Online Appendix to "Dispersed Behavior and Perceptions in Assortative Societies"

Mira Frick, Ryota Iijima, and Yuhta Ishii

D Omitted Proofs

D.1 Proofs for Appendix A

D.1.1 Proof of Lemma A.1

For the first point, note that for any $f \in L^1$,

$$\|T_C f\| = \int_0^1 |T_C f(x)|dx \leq \int_0^1 \int_0^1 c(x',x)|f(x')|dx'dx = \int_0^1 |f(x')|dx' = \|f\| < \infty.$$

Thus, $T_C : L^1 \to L^1$. Moreover, since T_C is clearly linear, the above ensures that it is also continuous.

For the second point, consider $f \in \mathcal{I}$. Since C is assortative, $T_C f(x) \geq T_C f(x')$ for all $x \geq x'$, so that $T_C f$ is weakly increasing. To show that $T_C f$ is absolutely continuous, note that for each $x, x' \in (0,1)$,

$$\begin{aligned} T_C f(x) &= \int_0^1 c(y,x)f(y)dy = \int_0^1 \left(\int_{x'}^x c_2(y,z)dz + c(y,x')\right) f(y)dy \\ &= \int_{x'}^x \int_0^1 c_2(y,z)f(y)dydz + T_C f(x'), \end{aligned}$$

where c_2 denotes the partial derivative of c with respect to the second argument, which exists almost everywhere by the absolute continuity assumption on c. Thus $T_C f$ is absolutely continuous with $(T_C f)'(z) = \int_0^1 c_2(y,z)f(y)dy$ for each z.

Finally, for the third point, fix any $f \in L^1$ and $\gamma \in (-1,1)$. Then for any $\tau > \tau'$,

$$\|\sum_{t=0}^{\tau} \gamma^t (T_C)^t f - \sum_{t=0}^{\tau'} \gamma^t (T_C)^t f\| \leq \sum_{t=\tau'+1}^{\tau} |\gamma|^t \|(T_C)^t f\| \leq \sum_{t=\tau'+1}^{\tau} |\gamma|^t \|f\| \leq \frac{|\gamma|^{\tau'+1}}{1-\gamma}\|f\|,$$

which vanishes as $\tau' \to \infty$. Thus, the sequence is Cauchy. Since the space L^1 is complete, this yields the desired result. $\square$

D.1.2 Proof of Lemma A.3

$\succsim_m$-order: It is clear from the definition that $\succsim_m$ is reflexive and transitive; moreover, by Lemma A.2, $\succsim_m$ is linear. To check that $\succsim_m$ is continuous, take sequences $f_n \to f, g_n \to g$ in

$\mathcal{I}$ such that $f_n \succsim_m g_n$ for each n. For any $y \in (0,1)$, we have

$$|\int_y^1 f(x)dx - \int_y^1 f_n(x)dx| \leq \int_y^1 |f(x) - f_n(x)|dx \leq \|f - f_n\| \to 0$$

and likewise $|\int_y^1 g(x)dx - \int_y^1 g_n(x)dx| \to 0$. Since $\int_y^1 f_n(x)dx \geq \int_y^1 g_n(x)dx$ and $\int_0^1 f_n(x)dx = \int_0^1 g_n(x)dx$ for each n, this implies $\int_y^1 f(x)dx \geq \int_y^1 g(x)dx$ and $\int_0^1 f(x)dx = \int_0^1 g(x)dx$. Thus, $f \succsim_m g$ by Lemma A.2.

To verify that $\succsim_m$ is isotone, take any $f, g \in \mathcal{I}$ such that $f \succsim_m g$ and set $h := f - g$. Note that $\int_0^1 h(x)dx = \int_0^1 T_C h(x)dx = 0$. It suffices to show that $\int_y^1 T_C h(x)dx \geq 0$ for all $y \in (0,1)$. To see this, note that $\int_y^1 T_C h(x)dx$ is given by

$$\begin{aligned}
\int_y^1 \int_0^1 h(z)c(z|x)dzdx &= \int_0^1 \int_y^1 c(z|x)dxh(z)dz = \int_0^1 (1 - C(y|z))h(z)dz \\
&= -\int_0^1 \frac{\partial 1 - C(y|z)}{\partial z} \int_0^z h(z')dz'dz + \left[(1 - C(y|z)) \int_0^z h(z')dz'\right]_0^1 \\
&= \int_0^1 \frac{\partial C(y|z)}{\partial z} \int_0^z h(z')dz'dz \geq 0,
\end{aligned}$$

where the second equality uses $\int_y^1 c(z|x)dx = \int_y^1 c(x|z)dx = 1 - C(y|z)$, the third holds by integration by parts (using absolute continuity of c), the fourth uses $\int_0^1 h(z)dz = 0$, and the final inequality uses $\int_0^z h(z')dz' \leq 0$ (by $f \succsim_m g$) and assortativity of C.

