Online Appendix to “Dispersed Behavior and Perceptions in Assortative Societies”

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D Omitted Proofs

D.1 Proofs for Appendix A

D.1.1 Proof of Lemma A.1

For the first point, note that for any \( f \in L^1 \),
\[
\|T_C f\| = \int_0^1 |T_C f(x)| dx \leq \int_0^1 \int_0^1 c(x', x)|f(x')| dx' dx = \int_0^1 |f(x')| dx' = \| f \| < \infty.
\]
Thus, \( T_C : L^1 \to L^1 \). Moreover, since \( T_C \) is clearly linear, the above ensures that it is also continuous.

For the second point, consider \( f \in I \). Since \( C \) is assortative, \( T_C f(x) \geq T_C f(x') \) for all \( x \geq x' \), so that \( T_C f \) is weakly increasing. To show that \( T_C f \) is absolutely continuous, note that for each \( x,x' \in (0,1) \),
\[
T_C f(x) = \int_0^1 c(y, x)f(y)dy = \int_0^1 \left( \int_{x'}^x c_2(y, z)dz + c(y, x') \right) f(y)dy
\]
\[
= \int_{x'}^x \int_0^1 c_2(y, z)f(y)dy dz + T_C f(x'),
\]
where \( c_2 \) denotes the partial derivative of \( c \) with respect to the second argument, which exists almost everywhere by the absolute continuity assumption on \( c \). Thus \( T_C f \) is absolutely continuous with \( (T_C f)'(z) = \int_0^1 c_2(y, z)f(y)dy \) for each \( z \).

Finally, for the third point, fix any \( f \in L^1 \) and \( \gamma \in (-1,1) \). Then for any \( \tau > \tau' \),
\[
\| \sum_{t=0}^\tau \gamma^t(T_C)^tf - \sum_{t=0}^{\tau'} \gamma^t(T_C)^tf \| \leq \sum_{t=\tau'+1}^\tau |\gamma|^t \| (T_C)^tf \| \leq \sum_{t=\tau'+1}^\tau |\gamma|^t \| f \| \leq \frac{|\gamma|^{\tau'+1}}{1-\gamma} \| f \|,
\]
which vanishes as \( \tau' \to \infty \). Thus, the sequence is Cauchy. Since the space \( L^1 \) is complete, this yields the desired result. \( \square \)

D.1.2 Proof of Lemma A.3

\( \succsim_m \)-order: It is clear from the definition that \( \succsim_m \) is reflexive and transitive; moreover, by Lemma A.2, \( \succsim_m \) is linear. To check that \( \succsim_m \) is continuous, take sequences \( f_n \to f, g_n \to g \) in
\[ I \text{ such that } f_n \succ_m g_n \text{ for each } n. \text{ For any } y \in (0,1), \text{ we have} \]
\[ |\int_y^1 f(x)dx - \int_y^1 f_n(x)dx| \leq \int_y^1 |f(x) - f_n(x)|dx \leq \|f - f_n\| \to 0 \]
and likewise \[ |\int_y^1 g(x)dx - \int_y^1 g_n(x)dx| \to 0. \] Since \[ \int_y^1 f_n(x)dx \geq \int_y^1 g_n(x)dx \text{ and } \int_0^1 f_n(x)dx = \int_0^1 g_n(x)dx \text{ for each } n, \] this implies \[ \int_y^1 f(x)dx \geq \int_y^1 g(x)dx \text{ and } \int_0^1 f(x)dx = \int_0^1 g(x)dx. \] Thus, \[ f \succ_m g \text{ by Lemma A.2.} \]

To verify that \[ \succ_m \text{ is isotone}, \] take any \( f, g \in I \) such that \( f \succ_m g \) and set \( h := f - g \). Note that \[ \int_0^1 h(x)dx = \int_0^1 T_C h(x)dx = 0. \] It suffices to show that \[ \int_y^1 T_C h(x)dx \geq 0 \text{ for all } y \in (0,1). \]

To see this, note that \[ \int_y^1 T_C h(x)dx \text{ is given by} \]
\[ \int_y^1 \int_0^1 h(z)c(z|x)dzdx = \int_0^1 \int_y^1 c(z|x)dxh(z)dz = \int_0^1 (1 - C(y|z))h(z)dz \]
\[ = - \int_0^1 \frac{\partial 1 - C(y|z)}{\partial z} \int_0^z h(z')dz'dz + \left[ (1 - C(y|z)) \int_0^z h(z')dz' \right]_0^1 \]
\[ = \int_0^1 \frac{\partial C(y|z)}{\partial z} \int_0^z h(z')dz'dz \geq 0, \]

where the second equality uses \( \int_y^1 c(z|x)dx = \int_y^1 c(x|z)dx = 1 - C(y|z), \) the third holds by integration by parts (using absolute continuity of \( c \)), the fourth uses \( \int_0^1 h(z)dz = 0, \) and the final inequality uses \( \int_0^z h(z')dz' \leq 0 \) (by \( f \succ_m g \)) and assortativity of \( C \).

