

Online Appendix

Dynamic Oligopoly and Price Stickiness

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A Dynamics after a Monetary Shock

A.1 Exact Dynamics

Proof of Proposition 1. If the consumer maximizes

$$\int e^{-\rho t} \left[\frac{C(t)^{1-\sigma}}{1-\sigma} - \frac{N(t)^{1+\psi}}{1+\psi} + \frac{m(t)^{1-\chi}}{1-\chi} \right] dt$$

we have

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= \frac{1}{\sigma} (R(t) - \pi(t) - \rho) \\ N(t)^\psi C(t)^\sigma &= \frac{W(t)}{P(t)} \Rightarrow \psi \frac{\dot{N}(t)}{N(t)} = \frac{\dot{W}(t)}{W(t)} - R(t) + \rho \\ M(t)^{-\chi} P(t)^\chi C(t)^\sigma &= R(t) \end{aligned}$$

We look for an equilibrium with constant nominal interest rate $R(t) = R$ and nominal wage $W(t) = W$ following a permanent shock to M . Suppose $\psi = 0$ then we get

$$\frac{\dot{W}(t)}{W(t)} = R - \rho$$

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To get constant wage $W(t) = W$ we need $R = \rho$ (this is necessary, otherwise we would get permanent wage inflation). The constant wage implies

$$P(t)C(t)^\sigma = W$$

Then the third equation gives

$$\rho M^\chi = P(t)^\chi C(t)^\sigma$$

So we need $\chi = 1$ for our guess to be indeed an equilibrium.

The representative consumer's expenditure in sector s at time t is

$$E_s(t) = P_s(t)^{1-\omega} [C(t)P(t)^\omega]$$

where $P(t)$ is the aggregate price level $(\int_s P_s(t)^{1-\omega} ds)^{\frac{1}{1-\omega}}$ hence the real demand vector in sector s is (given our within-sector CRS assumption as in Kimball)

$$D(\{p_{j,s}(t)\}, E_s(t)) = D(\{p_{j,s}(t)\}, 1) P_s(t)^{1-\omega} C(t) P(t)^\omega$$

where P_s is the sectoral price index. Denote the function of prices in sector s only

$$d(\{p_{j,s}\}) = D(\{p_{j,s}\}, 1) P_s^{1-\omega}$$

The nominal profit of firm i in sector s given all the other prices in the economy is

$$d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega \left[p_{i,s} - W(t) \frac{f^{-1}(d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega)}{d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega} \right]$$

where $p_{-i,s} = \{p_{j,s}\}_{j \neq i}$. Thus the real profit is

$$d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^{\omega-1} \left[p_{i,s} - W(t) \frac{f^{-1}(d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega)}{d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega} \right]$$

Firm i maximizes the present value of real profits discounted using the SDF $e^{-\rho t} C(t)^{-\sigma}$,

that is

$$\int e^{-\rho t} C(t)^{1-\sigma} P(t)^{\omega-1} d^i(p_{i,s}, p_{-i,s}) \left[p_{i,s} - W(t) \frac{f^{-1}(d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega)}{d^i(p_{i,s}, p_{-i,s}) C(t) P(t)^\omega} \right] dt.$$

With general σ (but linear disutility of labor and log-utility of real balances, that are needed to obtain constant nominal interest rate and wage) we have that

$$P(t)C(t)^\sigma = W,$$

therefore if

$$\omega\sigma = 1$$

then the terms

$$\begin{aligned} C(t)^{1-\sigma} P(t)^{\omega-1} &= W^{\frac{1}{\sigma}-1} P(t)^{\omega-\frac{1}{\sigma}} \\ C(t)P(t)^\omega &= W^{\frac{1}{\sigma}} P(t)^{\omega-\frac{1}{\sigma}} \end{aligned}$$

are constant. Denote $\hat{p}_s = (\frac{p_{1,s}}{1+\delta}, \dots, \frac{p_{n,s}}{1+\delta})$ the vector of normalized prices. The present discounted value of real profits is