<u>$\succsim_d$***-order:***</u> It is clear from the definition that $\succsim_d$ is reflexive, transitive, and linear. To check that it is continuous, take sequences $f_n \to f$ and $g_n \to g$ in $\mathcal{I}$ such that $f_n \succsim_d g_n$ for each n. By standard results (e.g., Theorem 13.6 in Aliprantis and Border (2006)), we can find subsequences $(f_{n_k})_{k\in\mathbb{N}}, (g_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k}(x) \to f(x)$ and $g_{n_k}(x) \to g(x)$ for almost all $x \in (0,1)$. This implies $f(x) - f(x') \geq g(x) - g(x')$ for almost all $x \geq x'$, which ensures $f \succsim_d g$ since f and g are continuous.

To show that $\succsim_d$ is isotone, first consider any bounded $f, g \in \mathcal{I}$ such that $f \succsim_d g$. Since f and g are absolutely continuous, there exist integrable functions $f', g' : (0,1) \to \mathbb{R}$ such that $f(x) = f(0) + \int_0^x f'(y)\,dy$ and $g(x) = g(0) + \int_0^x g'(y)\,dy$ for all $x \in (0,1)$. Then, for any $x \geq x'$ and $C \in \mathcal{C}$, integration by parts yields

$$\begin{aligned}
T_C f(x) - T_C f(x') &= \int_0^1 f(y)(c(y|x) - c(y|x'))dy \\
&= -\int_0^1 f'(y)(C(y|x) - C(y|x'))dy + [f(y)(C(y|x) - C(y|x'))]_0^1 \\
&= -\int_0^1 f'(y)(C(y|x) - C(y|x'))dy \geq -\int_0^1 g'(y)(C(y|x) - C(y|x'))dy \\
&= -\int_0^1 g'(y)(C(y|x) - C(y|x'))dy + [g(y)(C(y|x) - C(y|x'))]_0^1 \\
&= \int_0^1 g(y)(c(y|x) - c(y|x'))dy = T_C g(x) - T_C g(x').
\end{aligned}$$

Here, the inequality holds because the fact that $f \succsim_d g$ and $f, g \in \mathcal{I}$ implies $f'(y) \geq g'(y) \geq 0$ for almost all $y \in (0,1)$.

Next, consider arbitrary $f, g \in \mathcal{I}$ such that $f \succsim_d g$. By defining bounded functions

$$f_n(x) = \begin{cases} f(\frac{1}{n}) \text{ if } x \in (0, \frac{1}{n}) \\ f(x) \text{ if } x \in [\frac{1}{n}, \frac{n-1}{n}] \\ f(\frac{n-1}{n}) \text{ if } x \in (\frac{n-1}{n}, 1) \end{cases} \qquad g_n(x) = \begin{cases} g(\frac{1}{n}) \text{ if } x \in (0, \frac{1}{n}) \\ g(x) \text{ if } x \in [\frac{1}{n}, \frac{n-1}{n}] \\ g(\frac{n-1}{n}) \text{ if } x \in (\frac{n-1}{n}, 1) \end{cases} \tag{21}$$

for each $n \in \mathbb{N}$, we obtain $f_n \succsim_d g_n$ for each n and $f_n \to f, g_n \to g$. For any $C \in \mathcal{C}$, since T_C is a continuous operator, this implies $T_C f_n \to T_C f$ and $T_C g_n \to T_C g$. Thus, $T_C f \succsim_d T_C g$ by continuity of $\succsim_d$, as we already know that $T_C f_n \succsim_d T_C g_n$ from the previous part of the proof. □

D.1.3 Proof of Lemma A.4

The base case $t = 0$ holds because of the following result by Ryff (1963): Call a linear operator $T : L^1 \to L^1$ an $\mathfrak{S}$***-operator*** if $f \succsim_m Tf$ for all $F \in \mathcal{I}$. The representation theorem in Ryff (1963) implies that T is an $\mathfrak{S}$-operator if there exists some measurable function $K : [0,1]^2 \to \mathbb{R}$ such that $Tf(x) = \frac{d}{dx} \int_0^1 K(x,y) f(y) dy$ for all $f \in L^1$ and almost every x and the following conditions are met: (1) $K(0,y) = 0$ for all $0 \leq y \leq 1$; (2) $\text{essup}_y V(K(\cdot, y)) < \infty$, where $V(\cdot)$ denotes the total variation and essup the essential supremum; (3) $\int_0^1 K(x,y) f(y) dy$ is absolutely continuous in x for all $f \in L^1$; (4) $x = \int_0^1 K(x,y) dy$; (5) $x_1 < x_2 \implies K(x_1, \cdot) \leq K(x_2, \cdot)$; and (6) $K(1,y) = 1$ for all $y \in [0,1]$.

