Grounded by Gravity:
A Well-Behaved Trade Model with Industry-Level Economies of Scale

Online Appendix

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1. Formal Definitions

For convenience, let us repeat here our key system of complementary slackness conditions that equilibrium labor allocation in industry (i,k) must satisfy:

$$L_{i,k} \geq 0, \quad G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k) \geq 0, \quad L_{i,k} G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k) = 0. \tag{1}$$

It is clear that both functions $G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ and $L_{i,k} G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ which appear in the nonlinear complementarity problem (1) are well-defined for all positive wages and positive labor allocations, i.e., for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$ and $\boldsymbol{L}_k \in \mathbb{R}^N_{++}$. We are interested in extending the definitions of $G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ and $L_{i,k} G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ to the set of all non-negative labor allocations excluding the point with $L_{i,k} = 0$ for all i, i.e., to the set $\mathbb{R}^N_+ \setminus \{\mathbf{0}\}$. To this end, we allow for function $G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ to take infinite values. Formally, we consider the function $G_{i,k} : \mathbb{R}^N_{++} \times \mathbb{R}^N_+ \setminus \{\mathbf{0}\} \to \mathbb{R} \cup \{-\infty, +\infty\}$,[1] and for each given vector of wages $\boldsymbol{w} \in \mathbb{R}^N_{++}$ and vector of labor allocations $\boldsymbol{L}_k \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ we formally define $G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$

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[1] Here the set $\mathbb{R} \cup \{-\infty, +\infty\}$ is the extended real number system with symbols $-\infty$ and $+\infty$ following the standard conventions (see, for example, p. 11-12 in Rudin, 1976). In particular, for any $x \in \mathbb{R}$, $-\infty < x < +\infty$.

and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ by the limits

$$G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)\equiv\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\left[w_i-\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)\beta_{n,k}w_n\bar{L}_n\right]$$

and

$$L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)\equiv\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}x_i^t\left[w_i-\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)\beta_{n,k}w_n\bar{L}_n\right],$$

where $\left\{\boldsymbol{x}^t\right\}_{t=1}^{\infty}$ is any sequence converging to $\boldsymbol{L}_k$ such that $\boldsymbol{x}^t\in\mathbb{R}_{++}^N$ for $t=1,2,\ldots$.

Let us verify that functions $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ are well-defined. Since for all $\boldsymbol{L}_k\in\mathbb{R}_{++}^N$ functions $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ are well-defined and continuous, the above limits coincide with the values of these functions in the corresponding points.

Next, consider any sequence $\left\{\boldsymbol{x}^t\right\}_{t=1}^{\infty}$ with $\boldsymbol{x}^t\in\mathbb{R}_{++}^N$ for $t=1,2,\ldots$ and converging to $\boldsymbol{L}_k$. We have

$$\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)=\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\left[x_i^t\right]^{\alpha_k-1}\sum_n\frac{S_{i,k}\left(w_i\tau_{ni,k}\right)^{-\varepsilon_k}}{\sum_l S_{l,k}\left[x_l^t\right]^{\alpha_k}\left(w_l\tau_{nl,k}\right)^{-\varepsilon_k}}\beta_{n,k}w_n\bar{L}_n.$$

Then, since, $\boldsymbol{L}_k\neq\boldsymbol{0}$,

$$\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\sum_l S_{l,k}\left[x_l^t\right]^{\alpha_k}\left(w_l\tau_{nl,k}\right)^{-\varepsilon_k}=\sum_l S_{l,k}L_{l,k}^{\alpha_k}\left(w_l\tau_{nl,k}\right)^{-\varepsilon_k}>0\ \text{ for all } n.$$

Hence, $\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)=\infty$ if $L_{i,k}=0$ and $0\leq\alpha_k<1$, and $\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)$ is a positive number if $L_{i,k}>0$ or if $\alpha_k\geq 1$. This, in turn, implies that

$$\begin{aligned}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)&=\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\left[w_i-\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)\beta_{n,k}w_n\bar{L}_n\right]\\&=\begin{cases}-\infty, & \text{if } L_{i,k}=0 \text{ and } 0\leq\alpha_k<1,\\ \text{finite number}, & \text{if } L_{i,k}>0 \text{ or } \alpha_k\geq 1.\end{cases}\end{aligned}$$

So, the limit always exists and is either $-\infty$ or a finite number. Hence, function $G_{i,k}$ is

well-defined with its codomain given by the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$.

Similarly, it is easy to verify that $\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} x_i^t \left[w_i - \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t) \beta_{n,k} w_n \bar{L}_n \right]$ always exists. Moreover, this limit is always a finite number. Hence, function $L_{i,k} G_{i,k}$ is also well-defined.

2. Characterizing Corner Allocations in Economic Geography Models

Consider an Allen and Arkolakis (2014) setup with N locations indexed by i, n, and j. The equilibrium system of equations is given by

$$L_i \geq 0, \quad \bar{U} - U_i \geq 0, \quad L_i \left(\bar{U} - U_i \right) = 0,$$

$$w_i L_i = \sum_{n=1}^{N} L_i^{\alpha} \left(\frac{w_i \tau_{ni}}{\bar{A}_i} \right)^{1-\sigma} P_n^{\sigma-1} w_n L_n, \tag{2}$$

$$\sum_{i=1}^{N} L_i = \bar{L},$$

$$U_i = \frac{w_i}{P_i} \bar{u}_i L_i^{-\delta}, \tag{3}$$

$$P_n^{1-\sigma} = \sum_j L_j^{\alpha} \left(\frac{w_j \tau_{nj}}{\bar{A}_j} \right)^{1-\sigma},$$

where L_i is employment, U_i is welfare, P_i is the price index, w_i is the wage, $\bar{A}_i$ is the exogenous productivity, and $\bar{u}_i$ is an exogenous amenity value – all in location i. τ_{ni} is the iceberg trade cost between locations n and i. σ is the elasticity of substitution between goods produced in different locations, $\alpha = \phi(\sigma - 1)$ with $\phi > 0$ being the scale elasticity (as in our paper), and $\delta > 0$ governs the strength of congestion externalities. $\bar{L}$ is the total amount of labor in the economy. How do we check if $L_i = 0$ is possible in equilibrium? Following Allen and Arkolakis, we start by considering $L_i > 0$, solving for w_i from (2) to get

$$w_i = L_i^{\frac{\alpha-1}{\sigma}} \left(\sum_{n=1}^{N} \left(\frac{\tau_{ni}}{\bar{A}_i} \right)^{1-\sigma} P_n^{\sigma-1} w_n L_n \right)^{\frac{1}{\sigma}}$$

substituting it into (3) to get

$$U_i = \left(\sum_{n=1}^{N} \left(\frac{\tau_{ni}}{\bar{A}_i} \right)^{1-\sigma} P_n^{\sigma-1} w_n L_n \right)^{\frac{1}{\sigma}} \frac{\bar{u}_i}{P_i} L_i^{\frac{\alpha-1}{\sigma}-\delta},$$

and then taking the limit $L_i \to 0$. This argument leads us to the conclusion that $L_i = 0$ is possible if and only if $\frac{\alpha-1}{\sigma} \geq \delta$, as shown by Allen and Arkolakis. If $\delta = 0$ then this condition is $\alpha \geq 1$, just as in our trade model.

3. An Armington Model with Heterogeneous Labor Supply

In this section we present a version of an Armington model with external economies of scale and heterogeneous labor supply as in Galle, Rodríguez-Clare and Yi (2022). Similarly to our baseline model of Section 2 of the main text, each country $i = 1, \ldots, N$ produces are differentiated good in each industry $k = 1, \ldots, K$. Labor productivity in good (i,k) is $A_{i,k} E_{i,k}^{\phi_k}$, where $E_{i,k}$ is the amount of efficiency units of labor employed in industry (i,k) and $A_{i,k}$ is an exogenous productivity parameter. Parameter ϕ_k plays the role of the strength of economies of scale in terms of efficiency units of labor. A worker in country i can supply $a_{i,k}$ efficiency units to industry k, with $a_{i,k}$ drawn from a Fréchet distribution with shape parameter $\kappa > 1$ and scale parameter $\mathcal{T}_{i,k}$. The total number of workers in country i is $\bar{L}_i$.

Since workers are heterogeneous in their sector productivities, wages can differ across sectors. Denote by $w_{i,k}$ wage per efficiency unit in industry (i,k). The industry-level bilateral trade shares are given by

$$\lambda_{ni,k}\left(\boldsymbol{w}_k, \boldsymbol{E}_k\right) = \frac{A_{i,k}^{\sigma_k-1} \left(\tau_{ni,k} w_{i,k} / E_{i,k}^{\phi_k} \right)^{-(\sigma_k-1)}}{\sum_l A_{l,k}^{\sigma_k-1} \left(\tau_{nl,k} w_{l,k} / E_{l,k}^{\phi_k} \right)^{-(\sigma_k-1)}}, \tag{4}$$

where $\boldsymbol{w}_k \equiv (w_{1,k}, \ldots, w_{N,k})$ and $\boldsymbol{E}_k \equiv (E_{1,k}, \ldots, E_{N,k})$. Equilibrium allocations of efficiency units in industry (i,k) satisfy the following complementary slackness condition:

$$\frac{w_{i,k} E_{i,k}}{\Phi_i} \geq 0, \quad G_{i,k}\left(\boldsymbol{w}_k, \boldsymbol{E}_k\right) \geq 0, \quad \frac{w_{i,k} E_{i,k}}{\Phi_i} G_{i,k}\left(\boldsymbol{w}_k, \boldsymbol{E}_k\right) = 0, \tag{5}$$

where

$$G_{i,k}\left(\boldsymbol{w}_k, \boldsymbol{E}_k\right) = \Phi_i - \frac{\Phi_i}{w_{i,k}E_{i,k}} \sum_n \lambda_{ni,k}\left(\boldsymbol{w}_k, \boldsymbol{E}_k\right) \beta_{n,k}\eta\Phi_n\bar{L}_n, \tag{6}$$

$$\Phi_i \equiv \left(\sum_s \mathcal{T}_{i,s} w_{i,s}^{\kappa}\right)^{\frac{1}{\kappa}}, \tag{7}$$

and $\eta > 0$ is a constant. Since $w_{i,k}E_{i,k}/\Phi_i$ is proportional to employment in sector (i,k) and Φ_i is proportional to the wage per person in country i, this definition of function $G_{i,k}$ is a straightforward extension of the one used for the baseline model with homogeneous labor.[2] Supply of efficiency units in industry (i,k) is given by

$$E_{i,k} = \eta \frac{\mathcal{T}_{i,k} w_{i,k}^{\kappa-1}}{\Phi_i^{\kappa-1}} \bar{L}_i, \tag{8}$$

and the total income in country i can be calculated as

$$Y_i = \sum_k w_{i,k}E_{i,k} = \eta\Phi_i\bar{L}_i. \tag{9}$$

The equilibrium of the model is given by sector wages $\boldsymbol{w}_k$ and efficiency labor allocations $\boldsymbol{E}_k$, for $k = 1, \ldots, K$, that satisfy the equilibrium system given by (4)-(9). In order to show equivalence of this equilibrium system to the one in the common framework of Section 3 of the main text, use (8) to express $w_{i,k} = \left(\eta\bar{L}_i\mathcal{T}_{i,k}\right)^{\frac{1}{1-\kappa}} E_{i,k}^{\frac{1}{\kappa-1}} \Phi_i$, relabel Φ_i as w_i, and let

$$L_{i,k} \equiv \eta^{-1} \frac{w_{i,k}E_{i,k}}{w_i} = \eta^{\frac{\kappa}{1-\kappa}} \left(\bar{L}_i\mathcal{T}_{i,k}\right)^{\frac{1}{1-\kappa}} E_{i,k}^{\frac{\kappa}{\kappa-1}}.$$

[2] Specifically, as we show below, the number of workers employed in sector (i,k) is proportional to $w_{i,k}E_{i,k}/\Phi_i$, while Φ_i is proportional to the wage in country i. It is easy to check that for $\tilde{\alpha}_k \in (1 - 1/\kappa, 1]$ the alternative definition $G_{i,k} = w_{i,k} - \frac{1}{E_{i,k}} \sum_n \lambda_{ni,k}\beta_{n,k}Y_n$ with complementary slackness condition (5) changed accordingly, would generate corner labor allocations that satisfy all conditions (4)-(9) and, nevertheless, are not equilibrium allocations.

In this new notation $E_{i,k} = \eta \left(\bar{L}_i \mathcal{T}_{i,k}\right)^{\frac{1}{\kappa}} L_{i,k}^{\frac{\kappa-1}{\kappa}}$ and trade shares are given by

$$\tilde{\lambda}_{ni,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) = \frac{S_{i,k}\left(\tau_{ni,k} w_i\right)^{-(\sigma_k-1)} L_{i,k}^{\tilde{\alpha}_k}}{\sum_l S_{l,k}\left(\tau_{nl,k} w_l\right)^{-(\sigma_k-1)} L_{l,k}^{\tilde{\alpha}_k}},$$

where $S_{i,k} \equiv \left(A_{i,k}\eta^{\phi_k}\left(\bar{L}_i \mathcal{T}_{i,k}\right)^{\frac{1+\phi_k}{\kappa}}\right)^{\sigma_k-1}$ and $\tilde{\alpha}_k \equiv (\sigma_k - 1)\left(\phi_k - \frac{1}{\kappa-1}\right)\frac{\kappa-1}{\kappa}$. Goods market clearing conditions (5)-(6) can be rewritten as

$$L_{i,k} \geq 0, \quad \tilde{G}_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) \geq 0, \quad L_{i,k}\tilde{G}_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) = 0,$$

where

$$\tilde{G}_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) = w_i - \frac{1}{L_{i,k}} \sum_n \tilde{\lambda}_{ni,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) \beta_{n,k} w_n \bar{L}_n.$$

Finally, expression (7) can be manipulated to yield the labor market clearing condition:

$$\begin{aligned} w_i &= \left(\sum_s \mathcal{T}_{i,s} w_{i,s}^{\kappa}\right)^{\frac{1}{\kappa}} = \left(\sum_s \mathcal{T}_{i,s}\left(\eta \bar{L}_i \mathcal{T}_{i,s}\right)^{\frac{\kappa}{1-\kappa}} E_{i,s}^{\frac{\kappa}{\kappa-1}} w_i^{\kappa}\right)^{\frac{1}{\kappa}} \\ &= \left(\sum_s \mathcal{T}_{i,s}\left(\bar{L}_i \mathcal{T}_{i,s}\right)^{-1} L_{i,s}\right)^{\frac{1}{\kappa}} w_i, \end{aligned}$$

which is equivalent to $\sum_s L_{i,s} = \bar{L}_i$. Hence, indeed, we get the equilibrium system that is equivalent to the one for the common framework of Section 3 of the main text. Condition for uniqueness in the common framework, $\tilde{\alpha}_k \leq 1$, is equivalent to $\left(\phi_k - \frac{1}{\kappa-1}\right)\frac{\kappa-1}{\kappa} \leq \frac{1}{\sigma_k - 1}$. As $\kappa \to \infty$, we get the same condition in terms of parameters σ_k and ϕ_k as in our baseline Armington model of Section 2 of the main text, but in general we can allow for stronger economies of scale in the model with heterogenous labor — given by ϕ_k — while still having condition $\tilde{\alpha}_k \leq 1$ satisfied.