**\( \succ_d \)-order:** It is clear from the definition that \( \succ_d \) is reflexive, transitive, and linear. To check that it is continuous, take sequences \( f_n \to f \) and \( g_n \to g \) in \( I \) such that \( f_n \succ_d g_n \) for each \( n \). By standard results (e.g., Theorem 13.6 in Aliprantis and Border (2006)), we can find subsequences \( (f_{n_k})_{k \in \mathbb{N}}, (g_{n_k})_{k \in \mathbb{N}} \) such that \( f_{n_k}(x) \to f(x) \) and \( g_{n_k}(x) \to g(x) \) for almost all \( x \in (0,1) \). This implies \( f(x) - f(x') \geq g(x) - g(x') \) for almost all \( x \geq x' \), which ensures \( f \succ_d g \) since \( f \) and \( g \) are continuous.

To show that \( \succ_d \) is isotone, first consider any bounded \( f, g \in I \) such that \( f \succ_d g \). Since \( f \) and \( g \) are absolutely continuous, there exist integrable functions \( f', g' : (0,1) \to \mathbb{R} \) such that \( f(x) = f(0) + \int_0^x f'(y)dy \) and \( g(x) = g(0) + \int_0^x g'(y)dy \) for all \( x \in (0,1) \). Then, for any \( x \geq x' \) and \( C \in \mathcal{C} \), integration by parts yields
\[ T_C f(x) - T_C f(x') = \int_0^1 f(y)(c(y|x) - c(y|x'))dy \]
\[ = - \int_0^1 f'(y)(C(y|x) - C(y|x'))dy + \left[ f(y)(C(y|x) - C(y|x')) \right]_0^1 \]
\[ = - \int_0^1 f'(y)(C(y|x) - C(y|x'))dy \geq - \int_0^1 g'(y)(C(y|x) - C(y|x'))dy \]
\[ = - \int_0^1 g'(y)(C(y|x) - C(y|x'))dy + \left[ g(y)(C(y|x) - C(y|x')) \right]_0^1 \]
\[ = \int_0^1 g(y)(c(y|x) - c(y|x'))dy = T_C g(x) - T_C g(x'). \]
Here, the inequality holds because the fact that \( f \succeq_d g \) and \( f, g \in \mathcal{I} \) implies \( f'(y) \geq g'(y) \geq 0 \) for almost all \( y \in (0, 1) \).

Next, consider arbitrary \( f, g \in \mathcal{I} \) such that \( f \succeq_d g \). By defining bounded functions

\[
 f_n(x) = \begin{cases} 
  f(\frac{1}{n}) & \text{if } x \in (0, \frac{1}{n}) \\
  f(x) & \text{if } x \in [\frac{1}{n}, \frac{n-1}{n}] \\
  f(\frac{n-1}{n}) & \text{if } x \in (\frac{n-1}{n}, 1) 
\end{cases} 
\]

\[
 g_n(x) = \begin{cases} 
  g(\frac{1}{n}) & \text{if } x \in (0, \frac{1}{n}) \\
  g(x) & \text{if } x \in [\frac{1}{n}, \frac{n-1}{n}] \\
  g(\frac{n-1}{n}) & \text{if } x \in (\frac{n-1}{n}, 1) 
\end{cases}
\]

(21) for each \( n \in \mathbb{N} \), we obtain \( f_n \succeq_d g_n \) for each \( n \) and \( f_n \rightarrow f, g_n \rightarrow g \). For any \( C \in \mathcal{C} \), since \( T_C \) is a continuous operator, this implies \( T_C f_n \rightarrow T_C f \) and \( T_C g_n \rightarrow T_C g \). Thus, \( T_C f \succeq_d T_C g \) by continuity of \( \succeq_d \), as we already know that \( T_C f_n \succeq_d T_C g_n \) from the previous part of the proof.