Since $C \in \mathcal{C}$, it is easy to see that T_C satisfies these conditions with $K(x,y) := C(x \mid y)$ for all x, y, so that T_C is an $\mathfrak{S}$-operator. Thus, $f \succsim_m T_C f$, proving the base case. The inductive step then follows from isotonicity of $\succsim_m$ (Lemma A.3). □

D.2 Proofs for Appendix C

D.2.1 Proof of Proposition C.1

Let $\mu := \mathbb{E}_F[\theta]$. Consider strategy profiles g_a^α and g_c^α of assortativity neglect and correct agents expressed as functions of quantiles. Write $g^\alpha := \alpha g_a^\alpha + (1-\alpha) g_c^\alpha$. In an α-ANE, we must have

$$g_a^\alpha(x) = F^{-1}(x) + (\beta + \gamma) T_C g^\alpha(x), \qquad g_c^\alpha(x) = F^{-1}(x) + \gamma T_C g^\alpha(x) + \beta \int_0^1 g^\alpha(y) dy$$

for each $x \in (0,1)$. Since $g^\alpha = \alpha g_a^\alpha + (1-\alpha) g_c^\alpha$, it follows that

$$g^\alpha(x) = F^{-1}(x) + (\gamma + \alpha\beta) T_C g^\alpha(x) + (1-\alpha)\beta \int_0^1 g^\alpha(y) dy$$

for each x, which implies $\int_0^1 g^\alpha(y)dy = \frac{\mu}{1-\beta-\gamma}$ by integrating both sides over x. Moreover, iterating the above equation we obtain

$$g^\alpha(x) = \sum_{t\geq 0}(\gamma+\alpha\beta)^t (T_C)^t F^{-1}(x) + \frac{(1-\alpha)\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)},$$

where the convergence of the RHS can be shown as in the proof of Lemma 1. Note that this uniquely determines g^α for any α. By the best-response conditions, we obtain

$$\begin{aligned} g^\alpha_a(x) &= F^{-1}(x) + (\beta+\gamma)T_C g^\alpha(x) \\ &= F^{-1}(x) + (\beta+\gamma)\sum_{t\geq 1}(\gamma+\alpha\beta)^{t-1}(T_C)^t F^{-1}(x) + \frac{(\beta+\gamma)(1-\alpha)\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)}, \\ g^\alpha_c(x) &= F^{-1}(x) + \gamma T_C g^\alpha(x) + \beta\int_0^1 g^\alpha(y)dy \\ &= F^{-1}(x) + \gamma\sum_{t\geq 1}(\gamma+\alpha\beta)^{t-1}(T_C)^t F^{-1}(x) + \frac{(1-\alpha(\beta+\gamma))\,\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)} \end{aligned}$$

for each x, yielding (18)-(19). Then the claim $g^\alpha_a \succsim_d g^\alpha_c$ and the comparative statics with respect to α can be verified using linearity and continuity of $\succsim_d$. □

D.2.2 Proof of Proposition C.2

Write $P=(F,C)$. Consider any PANE s with $\int a\, d\hat{G}_\theta(a)$ absolutely continuous in θ. Then $s(\theta) = \theta + \beta\int a\, d\hat{G}_\theta(a) + \gamma\mathbb{E}_P[s(\theta')|\theta]$ for each θ. Thus, s is the Nash equilibrium in environment $(\tilde{F}, C, \tilde{\beta}, \gamma)$, where $\tilde{\beta}=0$ and $\tilde{F}^{-1}(x) = F^{-1}(x) + \beta\int a\, d\hat{G}_{F^{-1}(x)}(a)$ for each x (note that $\tilde{F}\in\mathcal{F}$, as $\int a d\hat{G}_\theta(a)$ is increasing and absolutely continuous in θ). Since $\tilde{F}$ is more dispersive than F (and the global complementarity parameter does not affect Nash action dispersion by Proposition 4), Proposition 3 implies that $G^{s,P}$ is more dispersive than the Nash global action distribution in environment (P,β,γ). □

