s1})\}_{s=1}^S \) and whose expected value is guaranteed to be contained in \([\min_s \beta_s, \max_s \beta_s]\) for any heterogeneous model and any \( \{x_{s0}\}_{s=1}^S \).

**Proof.** It is sufficient to establish this claim for a special case with \( S = 2 \), some \( x_{s0} \)'s with \( 0 < x_{20} \leq x_{10} < 1 \), \( \beta_1 < \beta_2 \), and \( \delta_0 \) known to be zero. The model for the data is then

\[
\begin{align*}
y_{s0} &= \alpha_s + \beta_s \cdot x_{s0} + \varepsilon_{s0} \\
y_{s1} &= \alpha_s + \delta_1 + \beta_s + \varepsilon_{s1}
\end{align*}
\]

with parameters \( \theta = \{(\alpha_s, \beta_s)\}_{s=1}^2, \delta_1, F_{\varepsilon|X} \), for \( F_{\varepsilon|X} \) the distribution of \((\varepsilon_{s0}, \varepsilon_{s1})\) conditional on \( x_{s0} \). Pick some estimator \( \hat{\beta}' \). Given any parameter \( \theta \), define the distinct parameter \( \theta' = \{(\alpha'_s, \beta'_s)\}_{s=1}^2, \delta'_1, F_{\varepsilon|X} \) given by

\[
\theta' = \left\{ \left( \alpha_s + \frac{\Delta \cdot x_{s0}}{1-x_{s0}}, \beta_s - \frac{\Delta}{1-x_{s0}} \right) \right\}_{s=1}^2, \delta_1 + \Delta, F_{\varepsilon|X} \)
\]

for some \( \Delta > (\beta_2 - \beta_1) \cdot (1-x_{20}) > 0 \).

We show that the two parameter values \( \theta \) and \( \theta' \) are observationally equivalent, which means the expected value of \( \hat{\beta}' \) must be the same under \( \theta \) and \( \theta' \). To see this, note that the distribution of \((y_{s0}, y_{s1})\) conditional on \( x_{s0} \) is the same under \( \theta \) and \( \theta' \):
However, the $\Delta$ is chosen such that 

$$\beta'_1 = \beta_1 - \frac{\Delta}{1-x_{10}} < \beta_2 - \frac{\Delta}{1-x_{20}} = \beta'_2 < \beta_1 < \beta_2.$$ 

Therefore the expected value of $\hat{\beta}'$ cannot be contained in both $[\beta_1, \beta_2]$ and $[\beta'_1, \beta'_2]$, because these intervals do not intersect. 

**Claim 3.** In the setting of Section “A Difference-in-Differences Perspective,” the exposure-adjusted difference-in-differences estimator $\hat{\beta}_{s,s'}^{DID}$ is equivalent to the TWFE estimator $\hat{\beta}$ based on the two states $s$ and $s'$. Moreover, the expected value of $\hat{\beta}_{s,s'}^{DID}$, given the data $x = \{x_s, x_{s'}\}$ for states $s$ and $s'$, is given by

$$E(\hat{\beta}_{s,s'}^{DID} | x) = \frac{(1 - x_{s0}) \beta_s - (1 - x_{s'0}) \beta_{s'}}{x_{s'0} - x_{s0}},$$

where the expectation $E(\hat{\beta}_{s,s'}^{DID} | x)$ is taken with respect to the distribution of the errors $\varepsilon_{st}$ conditional on the data $x = \{x_{s0}, x_{s'0}\}$.

**Proof.** For the first part of the claim, note that from the proof of Claim 1 we have

$$\hat{\beta} = \frac{Cov(y_{s1} - y_{s0}, 1 - x_{s0})}{Var(1 - x_{s0})},$$

where $\text{Cov}(\cdot, \cdot)$ and $\text{Var}(\cdot)$ denote the sample covariance and variance, respectively.
Since the sample includes only two states \(s\) and \(s'\), for the numerator we have

\[
\text{Cov} \left( \frac{y_{s1} - y_{s0}}{1 - x_{s0}}, \frac{1 - x_{s0}}{1 - x_{s'0}} \right) = \frac{1}{4} \left( (y_{s1} - y_{s0}) - (y_{s'1} - y_{s'0}) \right) \left( 1 - x_{s0} \right) + \frac{1}{4} \left( (y_{s'1} - y_{s'0}) - (y_{s1} - y_{s0}) \right) \left( 1 - x_{s'0} \right)
\]

\[
= \frac{1}{4} \left( (1 - x_{s0}) - (1 - x_{s'0}) \right) \left( (y_{s1} - y_{s0}) - (y_{s'1} - y_{s'0}) \right)
\]

where the first equality applies the definition of sample covariance and \(a - \frac{a+b}{2} = \frac{a-b}{2}\).

Similarly, for the denominator we have

\[
\text{Var} \left( \frac{1 - x_{s0}}{1 - x_{s'0}} \right) = \frac{1}{4} \left( (1 - x_{s0}) - (1 - x_{s'0}) \right)^2.
\]

Plugging the above expressions into \(\hat{\beta}\) gives the equivalence to \(\hat{\beta}_{\text{DID}}^{s,s'}\).

Given the equivalence between \(\hat{\beta}\) and \(\hat{\beta}_{\text{DID}}^{s,s'}\) when the sample includes only two states \(s\) and \(s'\), we apply Claim 1 to derive the expected value of \(\hat{\beta}_{\text{DID}}^{s,s'}\). Specifically, Claim 1 implies that given the data \(x = \{x_{s0}, x_{s'0}\}\) for states \(s\) and \(s'\), we have

\[
E \left( \hat{\beta}_{\text{DID}}^{s,s'} \middle| x \right) = \frac{\text{Cov} \left( \frac{\beta_s (1 - x_{s0})}{1 - x_{s0}}, \frac{1 - x_{s0}}{1 - x_{s'0}} \right)}{\text{Var} \left( \frac{1 - x_{s0}}{1 - x_{s'0}} \right)}.
\]

Based on a similar simplification to the expression of \(\hat{\beta}_{\text{DID}}^{s,s'}\), we have

\[
\text{Cov} \left( \frac{\beta_s (1 - x_{s0})}{1 - x_{s0}}, \frac{1 - x_{s0}}{1 - x_{s'0}} \right) = \frac{1}{4} \left( (1 - x_{s0}) - (1 - x_{s'0}) \right) \left( \frac{\beta_s - (1 - x_{s'0}) \beta_s'}{x_{s'0} - x_{s0}} \right)
\]

and therefore

\[
\frac{\text{Cov} \left( \frac{\beta_s (1 - x_{s0})}{1 - x_{s0}}, \frac{1 - x_{s0}}{1 - x_{s'0}} \right)}{\text{Var} \left( \frac{1 - x_{s0}}{1 - x_{s'0}} \right)} = \frac{(1 - x_{s0}) \beta_s - (1 - x_{s'0}) \beta_s'}{x_{s'0} - x_{s0}}.
\]

\(\square\)