$$\begin{aligned} & W^{1/\sigma-1} \int e^{-\rho t} d^i(\hat{p}_s) \left[p_{i,s} - W \frac{f^{-1}(d^i(p_{i,s}, p_{-i,s}) W^{1/\sigma})}{d^i(p_{i,s}, p_{-i,s}) W^{1/\sigma}} \right] dt \\ &= W_-^{1/\sigma-1} (1+\delta)^{1/\sigma-\omega} \int e^{-\rho t} d^i(\hat{p}_s) \left[\hat{p}_{i,s} - W_- \frac{f^{-1}(d^i(\hat{p}_s) W_-^{1/\sigma} (1+\delta)^{1/\sigma-\omega})}{d^i(\hat{p}_s) W_-^{1/\sigma} (1+\delta)^{1/\sigma-\omega}} \right] dt \\ &= W_-^{1/\sigma-1} \int e^{-\rho t} d^i(\hat{p}_s) \left[\hat{p}_{i,s} - W_- \frac{f^{-1}(d^i(\hat{p}_s) W_-^{1/\sigma})}{d^i(\hat{p}_s) W_-^{1/\sigma}} \right] dt \end{aligned}$$

which is exactly the same as before the shock up to the change of variables $p \rightarrow \hat{p}$.

A.2 Approximate Dynamics

Proof of Proposition 2. Fix n and a sector $s \in [0, 1]$. Define the state $v_s(t)$ as

$$v_s = (z_1, \dots, z_n)'$$

where $z_i = \log p_i - \log \bar{p}$. Denote the first-order expansion of the best response $p'_i = g(p_{-i}, P)$ by

$$z'_i = \alpha Z + \beta \left(\sum_{j \neq i} z_j \right)$$

where $Z(t) = \log P(t) - \log \bar{p}$ is the log deviation of the aggregate price level. Proposition 1 shows that $\alpha = 0$ if $\omega\sigma = 1$; otherwise α will be non-zero and we derive the aggregation in the general case.

When firm i adjusts its price, the state of sector s changes to

$$v'_s(t) = \alpha Z(t) u_i + M_i v_s(t)$$

where u_i is the vector $(0, \dots, 0, \underset{\uparrow}{0}, 0, \dots, 0)$ M_i is the identity matrix except for row i which is equal to $(\beta, \dots, \beta, \underset{\uparrow}{0}, \beta, \dots, \beta)$.

First suppose that all sectors are identical. Define the aggregate state variable

$$V(t) = \int_{s \in [0,1]} v_s(t) ds \in \mathbb{R}^n$$

Between t and $t + \Delta t$, a mass $n\lambda\Delta t$ of firms adjusts prices so V evolves as

$$\begin{aligned} V(t + \Delta t) &= (1 - n\lambda\Delta t)V(t) + \int_{\text{a firm in } s \text{ adjusts}} v_s(t + \Delta t) ds \\ &= (1 - n\lambda\Delta t)V(t) + (\lambda n\Delta t) \left[\alpha Z(t) \frac{\sum_i u_i}{n} + \frac{\sum_i M_i}{n} V(t) \right] \end{aligned}$$

therefore in the limit $\Delta t \rightarrow 0$

$$\dot{V}(t) = \lambda \alpha Z(t) U + n\lambda \left(\frac{\sum_i M_i}{n} - I_n \right) V(t)$$

where $U = \sum_i u_i = (1, \dots, 1)'$ and

$$\frac{\sum_i M_i}{n} - I_n = \begin{pmatrix} \frac{-1}{n} & \frac{\beta}{n} & \dots & \frac{\beta}{n} \\ \frac{\beta}{n} & \frac{-1}{n} & \dots & \frac{\beta}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta}{n} & \frac{\beta}{n} & \dots & \frac{-1}{n} \end{pmatrix}$$

The aggregate price level is then $Z(t) = LV_t$ where $L = \frac{1}{n}(1, \dots, 1)$. The eigenvalues

of $n\lambda \left(\frac{\sum_i M_i}{n} - I_n \right)$ are:

- $\mu_1 = -\lambda(1 + \beta)$ with multiplicity $n - 1$,
- $\mu_2 = -\lambda[1 - (n - 1)\beta]$ with multiplicity 1.

The vector U is an eigenvector associated with μ_2 , so if we start from symmetric initial conditions $V(0) = (\log p_0 - \log \bar{p}) U$ we have

$$V(t) = V(0) e^{(\lambda\alpha + \mu_2)t}$$

hence finally, the price index evolves to first order in δ as:

$$\begin{aligned} \log \left(\frac{P(t)}{\bar{P}} \right) &= \log \left(\frac{P(0)}{\bar{P}} \right) e^{-\lambda[1 - \alpha - (n-1)\beta]t} \\ &= -\delta e^{-\lambda[1 - \alpha - (n-1)\beta]t} \end{aligned}$$

With heterogeneous sectors s the aggregation across sectors yields

$$\log \left(\frac{P(t)}{\bar{P}} \right) = -\delta \int_s \zeta_s e^{-\lambda_s[1 - \alpha_s - (n_s - 1)\beta_s]t} ds$$

where ζ_s is the steady state expenditure share of sector s .