D.2.3 Details for Example C.1

Fix any $\hat{\rho}\in[0,\rho]$. We verify that, for the expressions in Example C.1, s^* is a PANE and $(\hat{P}_\theta, \hat{s}_\theta)$ are associated coherent perceptions. Let $x := \frac{1}{1-\gamma\rho-\beta\frac{\rho-\hat{\rho}}{1-\hat{\rho}}}$ and $\hat{x} := \frac{1}{1-\gamma\hat{\rho}}$, so that $s^*(\theta) = x(\theta-\mu) + \frac{\mu}{1-\beta-\gamma}$ and $\hat{s}_\theta(\theta') = \hat{x}(\theta'-\hat{\mu}_\theta) + \frac{\hat{\mu}_\theta}{1-\beta-\gamma}$ for all θ,θ'. Since $P(\cdot|\theta)$ is distributed $\mathcal{N}(\rho\theta+(1-\rho)\mu, (1-\rho^2)\sigma^2)$, θ's true local action distribution $L^{s^*,P}_\theta$ is distributed $\mathcal{N}(x\rho(\theta-\mu)+\frac{\mu}{1-\beta-\gamma}, x^2(1-\rho^2)\sigma^2)$. Since $\hat{P}_\theta(\cdot|\theta)$ is distributed $\mathcal{N}(\hat{\rho}\theta+(1-\hat{\rho})\hat{\mu}_\theta, (1-\hat{\rho}^2)\hat{\sigma}^2)$, θ's perceived local action distribution $L^{\hat{s}_\theta,\hat{P}_\theta}_\theta$ is distributed $\mathcal{N}(\hat{x}\hat{\rho}(\theta-\hat{\mu}_\theta)+\frac{\hat{\mu}_\theta}{1-\beta-\gamma}, \hat{x}^2(1-\hat{\rho}^2)\hat{\sigma}^2)$. Thus, condition 1(a) of coherency can be verified by observing that, by construction, the mean and variance of $L^{s^*,P}_\theta$ and $L^{\hat{s}_\theta,\hat{P}_\theta}_\theta$ are equal.

To verify condition 2, note that, by construction, θ's perceived strategy profile $\hat{s}_\theta$ is the Nash equilibrium in society $\hat{P}_\theta = (\hat{\mu}_\theta, \hat{\sigma}^2, \hat{\rho})$ (see Example 1).

Finally, we verify that s^* is a PANE with perceived global action distributions $\hat{G}_\theta = \hat{G}^{\hat{s}_\theta,\hat{P}_\theta}$, as required by condition 1(b). Note first that, by construction, $s^*(\theta) = \hat{s}_\theta(\theta)$ for all θ. Thus, conditions 1(a) and 2 imply that $s^*(\theta) \in \mathrm{BR}_\theta(\hat{G}^{\hat{s}_\theta,\hat{P}_\theta}, L_\theta^{s^*,P})$. It remains to check that $\hat{G}^{\hat{s}_\theta,\hat{P}_\theta}$ is FOSD-increasing in θ. This holds because $\hat{G}^{\hat{s}_\theta,\hat{P}_\theta}$ is distributed $\mathcal{N}(\frac{\hat{\mu}_\theta}{1-\beta-\gamma}, \hat{x}^2\hat{\sigma}^2)$ and because $\hat{\rho} \le \rho$ ensures that $\hat{\mu}_\theta$ is increasing in θ.

D.2.4 Proof of Proposition C.3

We only consider Nash equilibrium, as ANE at (P, β, γ) corresponds to Nash equilibrium at $(P, 0, \beta + \gamma)$. Let $\mu := \mathbb{E}_F[\theta]$ and, for each $x \in (0, 1)$, define

$$h(x) := \sum_{t\ge 0} \gamma^t (T_C)^t F^{-1}(x) + \frac{\beta\mu}{(1-\gamma)(1-\beta-\gamma)},$$

which is a well-defined function in L^1 as $|\gamma| < 1$. Following the same argument as in the proof of Lemma 1, the strategy profile defined by $s^{NE}(\theta) = h(F^{-1}(\theta))$ for each θ is the unique Nash equilibrium and satisfies (6).