D.1.3 Proof of Lemma A.4

The base case \( t = 0 \) holds because of the following result by Ryff (1963): Call a linear operator \( T : L^1 \rightarrow L^1 \) an \( \mathcal{S} \)-operator if \( f \succeq_m T f \) for all \( f \in \mathcal{I} \). The representation theorem in Ryff (1963) implies that \( T \) is an \( \mathcal{S} \)-operator if there exists some measurable function \( K : [0, 1]^2 \rightarrow \mathbb{R} \) such that \( T f(x) = \frac{d}{dt} \int_0^1 K(x, y)f(y)dy \) for all \( f \in L^1 \) and almost every \( x \) and the following conditions are met: (1) \( K(0, y) = 0 \) for all \( 0 \leq y \leq 1 \); (2) \( \text{essup}_y V(K(\cdot, y)) < \infty \), where \( V(\cdot) \) denotes the total variation and \( \text{essup} \) the essential supremum; (3) \( \int_0^1 K(x, y)f(y)dy \) is absolutely continuous in \( x \) for all \( f \in L^1 \); (4) \( x = \int_0^1 K(x, y)dy \); (5) \( x_1 < x_2 \implies K(x_1, \cdot) \leq K(x_2, \cdot) \); and (6) \( K(1, y) = 1 \) for all \( y \in [0, 1] \).

Since \( C \in \mathcal{C} \), it is easy to see that \( T_C \) satisfies these conditions with \( K(x, y) := C(x \mid y) \) for all \( x, y \), so that \( T_C \) is an \( \mathcal{S} \)-operator. Thus, \( f \succeq_m T_C f \), proving the base case. The inductive step then follows from isotonicity of \( \succeq_m \) (Lemma A.3).

D.2 Proofs for Appendix C

D.2.1 Proof of Proposition C.1

Let \( \mu := \mathbb{E}_F[\theta] \). Consider strategy profiles \( g_a^\alpha \) and \( g_c^\alpha \) of assortativity neglect and correct agents expressed as functions of quantiles. Write \( g^\alpha := \alpha g_a^\alpha + (1 - \alpha) g_c^\alpha \). In an \( \alpha \)-ANE, we must have

\[
 g_a^\alpha(x) = F^{-1}(x) + (\beta + \gamma) T_C g^\alpha(x), \quad g_c^\alpha(x) = F^{-1}(x) + \gamma T_C g^\alpha(x) + \beta \int_0^1 g^\alpha(y)dy
\]

for each \( x \in (0, 1) \). Since \( g^\alpha = \alpha g_a^\alpha + (1 - \alpha) g_c^\alpha \), it follows that

\[
 g^\alpha(x) = F^{-1}(x) + (\gamma + \alpha \beta) T_C g^\alpha(x) + (1 - \alpha) \beta \int_0^1 g^\alpha(y)dy
\]
for each \( x \), which implies \( \int_0^1 g^\alpha(y)dy = \frac{\mu}{1-\beta-\gamma} \) by integrating both sides over \( x \). Moreover, iterating the above equation we obtain
\[
g^\alpha(x) = \sum_{t\geq 0} (\gamma + \alpha\beta)^t(T_C)^tF^{-1}(x) + \frac{(1-\alpha)\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)},
\]
where the convergence of the RHS can be shown as in the proof of Lemma 1. Note that this uniquely determines \( g^\alpha \) for any \( \alpha \). By the best-response conditions, we obtain
\[
g^\alpha_a(x) = F^{-1}(x) + (\beta + \gamma)T_Cg^\alpha(x)
= F^{-1}(x) + (\beta + \gamma)\sum_{t\geq 1} (\gamma + \alpha\beta)^{t-1}(T_C)^tF^{-1}(x) + \frac{(\beta + \gamma)(1-\alpha)\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)},
\]
\[
g^\alpha_c(x) = F^{-1}(x) + \gamma T_Cg^\alpha(x) + \beta \int_0^1 g^\alpha(y)dy
= F^{-1}(x) + \gamma \sum_{t\geq 1} (\gamma + \alpha\beta)^{t-1}(T_C)^tF^{-1}(x) + \frac{(1-\alpha(\beta + \gamma))\beta\mu}{(1-\gamma-\alpha\beta)(1-\beta-\gamma)}
\]
for each \( x \), yielding (18)-(19). Then the claim \( g^\alpha_a \succeq_d g^\alpha_c \) and the comparative statics with respect to \( \alpha \) can be verified using linearity and continuity of \( \succeq_d \).