B Markov Equilibrium and Sufficient Statistics

Let $V^{i,s}(p; t)$ denote the value function for firm i , where p is the vector of n_s prices. We focus on equilibria with differentiable g and V satisfying the Bellman equation

$$R(t)V^{i,s}(p; t) = \Pi^{i,s}(p; t) + \lambda_s \sum_{j \in I_s} \left(V^{i,s}(g^{j,s}(p_{-j}; t), p_{-j}; t) - V^{i,s}(p; t) \right) + \frac{\partial V^{i,s}}{\partial t}(p; t) \quad (\text{A.1})$$

where $g^{j,s}(p_{-j}; t)$ satisfies the optimality condition $g^{j,s}(p_{-j}; t) \in \arg \max_{p_j} V^{j,s}(p_j, p_{-j}; t)$ with first-order necessary condition

$$V_{p_j}^{j,s}(g^{j,s}(p_{-j}; t), p_{-j}; t) = 0 \quad (\text{A.2})$$

for all j .

Proof of Proposition 4. Differentiating the Bellman equation (A.1) and making use of symmetry, we obtain at the steady state \bar{p} of a symmetric equilibrium:

$$0 = \Pi_{p_i}^i(\bar{p}) + \lambda \sum_{j \neq i} \left[V_{p_j}^i(\bar{p}) \frac{\partial g^j}{\partial p_i}(\bar{p}) \right]$$

$$V_{p_j}^i(\bar{p}) = \frac{\Pi_{p_j}^i(\bar{p})}{\rho + \lambda} + \frac{\lambda}{\rho + \lambda} \sum_{k \neq i, j} \left[V_{p_k}^i(\bar{p}) \frac{\partial g^k}{\partial p_j}(\bar{p}) \right] \quad \forall j \neq i$$

Using $\sum_j \sum_{k \neq i, j} V_{p_k}^i(\bar{p}) = (n-2) \sum_{j \neq i} V_{p_j}^i(\bar{p})$, the second condition becomes

$$\sum_{k \neq i} V_{p_k}^i(\bar{p}) = \frac{\sum_{k \neq i} \frac{\Pi_{p_k}(\bar{p})}{\rho + \lambda}}{1 - \frac{\lambda(n-2)\beta_n}{\rho + \lambda}}$$

Hence the first condition becomes

$$0 = \Pi_{p_i}(\bar{p}) + \frac{\lambda \beta_n}{\rho + \lambda [1 - (n-2)\beta_n]} \sum_{k \neq i} \Pi_{p_k}(\bar{p})$$

and the symmetry of $\Pi_{p_j}^i$ across $j \neq i$, we obtain

$$0 = \Pi_i^i(\bar{p}) + \frac{\lambda(n-1)\beta}{\rho + \lambda [1 - (n-2)\beta]} \Pi_j^i(\bar{p})$$

thus the formula for $B = (n-1)\beta$ is

$$B = \frac{\rho + \lambda}{\lambda} \frac{1}{\frac{n-2}{n-1} + \left(\frac{\Pi_j^i}{-\Pi_i^i} \right)}. \quad (\text{A.3})$$

We can reexpress

$$\frac{\Pi_j^i}{-\Pi_i^i} = \frac{\epsilon_j^i (1 - 1/\mu)}{-\epsilon_i^i (1 - 1/\mu) - 1}$$

where $\mu = \frac{\bar{p}}{W/f'(f^{-1}(d^i(\bar{p})))}$ is the steady state markup (the denominator is the marginal cost) to rewrite (A.3) in terms of demand own-elasticity $\epsilon_i^i = \frac{\partial \log d^i}{\partial \log p_i}$ and cross-elasticity

$$\epsilon_j^i = \frac{\partial \log d^i}{\partial \log p_j}:$$

$$B = \frac{\rho + \lambda}{\lambda} \frac{1}{\frac{n-2}{n-1} + \frac{\epsilon_j^i}{-\epsilon_i^i - \frac{\mu}{\mu-1}}}.$$

Homothetic preferences imply that the cross-elasticity is related to the own-elasticity through $(n-1)\epsilon_j^i = -(\omega + \epsilon_i^i)$.

$$B = \frac{\lambda + \rho}{\lambda} \frac{1}{1 + \frac{1 - (\mu-1)(\omega-1)}{(n-1)[(\epsilon-1)(\mu-1)-1]}}$$

where $\epsilon = |\epsilon_i^i|$.