4. Properties of Equilibrium

For brevity of exposition of the results in this and the following sections, it is useful to formally introduce the following assumption

Assumption OA.1. *The matrix*

$$\begin{pmatrix} \tau_{11,k}^{-\varepsilon_k} & \cdots & \tau_{1N,k}^{-\varepsilon_k} \\ \vdots & & \vdots \\ \tau_{N1,k}^{-\varepsilon_k} & \cdots & \tau_{NN,k}^{-\varepsilon_k} \end{pmatrix}$$

is non-singular.

4.1. Continuity of Labor Allocations With Respect To α

Here we use notation used in the proof of Lemmas 1-3 in Appendix B in the main text.

One might wonder if the equilibrium labor allocation is continuous in α as we approach $\alpha = 1$ from below. Economically speaking, one would expect this to be the case, so that if at $\alpha = 1$ we have a corner allocation with $x_i = 0$ for some country i then $x_i(\alpha) > 0$ for all $\alpha < 1$ but $\lim_{\alpha \uparrow 1} x_i(\alpha) \to 0$. Mathematically, however, this result is not trivial because the function $\boldsymbol{G}$ is not jointly continuous in $\boldsymbol{x}$ and α for $\alpha = 1$ and points $\boldsymbol{x}$ with $x_i = 0$ for some i. Still, thanks to the optimization approach followed in the previous lemmas, we can establish the left continuity of $\boldsymbol{x}(\alpha)$.

Lemma OA.1. *If Assumption OA.1 holds, then $\boldsymbol{x}(\alpha)$ is continuous as a function of α for all $\alpha \in (0,1]$. In particular,* $\lim_{\alpha \uparrow 1} \boldsymbol{x}(\alpha) = \boldsymbol{x}(1)$.

Proof. Let us formally bring argument α into the notation of function F defined in Appendix B of the main text as

$$F(\boldsymbol{x}) \equiv \alpha \sum_n x_n - \sum_n b_n \ln \left(\sum_i a_{ni} x_i^{\alpha} \right). \tag{10}$$

That is, consider the function $F(\boldsymbol{x};\alpha)$. Lemma 1 in Appendix B in the main text establishes that under Assumption OA.1 the solution to the optimization problem $\min_{x \in \Gamma} F(\boldsymbol{x};\alpha)$ defines a function $\boldsymbol{x} : (0,1] \to \mathbb{R}_+^N \setminus \{\mathbf{0}\}$. Clearly, $F(\boldsymbol{x};\alpha)$ is continuous for all $\boldsymbol{x} \in \mathbb{R}_+^N \setminus \{\mathbf{0}\}$ and $\alpha \in (0,1]$. Γ is a compact set which is the same for all $\alpha \in (0,1]$. Thus, all conditions for Theorem of the Maximum (Theorem 3.6) from Stokey, Lucas and Prescott (1989) are statisfied, and $\boldsymbol{x}(\alpha)$ is continuous for all $\alpha \in (0,1]$. □

4.2. Existence of Equilibrium

In the cases with $0 \leq \alpha_k < 1$ or with $\alpha_k = 1$ and Assumption OA.1 satisfied, Proposition 1 in the main text implies that the solution of the system of complementary slackness conditions (1) determines a univalent function from wages to labor allocations, $\boldsymbol{L}_k(\boldsymbol{w})$ for $\boldsymbol{w} \in \mathbb{R}^N_{++}$. When we prove uniqueness of equilibrium below, we use this function to construct the labor excess demand system and to show that there is only one wage vector that clears all labor markets. Assumption OA.1 is a regularity assumption that helps us establish uniqueness in the case of $\alpha_k = 1$, but its violation does not affect existence of equilibrium. In fact, it is possible to show equilibrium existence for any set of non-negative α_k without relying on additional assumptions. The key result that can be invoked to establish existence is Theorem 8 from Debreu (1982). Applying this theorem requires checking a number of (standard) technical conditions about the labor excess demand system. Among these conditions the key one is upper-hemicontinuity. For industries with $\alpha_k \in [0,1]$ we prove upper-hemicontinuity by exploiting the equivalence between the system in (1) and a constrained optimization problem and invoking the Theorem of the Maximum from Stokey, Lucas and Prescott (1989). For each industry with $\alpha_k \geq 1$ we explicitly construct an equilibrium labor allocation with the required properties. Our existence result is summarized in the following proposition:[3]

Proposition OA.1. *If $\alpha_k \geq 0$ for all k then an equilibrium exists.*

We start the proof of Proposition OA.1 with two lemmas:

Lemma OA.2. *If either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption OA.1 holds, then the function $\boldsymbol{L}_k(\boldsymbol{w})$ is continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$.*

Lemma OA.3. *If $\alpha_k = 1$, then the solution to (1) determines a non-empty convex-valued upper hemi-continuous correspondence $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$.*

Proof of Lemmas OA.2 and OA.3. We prove Lemmas OA.2 and OA.3 simultaneously. The case with $\alpha_k = 0$ is trivial because labor allocations are explicitly obtained from the goods market clearing conditions $L_{i,k}G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k) = 0$, and the resulting expressions for $L_{i,k}(\boldsymbol{w})$ are obviously continuous. Below we focus on the case with $\alpha_k \in (0,1]$.

[3] See Somale (2021) for related existence results.

Define a multi-valued correspondence $\Gamma_k : \mathbb{R}^N_{++} \to \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ by

$$\Gamma_k(\boldsymbol{w}) = \left\{ \boldsymbol{L}_k \in \mathbb{R}^N \mid L_{i,k} \geq 0, \sum_i w_i L_{i,k} = \sum_i \beta_{i,k} w_i \bar{L}_i \right\}.$$

Define function $F_k : \left(\mathbb{R}^N_+ \setminus \{\mathbf{0}\}\right) \times \mathbb{R}^N_{++} \to \mathbb{R}$ by

$$F_k(\boldsymbol{L}_k; \boldsymbol{w}) = \alpha_k \sum_n w_n L_{n,k} - \sum_n \beta_{n,k} w_n \bar{L}_n \ln \left(\sum_i S_{i,k} L_{i,k}^{\alpha_k} \left(w_i \tau_{ni,k}\right)^{-\varepsilon_k} \right).$$

Denote the set of labor allocations at which $F_k(\boldsymbol{L}_k; \boldsymbol{w})$ achieves its minimum on $\Gamma_k(\boldsymbol{w})$ by $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w}) \equiv \arg\min_{\boldsymbol{L}_k \in \Gamma_k(\boldsymbol{w})} F_k(\boldsymbol{L}_k; \boldsymbol{w})$. It is straightforward to show that $\Gamma_k(\boldsymbol{w})$ is both lower hemi-continuous and upper hemi-continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$ (see the corresponding definitions in Nancy L. Stokey, Robert E. Lucas, Jr. and Edward C. Prescott, 1989). Hence, $\Gamma_k(\boldsymbol{w})$ is continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Clearly, $\Gamma_k(\boldsymbol{w})$ is also compact-valued for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Function $F_k(\boldsymbol{L}_k; \boldsymbol{w})$ is continuous for all $\boldsymbol{L}_k \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ and $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Thus, all conditions for Theorem 3.6 (Theorem of the Maximum) from Stokey, Lucas and Prescott (1989) are satisfied, and the correspondence $\boldsymbol{\mathcal{L}}_k : \mathbb{R}^N_{++} \to \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ is nonempty and upper hemi-continuous.

Lemma 1 in Appendix B in the main text establishes that under conditions (a) or (b) $F_k(\boldsymbol{L}_k; \boldsymbol{w})$ is strictly convex in $\boldsymbol{L}_k$ and, hence, $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ is a singleton. In this case upper hemi-continuity of $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ simply means continuity. Lemma 3 in Appendix B in the main text implies that all global minima of $F_k(\cdot; \boldsymbol{w})$ on $\Gamma_k(\boldsymbol{w})$ are solutions to problem (1). Therefore, under conditions (a) or (b) the solution to (1) defines a continuous function $\boldsymbol{L}_k(\boldsymbol{w})$ from wages to labor allocations.

If $\alpha_k = 1$ and Assumption OA.1 does not hold, function $F_k(\boldsymbol{L}_k; \boldsymbol{w})$ is convex in $\boldsymbol{L}_k$, but not necessarily strictly convex. Then, since $\Gamma_k(\boldsymbol{w})$ is a convex set, $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ is also convex. Again, Lemma 3 in Appendix B in the main text implies that $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ consists of all solutions to problem (1). So, in this case, solution to (1) determines a correspondence $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ between wages and equilibrium labor allocations which is non-empty, convex-valued, and upper hemi-continuous. □

Proof of Proposition OA.1. Without loss of generality assume that $\alpha_k > 1$ for $k = 1, \ldots, K^*$ and $0 \leq \alpha_k \leq 1$ for $k = K^* + 1, \ldots, K$ and consider the following three cases:

$$\begin{pmatrix} 0 & \dots & 0 & L_{1,K^*+1} & \dots & L_{1,K} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & L_{N-1,K^*+1} & \dots & L_{N-1,K} \\ L_{N,1} & \dots & L_{N,K^*} & L_{N,K^*+1} & \dots & L_{N,K} \end{pmatrix}$$

Case (a)

$$\begin{pmatrix} \bar{L}_1 & & & \\ & \ddots & 0 & \\ & 0 & \ddots & \\ & & & \bar{L}_{K^*} \\ 0 & \dots & 0 & \bar{L}_{K^*+1} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \bar{L}_N \end{pmatrix}$$

Case (b)

$$\begin{pmatrix} \bar{L}_1 & & 0 & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & \bar{L}_{N-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & L_{N,N} & \dots & L_{N,K^*} \end{pmatrix}$$

Case (c)

Figure 1: Labor Allocation Patterns in Proposition OA.1

(a) $0 \leq K^* < K$; (b) $K^* = K$ and $K < N$; (c) $K^* = K$ and $K \geq N$. In what follows, refer to Figure 1 for an illustration of the patterns of labor allocations that we choose for the cases (a)-(c). In this figure, rows of matrices correspond to countries and columns correspond to industries. In the next paragraph we formally define these patterns.

If we are in case (a), then for $i = 1, \dots, N-1$ and $k = 1, \dots, K^*$ set $L_{i,k} = 0$. In this case industries $k = 1, \dots, K^*$ are arbitrary chosen to be supplied by country N only. Next, if we are in case (b), then for $i = 1, \dots, K^*$ and $k = 1, \dots, K^*$ set $L_{i,k} = 0$ if $i \neq k$; and for $i = K^* + 1, \dots, N$ and $k = 1, \dots, K^* - 1$ set $L_{i,k} = 0$. In this case we arbitrary assign each country with index $i = 1, \dots, K^* - 1$ to be the only supplier of the industry with the corresponding index $k = 1, \dots, K^* - 1$, while the remaining countries allocate all their labor to industry K^*. Finally, if we are in case (c), then for $i = 1, \dots, N-1$ and $k = 1, \dots, K$ set $L_{i,k} = 0$ if $i \neq k$; and for $k = 1, \dots, N-1$ set $L_{N,k} = 0$. In this case, similarly to case (b), each country with index $i = 1, \dots, N-1$ is the only supplier of the industry with the corresponding index $k = 1, \dots, N-1$, while the remaining industries are all supplied by country N only.

In cases (b) and (c), when some country i allocates all its labor to only one industry, we label the corresponding entries of the labor allocation matrices in Figure 1 by $\bar{L}_i$. At

the same time, in the formal definitions of the labor allocation patterns above we do not explicitly set labor allocations in the corresponding cases to the full labor endowments. The reason is that we are going to use the two-step definition of equilibrium from the main text to prove existence. In the first step we fix wages and derive equilibrium labor allocations, and in the second step we find wages that clear labor markets. So, labeling of non-zero entries in Figure 1 shall be understood as equilibrium outcomes rather than predetermined allocations.

It is easy to verify that for all country-industry pairs (i,k) for which we assigned $L_{i,k} = 0$ the corresponding complementary slackness conditions (1) are satisfied for any positive vector of wages. This is because in all these cases we have $\alpha_k > 1$. For all other cases we can either explicitly (for $\alpha_k = 0$ or $\alpha_k > 1$) or implicitly (for $0 < \alpha_k \leq 1$) solve (1) to find (first-step) equilibrium labor allocations.

Importantly for what follows, the allocations described in cases (a)-(c) imply that for any country i we have the following three mutually exclusive possibilities:

1. There is some industry k for which country i is the only supplier. In this case country i's equilibrium labor allocation in industry k is given by

$$L_{i,k}(\boldsymbol{w}) = \frac{1}{w_i} \sum_n \beta_{n,k} w_n \bar{L}_n. \tag{11}$$

2. Country i allocates all its labor to some industry k that is supplied by multiple countries each of which allocates all its labor to this industry — this happens in case (b) above if $i \geq K^*$. The equilibrium labor allocation in industry k is given by

$$L_{i,k}(\boldsymbol{w}) = \sum_n \frac{S_{i,K^*} \left[\tau_{ni,K^*}\right]^{-\varepsilon_{K^*}} \bar{L}_i^{\alpha_{K^*}} w_i^{-\varepsilon_{K^*}-1}}{\sum_{l \geq K^*} S_{l,K^*} \left[\tau_{nl,K^*}\right]^{-\varepsilon_{K^*}} \bar{L}_l^{\alpha_{K^*}} w_l^{-\varepsilon_{K^*}}} \beta_{n,K^*} w_n \bar{L}_n, \tag{12}$$

   while $L_{i,k'}(\boldsymbol{w}) = 0$ for all $k' \neq k$.