D.2.2 Proof of Proposition C.2

Write \( P = (F,C) \). Consider any PANE \( s \) with \( \int a d\hat{G}_\theta(a) \) absolutely continuous in \( \theta \). Then \( s(\theta) = \theta + \beta \int a d\hat{G}_\theta(a) + \gamma E_P[s(\theta')|\theta] \) for each \( \theta \). Thus, \( s \) is the Nash equilibrium in environment \((\hat{F},C,\tilde{\beta},\gamma)\), where \( \tilde{\beta} = 0 \) and \( \hat{F}^{-1}(x) = F^{-1}(x) + \beta \int a d\hat{G}_{F^{-1}}(a) \) for each \( x \) (note that \( \hat{F} \in F \), as \( \int a d\hat{G}_\theta(a) \) is increasing and absolutely continuous in \( \theta \)). Since \( \hat{F} \) is more dispersive than \( F \) (and the global complementarity parameter does not affect Nash action dispersion by Proposition 4), Proposition 3 implies that \( G^{s,P} \) is more dispersive than the Nash global action distribution in environment \((P,\beta,\gamma)\).

D.2.3 Details for Example C.1

Fix any \( \hat{\rho} \in [0,\rho] \). We verify that, for the expressions in Example C.1, \( s^* \) is a PANE and \((\hat{P}_\theta,\hat{s}_\theta)\) are associated coherent perceptions. Let \( x := \frac{1}{1-\gamma\rho - \hat{\rho} \frac{\mu}{1-\beta-\gamma}} \) and \( \hat{x} := \frac{1}{1-\gamma\hat{\rho}} \), so that \( s^*(\theta) = x(\theta - \mu) + \frac{\mu}{1-\beta-\gamma} \) and \( \hat{s}_\theta(\theta') = \hat{x}(\theta' - \hat{\mu}_\theta) + \frac{\hat{\mu}_\theta}{1-\beta-\gamma} \) for all \( \theta,\theta' \). Since \( P(\cdot|\theta) \) is distributed \( \mathcal{N}(\rho\theta + (1 - \rho)\mu, (1 - \rho^2)\sigma^2) \), \( \theta \)'s true local action distribution \( L^*_{\theta,P} \) is distributed \( \mathcal{N}(x\rho\theta - \mu + \frac{\mu}{1-\beta-\gamma}, x^2(1-\rho^2)\sigma^2) \). Since \( \hat{P}_\theta(\cdot|\theta) \) is distributed \( \mathcal{N}(\hat{\rho}\theta + (1 - \hat{\rho})\hat{\mu}_\theta, (1 - \hat{\rho}^2)\hat{\sigma}^2) \), \( \theta \)'s perceived local action distribution \( L^\hat{\theta}_{\theta,P} \) is distributed \( \mathcal{N}(\hat{x}\hat{\rho}(\theta - \hat{\mu}_\theta) + \frac{\hat{\mu}_\theta}{1-\beta-\gamma}, \hat{x}^2(1-\hat{\rho}^2)\hat{\sigma}^2) \). Thus, condition 1(a) of coherency can be verified by observing that, by construction, the mean and variance of \( L^*_{\theta,P} \) and \( L^\hat{\theta}_{\theta,P} \) are equal.

To verify condition 2, note that, by construction, \( \theta \)'s perceived strategy profile \( \hat{s}_\theta \) is the Nash equilibrium in society \( \hat{P}_\theta = (\hat{\mu}_\theta, \hat{\sigma}^2, \hat{\rho}) \) (see Example 1).

Finally, we verify that \( s^* \) is a PANE with perceived global action distributions \( \hat{G}_\theta = \hat{G}^\hat{s}_\theta,\hat{P}_\theta \), as required by condition 1(b). Note first that, by construction, \( s^*(\theta) = \hat{s}_\theta(\theta) \) for all \( \theta \). Thus,
conditions 1(a) and 2 imply that \( s^*(\theta) \in \text{BR}_\theta(\hat{G}^{\theta_0, \hat{P}_\theta}, L^s_\theta, \hat{P}_\theta) \). It remains to check that \( \hat{G}^{\theta_0, \hat{P}_\theta} \) is FOSD-increasing in \( \theta \). This holds because \( \hat{G}^{\theta_0, \hat{P}_\theta} \) is distributed \( \mathcal{N}(\hat{\mu}_\theta, \hat{x}^2\hat{\sigma}^2) \) and because \( \hat{\rho} \leq \rho \) ensures that \( \hat{\mu}_\theta \) is increasing in \( \theta \).