C Demand Elasticities

C.1 General non-parametric results

We first assume an outer elasticity $\omega = 1$. Differentiating the budget constraint, we have for any i and p

$$c^i + \sum_j p_j \frac{\partial c^j}{\partial p_i} = 0 \tag{A.4}$$

Then Slutsky symmetry and constant returns to scale imply

$$\epsilon_i^i + \sum_{j \neq i} \epsilon_j^i = -1 \tag{A.5}$$

where $\epsilon_j^i = \frac{\partial \log c^i}{\partial \log p_j}$. At a symmetric price, this becomes

$$\epsilon_j^i = -\frac{1 + \epsilon_i^i}{n-1} \tag{A.6}$$

so the convergence to Nash holds as long as the own elasticity ϵ_i^i is bounded. Call for any pair j, k

$$\epsilon_{jk}^i = \frac{\partial^2 \log d_i}{\partial \log p_k \partial \log p_j}$$

We can differentiate (A.5) with respect to $\log p_i$ to get

$$\epsilon_{ii}^i + \sum_{j \neq i} \epsilon_{ij}^i = 0$$

hence at a symmetric price,

$$\epsilon_{ii}^i + (n-1)\epsilon_{ij}^i = 0 \quad (\text{A.7})$$

Differentiating once more the budget constraint with respect to p_i

$$2 \frac{\partial c^i}{\partial p_i} + \sum_j \frac{\partial^2 c^j}{\partial p_i^2} = 0 \quad (\text{A.8})$$

Elasticities and second-derivatives are related by

$$\frac{\partial^2 c^i}{\partial p_k \partial p_j} = \frac{c^i}{p_k p_j} [\epsilon_{jk}^i + \epsilon_j^i \epsilon_k^i] \text{ for any } j \neq k$$

$$\frac{\partial^2 c^i}{\partial p_j^2} = \frac{c^i}{p_j^2} [\epsilon_{jj}^i - \epsilon_j^i + (\epsilon_j^i)^2] \text{ for any } j$$

At a symmetric price (using $\epsilon_{ii}^i = \epsilon_{jj}^i$), we have from (A.8)

$$\epsilon_{jj}^i = \epsilon_j^i (1 - \epsilon_j^i) - \frac{1}{n-1} [\epsilon_{ii}^i + \epsilon_i^i (1 + \epsilon_i^i)] \quad (\text{A.9})$$

Finally, differentiating (A.4) with respect to p_k for some $k \neq i$ gives

$$\frac{\partial c^i}{\partial p_k} + \frac{\partial c^k}{\partial p_i} + \sum_{j \neq i, k} p_j \frac{\partial^2 c^j}{\partial p_k \partial p_i} + p_i \frac{\partial^2 c^i}{\partial p_k \partial p_i} + p_k \frac{\partial^2 c^k}{\partial p_k \partial p_i} = 0$$

and at a symmetric price p

$$\frac{2}{p} \frac{\partial c^i}{\partial p_k} + (n-2) \frac{\partial^2 c^i}{\partial p_k \partial p_j} + 2 \frac{\partial^2 c^i}{\partial p_k \partial p_i} = 0$$

Therefore, in elasticities at a symmetric price,

$$2\epsilon_j^i + (n-2) [\epsilon_{jk}^i + (\epsilon_j^i)^2] + 2 [\epsilon_{ij}^i + \epsilon_j^i \epsilon_i^i] = 0 \quad (\text{A.10})$$

for $k \neq j, i, j \neq i$. The own-superelasticity is defined as the elasticity of (minus the) elasticity:

$$\Sigma = \frac{\partial \log(-\epsilon_i^i)}{\partial \log p_i} = \frac{\epsilon_{ii}^i}{\epsilon_i^i}$$

So in the end we have two degrees of freedom: $\{\epsilon_i^i, \epsilon_{ii}^i\}$ or equivalently $\{\epsilon, \Sigma\}$ to parametrize a symmetric steady state.