To show the "moreover" part, note that

$$h = \sum_{t\ge 0} \gamma^{2t} T_C^{2t}(F^{-1} + \gamma T_C F^{-1}) + \frac{\beta\mu}{(1-\gamma)(1-\beta-\gamma)}.$$

Since $\gamma > -1$, the additional assumption on P implies that $F^{-1} + \gamma T_C F^{-1}$ is strictly increasing. Therefore, h, and hence s^{NE}, is strictly increasing. □

D.2.5 Proof of Proposition C.4

We first show that, analogously to the relationship between $\succsim_{MA}$ and $\succsim_m$ (Lemma B.1), the strongly more-assortative order $\succsim_{SMA}$ is the "dual order" of the dispersiveness order $\succsim_d$:

Lemma D.1. *Fix any $C_1, C_2 \in \mathcal{C}$. Then $C_1 \succsim_{SMA} C_2$ if and only if $T_{C_1} f \succsim_d T_{C_2} f$ for all $f \in \mathcal{I}$.*

Proof. For the "only if" part, suppose that $C_1 \succsim_{SMA} C_2$. First consider any bounded $f \in \mathcal{I}$. Then there exists an integrable function $f' : (0,1) \to \mathbb{R}$ that is nonnegative almost everywhere such that $f(x) = f(0) + \int_0^x f'(y)dy$ for all $x \in (0, 1)$. Thus, for any $x \ge x'$, integration by parts yields

$$\begin{aligned}
T_{C_1} f(x) - T_{C_1} f(x') &= \int_0^1 f(y)(c_1(y|x) - c_1(y|x'))dy \\
&= -\int_0^1 f'(y)(C_1(y|x) - C_1(y|x'))dy + [f(y)(C_1(y|x) - C_1(y|x'))]_0^1 \\
&= -\int_0^1 f'(y)(C_1(y|x) - C_1(y|x'))dy \ge -\int_0^1 f'(y)(C_2(y|x) - C_2(y|x'))dy \\
&= -\int_0^1 f'(y)(C_2(y|x) - C_2(y|x'))dy + [f(y)(C_2(y|x) - C_2(y|x'))]_0^1 \\
&= \int_0^1 f(y)(c_2(y|x) - c_2(y|x'))dy = T_{C_2} f(x) - T_{C_2} f(x'),
\end{aligned}$$

where the inequality holds because $f'(y) \geq 0$ for almost all y. Hence, $T_{C_1} f \succsim_d T_{C_2} f$.

Next take an arbitrary $f \in \mathcal{I}$. Define the sequence of bounded functions (f_n) as in (21), so that $f_n \to f$. By the previous observation, we have $T_{C_1} f_n \succsim_d T_{C_2} f_n$ for each n. Since $T_{C_1} f_n \to T_{C_1} f$ and $T_{C_2} f_n \to T_{C_2} f$ by continuity of T_{C_1} and T_{C_2}, continuity of $\succsim_d$ then yields $T_{C_1} f \succsim_d T_{C_2} f$.

For the "if" part, we prove the contrapositive. Suppose that C_1 is not strongly more assortative than C_2. That is, there exist y and $x > x'$ such that

$$C_2(y|x) - C_2(y|x') < C_1(y|x) - C_1(y|x') \leq 0.$$

Since C_1 and C_2 admit densities, the above inequality holds throughout some interval $(y_1, y_2) \ni y$. Define $f \in \mathcal{I}$ by $f(z) = \int_0^z f'(y')dy'$ for all z, where f' is an integrable function given by $f'(y') = 1$ for $y' \in (y_1, y_2)$ and $f'(y') = 0$ for all $y' \notin (y_1, y_2)$. Using the same integration by parts argument as above, we obtain

$$T_{C_1} f(x) - T_{C_1} f(x') = -\int f'(y)(C_1(y|x) - C_1(y|x'))dy$$
$$< -\int f'(y)(C_2(y|x) - C_2(y|x'))dy = T_{C_2} f(x) - T_{C_2} f(x').$$

Thus, $T_{C_1} f \succsim_d T_{C_2} f$ fails. □

Proof of Proposition C.4. Note that Nash and ANE strategies are monotone by the assumption on P_i (Proposition C.3). We prove each part only for Nash, as the ANE at (P, β, γ) is the Nash at $(P, 0, \beta + \gamma)$. For each $f, g \in \mathcal{I}$, write $f \succsim_{dil} g$ iff $f \succsim_m g + \alpha$ for some constant function α. This order inherits linearity, isotonicity, and continuity from $\succsim_m$. Note that for $F, G \in \mathcal{F}$, F is a dilation of G iff $F^{-1} \succsim_{dil} G^{-1}$; moreover, the $\succsim_{dil}$ order is implied by the $\succsim_d$ order.