3. Country i allocates all its labor to industries with $\alpha_k \leq 1$. In this case country i's equilibrium labor allocations satisfy (1), which defines a function $L_{i,k}(\boldsymbol{w})$ if $\alpha_k < 1$ and a correspondence $\mathcal{L}_{i,k}(\boldsymbol{w})$ if $\alpha_k = 1$.

Let

$$\mathcal{Z}_i(\boldsymbol{w}) \equiv \left\{ \sum_k L_{i,k} - \bar{L}_i \middle| L_{i,k} = L_{i,k}(\boldsymbol{w}) \text{ if } \alpha_k \neq 1 \text{ and } L_{i,k} \in \mathcal{L}_{i,k}(\boldsymbol{w}) \text{ if } \alpha_k = 1 \right\}$$

be the excess labor demand correspondence in country i, and let

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{w}) \equiv (\mathcal{Z}_1(\boldsymbol{w}), \ldots, \mathcal{Z}_N(\boldsymbol{w}))\,.$$

We are going to use Theorem 8 from Debreu (1982) to show that there exists a positive vector of wages $\boldsymbol{w}$ such that $\mathbf{0} \in \boldsymbol{\mathcal{Z}}(\boldsymbol{w})$. For that we need to verify that $\boldsymbol{\mathcal{Z}}$ satisfies the following properties: (i) $\boldsymbol{\mathcal{Z}}$ is homogeneous of degree zero;[4] (ii) $\boldsymbol{\mathcal{Z}}$ is convex-valued; (iii) $\boldsymbol{\mathcal{Z}}$ is bounded below; (iv) $\boldsymbol{\mathcal{Z}}$ is upper hemi-continuous; (v) (Walras' Law) $\sum w_i Z_i = 0$ for any $\boldsymbol{w} \in \mathbb{R}^N_{++}$ and any $(Z_1, \ldots, Z_N) \in \boldsymbol{\mathcal{Z}}(\boldsymbol{w})$; (vi) (Boundary Condition) if $\left\{\boldsymbol{w}^t\right\}_{t=1}^{\infty}$ is a wage sequence such that $\boldsymbol{w}^t \to \boldsymbol{w}$ as $t \to \infty$, where $\boldsymbol{w} \neq 0$ is a finite vector of wages and $w_i = 0$ for some i, then for any sequence $\left(Z_1^t, \ldots, Z_N^t\right) \in \boldsymbol{\mathcal{Z}}(\boldsymbol{w}^t)$ for $t = 1, 2, \ldots$, we have $\max\left\{Z_1^t, \ldots, Z_N^t\right\} \to \infty$ as $t \to \infty$.

It is immediate to see that $\boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ is homogeneous of degree zero, that Walras' Law is satisfied for any positive $\boldsymbol{w}$, and that $Z_i > -\bar{L}_i$ for any $(Z_1, \ldots, Z_N) \in \boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ and all positive $\boldsymbol{w}$. The property that $\boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ is convex-valued follows from the fact that $\boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ consists of the sum of functions $L_{i,k}(\boldsymbol{w})$ and correspondences $\mathcal{L}_{i,k}(\boldsymbol{w})$ which are convex-valued by virtue of Lemma OA.3. Upper hemi-continuity of $\boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ follows from upper hemi-continuity of $\mathcal{L}_{i,k}(\boldsymbol{w})$ established in Lemma OA.3 and from the fact $L_{i,k}(\boldsymbol{w})$ are given by (11), or by (12), or by the solution to (1), which is continuous by Lemma OA.2.

The only non-trivial condition to check is the boundary condition (vi). Consider any wage sequence from this condition. Let index j be such that wage w_j^t converges to 0 weakly "faster" than other wages. Formally, index j is such that the limit $\lim_{t\to\infty} w_j^t / w_l^t$ is finite for all l. Such index always exists because there is a finite number of indices.

Consider the three possibilities above. Under the first possibility, country j is the only supplier of some industry k, therefore by expression (11) we have $L_{j,k}\left[\boldsymbol{w}^t\right] = \sum_n \beta_{n,k}\left[w_n^t / w_j^t\right]\bar{L}_n$. This converges to ∞ as $t \to \infty$, and so $\max\left\{Z_1^t, \ldots, Z_N^t\right\} \to$

[4] Homogeneity of degree zero is not explicitly mentioned in Theorem 8 in Debreu (1982). Instead, the excess demand correspondence is assumed to be defined on a simplex of prices. For our purposes this is the same as assuming homogeneity of degree zero.

∞ for any sequence $\left(Z_1^t, \ldots, Z_N^t\right) \in \boldsymbol{\mathcal{Z}}(\boldsymbol{w}^t)$ for $t = 1, 2, \ldots$.

Under the second possibility, country j's excess labor demand function is given by

$$Z_j(\boldsymbol{w}) = \sum_n \frac{S_{j,K^*} \left[\tau_{nj,K^*}\right]^{-\varepsilon_{K^*}} \bar{L}_j^{\alpha_{K^*}} (w_j^t)^{-1}}{\sum\limits_{l \geq K^*} S_{l,K^*} \left[\tau_{nl,K^*}\right]^{-\varepsilon_{K^*}} \bar{L}_l^{\alpha_{K^*}} \left[w_j^t / w_l^t\right]^{\varepsilon_{K^*}}} \beta_{n,K^*} w_n^t \bar{L}_n - \bar{L}_i.$$

The denominator of any term in the above sum converges to a finite positive number as $t \to \infty$. The numerator converges either to a finite positive number or to infinity. Moreover, since for at least one index n wage w_n^t converges to a positive number and w_j^t converges to 0, we have that for at least one index n the numerator of the corresponding term in the above sum converges to ∞. Hence, the whole sum converges to ∞. Therefore, again, the boundary condition is satisfied.

Finally, under the third possibility country j supplies all its labor to industries with $\alpha_k \leq 1$. Pick any such industry k. Equilibrium labor allocations to industry k in all countries satisfy (1) (see case (a) in Figure 1). Let us use the general notation $\mathcal{L}_{i,k}(\boldsymbol{w})$ for all such labor allocations (if $\alpha_k = 1$, then $\mathcal{L}_{i,k}(\boldsymbol{w})$ is a singleton). If there is some country i with the corresponding sequence of sets $\mathcal{L}_{i,k}(\boldsymbol{w}^t)$ such that any sequence $L_{i,k}^t \in \mathcal{L}_{i,k}(\boldsymbol{w}^t)$ converges to ∞ as $t \to \infty$, then the boundary condition is satisfied. Let us show that there always exists such a country by supposing the contrary. That is, suppose that for any country i there exists a sequence $L_{i,k}^t \in \mathcal{L}_{i,k}(\boldsymbol{w}^t)$ converging to a finite number as $t \to \infty$. That means that there exists a sequence $\left(L_{1,k}^t, \ldots, L_{N,k}^t\right) \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w}^t)$ converging to a finite vector as $t \to \infty$. Consider this sequence. Let us focus again on the country j for which the wage converges to 0 weakly "faster" than for any other country. For any t, $L_{j,k}^t$ satisfies (1) and, in particular, $L_{j,k}^t$ satisfies the inequality

$$w_j^t \geq \sum_n \frac{S_{j,k} \left[L_{j,k}^t\right]^{\alpha_k - 1} \left[w_j^t \tau_{nj,k}\right]^{-\varepsilon_k}}{\sum\limits_l S_{l,k} \left[L_{l,k}^t\right]^{\alpha_k} \left[w_l^t \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} w_n^t \bar{L}_n.$$

This inequality can be equivalently rewritten as

$$w_j^t \geq \sum_{n=1}^N \frac{S_{j,k} \left[L_{j,k}^t\right]^{\alpha_k - 1} \tau_{nj,k}^{-\varepsilon_k}}{\sum\limits_l S_{l,k} \left[L_{l,k}^t\right]^{\alpha_k} \tau_{nl,k}^{-\varepsilon_k} \left[w_j^t / w_l^t\right]^{\varepsilon_k}} w_n^t \bar{L}_n. \tag{13}$$

The denominator of any term in the above summation (13) converges to a finite number (which can be either positive or zero). The numerator of any term in the summation (13) converges to either a finite positive number or to infinity. Also, there exists at least one index n such that $\lim_{t\to\infty} w_n^t > 0$. Then, for this index n the corresponding term in the summation (13) converges to either a finite positive number or to ∞. This, in turn, implies that the whole sum in (13) converges to either a finite positive number or to ∞. At the same time, the left-hand side of inequality (13) converges to 0. A contradiction. □

4.3. Sufficient Conditions for Assumption OA.1

While it is easy to check if Assumption OA.1 is satisfied for a particular parametrization, we can say a little bit more about the conditions which guarantee that this assumption holds. Behrens et al. (2004) invoke classical results by Schoenberg (1938) to show that, if trade costs τ_{ni} correspond to the Euclidean distance between countries n and i, then the matrix in Assumption OA.1 is positive definite (and, hence, non-singular) as long as all countries are at distinct locations. In fact, any three distinct numbers that satisfy the triangle inequality can be mapped to lengths of sides of a triangle in $\mathbb{R}^2$, which means that any such numbers correspond to Euclidean distances between vertices of a triangle in $\mathbb{R}^2$. Together with the results from Schoenberg (1938), this observation implies that for $N = 3$ the matrix in Assumption OA.1 is positive definite if (i) the iceberg trade costs are symmetric, (ii) greater than 1 for different countries, and (iii) satisfy the triangle inequality. For $N > 3$, conditions (i)-(iii) do not generally imply that the iceberg trade costs correspond to distances in an Euclidean space. Still, extensive simulations for trade-freeness matrices for $N = 4, 5, 6$ lead us to conjecture that conditions (i)-(iii) guarantee that the matrix in Assumption OA.1 is positive definite. Moreover, we conjecture that we can even dispense with the symmetry condition (i) — in this case it is the sum of the matrix in Assumption OA.1 with its transpose that is positive definite.

4.4. Necessary Condition for Uniqueness of Equilibrium

Proposition OA.2. *If there is an industry k with $\alpha_k > 1$, then there are multiple equilibria.*

Proof. If $\alpha_k > 1$ for some k, then in the proof of Proposition OA.1 $K^* > 0$. This implies

that there are different allocations that we can assign (i.e., one for each country), and since there is an equilibrium for each one, this immediately establishes that there are multiple equilibria. □

Proposition OA.2 implies that $\alpha_k \leq 1$ for all k is a necessary condition for uniqueness.

4.5. Uniqueness in the Case of Two Countries

Proposition OA.3. *Assume that $N = 2$ and that for all k either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption OA.1 holds. Then there is a unique equilibrium.*

Proof. The proof proceeds by showing that $\boldsymbol{Z}(\boldsymbol{w})$ satisfies the gross substitutes property (GSP). Uniqueness of wages then follows from Proposition 17.F.3 from Mas-Colell, Whinston and Green (1995).

Consider any particular industry k. Let us separately analyze the two possibilities $0 \leq \alpha_k < 1$ and $\alpha_k = 1$.

If $0 \leq \alpha_k < 1$, then for any i we have that $L_{i,k}(\boldsymbol{w}) > 0$ for any wage vector $\boldsymbol{w} \in \mathbb{R}^N_{++}$, and $L_{i,k}(\boldsymbol{w})$ solves:

$$w_i L_{i,k}(\boldsymbol{w}) = \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{L}_k(\boldsymbol{w}))\beta_{n,k} w_n \bar{L}_n.$$

By differentiating both sides of this expression w.r.t. wages, we can get a linear system of equations which determines the effect of wages on labor allocations. Let us introduce additional notation to write in matrix form this effect. Denote $x_{ij,k} \equiv \dfrac{d \ln L_{i,k}(\boldsymbol{w})}{d \ln w_j}$, $q_{i,k} \equiv w_i L_{i,k}(\boldsymbol{w})$, and $b_{i,k} \equiv \beta_{i,k} w_i \bar{L}_i$. Let B_k denote the diagonal matrix with elements $b_{i,k}$ along the diagonal, Q_k the diagonal matrix with elements $q_{i,k}$ along the diagonal, Λ_k the matrix of sector level expenditure shares $\lambda_{ij,k}$, and X_k the matrix of partials $x_{ij,k}$. Finally, let $U_k \equiv \left((1-\alpha_k) Q_k + \alpha_k \Lambda_k^T B_k \Lambda_k\right)$ and $V_k \equiv \left(\Lambda_k^T B_k + \varepsilon_k \Lambda_k^T B_k \Lambda_k - (1+\varepsilon_k) Q_k\right)$. In this notation the effect of wages on labor allocations is obtained from the system:

$$U_k X_k = V_k.$$

It straightforward to check that matrix U_k is a positive definite matrix with all positive elements, and matrix V_k has negative diagonal and positive off-diagonal elements.

Since U_k is positive definite, the inverse exists and its determinant is positive. Moreover, $U_k^{-1} = \frac{1}{\det(U_k)} C_k^T$, where C_k^T is the transpose of the matrix of cofactors C_k of U_k. Since all the elements of U_k are positive, then for $N = 2$, C_k is a 2×2 matrix consisting of positive diagonal elements and negative off-diagonal elements.[5] Therefore, U_k^{-1} has this property as well. One can then readily verify that $U_k^{-1} V_k$ is a matrix with the same properties as V_k — it has negative diagonal and positive off diagonal elements. Thus the Jacobian matrix of wages effects on labor allocations in industry k with $0 \leq \alpha_k < 1$ satisfies the GSP.

If $\alpha_k = 1$, then $L_{i,k}(\boldsymbol{w})$ can be equal to 0 for some i, and we cannot establish differentiability of labor allocations in that region. We are going to check directly what happens to labor allocations as wages change. To that end, assume without loss of generality that $\boldsymbol{w'}$ and $\boldsymbol{w''}$ are such that $w_1'' > w_1'$ and $w_2'' = w_2' = 1$. Let us show that $L_{2,k}(\boldsymbol{w''}) \geq L_{2,k}(\boldsymbol{w'})$ for all k and there is some industry $\tilde{k}$ such that $L_{2,\tilde{k}}(\boldsymbol{w''}) > L_{2,\tilde{k}}(\boldsymbol{w'})$.

In general, given wage $\boldsymbol{w}'$ there are three cases: (a) $L_{1,k}(\boldsymbol{w}') = 0$ and $L_{2,k}(\boldsymbol{w}') = \beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2$; (b) $L_{i,k}(\boldsymbol{w}') > 0$ for $i = 1, 2$; (c) $L_{2,k}(\boldsymbol{w}') = 0$ and $L_{1,k}(\boldsymbol{w}') = \dfrac{1}{w_1'} \left(\beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2\right)$.