In the non-Cobb-Douglas case $\omega \neq 1$, all the steps are almost the same except that we start from the sectoral budget constraint

$$\sum_{i \in I_s} p_i d^i = \mathcal{P}_s^{1-\omega}$$

where \mathcal{P}_s is the sectoral price index. As a result the elasticities at a symmetric price satisfy (A.7), (A.10) as before, but (A.6) and (A.9) become respectively

$$\begin{aligned} \epsilon_j^i &= -\frac{\omega + \epsilon_i^i}{n-1} \\ \epsilon_{jj}^i &= \epsilon_j^i (1 + \epsilon_j^i) - \frac{1}{n-1} \left[\epsilon_{ii}^i + \epsilon_i^i (\omega + \epsilon_i^i) \right]. \end{aligned}$$

Special case: $n = 2$. If $n = 2$ there is only 1 degree of freedom, so CES is without loss of generality (locally), even when the outer aggregation is not Cobb-Douglas (i.e., $\omega \neq 1$). From (A.10), the cross-superelasticity ϵ_{ij}^i is determined by elasticities, hence so is $\epsilon_{ii}^i = -(n-1)\epsilon_{ij}^i$.

C.2 Closed-form elasticities with Kimball Demand

Here again we outline the steps under Cobb-Douglas preferences across sectors, $\omega = 1$, but give the general expressions with $\omega \neq 1$ below.

Start with a general Kimball (1995) aggregator that defines C as

$$\frac{1}{n} \sum_i \Psi \left(\frac{c_i}{C} \right) = 1 \tag{A.11}$$

where Ψ is increasing, concave, and $\Psi(1) = 1$ which ensures the convention that at a

symmetric basket $c_i = c$, we have $C = c$. The consumer's problem is

$$\min_{\{c_i\}} \sum_i p_i c_i \text{ s.t. } \frac{1}{n} \sum_i \Psi \left(\frac{c_i}{C} \right) = 1$$

There exists a Lagrange multiplier $\lambda > 0$ such that for all i

$$p_i = \lambda \Psi' \left(\frac{c_i}{C} \right) \frac{1}{C} \quad (\text{A.12})$$

If we define the Kimball sectoral price index P (which differs from the ideal price index except under CES) by

$$\frac{1}{n} \sum_i \varphi \left(\Psi'(1) \frac{p_i}{P} \right) = 1$$

where

$$\varphi = \Psi \circ (\Psi')^{-1}$$

then at a symmetric price $p_i = p$ we have $P = p$, and $\lambda \Psi'(1) = PC$ so we can rewrite (A.12) as

$$\frac{p_i}{P} \Psi'(1) = \Psi' \left(\frac{c_i}{C} \right)$$

Taking logs and differentiating (A.12) with respect to $\log p_i$ yields

$$1 = \frac{\partial \log P}{\partial \log p_i} + \frac{\Psi'' \left(\frac{c_i}{C} \right) c_i}{\Psi' \left(\frac{c_i}{C} \right) C} \left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i} \right]$$

Differentiating (A.11) yields

$$\sum_j \Psi' \left(\frac{c_j}{C} \right) \frac{c_j}{C} \left[\frac{\partial \log c_j}{\partial \log p_i} - \frac{\partial \log C}{\partial \log p_i} \right] = 0$$

hence

$$\frac{\partial \log C}{\partial \log p_i} = \frac{\sum_j \Psi' \left(\frac{c_j}{C} \right) \frac{c_j}{C} \epsilon_i^j}{\sum_j \Psi' \left(\frac{c_j}{C} \right) \frac{c_j}{C}}$$

Using Slutsky symmetry $p_j \epsilon_i^j = p_i \epsilon_j^i$ to express this using demand elasticities for good

i only, we can reexpress as

$$\frac{\partial \log C}{\partial \log p_i} = \frac{\sum_j \Psi' \left(\frac{c_j}{C} \right) \frac{c_j p_i}{C} \epsilon_j^i}{\sum_j \Psi' \left(\frac{c_j}{C} \right) \frac{c_j}{C}}$$

At a symmetric price, budget exhaustion with constant returns implies

$$\frac{\partial \log C}{\partial \log p_i} = \frac{1}{n} \sum_j \epsilon_j^i = \frac{-1}{n}$$

For any $k \neq i$ we can differentiate

$$\log \Psi' \left(\frac{c^i}{C} \right) - \log \Psi' \left(\frac{c^k}{C} \right) = \log p_i - \log p_k$$

with respect to $\log p_i$ to get

$$\frac{\Psi'' \left(\frac{c^i}{C} \right)}{\Psi' \left(\frac{c^i}{C} \right)} \left(\frac{c^i}{C} \right) \frac{\partial}{\partial \log p_i} [\log c^i - \log C] - \frac{\Psi'' \left(\frac{c^k}{C} \right)}{\Psi' \left(\frac{c^k}{C} \right)} \left(\frac{c^k}{C} \right) \frac{\partial}{\partial \log p_i} [\log c^k - \log C] = 1$$

or, defining

$$R(x) = -\frac{x\Psi''(x)}{\Psi'(x)}$$

We have

$$R \left(\frac{c^k}{C} \right) \left[\epsilon_i^k - \frac{\partial \log C}{\partial \log p_i} \right] - R \left(\frac{c^i}{C} \right) \left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i} \right] = 1 \quad (\text{A.13})$$