Second part: Let $\beta := \beta_1 = \beta_2$, $\gamma := \gamma_1 = \gamma_2$, $C := C_1 = C_2$. The proof of Proposition 3 carries over to the case $\gamma \geq 0$, so we focus on the case $\gamma < 0$. Since β only shifts the action mean without affecting the dilation order, we also assume $\beta = 0$ without loss.

For each $i = 1, 2$, define an operator $\Gamma_i : \mathcal{I} \to \mathcal{I}$ by $\Gamma_i f = F_i^{-1} + \gamma T_C F_i^{-1} + \gamma^2 T_C^2 f$ for each $f \in \mathcal{I}$. Note that $\Gamma_i(\cdot)$ is increasing, as $(1 + \gamma T_C) F_i^{-1}$ is increasing by the assumption on P_i. We make two preliminary observations:

1. For $i = 1, 2$, $\Gamma_i f \succsim_{dil} \Gamma_i g$ whenever $f \succsim_{dil} g$.

   This follows from isotonicity of $\succsim_{dil}$.

2. $\Gamma_1 f \succsim_{dil} \Gamma_2 f$ for each $f \in \mathcal{I}$.

   To see this, note that $F_1^{-1} \succsim_d F_2^{-1}$ implies $F_1^{-1} - F_2^{-1} \in \mathcal{I}$. Thus,

   $$F_1^{-1} - F_2^{-1} \succsim_m T_C(F_1^{-1} - F_2^{-1}) \succsim_{dil} -\gamma T_C(F_1^{-1} - F_2^{-1}),$$

   where the first comparison uses Lemma A.4 and the second uses $-1 < \gamma \leq 0$. Therefore, $F_1^{-1} + \gamma T_C F_1^{-1} \succsim_{dil} F_2^{-1} + \gamma T_C F_2^{-1}$, and thus $\Gamma_1 f \succsim_{dil} \Gamma_2 f$ for each $f \in \mathcal{I}$.

Now, fix any $f \in \mathcal{I}$. Let

$$g_i := \sum_{t \geq 0} \gamma^t T_C^t F_i = \lim_{t \to \infty} \Gamma_i^t(f).$$

This is the inverse cdf of G_i^{NE}, as s_i^{NE} is increasing. By induction, we show that $\Gamma_1^t f \succsim_{dil} \Gamma_2^t f$ for all t. The base case $t = 1$ holds by the second observation above. Moreover, if $\Gamma_1^{t-1} f \succsim_{dil} \Gamma_2^{t-1} f$, then

$$\Gamma_1^t f \succsim_{dil} \Gamma_2 \Gamma_1^{t-1} f \succsim_{dil} \Gamma_2^t f$$

holds by observations 1-2. Given this, $g_1 \succsim_{dil} g_2$ follows by continuity of $\succsim_{dil}$.

First part: Let $F := F_1 = F_2$, $\beta := \beta_1 = \beta_2$, $\gamma := \gamma_1 = \gamma_2$. The proof of Proposition 2 carries over to the case $\gamma \leq 0$, so we focus on the case $\gamma < 0$. Since β only shifts the action mean without affecting the dilation order, we also assume $\beta = 0$ without loss. Let $g_i := \sum_{t \geq 0} \gamma^t T_{C_i}^t F^{-1}$; this is the inverse cdf of G_i^{NE} since s_i^{NE} is monotone.

For each $i = 1, 2$ and any $f \in L^1$, the linearity of the operators $T_{C_i}^t$ implies

$$(\mathbf{1} - \gamma_i T_{C_i}) \sum_{t \geq 0} \gamma_i^t T_{C_i}^t f = \sum_{t \geq 0} (\gamma_i^t T_{C_i}^t)(\mathbf{1} - \gamma_i T_{C_i}) f = f, \tag{22}$$

where $\mathbf{1}$ denotes the identity operator. Observe that

$$g_2 = \sum_{t \geq 0} \gamma^t T_{C_2}^t F^{-1} = \sum_{t \geq 0} \gamma^t T_{C_2}^t (\mathbf{1} - \gamma T_{C_1}) g_1,$$

where the second equality uses (22) with $i = 1$ and $f = F^{-1}$. Likewise,

$$g_1 = \sum_{t \geq 0} \gamma^t T_{C_2}^t (\mathbf{1} - \gamma T_{C_2}) g_1,$$

by the second equality in (22) with $i = 2$ and $f = g_1$. This shows that g_1 and g_2 correspond to the inverse cdfs of the Nash action distributions in two modified environments that share a common interaction structure C_2 and complementarity parameters $(0, \gamma)$ and have type distributions $\tilde{F}_1$ and $\tilde{F}_2$ with inverse cdfs $\tilde{F}_1^{-1} := (\mathbf{1} - \gamma T_{C_2}) g_1$ and $\tilde{F}_2^{-1} := (\mathbf{1} - \gamma T_{C_1}) g_1$, respectively. Since $g_1 \in \mathcal{I}$, $\gamma < 0$, and $C_1 \succsim_{SMA} C_2$, Lemma D.1 implies $\tilde{F}_2^{-1} \succsim_d \tilde{F}_1^{-1}$.