Let us consider these different cases.

Case (a). In this case we have $G_{1,k}(\boldsymbol{w}') \geq 0$ and $G_{1,k}(\boldsymbol{w}')$ simplifies to:

$$G_{1,k}(\boldsymbol{w}') = w_1 - \frac{S_{1,k} \left[w_1'\right]^{-\varepsilon_k}}{S_{2,k} L_{2,k}(\boldsymbol{w}') \tau_{12,k}^{-\varepsilon_k}} \beta_{1,k} w_1' \bar{L}_1 - \frac{S_{1,k} \left(w_1' \tau_{21,k}\right)^{-\varepsilon_k}}{S_{2,k} L_{2,k}(\boldsymbol{w}')} \beta_{2,k} \bar{L}_2.$$

After substituting $L_{2,k}(\boldsymbol{w}') = \beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2$ into the above expression for $G_{1,k}(\boldsymbol{w}')$, and dividing both sides of this expression by w_1', we get:

$$\frac{G_{1,k}(\boldsymbol{w}')}{w_1'} = 1 - \frac{S_{1,k}}{S_{2,k} \tau_{12,k}^{-\varepsilon_k}} \cdot \frac{\left[w_1'\right]^{-\varepsilon_k} \beta_{1,k} \bar{L}_1}{\beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2} - \frac{S_{1,k} \tau_{21,k}^{-\varepsilon_k}}{S_{2,k}} \cdot \frac{\left[w_1'\right]^{-1-\varepsilon_k} \beta_{2,k} \bar{L}_2}{\beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2}.$$

Clearly, the right-hand side of this expression is increasing in w_1'. Hence, $G_{1,k}(\boldsymbol{w''})/w_1'' > G_{1,k}(\boldsymbol{w'})/w_1' \geq 0$, which in turn implies that $G_{1,k}(\boldsymbol{w''}) > 0$. Therefore, $L_{1,k}(\boldsymbol{w''}) = 0$ and $L_{2,k}(\boldsymbol{w''}) = \beta_{1,k} w_1'' \bar{L}_1 + \beta_{2,k} \bar{L}_2$ solve the complementary slackness problem (1). In other words, we still remain in case (a) after we increase the wage of the country 1 from w_1' to w_1''. Clearly, in this case $L_{2,k}(\cdot)$ is a strictly increasing function of the wage of the first

[5] This is no longer true with $N > 2$.

country, $L_{2,k}(\boldsymbol{w}'') > L_{2,k}(\boldsymbol{w}')$.

Case (b). We know that, as long as we are in case (b), $L_{1,k}(\cdot)$ is a decreasing function and $L_{2,k}(\cdot)$ is an increasing function of w_1. Therefore, starting in case (b) with $\boldsymbol{w}'$ and gradually increasing w_1 from w_1' to w_1'', we either remain in case (b) or switch to case (a) at some point. The above argument for case (a) implies that, once we switch to case (a), we will remain in case (a) as we keep increasing w_1. Thus, for $\boldsymbol{w}''$ we can either be in case (a) or in case (b), but not in case (c), and since in both cases (a) and (b) $L_{2,k}(\cdot)$ is a strictly increasing function of w_1, we must have $L_{2,k}(\boldsymbol{w}'') > L_{2,k}(\boldsymbol{w}')$.

Case (c). In this case, we can be in any of the cases (a)-(c) for $\boldsymbol{w}''$. If we are in cases (a) or (b) for $\boldsymbol{w}''$, then $L_{2,k}(\boldsymbol{w}'') > L_{2,k}(\boldsymbol{w}') = 0$. If we are in case (c) for $\boldsymbol{w}''$, then $L_{2,k}(\boldsymbol{w}'') = L_{2,k}(\boldsymbol{w}') = 0$, but there must exist some industry $\tilde{k}$, for which we are in case (a) or (b) for $\boldsymbol{w}'$ (regardless of the value of $\alpha_{\tilde{k}}$ in this industry). Applying the arguments above, for any such industry we have $L_{2,\tilde{k}}(\boldsymbol{w}'') > L_{2,\tilde{k}}(\boldsymbol{w}')$.

Since the effect of changes in wages on $\boldsymbol{Z}(\boldsymbol{w})$ consists of the sum (across industries) of effects on industry-level labor allocations, we conclude that $\boldsymbol{Z}(\boldsymbol{w})$ satisfies the GSP.

□

4.6. Proof of Proposition 2 in the Main Text: Uniqueness under Free Trade

For convenience, let us repeat the statement of Proposition 2 from the main text.

Proposition 2. *If $0 \leq \alpha_k \leq 1$ and trade is frictionless in all industries, then there is a unique wage equilibrum.*

We prove this proposition in three steps. First, we show that the demand for labor in industries with $0 \leq \alpha_k < 1$ has the gross substitutes property. Then we show that the demand for labor in industries with $\alpha_k = 1$ has a version of the gross substitutes property that is adapted to incorporate potential multiplicity of labor allocations given wages. Finally, we prove the uniqueness of wages by adapting the standard proof (as, for example, the proof of Proposition 17.F.3 from Mas-Colell, Whinston and Green (1995)) of uniqueness of prices given that the excess demand system has the gross substitutes property.

4.6.1. Labor allocations in industries with $0 \leq \alpha_k < 1$

Consider any industry k with $0 \leq \alpha_k < 1$. Let L_k be a diagonal matrix with elements $L_{i,k}$ along the diagonal, and $D\boldsymbol{L}_k(\boldsymbol{w})$ be the Jacobian matrix of industry-level labor allocations with elements $\partial L_{i,k}(\boldsymbol{w})/\partial w_j$.

The following proposition formally states that the labor demand in industry k has the gross substitutes property:

Proposition OA.4. *In each industry k with $0 \leq \alpha_k < 1$ matrix $D\boldsymbol{L}_k(\boldsymbol{w})$ has the following properties: (i) entries in each row add up to 0; (ii) diagonal entries are negative; (iii) off-diagonal entries are positive.*

In the proof of this proposition we will use matrices B_k, Q_k, Λ_k, U_k, and V_k defined in the proof of Proposition OA.3. In addition to that, let W be a diagonal matrix with elements w_i along the diagonal.

We have

$$D\boldsymbol{L}_k(\boldsymbol{w}) = L_k U_k^{-1} V_k W^{-1}.$$

Matrix V_k has the following properties: (i) entries in each row add up to 0; (ii) diagonal entries are negative; (iii) off-diagonal entries are positive. If $\alpha_k = 0$, then matrix U_k reduces to diagonal matrix Q_k with positive diagonal elements. Therefore, we can immediately conclude that in this case $D\boldsymbol{L}_k(\boldsymbol{w})$ has properties (i)-(iii) as well. The rest of this appendix section is devoted to proving that matrix $D\boldsymbol{L}_k(\boldsymbol{w})$ also has properties (i)-(iii) under free trade for $0 < \alpha_k < 1$. For brevity of notation we drop the industry index k in the rest of this proof.

According to Proposition 1 in the main text, all industry-level labor allocations are interior, and so, $L_i > 0$, $\lambda_{ii} > 0$, $q_i > 0$ for all i. We start with three lemmas which apply to the general case of costly trade.

Lemma OA.4. *Let $\mu_1, \ldots, \mu_N$ be eigenvalues of matrix $Q^{-1}\Lambda^T B \Lambda$. Then μ_i is real and $0 \leq \mu_i \leq 1$ for each i.*

Proof. Consider matrix $Q^{-1/2}\Lambda^T B \Lambda Q^{-1/2}$, and let μ be any eigenvalue of this matrix with the corresponding eigenvector v. By definition of an eigenvalue, $Q^{-1/2}\Lambda^T B \Lambda Q^{-1/2} v = \mu v$. This is equivalent to $Q^{-1}\Lambda^T B \Lambda \left(Q^{-1/2} v\right) = \mu \left(Q^{-1/2} v\right)$.

Hence, μ is an eigenvalue of $Q^{-1}\Lambda^T B\Lambda$ with the corresponding eigenvector $Q^{-1/2}\upsilon$. Therefore, matrices $Q^{-1}\Lambda^T B\Lambda$ and $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$ have the same eigenvalues, and so $\mu_1,\ldots,\mu_N$ are eigenvalues of $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$.

Clearly, matrix $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$ is positive semi-definite. Hence, all its eigenvalues are real and nonnegative, i.e., μ_i is real and $\mu_i \geq 0$ for each i. Next, matrix $Q^{-1}\Lambda^T B\Lambda$ is a positive stochastic matrix (its entries in each row add up to 1). Therefore, the Perron-Frobenius theorem implies that 1 is its eigenvalue with algebraic multiplicity one and $|\mu_i| < 1$ for any $|\mu_i| \neq 1$. Since $\mu_i \geq 0$ for all i, we have the statement of the lemma. □

Lemma OA.5. $\lim_{t\to\infty} \alpha^t \left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t = 0$.

Proof. Eigenvalues of matrix $I_N - Q^{-1}\Lambda^T B\Lambda$ are $1-\mu_1,\ldots,1-\mu_N$, where $\mu_1,\ldots,\mu_N$ are eigenvalues of $Q^{-1}\Lambda^T B\Lambda$. Lemma OA.4 implies that $0 \leq 1-\mu_i \leq 1$ for all i. Then, since eigenvalues of matrix $\alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)$ are $\alpha(1-\mu_1),\ldots,\alpha(1-\mu_1)$, we have that $\rho\left(\alpha\left[I_N - Q^{-1}\Lambda^T B\Lambda\right]\right) < 1$, where $\rho(\cdot)$ is the spectral radius of a matrix. Therefore, $\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t \to 0$ as $t\to\infty$ (see, for example, Theorem 5.6.12 in Horn and Johnson, 2013). □

Lemma OA.6. $U^{-1}V = \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[\left(1+\varepsilon\alpha^{-1}\right)I_N - Q^{-1}\Lambda^T B\right]$.

Proof. Consider U^{-1}:

$$\begin{aligned} U^{-1} &= \left[(1-\alpha)Q + \alpha\Lambda^T B\Lambda\right]^{-1} = \left[(1-\alpha)I_N + \alpha Q^{-1}\Lambda^T B\Lambda\right]^{-1} Q^{-1} \\ &= \left[I_N - \alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)\right]^{-1} Q^{-1}. \end{aligned}$$

Lemma OA.5 implies that we can write

$$\left[I_N - \alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)\right]^{-1} = \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t$$

(see, for example, Corollary 5.6.15 in Horn and Johnson, 2013). Then

$$\begin{aligned} U^{-1}V &= -\sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[(1+\varepsilon)I_N - Q^{-1}\Lambda^T B - \varepsilon Q^{-1}\Lambda^T B\Lambda\right] \\ &= -\varepsilon\sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^{t+1} - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[I_N - Q^{-1}\Lambda^T B\right] \\ &= \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[\left(1+\varepsilon\alpha^{-1}\right)I_N - Q^{-1}\Lambda^T B\right]. \end{aligned}$$

□

Let us now consider the case of frictionless trade. In this case the matrix of trade shares, Λ, has the same entries in each column:

$$\Lambda \equiv \begin{pmatrix} \lambda_{11} & \dots & \lambda_{NN} \\ \vdots & & \vdots \\ \lambda_{11} & \dots & \lambda_{NN} \end{pmatrix}.$$

So, it can be represented (with a slight abuse of notation) as $O\Lambda$ where O is an $N \times N$ matrix of ones (i.e., $O = \iota \cdot \iota^T$ with $\iota^T \equiv (1, \dots, 1)$) and

$$\Lambda \equiv \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}.$$

Then, in this notation

$$U = (1-\alpha)\, Q + \alpha \Lambda O B O \Lambda \quad \text{and} \quad V = \Lambda O B + \varepsilon \Lambda O B O \Lambda - (1+\varepsilon)\, Q.$$

Denote $b \equiv \sum_n b_n$ and observe that $OBO = bO$. Also, since L_i satisfies the goods market clearing condition, $w_i L_i = \sum_n \lambda_i b_n = \lambda_i \sum_n b_n = b\lambda_i$. Then, since in our notation $q_{ii} = w_i L_i$, we have that $Q = WL = b\Lambda$. These equalities together with Lemma OA.6 allow us to write:

$$\begin{aligned} U^{-1}V &= \varepsilon\alpha^{-1} I_N - \sum_{t=0}^{\infty} \alpha^t \left(I_N - Q^{-1}\Lambda O B O \Lambda\right)^t \left[\left(1+\varepsilon\alpha^{-1}\right) I_N - Q^{-1}\Lambda O B\right] \\ &= \varepsilon\alpha^{-1} I_N - \sum_{t=0}^{\infty} \alpha^t \left(I_N - O\Lambda\right)^t \left[\left(1+\varepsilon\alpha^{-1}\right) I_N - b^{-1} O B\right]. \end{aligned}$$

Using the fact that $\sum_i \lambda_i = 1$ and, hence, $O\Lambda O = O$, we get:

$$\begin{aligned} \left(I_N - O\Lambda\right)\left[\left(1+\varepsilon\alpha_k^{-1}\right) I_N - b^{-1} O B\right] &= \left(1+\varepsilon\alpha^{-1}\right) I_N - b^{-1} O B \\ &\quad - \left(1+\varepsilon\alpha^{-1}\right) O\Lambda + b^{-1} O\Lambda O B \\ &= \left(1+\varepsilon\alpha^{-1}\right)\left(I_N - O\Lambda\right), \end{aligned}$$

and

$$(I_N - O\Lambda)(I_N - O\Lambda) = I_N - O\Lambda - O\Lambda + O\Lambda O\Lambda = I_N - O\Lambda.$$

Therefore,

$$\begin{aligned} U^{-1}V &= \varepsilon\alpha^{-1} I_N - \left[\left(1+\varepsilon\alpha^{-1}\right) I_N - b^{-1}OB\right] - \left(1+\varepsilon\alpha^{-1}\right)\sum_{t=1}^{\infty}\alpha^t (I_N - O\Lambda)^t \\ &= -\left(I_N - b^{-1}OB\right) - \left(1+\varepsilon\alpha^{-1}\right)(I_N - O\Lambda)\sum_{t=1}^{\infty}\alpha^t \\ &= \left(b^{-1}OB - I_N\right) + \frac{\alpha+\varepsilon}{1-\alpha}(O\Lambda - I_N). \end{aligned}$$

Observe that both matrices $\left(b^{-1}OB - I_N\right)$ and $(O\Lambda - I_N)$ have properties (i)-(iii) listed in the statement of Proposition OA.4. Hence, matrix $U^{-1}V$ has properties (i)-(iii) as well. This, in turn, implies that matrix $\Lambda U^{-1}VW^{-1}$ also has properties (i)-(iii). This concludes the proof of Proposition OA.4.