D.2.4 Proof of Proposition C.3

We only consider Nash equilibrium, as ANE at \((P, \beta, \gamma)\) corresponds to Nash equilibrium at \((P, 0, \beta + \gamma)\). Let \( \mu := \mathbb{E}_F[\theta] \) and, for each \( x \in (0, 1) \), define

\[
h(x) := \sum_{t \geq 0} \gamma^t(T_C)^t F^{-1}(x) + \frac{\beta \mu}{(1 - \gamma)(1 - \beta - \gamma)},
\]

which is a well-defined function in \( L^1 \) as \( |\gamma| < 1 \). Following the same argument as in the proof of Lemma 1, the strategy profile defined by \( s^{NE}(\theta) = h(F^{-1}(\theta)) \) for each \( \theta \) is the unique Nash equilibrium and satisfies (6).

To show the “moreover” part, note that

\[
h = \sum_{t \geq 0} \gamma^t T_C^t (F^{-1} + \gamma T_C F^{-1}) + \frac{\beta \mu}{(1 - \gamma)(1 - \beta - \gamma)}.
\]

Since \( \gamma > -1 \), the additional assumption on \( P \) implies that \( F^{-1} + \gamma T_C F^{-1} \) is strictly increasing. Therefore, \( h \), and hence \( s^{NE} \), is strictly increasing.

D.2.5 Proof of Proposition C.4

We first show that, analogously to the relationship between \( \preceq_{MA} \) and \( \preceq_m \) (Lemma B.1), the strongly more-assortative order \( \preceq_{SMA} \) is the “dual order” of the dispersiveness order \( \preceq_d \):

**Lemma D.1.** Fix any \( C_1, C_2 \in \mathcal{C} \). Then \( C_1 \preceq_{SMA} C_2 \) if and only if \( T_{C_1} f \preceq_d T_{C_2} f \) for all \( f \in \mathcal{I} \).

**Proof.** For the “only if” part, suppose that \( C_1 \preceq_{SMA} C_2 \). First consider any bounded \( f \in \mathcal{I} \). Then there exists an integrable function \( f' : (0, 1) \to \mathbb{R} \) that is nonnegative almost everywhere such that \( f(x) = f(0) + \int_0^x f'(y)dy \) for all \( x \in (0, 1) \). Thus, for any \( x \geq x' \), integration by parts yields

\[
T_{C_1} f(x) - T_{C_1} f(x') = \int_0^1 f(y)(c_1(y|x) - c_1(y|x'))dy
\]

\[
= - \int_0^1 f'(y)(C_1(y|x) - C_1(y|x'))dy + [f(y)(C_1(y|x) - C_1(y|x'))]_0^1
\]

\[
= - \int_0^1 f'(y)(C_1(y|x) - C_1(y|x'))dy \geq - \int_0^1 f'(y)(C_2(y|x) - C_2(y|x'))dy
\]

\[
= - \int_0^1 f'(y)(C_2(y|x) - C_2(y|x'))dy + [f(y)(C_2(y|x) - C_2(y|x'))]_0^1
\]

\[
= \int_0^1 f(y)(c_2(y|x) - c_2(y|x'))dy = T_{C_2} f(x) - T_{C_2} f(x'),
\]

Since \( \gamma > -1 \), the additional assumption on \( P \) implies that \( F^{-1} + \gamma T_C F^{-1} \) is strictly increasing. Therefore, \( h \), and hence \( s^{NE} \), is strictly increasing.
Thus, mean without affecting the dilation order, we also assume carries over to the case $f \in I$. Let

Second part:

Proof of Proposition C.4. Note that Nash and ANE strategies are monotone by the assumption on $P_i$ (Proposition C.3). We prove each part only for Nash, as the ANE at $(P, \beta, \gamma)$ is the Nash at $(P, 0, \beta + \gamma)$. For each $f, g \in I$, write $f \succeq_{\text{dil}} g$ iff $f - g$ is a dilation of $g$ iff $F^{-1} \succeq_{\text{dil}} G^{-1}$; moreover, the $\succeq_{\text{dil}}$ order is implied by the $\succeq_d$ order.

Second part: Let $\beta := \beta_1 = \beta_2$, $\gamma := \gamma_1 = \gamma_2$, $C := C_1 = C_2$. The proof of Proposition 3 carries over to the case $\gamma \geq 0$, so we focus on the case $\gamma < 0$. Since $\beta$ only shifts the action mean without affecting the dilation order, we also assume $\beta = 0$ without loss.