Given this, the arguments in part 2 above imply that $g_2 \succsim_{dil} g_1$, provided we can show that $(\mathbf{1} + \gamma T_{C_2}) \tilde{F}_i^{-1}$ is increasing for $i = 1, 2$ (which ensures that the corresponding operators $\Gamma_i(\cdot)$ in the two modified societies are increasing). For $i = 2$, note that $(\mathbf{1} + \gamma T_{C_2}) \tilde{F}_2^{-1} := (\mathbf{1} + \gamma T_{C_2})(\mathbf{1} - \gamma T_{C_1}) g_1 = (\mathbf{1} + \gamma T_{C_2}) F^{-1}$ by (22), which is increasing by the assumption on P_2 and since $\gamma > -1$. For $i = 1$, note that (i) $(\mathbf{1} - \gamma^2 T_{C_1}^2) g_1 = (\mathbf{1} + \gamma T_{C_1}) F^{-1}$ is increasing (by the assumption on P_1 and since $\gamma > -1$), and (ii) $\gamma^2 T_{C_1}^2 g_1 \succsim_d \gamma^2 T_{C_2}^2 g_1$ since $C_1 \succsim_{SMA} C_2$ (Lemma D.1). Combining (i) and (ii) yields that $(\mathbf{1} + \gamma T_{C_2}) \tilde{F}_1^{-1} := (\mathbf{1} - \gamma^2 T_{C_2}^2) g_1$ is increasing, as required.

Third part: Let $F := F_1 = F_2$, $C := C_1 = C_2$. The proof of Proposition 4 carries over to the case $\gamma_i \geq 0$ for $i = 1, 2$. Thus, by the transitivity of the dilation order, we can focus on the case $\gamma_i \leq 0$ for $i = 1, 2$. Since β only shifts the action mean without affecting the dilation order, we also assume $\beta_1 = \beta_2 = 0$ without loss. Let $g_i := \sum_{t \geq 0} \gamma_i^t T_C^t F^{-1}$; this is the inverse cdf of G_i^{NE} since s_i^{NE} is monotone. Observe that

$$g_1 = \sum_{t \geq 0} \gamma_1^t T_C^t F^{-1} = \sum_{t \geq 0} \gamma_1^t T_C^t (\mathbf{1} - \gamma_2 T_C) g_2,$$

where the second equality uses (22) with $i = 1$ and $f = F^{-1}$. Likewise,

$$g_2 = \sum_{t \geq 0} \gamma_1^t T_C^t (\mathbf{1} - \gamma_1 T_C) g_2,$$

by the second equality in (22) with $i = 1$ and $f = g_2$. This shows that g_1 and g_2 can be seen as inverse cdfs of Nash action distributions in two modified environments that share a common interaction structure C and complementarity parameters $(0, \gamma_1)$ and have type distributions $\tilde{F}_1$ and $\tilde{F}_2$ with inverse cdfs $\tilde{F}_1^{-1} := (\mathbf{1} - \gamma_2 T_C) g_2$ and $\tilde{F}_2^{-1} := (\mathbf{1} - \gamma_1 T_C) g_2$, respectively. Since $0 \geq \gamma_1 \geq \gamma_2$, we have $\tilde{F}_1^{-1} \succsim_d \tilde{F}_2^{-1}$.

Given this, the arguments in part 2 above imply that $g_1 \succsim_{dil} g_2$, provided we can show that $(\mathbf{1} + \gamma_1 T_C) \tilde{F}_i^{-1}$ is increasing for $i = 1, 2$ (which ensures that the corresponding operators $\Gamma_i(\cdot)$ in the two modified societies are increasing). For $i = 1$, note that $(\mathbf{1} + \gamma_1 T_C) \tilde{F}_2^{-1} := (\mathbf{1} + \gamma_1 T_C)(\mathbf{1} - \gamma_2 T_C) g_2 = (\mathbf{1} + \gamma_1 T_C) F^{-1}$, which is increasing by the assumption on P_i and $\gamma_1 > -1$. For $i = 2$, note that (i) $(\mathbf{1} - \gamma_2^2 T_C^2) g_2 = (\mathbf{1} + \gamma_2 T_C) F^{-1}$ is increasing (by the assumption on P_i and since $\gamma_2 > -1$), and (ii) $\gamma_2^2 T_C^2 g_2 \succsim_d \gamma_1^2 T_C^2 g_2$ as $0 \geq \gamma_1 \geq \gamma_2$. Combining (i) and (ii) yields that $(\mathbf{1} + \gamma_1 T_C) \tilde{F}_1^{-1} := (\mathbf{1} - \gamma_1^2 T_C^2) g_2$ is increasing, as required. □