□

4.6.2. Labor allocations in industries with $\alpha_k = 1$

Consider any industry k with $\alpha_k = 1$. Using the same argument as in Lemmas 1 and 3 in Appendix B of the main text, we can show that the labor allocations $L_{i,k}$ that solve the complementary slackness problem (1) given wages $\boldsymbol{w}$ are non-zero only for countries i such that $S_{i,k}w_i^{-\varepsilon_k-1} \geq S_{j,k}w_j^{-\varepsilon_k-1}$ for all j.[6] Denote by $\mathcal{N}_k(\boldsymbol{w})$ the set of such countries corresponding to wage vector $\boldsymbol{w}$. Clearly, $S_{i',k}w_{i'}^{-\varepsilon_k-1} = S_{i'',k}w_{i''}^{-\varepsilon_k-1}$ for any $i', i'' \in \mathcal{N}_k(\boldsymbol{w})$. The set of non-zero labor allocations corresponding to wages $\boldsymbol{w}$ consists of all $L_{i,k}$ with $i \in \mathcal{N}_k(\boldsymbol{w})$ that satisfy $\sum_{i\in\mathcal{N}_k(\boldsymbol{w})} w_i L_{i,k} = \sum_{n=1}^{N}\beta_{n,k}w_n\bar{L}_n$. In particular, if $\mathcal{N}_k(\boldsymbol{w})$ has more than one country, then there is a continuum of labor allocations corresponding to $\boldsymbol{w}$. Let $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w}) \equiv \{\boldsymbol{L}_k \mid \boldsymbol{L}_k = (L_{1,k},\ldots,L_{N,k})$ solves (1) given $\boldsymbol{w}\}$.

The following proposition adapts the gross substitues property of labor demand by taking into account the possibility of multiple labor allocations given wages:

Proposition OA.5. *Consider any positive wage vectors $\boldsymbol{w}'$ and $\boldsymbol{w}''$ such that $w_i'' = w_i'$ for $i < I$ and $w_i'' > w_i'$ for $i \geq I$ with $2 \leq I \leq N$. Then for any industry k with $\alpha_k = 1$ and any*

[6] In notation of Lemmas 1 and 3, the labor allocations corresponding to a wage vector $\boldsymbol{w}$ can be shown to maximize the linear function $\sum_i a_i x_i$ subject to constraints $\sum_i x_i = B$ and $x_i \geq 0$ for all i.

$\boldsymbol{L'}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$ we have that $\sum_{i=1}^{I-1} w''_i L''_{i,k} \geq \sum_{i=1}^{I-1} w'_i L'_{i,k}$. Moreover, for any $\boldsymbol{L'}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$ such that $\sum_{i=1}^{I-1} w''_i L''_{i,k} = \sum_{i=1}^{I-1} w'_i L'_{i,k}$, we have that $\sum_{i=I}^{N} w''_i L''_{i,k} > \sum_{i=I}^{N} w'_i L'_{i,k}$.

Proof. There are several cases possible depending on which countries from $\{1,\ldots,I-1\}$ and $\{I,\ldots,N\}$ are in the sets $\mathcal{N}_k(\boldsymbol{w'})$ and $\mathcal{N}_k(\boldsymbol{w''})$. We consider these cases below. In each of the cases, we will use the facts that $\sum_{i=1}^{N} w_i L_{i,k} = \sum_{n=1}^{N} \beta_{n,k} w_n \bar{L}_n$ for all $\boldsymbol{L}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ and $\boldsymbol{w} = \boldsymbol{w'}, \boldsymbol{w''}$, and that $\sum_{n=1}^{N} \beta_{n,k} w''_n \bar{L}_n > \sum_{n=1}^{N} \beta_{n,k} {w'}_n \bar{L}_n$. Note that this last inequality corresponds to the effect of higher world demand when wages are higher. In each of the cases we will consider any $\boldsymbol{L'}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$.

(i) $\boldsymbol{\mathcal{N}_k(w') \subseteq \{1,\ldots,I-1\}}$. In this case $\mathcal{N}_k(\boldsymbol{w'}) = \mathcal{N}_k(\boldsymbol{w''})$, since $w''_i > w'_i$ for $i \geq I$, and so $S_{i,k}[w''_i]^{-\varepsilon_k - 1} < S_{i,k}[w'_i]^{-\varepsilon_k - 1}$ for $i \geq I$, while $S_{i,k}[w'_i]^{-\varepsilon_k - 1} < S_{j,k}\left[w'_j\right]^{-\varepsilon_k - 1}$ for any $i \geq I$ and $j \in \mathcal{N}_k(\boldsymbol{w'})$. Then, using $L'_{i,k} = L''_{i,k} = 0$ for all $i \notin \mathcal{N}_k(\boldsymbol{w'}) = \mathcal{N}_k(\boldsymbol{w''})$, this implies that

$$\sum_{i=1}^{I-1} w''_i L''_{i,k} = \sum_{n=1}^{N} \beta_{n,k} w''_n \bar{L}_n > \sum_{n=1}^{N} \beta_{n,k} w'_n \bar{L}_n = \sum_{i=1}^{I-1} w'_i L'_{i,k}.$$

(ii) $\boldsymbol{\mathcal{N}_k(w') \subseteq \{I,\ldots,N\}}$ and $\boldsymbol{\mathcal{N}_k(w'') \subseteq \{I,\ldots,N\}}$. We have that $L'_{i,k} = L''_{i,k} = 0$ for $i = 1,\ldots,I-1$, and, therefore, trivially, $\sum_{i=1}^{I-1} w''_i L''_{i,k} = \sum_{i=1}^{I-1} w'_i L'_{i,k} = 0$. At the same time, $\sum_{i=I}^{N} w'_i L'_{i,k} = \sum_{n=1}^{N} \beta_{n,k} {w'}_n \bar{L}_n$ and $\sum_{i=I}^{N} w''_i L''_{i,k} = \sum_{n=1}^{N} \beta_{n,k} w''_n \bar{L}_n$, and, therefore, $\sum_{i=I}^{N} w''_i L''_{i,k} > \sum_{i=I}^{N} w'_i L'_{i,k}$.

(iii) $\boldsymbol{\mathcal{N}_k(w') \subseteq \{I,\ldots,N\}}$ and $\boldsymbol{\mathcal{N}_k(w'') \cap \{1,\ldots,I-1\} \neq \emptyset}$ and $\boldsymbol{\mathcal{N}_k(w'') \cap \{I,\ldots,N\} \neq \emptyset}$. In this case, $\sum_{i=1}^{I-1} w''_i L''_{i,k} \geq \sum_{i=1}^{I-1} w'_i L'_{i,k} = 0$.[7] At the same time, if $L''_{i,k} = 0$ for all $i < I$, then

$$\sum_{i=I}^{N} w''_i L''_{i,k} = \sum_{n=1}^{N} \beta_{n,k} {w''}_n \bar{L}_n > \sum_{n=1}^{N} \beta_{n,k} {w'}_n \bar{L}_n = \sum_{i=I}^{N} w'_i L'_{i,k}$$

Overall, the conclusion of this case is that we either have $\sum_{i=1}^{I-1} w''_i L''_{i,k} > \sum_{i=1}^{I-1} w'_i L'_{i,k}$ or $\sum_{i=I}^{N} w''_i L''_{i,k} > \sum_{i=I}^{N} w'_i L'_{i,k}$ (or both).

(iv) $\boldsymbol{\mathcal{N}_k(w') \cap \{1,\ldots,I-1\} \neq \emptyset}$ and $\boldsymbol{\mathcal{N}_k(w') \cap \{I,\ldots,N\} \neq \emptyset}$. In this case, the only possibility is that $\mathcal{N}_k(\boldsymbol{w''}) \subseteq \{1,\ldots,I-1\}$ (the argument is similar to the one

[7] Observe that the inequality is weak because, due to multiplicity of labor allocations given wages, for some $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$ we can have $L''_{i,k} = 0$ for all $i < I$ even though $\mathcal{N}_k(\boldsymbol{w''})$ contains some $i < I$.

used in case (i)). Then we have that[8]

$$\sum_{i=1}^{I-1} w'_i L'_{i,k} \leq \sum_{n=1}^{N} \beta_{n,k} w'_n \bar{L}_n < \sum_{n=1}^{N} \beta_{n,k} w''_n \bar{L}_n = \sum_{i=1}^{I-1} w''_i L''_{i,k}.$$

□

4.6.3. Uniqueness of wages

As in the previous subsection, for any positive wage vector $\boldsymbol{w}$ and industry k, let

$$\boldsymbol{\mathcal{L}}_k(\boldsymbol{w}) \equiv \{\boldsymbol{L}_k \mid \boldsymbol{L}_k = (L_{1,k}, \ldots, L_{N,k}) \text{ solves (1) given } \boldsymbol{w}\}.$$

Propostion 1 in the main text implies that $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w})$ is a singleton for all industries k with $0 \leq \alpha_k < 1$, but having this more general notation is useful for the sake of the proof of uniqueness of wages. Let

$$\boldsymbol{\mathcal{Z}}(\boldsymbol{w}) \equiv \left\{ \left(\sum_{k=1}^{K} L_{1,k} - \boldsymbol{L}_1, \ldots, \sum_{k=1}^{K} L_{N,k} - \boldsymbol{L}_N \right) \middle| (L_{1,k}, \ldots, L_{N,k}) \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w}) \text{ for } k = 1, \ldots, K \right\}.$$

In the proof of Proposition OA.1 we showed that $\boldsymbol{\mathcal{Z}}(\boldsymbol{w})$ satisfies the boundary condition, which implies that any equilibrium wage vector $\boldsymbol{w}$ is positive.

Consider any positive wage vectors $\boldsymbol{w'}$ and $\boldsymbol{w''}$ such that $w''_i = w'_i$ for $i < I$ and $w''_i > w'_i$ for $i \geq I$ with $2 \leq I \leq N$. Consider any industry k with $0 \leq \alpha_k < 1$ and $\boldsymbol{L'}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$ (in this case, $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$ are singletones). Observe that vector $\boldsymbol{w''}$ can be obtained from vector $\boldsymbol{w'}$ by iteratively rising each of its ith components from w'_i to w''_i for $i = I, \ldots, N$. Proposition OA.4 implies that, at each step of this procedure, we will be increasing $L_{i,k}$. This implies that $L''_{i,k} > L'_{i,k}$ for $i < I$, which is equivalent to $w''_i L''_{i,k} > w'_i L'_{i,k}$, because $w'_i = w''_i$ for $i < I$. Thus, we get that $\sum_{i=1}^{I-1} w''_i L''_{i,k} > \sum_{i=1}^{I-1} w'_i L'_{i,k}$.

Now consider industries with $\alpha_k = 1$ and *any* $\boldsymbol{L'}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w'})$ and $\boldsymbol{L''}_k \in \boldsymbol{\mathcal{L}}_k(\boldsymbol{w''})$. Proposition OA.5 implies that in all such such industries, $\sum_{i=1}^{I-1} w''_i L''_{i,k} \geq \sum_{i=1}^{I-1} w'_i L'_{i,k}$. Then, using the point shown just above, if there is at least one industry k' with $0 \leq \alpha_{k'} <$

[8] Again, the first inequality is weak, because, due to multiplicity of labor allocations given wages, it can be that $L'_{i,k} = 0$ for all $I \leq i \leq N$ even though $\mathcal{N}_k(\boldsymbol{w'})$ contains some $I \leq i \leq N$.

1, we necessarily have that

$$\sum_{i=1}^{I-1} w_i'' \mathcal{Z}_i\left(\boldsymbol{w''}\right) = \sum_{k=1}^{K}\sum_{i=1}^{I-1} w_i'' L_{i,k}'' - \sum_{i=1}^{I-1} w_i'' \bar{L}_i > \sum_{k=1}^{K}\sum_{i=1}^{I-1} w_i' L_{i,k}' - \sum_{i=1}^{I-1} w_i' \bar{L}_i = \sum_{i=1}^{I-1} w_i' \mathcal{Z}_i\left(\boldsymbol{w'}\right), \tag{14}$$

where we have used the fact that $\sum_{i=1}^{I-1} w_i'' \bar{L}_i = \sum_{i=1}^{I-1} w' \bar{L}_i$ since $w_i'' = w_i'$ for all $i < I$. This contradicts the fact that $\mathcal{Z}_i\left(\boldsymbol{w'}\right) = 0$ and $\mathcal{Z}_i\left(\boldsymbol{w''}\right) = 0$ for all i.