Hence at a symmetric steady state, using $\epsilon_i^k = \epsilon_k^i = -\frac{1+\epsilon_i^i}{n-1}$ we have

$$\epsilon_i^i = -\left(\frac{n-1}{n} \frac{1}{R(1)} + \frac{1}{n} \right)$$

Differentiating once more with respect to $\log p_i$,

$$-R' \left(\frac{c^i}{C} \right) \left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i} \right]^2 + R' \left(\frac{c^k}{C} \right) \left[\epsilon_i^k - \frac{\partial \log C}{\partial \log p_i} \right]^2 - R \left(\frac{c^i}{C} \right) \left[\epsilon_{ii}^i - \frac{\partial^2 \log C}{\partial^2 \log p_i} \right] + R \left(\frac{c^k}{C} \right) \left[\epsilon_{ii}^k - \frac{\partial^2 \log C}{\partial^2 \log p_i} \right] = 0$$

At a symmetric steady state,

$$-R'(1) \left[\epsilon_i^i + \frac{1}{n} \right]^2 + R'(1) \left[\epsilon_i^k + \frac{1}{n} \right]^2 - R(1) \left[\epsilon_{ii}^i - \epsilon_{ii}^k \right] = 0$$

$$-R'(1) \left[\epsilon_i^i + \frac{1}{n} \right]^2 + R'(1) \left[\epsilon_i^k + \frac{1}{n} \right]^2 - R(1) \left[\epsilon_{ii}^i - \epsilon_{jj}^i \right] = 0$$

Using (A.9) we get

$$-R'(1) \left[\frac{n-1}{n} \frac{1}{R(1)} \right]^2 + R'(1) \left[-\frac{1+\epsilon_i^i}{n-1} + \frac{1}{n} \right]^2 - R(1) \left[\epsilon_{ii}^i \frac{n}{n-1} - \epsilon_j^i (1-\epsilon_j^i) + \frac{1}{n-1} \left[\epsilon_j^i (1+\epsilon_j^i) \right] \right] = 0$$

Now differentiating (A.13) with respect to $\log p_j$ for some $j \neq i, k$

$$\begin{aligned} & R' \left(\frac{c^i}{C} \right) \left[\epsilon_j^i - \frac{\partial \log C}{\partial \log p_j} \right] \left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i} \right] + R \left(\frac{c^i}{C} \right) \left[\epsilon_{ij}^i - \frac{\partial^2 \log C}{\partial \log p_i \partial \log p_j} \right] \\ & - R' \left(\frac{c^k}{C} \right) \left[\epsilon_i^k - \frac{\partial \log C}{\partial \log p_i} \right] \left[\epsilon_j^k - \frac{\partial \log C}{\partial \log p_j} \right] - R \left(\frac{c^k}{C} \right) \left[\epsilon_{ij}^k - \frac{\partial^2 \log C}{\partial \log p_i \partial \log p_j} \right] = 0 \end{aligned}$$

At a symmetric price,

$$R'(1) \left[\epsilon_j^i + \frac{1}{n} \right] \left[\epsilon_i^i + \frac{1}{n} \right] + R(1) \epsilon_{ij}^i = R'(1) \left[\epsilon_j^i + \frac{1}{n} \right]^2 + R(1) \epsilon_{jk}^i$$

Therefore, using (A.10) we have

$$\begin{aligned} \epsilon_i^i &= - \left[\left(\frac{n-1}{n} \right) \frac{1}{R(1)} + \frac{1}{n} \right] & (A.14) \\ \epsilon_j^i &= \frac{\frac{1}{R(1)} - 1}{n} \\ \epsilon_{ii}^i &= - \frac{n-1}{n^2} \left[\frac{R(1) [1 - R(1)]^2 + (n-2)R'(1)}{R(1)^3} \right] \\ \epsilon_{ij}^i &= \frac{R(1) [1 - R(1)]^2 + (n-2)R'(1)}{n^2 R(1)^3} \quad (j \neq i) \\ \epsilon_{jj}^i &= \frac{-(n-1)R(1) [1 - R(1)]^2 + (n-2)R'(1)}{n^2 R(1)^3} \quad (j \neq i) \\ \epsilon_{jk}^i &= \frac{R(1) [1 - R(1)]^2 - 2R'(1)}{n^2 R(1)^3} \quad (j \neq k, n \geq 3) \end{aligned}$$