For each $i = 1, 2$, define an operator $\Gamma_i : I \rightarrow I$ by $\Gamma_i f = F^{-1}_i + \gamma T_C F^{-1}_i + \gamma^2 T_C^2 f$ for each $f \in I$. Note that $\Gamma_i(\cdot)$ is increasing, as $(1 + \gamma T_C) F^{-1}_i$ is increasing by the assumption on $P_i$. We make two preliminary observations:

1. For $i = 1, 2$, $\Gamma_i f \succeq_{\text{dil}} \Gamma_i g$ whenever $f \succeq_{\text{dil}} g$.

   This follows from isotonicity of $\succeq_{\text{dil}}$.

2. $\Gamma_1 f \succeq_{\text{dil}} \Gamma_2 f$ for each $f \in I$.

   To see this, note that $F^{-1}_1 \succeq_d F^{-1}_2$ implies $F^{-1}_1 - F^{-1}_2 \in I$. Thus,

   $F^{-1}_1 - F^{-1}_2 \succeq_m T_C(F^{-1}_1 - F^{-1}_2) \succeq_{\text{dil}} -\gamma T_C(F^{-1}_1 - F^{-1}_2)$,

   where the first comparison uses Lemma A.4 and the second uses $-1 < \gamma \leq 0$. Therefore, $F^{-1}_1 + \gamma T_C F^{-1}_1 \succeq_{\text{dil}} F^{-1}_2 + \gamma T_C F^{-1}_2$, and thus $\Gamma_1 f \succeq_{\text{dil}} \Gamma_2 f$ for each $f \in I$.

Now, fix any $f \in I$. Let

$$g_i := \sum_{t \geq 0} \gamma^t T_C^t F_i = \lim_{t \to \infty} \Gamma_i^t(f).$$
This is the inverse cdf of $G_{i}^{NE}$, as $s_{i}^{NE}$ is increasing. By induction, we show that $\Gamma_{1}^{t}f \gtrless_{dil} \Gamma_{2}^{t}f$ for all $t$. The base case $t = 1$ holds by the second observation above. Moreover, if $\Gamma_{1}^{t-1}f \gtrless_{dil} \Gamma_{2}^{t-1}f$, then

$$\Gamma_{1}^{t}f \gtrless_{dil} \Gamma_{2}^{t-1}f \gtrless_{dil} \Gamma_{2}^{t}f$$

holds by observations 1-2. Given this, $g_{1} \gtrless_{dil} g_{2}$ follows by continuity of $\gtrless_{dil}$.

**First part:** Let $F := F_{1} = F_{2}$, $\beta := \beta_{1} = \beta_{2}$, $\gamma := \gamma_{1} = \gamma_{2}$. The proof of Proposition 2 carries over to the case $\gamma \leq 0$, so we focus on the case $\gamma < 0$. Since $\beta$ only shifts the action mean without affecting the dilation order, we also assume $\beta = 0$ without loss. Let $g_{i} := \sum_{t \geq 0} \gamma_{i}^{T}C_{1}F^{-1}$; this is the inverse cdf of $G_{i}^{NE}$ since $s_{i}^{NE}$ is monotone.

For each $i = 1, 2$ and any $f \in L^{1}$, the linearity of the operators $T_{C_{i}}^{t}$ implies

$$(1 - \gamma_{i}T_{C_{i}})(\sum_{t \geq 0} \gamma_{i}^{T}C_{1}f = \sum_{t \geq 0} (\gamma_{i}^{T}C_{1})(1 - \gamma_{i}T_{C_{1}})f = f,$$ (22)

where 1 denotes the identity operator. Observe that

$$g_{2} = \sum_{t \geq 0} \gamma^{T}C_{1}F^{-1} = \sum_{t \geq 0} \gamma^{T}C_{1}(1 - \gamma T_{C_{1}})g_{1},$$

where the second equality uses (22) with $i = 1$ and $f = F^{-1}$. Likewise,

$$g_{1} = \sum_{t \geq 0} \gamma^{T}C_{1}(1 - \gamma T_{C_{2}})g_{1},$$

by the second equality in (22) with $i = 2$ and $f = g_{1}$. This shows that $g_{1}$ and $g_{2}$ correspond to the inverse cdfs of the Nash action distributions in two modified environments that share a common interaction structure $C_{2}$ and complementarity parameters $(0, \gamma)$ and have type distributions $\bar{F}_{1}$ and $\bar{F}_{2}$ with inverse cdfs $\bar{F}_{1}^{-1} := (1 - \gamma T_{C_{1}})g_{1}$ and $\bar{F}_{2}^{-1} := (1 - \gamma T_{C_{1}})g_{1}$, respectively. Since $g_{1} \in \mathcal{I}$, $\gamma < 0$, and $C_{1} \gtrsim_{SMA} C_{2}$, Lemma D.1 implies $\bar{F}_{2}^{-1} \gtrsim_{dil} \bar{F}_{1}^{-1}$.