Suppose that there are no industries k' with $0 \leq \alpha_{k'} < 1$. If there is at least one industry k' such that $\sum_{i=1}^{I-1} w_i'' L_{i,k'}'' > \sum_{i=1}^{I-1} w_i' L_{i,k'}'$, then again, we get inequality (14), which leads to a contradiction. If, on the other hand, we have $\sum_{i=1}^{I-1} w_i'' L_{i,k}'' = \sum_{i=1}^{I-1} w_i' L_{i,k}'$ for all k, then Proposition OA.5 implies that we have strong inequalities $\sum_{i=I}^{N} w_i'' L_{i,k}'' > \sum_{i=I}^{N} w_i' L_{i,k}'$ for all k. This, in turn, gives an inequality similar to inequality (14) (with the summation over countries running from I to N), which leads to a contradiction. This concludes the proof of Proposition 2. □

4.7. Uniqueness in the Case of a Small Open Economy

Proposition OA.6. *Assume that $0 \leq \alpha_k \leq 1$ for all k. Furthermore, assume that a particular country is a small open economy in the sense that changes in its labor allocations and wage do not impact labor allocations, price indices, and wages in other countries. Then the economy of this country has a unique equilibrium.*

Proof. Fix country i and let it be a small open economy in the sense that changes in its labor allocations and wage do not impact labor allocations, price indices, and wages in other countries. The equilbrium system for country i's economy consists of $K+1$ conditions that are a subset of equilibrium conditions (3)-(5) defined in the main text for the world economy. Specifically, country i's equilibrium conditions are K goods market clearing conditions

$$L_{i,k} \geq 0, \quad G_{i,k}\left(w_i, L_{i,k}\right) \geq 0, \quad L_{i,k} G_{i,k}\left(w_i, L_{i,k}\right) = 0, \quad \text{for } k = 1, \ldots, K, \tag{15}$$

and one labor market clearing condition

$$\sum_{k} L_{i,k} - \bar{L}_i = 0, \tag{16}$$

where

$$G_{i,k}\left(w_i, L_{i,k}\right) \equiv w_i - \frac{S_{i,k}\beta_{i,k}\bar{L}_i}{S_{i,k}L_{i,k}^{\alpha_k}w_i^{-\varepsilon_k} + A_{i,k}} L_{i,k}^{\alpha_k - 1} w_i^{-\varepsilon_k + 1} - B_{i,k} S_{i,k} w_i^{-\varepsilon_k} L_{i,k}^{\alpha_k - 1}, \tag{17}$$

with

$$A_{i,k} \equiv \sum_{l \neq i} S_{l,k} L_{l,k}^{\alpha_k} \left(w_l \tau_{il,k}\right)^{-\varepsilon_k},$$
$$B_{i,k} \equiv \mu_k^{-\varepsilon_k} \sum_{n \neq i} \tau_{ni,k}^{-\varepsilon_k} P_{n,k}^{\varepsilon_k} \beta_{n,k} w_n \bar{L}_n.$$

The equilibrium of country i's economy is given by wage w_i and labor allocations $L_{i,k}$ for $k = 1, \ldots, K$ that solve (15)-(17). Here we slightly abuse the notation used in conditions (3)-(5) defined in the main text by dropping the dependence of $G_{i,k}$ on wages and labor allocations of countries different from i. This is done to emphasize the small open economy assumption.[9]

In what follows, drop country i's index from notation for brevity. Fix wage w and focus on the complementary slackness conditions (15). Consider first any industry k with $\alpha_k < 1$. Condition $G_k\left(w, L_k\right) \geq 0$ can be written as:

$$1 \geq \frac{S_k \beta_k \bar{L}}{S_k L_k^{\alpha_k} w^{-\varepsilon_k} + A_k} L_k^{\alpha_k - 1} w^{-\varepsilon_k} + B_k S_k w^{-\varepsilon_k - 1} L_k^{\alpha_k - 1}. \tag{18}$$

Since $\alpha_k < 1$, the right-hand side of (18) goes to ∞ as $L_k \to 0$, while the left-hand side does not depend on L_k. Hence, for a fixed wage w, only positive labor allocations can satisfy (18). The complementarity slackness condition (15) then implies that condition $G_k\left(w, L_k\right) \geq 0$ holds with equality. Next, the right-hand side of (18) is a decreasing function of L_k, and it falls from ∞ to 0 as L_k increases from 0 and ∞. Hence, for each fixed wage w there is a unique solution to $G_k\left(w, L_k\right) = 0$. In other words, equation $G_k\left(w, L_k\right) = 0$ defines an implicit function from wages to labor allocations, $L_k\left(w\right)$. Since the right-hand side of (18) is also a decreasing function of wages, the implicit function theorem implies that $L_k\left(\cdot\right)$ is a decreasing function. We can easily show that

[9] Observe that we assume that country i can impact its own industry-level price indices given by $\mu_k \left(S_{i,k} L_{i,k}^{\alpha_k} w_i^{-\varepsilon_k} + A_{i,k}\right)^{-\frac{1}{\varepsilon_k}}$. This guarantees that, as we increase trade costs to infinity, country i's production pattern converges to the production pattern in autarky.

this function ranges from 0 to ∞ for $w \in (0, \infty)$. For that, rewrite $G_k(w, L_k) = 0$ as

$$L_k^{1-\alpha_k} = \frac{S_k \beta_k \bar{L}}{S_k L_k^{\alpha_k} w^{-\varepsilon_k} + A_k} w^{-\varepsilon_k} + B_k S_k w^{-\varepsilon_k - 1}.$$

As w goes to ∞, the right-hand side of the above expression converges to 0, and, hence, $L_k(w)$ converges to 0. Similarly, as w goes to 0, the right-hand side of the above expression converges to ∞, and, hence, $L_k(w)$ converges to ∞. By the implicit function theorem $L_k(\cdot)$ is a continuous function, and, therefore, it takes the whole range of values from 0 to ∞ as w ranges from 0 to ∞.

Next, consider an industry k with $\alpha_k = 1$. Write condition $G_k(w, L_k) \geq 0$ as

$$1 \geq \frac{S_k \beta_k \bar{L}}{S_k L_k w^{-\varepsilon_k} + A_k} w^{-\varepsilon_k} + S_k B_k w^{-\varepsilon_k - 1}. \tag{19}$$

If $S_k B_k w^{-\varepsilon_k - 1} \geq 1$ or, equivalently, if $w \leq (S_k B_k)^{\frac{1}{1+\varepsilon_k}}$, then (19) cannot be satisfied for any finite $L_k \geq 0$. So, in any equilibrium we must have $w > \underline{w}_k$ with $\underline{w}_k \equiv (S_k B_k)^{\frac{1}{1+\varepsilon_k}}$. Now consider equation $G_k(w, 0) = 0$:

$$1 = \frac{S_k \beta_k \bar{L}}{A_k} w^{-\varepsilon_k} + S_k B_k w^{-\varepsilon_k - 1}. \tag{20}$$

The left-hand side of (20) does not depend on w, while the right-hand side is a decreasing function of w, which falls from ∞ to 0 as w increases from 0 to ∞. Hence, there is a unique w — denote it by $\overline{w}_k$ — that solves (20), and $\overline{w}_k > \underline{w}_k$.

Next, consider any $w \in (\underline{w}_k, \overline{w}_k)$. The definition of $\overline{w}_k$ implies that, if $L_k = 0$, then the right-hand side of (19) is higher than 1. Therefore, we must have $L_k > 0$, and, hence, the complementarity slackness condition (15) has to hold with equality. Since the right-hand side of (19) is a decreasing function of L_k and since $w > \underline{w}_k$, there exists a unique L_k that solves $G_k(w, L_k) = 0$ for a given w. In other words, for $w \in (\underline{w}_k, \overline{w}_k)$, condition $G_k(w, L_k) = 0$ defines an implicit function $L_k(w)$ — the same as in the case with $\alpha_k < 1$. Moreover, as in the case with $\alpha_k < 1$, $L_k(\cdot)$ is a decreasing function. Importantly, $L_k(w) \to \infty$ as $w \to \underline{w}_k$.

Now consider $w \geq \overline{w}_k$. For such w the right-hand side of (20) is weakly smaller than 1. Therefore, any positive L_k will make the right-hand side of (19) strictly smaller than 1, while the complementary slackness condition (15) requires that for $L_k > 0$ condi-

tion (19) holds with equality. Hence, the only possibility to satisfy (15) for $w \geq \overline{w}_k$ is to have $L_k = 0$, which is the unique solution of (15) in this case. Furthermore, $L_k(w) \to 0$ as w converges to $\overline{w}_k$ from the left.

The arguments in the above two paragraphs imply that for industries k with $\alpha_k = 1$ condition (15) defines a function $L_k(w)$ for $w > \underline{w}_k$. This function is decreasing for $w \in (\underline{w}_k, \overline{w}_k)$, is zero for all $w \geq \overline{w}_k$, and it takes the full range of values from 0 to ∞ as w varies from $\underline{w}_k$ to ∞.

Let us now turn to the labor market clearing condition (16). We can write the excess demand for labor as a function of the wage:

$$Z(w) = \sum_k L_k(w) - \bar{L} \quad \text{for } w > \underline{w},$$

where $\underline{w} \equiv \max\{\underline{w}_k \mid \text{for } k \text{ such that } \alpha_k = 1\}$ if there are industries k with $\alpha_k = 1$ and $\underline{w} \equiv 0$ if there are no such industries. $Z(w)$ is a decreasing function w, and it falls from ∞ to $-\bar{L}$ as w increases from $\underline{w}$ to ∞. Hence, there is a unique wage that solves $Z(w) = 0$.

□

4.8. Uniqueness in a Multi-Industry Closed Economy with CES Consumption across Industries

Consider an autarky version of the model in this paper, with the only exception being that we now assume CES preferences across industries. Without loss of generality normalize the wage in this economy to one. The equilibrium system is then given by

$$L_k = \frac{P_k^{1-\sigma}}{\sum_{s=1}^K P_s^{1-\sigma}} \bar{L} = \frac{\left(1/\left[S_k^{1/(\sigma-1)} L_k^{\psi_k}\right]\right)^{1-\sigma}}{\sum_{s=1}^K \left(1/\left[S_s^{1/(\sigma-1)} L_s^{\psi_s}\right]\right)^{1-\sigma}} \bar{L} = \frac{a_k L_k^{\alpha_k}}{\sum_{s=1}^K a_s L_s^{\alpha_s}} \bar{L},$$

where $a_k \equiv S_k$ and $\alpha_s \equiv (\sigma - 1)\psi_s$. Observe also that

$$\sum_{k=1}^K \frac{a_k L_k^{\alpha_k}}{\sum_{s=1}^K a_s L_s^{\alpha_s}} \bar{L} = \bar{L}.$$

Hence, any solution to the above equilibrium system clears the labor market.

If $\alpha_k = 0$ for all k, then we immediately have closed-form solutions for all industry-level labor allocations. Suppose, without loss of generality, that $0 < \alpha_k < 1$ for $k = 1, \ldots, \widetilde{K}$, and $\alpha_k = 0$ for $k = \widetilde{K}+1, \ldots, K$, with $1 < \widetilde{K} \leq K$. Given values for $L_1, \ldots, L_{\widetilde{K}}$, we can find the rest of labor allocations in closed form using expressions:

$$L_k = \frac{a_k}{\sum_{s=1}^{\widetilde{K}} a_s L_s^{\alpha_s} + b} \bar{L}, \quad k = \widetilde{K}+1, \ldots, K,$$

where $b \equiv \sum_{s=\widetilde{K}+1}^{K} a_s > 0$ if $\widetilde{K} < K$, and $b = 0$ if $\widetilde{K} = K$.

We now proceed to show that there exist a unique set of equilibrium industry-level labor allocations for industries $k = 1, \ldots, \widetilde{K}$. Note that restriction $\alpha \in [0, 1)$ implies that in equilibrium all industries are in operation, i.e., $L_k > 0$ for all k. Thus, we can rewrite the equilibrium system of industry-level labor allocations in the following form:

$$1 - \frac{a_k L_k^{\alpha_k - 1}}{\sum_{s=1}^{\widetilde{K}} a_s L_s^{\alpha_s} + b} \bar{L} = 0, \quad k = 1, \ldots, \widetilde{K}. \tag{21}$$

Let set Γ be defined by $\Gamma \equiv \left\{ \left(L_1, \ldots, L_{\widetilde{K}}\right) | L_k \geq 0,\, k = 1, \ldots, \widetilde{K};\, \sum_{k=1}^{K} L_k \leq \bar{L} \right\}$ if $\widetilde{K} < K$, and $\Gamma \equiv \left\{ (L_1, \ldots, L_K) | L_k \geq 0,\, k = 1, \ldots, K;\, \sum_{k=1}^{K} L_k = \bar{L} \right\}$ if $\widetilde{K} = K$, and consider the function

$$F\left(L_1, \ldots, L_{\widetilde{K}}\right) = \sum_{s=1}^{\widetilde{K}} \alpha_s L_s - \ln\left(\sum_{s=1}^{\widetilde{K}} a_s L_s^{\alpha_s} + b \right) \bar{L}.$$

It is easy to see that F is a well-defined continuous function on Γ, because the argument of the logarithm term is never zero on Γ. Indeed, if $\widetilde{K} < K$, then we necessarily have that $b > 0$; and if $\widetilde{K} = K$, then $\sum_{k=1}^{K} L_k = \bar{L} > 0$ and so $L_k > 0$ for at least one k. Therefore, since Γ is a compact set, $F(\cdot)$ has a global minimum on Γ.

Next, note that since $0 < \alpha_k < 1$, we have that for each $k = 1, \ldots, \widetilde{K}$ function $L_k^{\alpha_k}$ is strictly concave for $L_k \geq 0$. Then, since $\ln(\cdot)$ is a strictly increasing function, $\ln\left(\sum_{s=1}^{\widetilde{K}} a_s L_s^{\alpha_s} + b\right) \bar{L}$ is strictly concave in Γ. Hence, $F\left(L_1, \ldots, L_{\widetilde{K}}\right)$ is a strictly convex function in the set Γ. Since Γ is a convex set, $F(\cdot)$ has at most one global minimum $\boldsymbol{L^*} = \left(L_1^*, \ldots, L_{\widetilde{K}}^*\right)$ in this set. Using an argument similar to the one used in the proof of Lemma 2 in Appendix B in the main text, we can show that the unique global minimum

of $F(\cdot)$ on Γ is achieved at an interior point of Γ. That is, $L_k^* > 0$ for all k. Finally, observe that $\frac{1}{\alpha_k}\partial F\left(L_1,\ldots,L_{\widetilde{K}}\right)/\partial L_k$ is given by equation (21). So, applying the same argument as in the proof of Lemma 3 in Appendix B in the main text, we can show that $\boldsymbol{L}^*$ is a global minimum of $F(\cdot)$ on Γ if and only if $\boldsymbol{L}^*$ is a solution to the system in equation (21). Hence, there is a unique solution to the system in equation (21), which, in turn, implies that the economy has a unique equilibrium.

4.9. Applying Uniqueness Results in Allen, Arkolakis and Li (2021)

We can map the equilibrium system of our common framework into the system in Equation (3) in Allen, Arkolakis and Li (2021, henceforth AAL) and explore if their Theorem 2 can be invoked to establish uniqueness. Ignoring the inequality part of the complementary slackness conditions, the equilibrium conditions in the common framework can be written as

$$\begin{aligned} w_i^{1+\varepsilon_k} L_{i,k}^{1-\alpha_k} &= \sum_n S_{i,k}\tau_{ni,k}^{-\varepsilon_k} P_{n,k}^{\varepsilon_k}\beta_{n,k} w_n \bar{L}_n, \\ P_{n,k}^{-\varepsilon_k} &= \sum_l S_{l,k}\tau_{nl,k}^{-\varepsilon_k} w_l^{-\varepsilon_k} L_{l,k}^{\alpha_k}, \\ w_i &= \sum_s \frac{L_{i,s}}{\bar{L}_i} w_i. \end{aligned}$$

To turn this into the AAL structure, assume that $\varepsilon_k = \varepsilon$ and $\alpha_k = \alpha$ for all k, and let $x_{ik}^1 \equiv w_i$, $x_{ik}^2 \equiv L_{i,k}$, and $x_{ik}^3 \equiv P_{i,k}$. Then the above equilibrium conditions can be written as

$$\left(x_{ik}^1\right)^{1+\varepsilon}\left(x_{ik}^2\right)^{1-\alpha} = \sum_{ns} K_{ik,ns}^1 x_{ns}^1 \left(x_{ns}^3\right)^{\varepsilon}, \tag{22}$$

$$\left(x_{ik}^3\right)^{-\varepsilon} = \sum_{ns} K_{ik,ns}^2 \left(x_{ns}^1\right)^{-\varepsilon}\left(x_{ns}^2\right)^{\alpha}, \tag{23}$$

$$x_{ik}^1 = \sum_{ns} K_{ik,ns}^3 x_{ns}^1 x_{ns}^2, \tag{24}$$

where $K_{ik,ns}^1$, $K_{ik,ns}^2$, and $K_{ik,ns}^3$ are appropriate (nonnegative) constants. This system maps into the system of equations (3) in AAL with each "location" being an (i,s) pair.