In the general case $\omega \neq 1$, following similar steps these expressions generalize to

$$\begin{aligned}\epsilon_i^i &= - \left[\left(\frac{n-1}{n} \right) \frac{1}{R(1)} + \frac{1}{n} \omega \right] \\ \epsilon_j^j &= \frac{\frac{1}{R(1)} - \omega}{n} \\ \epsilon_{ii}^i &= - \frac{n-1}{n^2} \left[\frac{R(1) [1 - R(1)] [1 - R(1)\omega] + (n-2)R'(1)}{R(1)^3} \right] \\ \epsilon_{ij}^i &= \frac{R(1) [1 - R(1)] [1 - R(1)\omega] + (n-2)R'(1)}{n^2 R(1)^3} \quad (j \neq i) \\ \epsilon_{jj}^i &= \frac{-(n-1)R(1) [1 - R(1)] [1 - R(1)\omega] + (n-2)R'(1)}{n^2 R(1)^3} \quad (j \neq i) \\ \epsilon_{jk}^i &= \frac{R(1) [1 - R(1)] [1 - R(1)\omega] - 2R'(1)}{n^2 R(1)^3} \quad (j \neq k, n \geq 3)\end{aligned}$$

Equations (10)-(11) are written using the more convenient $\varphi(x) = 1/R(x)$.

Klenow and Willis (2016) use the functional form

$$\begin{aligned}\Psi'(x) &= \frac{\eta-1}{\eta} \exp\left(\frac{1-x^{\theta/\eta}}{\theta}\right) \\ \Psi''(x) &= -\frac{x^{\frac{\theta}{\eta}-1}}{\eta} \Psi'(x) \\ \Psi'''(x) &= \left[\left(\frac{x^{\frac{\theta}{\eta}-1}}{\eta} \right)^2 - \left(\frac{\theta-\eta}{\eta^2} \right) x^{\frac{\theta}{\eta}-2} \right] \Psi'(x)\end{aligned}$$

Therefore

$$\begin{aligned}R(1) &= \frac{1}{\eta} \\ R'(1) &= \frac{\theta}{\eta^2}\end{aligned}$$

so that this nests CES with $\theta = 0$. We thus have

$$\epsilon_i^i = -\frac{\eta(n-1) + \omega}{n} \quad (\text{A.15a})$$

$$\epsilon_j^i = \frac{\eta - \omega}{n} \quad (\text{A.15b})$$

$$\epsilon_{ii}^i = -\frac{(n-1)}{n^2} \left[\eta^2 - (1 + \omega)\eta + \omega + (n-2)\theta\eta \right] \quad (\text{A.15c})$$

$$\epsilon_{ij}^i = \frac{\eta^2 - (1 + \omega)\eta + \omega + (n-2)\theta\eta}{n^2} \quad (\text{A.15d})$$

$$\epsilon_{jj}^i = \frac{(n-2)\theta\eta - (\eta-1)(n-1)(\eta-\omega)}{n^2} \quad (\text{A.15e})$$

$$\epsilon_{jk}^i = \frac{\eta^2 - (1 + \omega)\eta + \omega - 2\theta\eta}{n^2} \quad (\text{A.15f})$$

With $\omega = 1$ as in the main text, the superelasticity, defined as $\Sigma = \frac{\epsilon_{ii}^i}{\epsilon_i^i}$, satisfies

$$\begin{aligned} \Sigma &= \frac{1}{\frac{S}{1-S} + \eta} \left[\theta\eta + \left((\eta-1)^2 - 2\theta\eta \right) S \right] \\ &\approx \theta + \left[\frac{(\eta-1)^2}{\eta} - 2\theta \right] S \end{aligned}$$

with $S = 1/n$ denoting the market share. The approximation in the second line holds if S is small relative to $\eta/(1+\eta)$, as is the case in a calibration with $\eta = 10$. With constant θ and η , the superelasticity is approximately linear in the Herfindahl index. If θ is lower than $\frac{(\eta-1)^2}{2\eta}$ which equals 4.05 when $\eta = 10$ (as in the CES case $\theta = 0$) then Σ increases with S . With high enough θ , it can actually decrease with S , but a high fixed θ is at odds with pass-through being larger for smaller firms.