D.2.6 Proof of Proposition C.5

Fix any ANE $s^{AN} =: s$ and θ. For each θ', set $\hat{s}_\theta(\theta') := \mathrm{BR}_{\theta'}(L_\theta^{s,P}, L_\theta^{s,P})$ and $\hat{F}_\theta(\theta') := L_\theta^{s,P}(\hat{s}_\theta(\theta'))$, and let $\hat{P}_\theta := \hat{F}_\theta \times \hat{F}_\theta$. To verify observational consistency, note that $L_\theta^{\hat{s}_\theta, \hat{P}_\theta}(a) = \hat{F}_\theta(\hat{s}_\theta^{-1}(a)) = L_\theta^{s,P}(a)$ for each a, where the first equality uses $\hat{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta$ and the inverse $\hat{s}_\theta^{-1}$ is well-defined and increasing by the surjectivity and monotonicity assumption on best-responses. To verify the perceived best-response condition, note that, for each θ',

$$\hat{s}_\theta(\theta') = \mathrm{BR}_{\theta'}(L_\theta^{s,P}, L_\theta^{s,P}) = \mathrm{BR}_{\theta'}(L_\theta^{\hat{s}_\theta, \hat{P}_\theta}, L_\theta^{\hat{s}_\theta, \hat{P}_\theta}) = \mathrm{BR}_{\theta'}(G^{\hat{s}_\theta, \hat{P}_\theta}, L_{\theta'}^{\hat{s}_\theta, \hat{P}_\theta}),$$

where the second equality uses observational consistency and the third uses non-assortativity of $\hat{P}_\theta$. Thus, $(\hat{P}_\theta, \hat{s}_\theta)$ is a coherent assortativity neglect perception for type θ.

To show uniqueness, consider any coherent assortativity neglect perception $(\hat{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta, \hat{s}_\theta)$ for θ. Then, for each θ', the perceived best-response condition, non-assortativity of $\hat{P}_\theta$, and observational consistency imply $\hat{s}_\theta(\theta') = \mathrm{BR}_{\theta'}(G^{\hat{s}_\theta, \hat{P}_\theta}, L_{\theta'}^{\hat{s}_\theta, \hat{P}_\theta}) = \mathrm{BR}_{\theta'}(L_\theta^{\hat{s}_\theta, \hat{P}_\theta}, L_\theta^{\hat{s}_\theta, \hat{P}_\theta}) = \mathrm{BR}_{\theta'}(L_\theta^{s,P}, L_\theta^{s,P})$. Moreover, $\hat{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta$ and observational consistency imply $\hat{F}_\theta(\hat{s}_\theta^{-1}(a)) = L_\theta^{\hat{s}_\theta, \hat{P}_\theta}(a) = L_\theta^{s,P}(a)$ for each a, which yields $\hat{F}_\theta(\theta') = L_\theta^{s,P}(\hat{s}_\theta(\theta'))$ for each θ'. Thus, $(\hat{P}_\theta, \hat{s}_\theta)$ coincides with the perceptions in the first paragraph. □

D.2.7 Proof of Proposition C.6

Consider any monotone ANE s^{AN} and any Nash equilibrium s^{NE}. For any types $\theta > \theta'$, the fact that ψ and ϕ are monotone yields

$$s^{AN}(\theta) - s^{AN}(\theta') = \phi(\theta) - \phi(\theta') + \psi(L_\theta^{s^{AN},P}) - \psi(L_{\theta'}^{s^{AN},P}) \geq \phi(\theta) - \phi(\theta') = s^{NE}(\theta) - s^{NE}(\theta') > 0,$$

where the first inequality holds because $L_\theta^{s^{AN},P}$ FOSD-dominates $L_{\theta'}^{s^{AN},P}$ (by monotonicity of s^{AN} and assortativity of P). Thus, $G^{s^{AN},P}$ is more dispersive than $G^{s^{NE},P}$. □