Following AAL's notation, we have

$$\Gamma = \begin{pmatrix} 1+\varepsilon & 1-\alpha & 0 \\ 0 & 0 & -\varepsilon \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & \varepsilon \\ -\varepsilon & \alpha & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Assuming that $\alpha \in [0,1)$, we have

$$\left|B\Gamma^{-1}\right| = \begin{pmatrix} 0 & 1 & 1 \\ \dfrac{\alpha}{1-\alpha} & 0 & \dfrac{\alpha+\varepsilon}{1-\alpha} \\ \dfrac{1}{1-\alpha} & 0 & \dfrac{\alpha+\varepsilon}{1-\alpha} \end{pmatrix}.$$

Each element of this matrix is a non-decreasing function of α and ε. Hence, the spectral radius of this matrix for arbitrary $\alpha \in [0,1)$ and $\varepsilon \geq 0$ is at least as large as the spectral radius of the same matrix with $\alpha = 0$ and $\varepsilon = 0$, i.e., of the matrix:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(see Corollary 8.1.19 in Horn and Johnson, 2013), which has a spectral radius of 1. In simulations we see that, in order to have the spectral radius of $\left|B\Gamma^{-1}\right|$ to be not larger than 1, we need to have a negative α. So, we cannot invoke AAL's Theorem 2 to establish uniqueness.

It is interesting to explore how the AAL approach can be used to establish uniqueness for labor allocations given wages. That would correspond to the case in which we take x^1_{ik} as given and ignore equation (24) in the system (22)-(24). Relabeling x^2_{ik} as y^1_{ik} and x^3_{ik} as y^2_{ik}, the relevant system can be written as

$$\left(x^1_{ik}\right)^{1-\alpha} = \sum_{ns} \tilde{K}^1_{ik,ns} \left(x^2_{ns}\right)^{\varepsilon},$$
$$\left(x^2_{ik}\right)^{-\varepsilon} = \sum_{ns} \tilde{K}^2_{ik,ns} \left(x^1_{ns}\right)^{\alpha}.$$

This entails

$$\Gamma = \begin{pmatrix} 1-\alpha & 0 \\ 0 & -\varepsilon \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \varepsilon \\ \alpha & 0 \end{pmatrix}.$$

Then for $\alpha \in [0,1)$ we have

$$\left|B\Gamma^{-1}\right| = \begin{pmatrix} 0 & 1 \\ \dfrac{\alpha}{1-\alpha} & 0 \end{pmatrix}.$$

The spectral radius of this matrix is $\sqrt{\frac{\alpha}{1-\alpha}}$, which is lower than 1 only if $\alpha < 1/2$. This condition is more stringent than the one in Proposition 1 in the main text.

5. Scale Economies, Welfare and Trade Flows

5.1. Proof of Proposition 4 in the Main Text

The labor allocation in industry 2 is given by (L_1, L_2) that solves

$$w_i L_i = \sum_n S_i L_i^{\alpha} \left(w_i \tau_{ni}\right)^{-\varepsilon} P_n^{\varepsilon} \beta_n w_n \bar{L}_n, \tag{25}$$

Letting $x_i \equiv w_i L_i$, $y_i \equiv P^{\varepsilon}$, $a_{ni} \equiv S_i w_i^{-\alpha-\varepsilon} \tau_{ni}^{-\varepsilon}$, $b_n \equiv \beta_n w_n \bar{L}_n$ and log-differentiating the system in (25) around an equilibrium point for some change in a_{ni} we get

$$d\ln x_i = \frac{1}{1-\alpha} \sum_n \chi_{ni} \left(d\ln a_{ni} + d\ln y_n\right) \quad \text{for} \quad i = 1, ..., N,$$

$$d\ln y_i = -\sum_j \lambda_{ij} \left(d\ln a_{ij} + \alpha d\ln x_j\right) \quad \text{for} \quad n = 1, ..., N,$$

where $\lambda_{ni} \equiv a_{ni} x_i^{\alpha} y_n$ are import shares and $\chi_{ij} \equiv \dfrac{a_{ij} x_j^{\alpha} y_i b_i}{\sum_n a_{nj} x_j^{\alpha} y_n b_n}$ are export shares (i.e. χ_{ij} is the share of total country j exports directed to country i).

Let $\mathcal{X}$ be the matrix of export shares with elements χ_{ni}, Λ be the matrix of import shares with elements λ_{ni}, let X and Y be column vectors with elements $d\ln x_i$ and $d\ln y_i$, let A be the matrix with typical element $d\ln a_{ni}$, and let matrix $\mathbf{1}$ be a column vector whose entries are all ones. We can rewrite the system in matrix form as

$$X = \frac{1}{1-\alpha} \left(\left[\mathcal{X}^T \circ A^T\right] \mathbf{1} + \mathcal{X}^T Y\right),$$

$$Y = -\left[\Lambda \circ A\right]\mathbf{1} - \alpha\Lambda X,$$

where the symbol "$\circ$" denotes the Hadamard product. Substituting the first equation into the second and rearranging we get

$$\left(\gamma I + \Lambda\mathcal{X}^T\right)Y = -\left(\gamma\left[\Lambda \circ A\right]\mathbf{1} + \Lambda\left[\mathcal{X}^T \circ A^T\right]\mathbf{1}\right), \tag{26}$$

where $\gamma = \dfrac{1-\alpha}{\alpha}$.

Since $\sum_n \lambda_{ni} b_n = x_i$ implies $\chi_{ij} = \dfrac{\lambda_{ij} b_i}{x_j}$, we can write $\mathcal{X} = B\Lambda L^{-1}$ and by extension $\Lambda\mathcal{X}^T = \Lambda L^{-1}\Lambda^T B$, where L is a diagonal matrix with elements x_i on the diagonal and B is a diagonal matrix with elements b_i on the diagonal. Observe that matrices $\Lambda L^{-1}\Lambda^T B$ and $\left(B^{\frac{1}{2}}\Lambda\right)L^{-1}\left(B^{\frac{1}{2}}\Lambda\right)^T$ have the same eigenvalues, and that matrix $\left(B^{\frac{1}{2}}\Lambda\right)L^{-1}\left(B^{\frac{1}{2}}\Lambda\right)^T$ is positive semidefinite. It then follows that all eigenvalues of $\Lambda L^{-1}\Lambda^T B$ are real and nonnegative, which, in turn, implies that eigenvalues of $\gamma I + \Lambda\mathcal{X}^T$ are real and positive for any $\gamma > 0$, and so $\det\left(\gamma I + \Lambda\mathcal{X}^T\right) > 0$ for $\gamma > 0$. Since we are interested only in the signs of entries of Y in expression (26), we can then focus on

$$-\det\left(\gamma I + \Lambda\mathcal{X}^T\right)Y = adj\left(\gamma I + \Lambda\mathcal{X}^T\right)\left(\gamma\left[\Lambda \circ A\right]\mathbf{1} + \Lambda\left[\mathcal{X}^T \circ A^T\right]\mathbf{1}\right), \tag{27}$$

where $adj\left(\cdot\right)$ is the adjugate of a matrix.

Consider now the case $N = 2$ and without loss of generality consider a unilateral trade liberalization for country 1. We are then interested in the sign of $\partial \ln y_1 / \partial \ln a_{12}$, and so for this case we have $d\ln a_{11} = d\ln a_{22} = d\ln a_{21} = 0$ and $d\ln a_{12} \neq 0$. Using the facts $adj\left(\gamma I + \Lambda\mathcal{X}^T\right) = \gamma I + adj\left(\Lambda\mathcal{X}^T\right)$ (this is true only in the case of 2×2 matrices), $adj\left(\Lambda\mathcal{X}^T\right) = adj\left(\mathcal{X}^T\right)adj\left(\Lambda\right)$ and $adj\left(\Lambda\right)\Lambda = \det\left(\Lambda\right)$, and applying the result in (27) together with some manipulation we have

$$-\frac{\det\left(\gamma I + \Lambda\mathcal{X}\right)}{\lambda_{12} d\ln a_{12}}Y = \gamma^2\begin{pmatrix}1\\0\end{pmatrix} + \gamma\begin{pmatrix}\lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12}\\ -\lambda_{21}\chi_{11} - \lambda_{22}\chi_{12} + \dfrac{\lambda_{22}}{\lambda_{12}}\chi_{12}\end{pmatrix} + \frac{\chi_{12}}{\lambda_{12}}\det\left(\Lambda\right)\begin{pmatrix}-\chi_{21}\\ \chi_{11}\end{pmatrix}.$$

Using the expression above together with some algebra one can then show that there exists $\bar{\gamma}^{1,\tau} > 0$ such that for any $\gamma \in (0,\infty)$ we have that $\partial \ln y_1 / \partial \ln a_{12}$ is negative if and

only if $\gamma > \bar{\gamma}^{1,\tau}$, with $\bar{\gamma}^{1,\tau}$ given by

$$\bar{\gamma}^{1,\tau} = \frac{\sqrt{D^{\tau}} - (\lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12})}{2} > 0,$$

where

$$D^{1,\tau} \equiv (\lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12})^2 + 4(\chi_{11}\chi_{22} - \chi_{12}\chi_{21})\lambda_{21}$$

is always positive.[10] Since $\gamma > \bar{\gamma}^{1,\tau} \Leftrightarrow \alpha < \bar{\alpha}^{1,\tau} = 1/(1+\bar{\gamma}^{1,\tau})$ and since

$$-\partial \ln P_1/\partial \ln \tau_{12} = \partial \ln P_1^{\varepsilon}/\partial \ln \tau_{12}^{-\varepsilon} = \partial \ln y_1/\partial \ln a_{12},$$

the result in the main text immediately follows.

Consider now a productivity increase in country 2. Here we are interested in the sign of $\partial \ln y_1/\partial \ln a_{22}$, and so for this case we have $d\ln a_{11} = d\ln a_{21} = 0$ and $d\ln a_{22} \neq 0$. Note also that we have $d\ln a_{12} = d\ln \tau_{12}^{-\varepsilon} + d\ln a_{22} = d\ln a_{22}$ since $d\ln \tau_{12}^{-\varepsilon} = 0$. Analogous to the trade liberalization exercise above one can readily show that $\partial \ln P_1/\partial \ln S_2 = \partial \ln P_1/\partial \ln a_{22} < 0$ if and only if $\bar{\gamma}_n^S \Leftrightarrow \alpha < \bar{\alpha}_n^S = 1/(1+\bar{\gamma}^{1,S})$, with $\bar{\gamma}^{1,S}$ given by

$$\bar{\gamma}^{1,S} \equiv \frac{\sqrt{D^{1,S}} - \left(\lambda_{21}\chi_{21} + \chi_{12} - \lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21}\right)}{2} > 0$$

where

$$\begin{aligned} D^{1,S} \equiv{}& \left(\lambda_{21}\chi_{21} + \chi_{12} - \lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21}\right)^2 + 4(\chi_{11}\chi_{22} - \chi_{12}\chi_{21})\lambda_{21} \\ &+ 4\left(\lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21} + \lambda_{22}\chi_{22}\right)\chi_{22}. \end{aligned}$$

The result in the main text then immediately follows. □

5.2. System in Changes and Algorithm for Section 6.2 in the Main Text

In this appendix we derive the system in changes and describe the algorithm used to perform counterfactual exercises in Section 6.2 in the main text.

[10] In particular, it is straightforward to verify that $\tau_{12}\tau_{21} \geq 1$ implies $\chi_{11}\chi_{22} - \chi_{12}\chi_{21} \geq 0$.

5.2.1. Derivation of System in Changes

In the presence of tariffs, total (*ad-valorem*) trade costs are given by $\tau_{ni,k}(1+t_{ni,k})$, where $t_{ni,k}$ is a tariff that importer n imposes on goods from exporter i's industry k, and $\tau_{ni,k}$ captures all other (iceberg) costs of trade. Since data on trade flows features trade imbalances, we assume that country n's value of net imports is given by D_n, which can be different from zero and satisfy $\sum_n D_n = 0$. The exact hat algebra approach by Dekle, Eaton and Kortum (2008) works as long as we start from an equilibrium that does not have corners, which is the case in our data as there are no (i,k) pairs with $L_{i,k} = 0$.