Given this, the arguments in part 2 above imply that $g_{2} \gtrless_{dil} g_{1}$, provided we can show that $(1 + \gamma T_{C_{2}})\bar{F}_{i}^{-1}$ is increasing for $i = 1, 2$ (which ensures that the corresponding operators $\Gamma_{i}(\cdot)$ in the two modified societies are increasing). For $i = 2$, note that $(1 + \gamma T_{C_{2}})\bar{F}_{2}^{-1} := (1 + \gamma T_{C_{2}})(1 - \gamma T_{C_{1}})g_{1} = (1 + \gamma T_{C_{2}})F^{-1}$ by (22), which is increasing by the assumption on $P_{2}$ and since $\gamma > -1$. For $i = 1$, note that (i) $(1 - \gamma T_{C_{2}})g_{1} = (1 + \gamma T_{C_{1}})F^{-1}$ is increasing (by the assumption on $P_{1}$ and since $\gamma > -1$), and (ii) $\gamma T_{C_{2}}^{2}g_{1} \gtrsim_{dil} \gamma T_{C_{2}}^{2}g_{1}$ since $C_{1} \gtrsim_{SMA} C_{2}$ (Lemma D.1). Combining (i) and (ii) yields that $(1 + \gamma T_{C_{2}})\bar{F}_{1}^{-1} := (1 - \gamma T_{C_{2}})g_{1}$ is increasing, as required.

**Third part:** Let $F := F_{1} = F_{2}$, $C := C_{1} = C_{2}$. The proof of Proposition 4 carries over to the case $\gamma_{i} \geq 0$ for $i = 1, 2$. Thus, by the transitivity of the dilation order, we can focus on the case $\gamma_{i} \leq 0$ for $i = 1, 2$. Since $\beta$ only shifts the action mean without affecting the dilation order, we also assume $\beta_{1} = \beta_{2} = 0$ without loss. Let $g_{i} := \sum_{t \geq 0} \gamma_{i}^{T}C_{1}F^{-1}$; this is the inverse cdf of $G_{i}^{NE}$ since $s_{i}^{NE}$ is monotone. Observe that

$$g_{1} = \sum_{t \geq 0} \gamma_{i}^{T}C_{1}F^{-1} = \sum_{t \geq 0} \gamma_{i}^{T}C_{1}(1 - \gamma_{2}T_{C})g_{2},$$
where the second equality uses (22) with \( i = 1 \) and \( f = F^{-1} \). Likewise,

\[
g_2 = \sum_{i \geq 0} \gamma_i T_C^i (1 - \gamma_i T_C) g_2,
\]

by the second equality in (22) with \( i = 1 \) and \( f = g_2 \). This shows that \( g_1 \) and \( g_2 \) can be seen as inverse cdfs of Nash action distributions in two modified environments that share a common interaction structure and complementarity parameters \((0, \gamma_i)\) and have type distributions \( \hat{F}_1 \) and \( \hat{F}_2 \) with inverse cdfs \( \hat{F}_1^{-1} := (1 - \gamma_2 T_C) g_2 \) and \( \hat{F}_2^{-1} := (1 - \gamma_1 T_C) g_2 \), respectively. Since \( 0 \geq \gamma_1 \geq \gamma_2 \), we have \( \hat{F}_1^{-1} \gtrapprox_d \hat{F}_2^{-1} \).

Given this, the arguments in part 2 above imply that \( g_1 \gtrapprox_d \hat{g}_2 \), provided we can show that \((1 + \gamma_1 T_C) \hat{F}_i^{-1} \) is increasing for \( i = 1, 2 \) (which ensures that the corresponding operators \( \Gamma_i(\cdot) \) in the two modified societies are increasing). For \( i = 1 \), note that \((1 + \gamma_1 T_C) \hat{F}_2^{-1} := (1 + \gamma_1 T_C)(1 - \gamma_2 T_C) g_2 = (1 + \gamma_1 T_C) F^{-1} \), which is increasing by the assumption on \( \hat{P}_i \) and \( \gamma_1 > -1 \). For \( i = 2 \), note that (i) \((1 - \gamma_2 T_C^2) g_2 = (1 + \gamma_2 T_C) F^{-1} \) is increasing (by the assumption on \( \hat{P}_i \) and since \( \gamma_2 > -1 \)), and (ii) \( \gamma_2 T_C^2 g_2 \gtrapprox_d \gamma_1 T_C^2 g_2 \) as \( 0 \geq \gamma_1 \geq \gamma_2 \). Combining (i) and (ii) yields that \((1 + \gamma_1 T_C) \hat{F}_1^{-1} := (1 - \gamma_1 T_C) g_2 \) is increasing, as required. \( \square \)

**D.2.6 Proof of Proposition C.5**

Fix any ANE \( s^{AN} =: s \) and \( \theta \). For each \( \theta' \), set \( \hat{s}_\theta(\theta') := \text{BR}_\theta(L_{\theta}^{s, P}, L_{\theta}^{s, P}) \) and \( \hat{F}_\theta(\theta') := L_{\theta}^{s, P}(\hat{s}_\theta(\theta')) \), and let \( \hat{F}_\theta := \hat{F}_\theta \times \hat{F}_\theta \). To verify observational consistency, note that \( \hat{L}_{\theta}^{s, P}(a) = \hat{F}_\theta(\hat{s}_\theta^{-1}(a)) = L_{\theta}^{s, P}(a) \) for each \( a \), where the first equality uses \( \hat{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta \) and the inverse \( \hat{s}_\theta^{-1} \) is well-defined and increasing by the surjectivity and monotonicity assumption on best-responses. To verify the perceived best-response condition, note that, for each \( \theta' \),

\[
\hat{s}_\theta(\theta') = \text{BR}_{\theta'}(L_{\theta}^{s, P}, L_{\theta}^{s, P}) = \text{BR}_\theta(L_{\theta}^{\hat{s}_\theta, \hat{P}_\theta}, L_{\theta}^{\hat{s}_\theta, \hat{P}_\theta}) = \text{BR}_{\theta'}(G_{\hat{s}_\theta, \hat{P}_\theta}, L_{\theta}^{\hat{\phi}, \hat{P}_\theta}),
\]

where the second equality uses observational consistency and the third uses non-assortativity of \( \hat{P}_\theta \). Thus, \((\hat{P}_\theta, \hat{s}_\theta)\) is a coherent assortativity neglect perception for type \( \theta \).

To show uniqueness, consider any coherent assortativity neglect perception \((\tilde{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta, \tilde{s}_\theta)\) for \( \theta \). Then, for each \( \theta' \), the perceived best-response condition, non-assortativity of \( \hat{P}_\theta \), and observational consistency imply \( \tilde{s}_\theta(\theta') = \text{BR}_\theta(G_{\tilde{s}_\theta, \tilde{P}_\theta}, L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}) = \text{BR}_\theta(L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}, L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}) = \text{BR}_{\theta'}(L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}, L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}) = \text{BR}_{\theta'}(L_{\theta}^{s, P}, L_{\theta}^{s, P}). \) Moreover, \( \hat{P}_\theta = \hat{F}_\theta \times \hat{F}_\theta \) and observational consistency imply \( \hat{F}_\theta(\tilde{s}_\theta^{-1}(a)) = L_{\theta}^{\tilde{s}_\theta, \tilde{P}_\theta}(a) = L_{\theta}^{s, P}(a) \) for each \( a \), which yields \( \hat{F}_\theta(\theta') = L_{\theta}^{s, P}(\tilde{s}_\theta(\theta')) \) for each \( \theta' \). Thus, \((\tilde{P}_\theta, \tilde{s}_\theta)\) coincides with the perceptions in the first paragraph. \( \square \)

**D.2.7 Proof of Proposition C.6**

Consider any monotone ANE \( s^{AN} \) and any Nash equilibrium \( s^{NE} \). For any types \( \theta > \theta' \), the fact that \( \psi \) and \( \phi \) are monotone yields

\[
s^{AN}(\theta) - s^{AN}(\theta') = \phi(\theta) - \phi(\theta') + \psi(L_{\theta}^{s^{AN}, P} - \psi(L_{\theta}^{s^{AN}, P}) \geq \phi(\theta) - \phi(\theta') = s^{NE}(\theta) - s^{NE}(\theta') > 0,
\]

where the first inequality holds because \( L_{\theta}^{s^{AN}, P} \) FOSD-dominates \( L_{\theta'}^{s^{AN}, P} \) (by monotonicity of \( s^{AN} \) and assortativity of \( P \)). Thus, \( G_{s^{AN}, P} \) is more dispersive than \( G_{s^{NE}, P} \). \( \square \)