Let us first derive the equilibrium system of equations for the version of our common framework that features tariffs and trade imbalances. Denote by E_n the total expenditure in country n. Then country n's expenditure on goods from industry (i,k) is given by $X_{ni,k} = \lambda_{ni,k}\beta_{n,k}E_n$, with trade shares given by

$$\lambda_{ni,k} = \frac{S_{i,k}L_{i,k}^{\alpha_k}\left(w_i\tau_{ni,k}\left(1+t_{ni,k}\right)\right)^{-\varepsilon_k}}{\sum_l S_{l,k}L_{l,k}^{\alpha_k}\left(w_l\tau_{nl,k}\left(1+t_{nl,k}\right)\right)^{-\varepsilon_k}}.$$

Budget balance requires that $E_n = w_n\bar{L}_n + D_n + R_n$, where D_n are trade imbalances in the data with $\sum_n D_n = 0$, and

$$R_n \equiv \sum_k \sum_i \frac{t_{ni,k}}{1+t_{ni,k}} X_{ni,k} = \sum_k \sum_i \frac{t_{ni,k}\lambda_{ni,k}}{1+t_{ni,k}}\beta_{n,k}E_n$$

denotes total tariff revenues in country n. Substituting E_n into the definition of R_n yields:

$$R_n = \frac{\pi_n}{1-\pi_n}\left(w_n\bar{L}_n + D_n\right),$$

where

$$\pi_n \equiv \sum_k \sum_i \frac{t_{ni,k}\lambda_{ni,k}}{1+t_{ni,k}}\beta_{n,k}.$$

Demand for goods from industry (i,k) is given by

$$\begin{aligned}\sum_n \frac{1}{1+t_{ni,k}} X_{ni,k} &= \sum_n \frac{\lambda_{ni,k}}{1+t_{ni,k}} \beta_{n,k} E_n \\ &= \sum_n \frac{\lambda_{ni,k}}{1+t_{ni,k}} \left(\frac{\beta_{n,k}}{1-\pi_n}\right) \left(w_n \bar{L}_n + D_n\right),\end{aligned}$$

and so the goods market clearing condition is

$$L_{i,k} \geq 0, \quad G_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) \geq 0, \quad L_{i,k} G_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) = 0,$$

with

$$\begin{aligned}G_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) &\equiv w_i - \frac{1}{L_{i,k}} \sum_n \frac{\lambda_{ni,k}}{1+t_{ni,k}} \beta_{n,k} E_n \\ &= w_i - \frac{1}{L_{i,k}} \sum_n \frac{\lambda_{ni,k}}{1+t_{ni,k}} \left(\frac{\beta_{n,k}}{1-\pi_n}\right) \left(w_n \bar{L}_n + D_n\right).\end{aligned}$$

Finally, the labor market clearing condition is the same as in the case without tariffs and trade imbalances:

$$\sum_k L_{i,k} = \bar{L}_i.$$

Now we can formulate the system in changes. For any observed variable x, denote its value in a counterfactual equilibrium by x' and the relative change in x by $\hat{x} \equiv x'/x$. Assuming that the observed equilibrium does not have corner labor allocations, we can use the hat notation to write the system of equations for a counterfactual equilibrium with new values for trade costs, tariffs, and productivities:

$$\begin{aligned}&\hat{L}_{i,k} L_{i,k} \geq 0, \quad \tilde{G}_{i,k}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}_k\right) \geq 0, \quad \hat{L}_{i,k} Y_{i,k} \tilde{G}_{i,k}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}_k\right) = 0, \\ &\sum_k \hat{L}_{i,k} Y_{i,k} = Y_i,\end{aligned}$$

with

$$\tilde{G}_{i,k}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}_k\right) \equiv \hat{w}_i - \frac{1}{\hat{L}_{i,k} Y_{i,k}} \sum_n \frac{\lambda'_{ni,k}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}_k\right)}{1+t'_{ni,k}} \cdot \frac{\beta_{n,k}\left(\hat{w}_n Y_n + D_n\right)}{1-\pi'_n\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}\right)},$$

and

$$\lambda'_{ni,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)=\frac{\hat{L}_{i,k}^{\alpha_k}\left(\hat{w}_i\left(1+t'_{ni,k}\right)\hat{\tau}_{ni,k}\right)^{-\varepsilon_k}\left(1+t_{ni,k}\right)^{\varepsilon_k}\lambda_{ni,k}}{\sum_l \hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l\left(1+t'_{nl,k}\right)\hat{\tau}_{nl,k}\right)^{-\varepsilon_k}\left(1+t_{nl,k}\right)^{\varepsilon_k}\lambda_{nl,k}},$$

$$\pi'_n\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}\right)=\sum_k\sum_i\frac{t'_{ni,k}\lambda'_{ni,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)}{1+t'_{ni,k}}\beta_{n,k},$$

where $Y_{i,k}\equiv w_i L_{i,k}$ and $Y_n\equiv w_n\bar{L}_n$.

The above system in changes still allows the counterfactual equilibrium to exhibit corner allocations. Therefore, we need to calculate changes in welfare explicitly as $\hat{I}_n/\hat{P}_n$, where $\hat{I}_n$ is the change in income given by

$$\hat{I}_n=\frac{w'_n\bar{L}_n+R'_n}{w_n\bar{L}_n+R_n}=\frac{w'_n\bar{L}_n+\pi'_nD_n}{w_n\bar{L}_n+\pi_nD_n}\cdot\frac{1-\pi_n}{1-\pi'_n},$$

and

$$\hat{P}_n=\prod_k\left(\sum_l\hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l(1+t'_{nl,k})\hat{\tau}_{nl,k}\right)^{-\varepsilon_k}\left(1+t_{nl,k}\right)^{\varepsilon_k}\lambda_{nl,k}\right)^{-\beta_{n,k}/\varepsilon_k}.$$

5.2.2. Algorithm for Counterfactuals in Section 6.2 in the Main Text

The algorithm consists of two logical parts: an inner loop and an outer loop. The inner loop keeps $\hat{\boldsymbol{L}}$ fixed and finds wages $\hat{\boldsymbol{w}}$ that clear labor markets. The outer loop finds labor allocations $\hat{\boldsymbol{L}}$ that clear goods markets.

The inner loop exploits the tatonnement process proposed by Alvarez and Lucas (2007). For any variable x calculated in the inner loop, let us denote the value of x on the t-th iteration of the inner loop by $x^{(t)}$ with $x^{(0)}$ denoting the value in the baseline equilibrium (corresponding by assumption to the data). Let us also use the hat notation for the change in $x^{(t)}$: $\hat{x}^{(t)}\equiv x^{(t)}/x^{(0)}$. The $(t+1)$-th inner loop iteration for wages can be written as

$$\hat{w}_i^{(t+1)}=\hat{w}_i^{(t)}+\nu\frac{\sum_k X_{i,k}^{(t)}-\hat{w}_i^{(t)}Y_i^{(0)}}{Y_i^{(0)}},$$

where ν is some small positive number that is a parameter of the algorithm,

$$X_{i,k}^{(t)} = \sum_n \frac{\lambda_{ni,k}^{(t)}}{1+t'_{ni,k}} \cdot \frac{\beta_{n,k}\left(\hat{w}_n^{(t)} Y_n^{(0)} + D_n^{(0)}\right)}{1 - \sum_k \sum_l \frac{t'_{nl,k}\lambda_{nl,k}^{(t)}}{1+t'_{nl,k}}\beta_{n,k}},$$

and

$$\lambda_{ni,k}^{(t)} = \frac{\hat{L}_{i,k}^{\alpha_k}\left(\hat{w}_i^{(t)}\right)^{-\varepsilon_k}\hat{\tau}_{ni,k}^{-\varepsilon_k}\left(\frac{1+t'_{ni,k}}{1+t_{ni,k}}\right)^{-\varepsilon_k}\lambda_{ni,k}^{(0)}}{\sum_l \hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l^{(t)}\right)^{-\varepsilon_k}\hat{\tau}_{nl,k}^{-\varepsilon_k}\left(\frac{1+t'_{nl,k}}{1+t_{nl,k}}\right)^{-\varepsilon_k}\lambda_{nl,k}^{(0)}}.$$

The inner loop iterates until there is no significant change between $\hat{\boldsymbol{w}}^{(t)}$ and $\hat{\boldsymbol{w}}^{(t+1)}$.

For a given $\hat{\boldsymbol{L}}$, the inner loop gives the set of wages $\hat{w}_i(\hat{\boldsymbol{L}})$ that clear labor markets. The outer loop iterates on $\hat{\boldsymbol{L}}$ using labor demand (in value) for sector (i,k). Denoting by $\hat{L}_{i,k}^{(l)}$ labor allocations on the l-th iteration of the outer loop, the $(l+1)$-th iteration of the outer loop can be written as:

$$\hat{L}_{i,k}^{(l+1)} = \frac{1}{\hat{w}_i(\hat{\boldsymbol{L}}^{(l)})Y_{ik}^{(0)}} \sum_n \lambda'_{ni,k}\left(\hat{\boldsymbol{w}}(\hat{\boldsymbol{L}}^{(l)}), \hat{\boldsymbol{L}}^{(l)}\right)\beta_{n,k}\hat{w}_n(\hat{\boldsymbol{L}}^{(l)})Y_n^{(0)}.$$

The outer loop iterates until there is no significant change between $\hat{\boldsymbol{L}}^{(l)}$ and $\hat{\boldsymbol{L}}^{(l+1)}$.

5.3. Derivation of Algebra for Section 7 in the Main Text

In this appendix we derive the system in changes for the counterfactuals in Section 7 in the main text.

Recall that we use the fact that if $\boldsymbol{L}$ is an equilibrium of the actual economy with scale economies then it is also an equilibrium of the economy with no scale economies given by

$$w_i L_{i,k} = \sum_{n=1}^{N} \frac{T_{i,k}\left(w_i\tau_{ni,k}\right)^{-\varepsilon_k}}{\sum_{l=1}^{N} T_{l,k}\left(w_l\tau_{nl,k}\right)^{-\varepsilon_k}}\beta_{n,k}(w_n\bar{L}_n + D_n) \tag{28}$$

and

$$\sum_{k=1}^{K} L_{i,k} = \bar{L}_i, \tag{29}$$

where $T_{i,k} \equiv S_{i,k}L_{i,k}^{\alpha_k}$ and where D_n are trade deficits satisfying $\sum_n D_n = 0$. Combining (28) and (29), and using $Y_i \equiv w_i\bar{L}_i$ together with shock $\hat{T}_{i,k} = (e_{i,k}/r_{i,k})^{\alpha_k}$, we get a

system in wage changes given by

$$\hat{w}_i Y_i = \sum_{k=1}^{K} \sum_{n=1}^{N} \frac{(e_{i,k}/r_{i,k})^{\alpha_k} (\hat{w}_i)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} (\hat{w}_l)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} (\hat{w}_n Y_n + D_n) .$$

The solution for $\hat{w}_i$ can then be used to get the implied hat change in the labor allocation from

$$\hat{L}_{i,k} = \frac{1}{\hat{w}_i Y_{i,k}} \sum_{n=1}^{N} \frac{(e_{i,k}/r_{i,k})^{\alpha_k} (\hat{w}_i)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} (\hat{w}_l)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} (\hat{w}_n Y_n + D_n) ,$$

where $Y_{i,k} \equiv w_i L_{i,k}$. Finally, we can then get the implied change in trade flows from

$$\hat{X}_{ni,k} = \frac{1}{X_{ni,k}} \cdot \frac{(e_{i,k}/r_{i,k})^{\alpha_k} (\hat{w}_i)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} (\hat{w}_l)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} (\hat{w}_n Y_n + D_n) .$$

6. Computation of Equilibrium

The analysis of Section 4 of the main text suggests two alternative approaches to numerically compute an equilibrium. First, one can use an algorithm that properly deals with the complementary slackness conditions in the system of equations (1) and $\sum_k L_{i,k} = \overline{L}_i$ for $(\boldsymbol{w}, \boldsymbol{L})$. This requires an algorithm for non-linear complementarity problems, such as the PATH solver (Michael C. Ferris and Todd S. Munson, 1999). Second, one can follow the approach used in Section 4 of the main text to prove existence and uniqueness of equilibrium and break the problem in two steps: first, for each wage vector $\boldsymbol{w}$ find $\boldsymbol{L}_k(\boldsymbol{w})$ for each k by solving the optimization problem associated with (10), and second, find the wage vector such that the excess labor demand $\boldsymbol{Z}(\boldsymbol{w}) \equiv \sum_k \boldsymbol{L}_k(\boldsymbol{w}) - \bar{\boldsymbol{L}}$ is zero using the *tatonnement* iterative procedure proposed by Alvarez and Lucas (2007).

It turns out, however, that a third approach does best. Consider the function $\boldsymbol{w}(\boldsymbol{T})$ that one would get simply by solving for wages in the standard multi-sector model with no scale economies and technology parameters $\boldsymbol{T} = \{T_{i,k}\}$, and let $L^d_{i,k}(\boldsymbol{T}, \boldsymbol{w})$ be labor demand as a function of technology parameters and wages also in that model. Let $\boldsymbol{T}(\boldsymbol{L})$ be defined by $T_{i,k}(\boldsymbol{L}) = S_{i,k} L_{i,k}^{\alpha_k}$ and let $\boldsymbol{H}(\boldsymbol{L}) \equiv \boldsymbol{L}^d(\boldsymbol{T}(\boldsymbol{L}), \boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L})))$. By definition of $\boldsymbol{w}(\boldsymbol{T})$ we must have $\sum_k L^d_{i,k}(\boldsymbol{T}(\boldsymbol{L}), \boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L}))) = \bar{L}_i$ for all i and $\boldsymbol{L}$. Thus, if $\boldsymbol{L}^*$ is an inte-

rior fixed point of the mapping $\boldsymbol{H}(\boldsymbol{L})$ then $(\boldsymbol{w}^*, \boldsymbol{L}^*) = (\boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L}^*), \boldsymbol{L}^*)$ is an equilibrium of our economy with economies of scale. Since $\boldsymbol{H}(\boldsymbol{L})$ is a continuous mapping from the compact set $\Lambda \equiv \{\boldsymbol{L} | \sum_k L_{i,k} = \bar{L}_i\}$ to itself, then we can use the iterative procedure given by $\boldsymbol{L_{t+1}} = \boldsymbol{H}(\boldsymbol{L}_t)$ to compute the equilibrium points.

We have used this algorithm for counterfactual analysis with many countries and sectors (see Subsection 6.2 and Section 7; also see Online Appendix 5.2 for a more detailed description this algorithm) and found that it can easily handle corners and that it is very robust. We have also used this algorithm on economies with three or four countries, two sectors, $\alpha = 0.9$ and randomly chosen values for all other parameters. Compared to a standard Newton method, it is slower but way more robust. We randomly generated more than a million economies with three countries and two sectors, and more than half a million economies with four countries and two sectors. In all cases the algorithm using the iterative procedure with $\boldsymbol{L_{t+1}} = \boldsymbol{H}(\boldsymbol{L}_t)$ found a solution, whereas the Newton method found a solution only for some initial conditions.

Because the Newton method is faster, we used it in combination with our iterative procedure in an effort to find examples with multiple equilibria. For each of the random economies mentioned above, we computed the equilibrium with the iterative procedure, and also with the Newton method with 400 different starting points. If there were multiple equilibria, we would likely have one of the solutions of the Newton method be different than the one found by the iterative procedure, but this never happened.[11] For the case with $\alpha = 0.9$ we also computed the sign of the determinant of the (negative of the) normalized excess labor demand evaluated at the equilibrium we found. By the Index Theorem, a negative value would imply multiplicity. We always found this sign to be positive.

[11]To check that this procedure delivers multiplicity when we know they exist, we used the same code for $\alpha = 2$. We find that this leads to multiple equilibria for randomly generated parameters with three economies and two sectors.

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