D Solution Method

Iteratively differentiating the Bellman equation (A.1) and the optimality condition (A.2) generates a system of equations relating the derivatives of the reaction function g' , g'' , and so on, to the steady state markup, demand elasticity ϵ_i^i , superelasticity ϵ_{ii}^i , and so on. Our formula (9) is one of such equations.

The standard interpretation of this system treats the sequence of derivatives of g as unknowns, and the infinite sequence of higher-order elasticities as given structural

parameters. Instead, we acknowledge that it is empirically impossible to know such fine properties of preferences or demand functions, since we can only estimate a finite number of elasticities. This leads us to take a dual view of the same system of equations: we still take low order elasticities as given, but choose the values of the unknown higher order elasticities to achieve some desired properties for the derivatives of g . In particular, we can find primitives such that the reaction function g is locally polynomial of order m , meaning that all its derivatives of higher order than m vanish when evaluated at the steady state.

Formally, let

$$\epsilon_{(1)} = \frac{\partial \log d^i}{\partial \log p_i}, \quad \epsilon_{(k)} = \frac{\partial \epsilon_{(k-1)}}{\partial \log p_i} \quad \forall k \geq 2.$$

Proposition 9. *For any order $m \geq 1$ and target elasticities $(\epsilon_{(1)}, \dots, \epsilon_{(m)})$, there exist Kimball within-sector preferences $\tilde{\phi}$ such that*

- (i) *the resulting elasticities up to order m match the target elasticities, and*
- (ii) *any MPE of the game with within-sector preferences $\tilde{\phi}$, strategy \tilde{g} and steady state \tilde{p} satisfies $\tilde{g}^{(k)}(\tilde{p}) = 0$ for $k \geq m$.*

Another interpretation is to view the infinite sequence of elasticities as structural: for instance, we could assume that preferences are exactly CES and compute the implied elasticities of any order. In this context our method is then an approximation of the exact solution given by the limit $m \rightarrow \infty$ where we can match all elasticities.

Under this interpretation we can evaluate the accuracy of the approximation by noting that for low n , we can compute the exact solution $m \rightarrow \infty$ using standard value function iteration. We then compare the resulting steady state price to what follows from our solution method with finite m . Figure D1 plots the steady state markup with $m = 1, 2, 3$ in the case of a duopoly, showing that $m = 2$ already provides an excellent approximation (within 1%) to the exact solution $m \rightarrow \infty$ and going to a higher order $m = 3$ improves the fit but not by much. Note that low n allows us to check numerically the accuracy of the approximation, but we know theoretically that the approximation should be even better as n grows, since all the orders m of approximation coincide with monopolistic competition as $n \rightarrow \infty$.

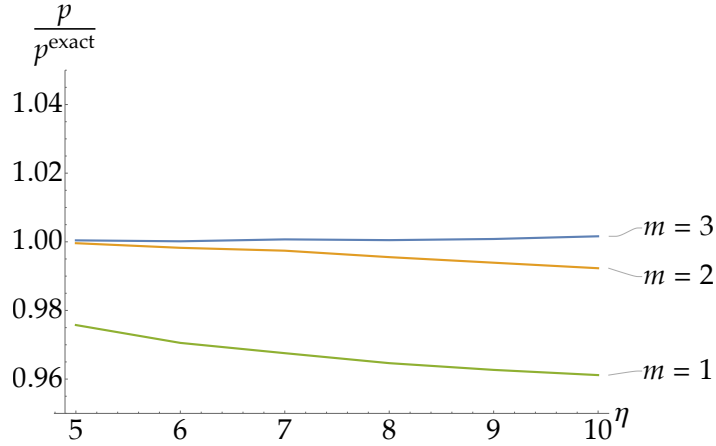


Figure D1: Steady state markup p with $n = 2$ firms, under our solution method with $m = 1, 2, 3$, relative to exact solution p^{exact} (which corresponds to $m \rightarrow \infty$).

Proof of Proposition 9. We start from the system that defines an MPE:

$$(\rho + n\lambda) V(p) = \Pi(p) + \lambda \sum_j V(g(p_{-j}), p_{-j}) \quad (\text{A.16})$$

$$V_p(g(p_{-i}), p_{-i}) = 0 \quad (\text{A.17})$$

Differentiating k times the Bellman equation (A.16) gives us for each $k \geq 1$ a linear system in the k th-derivatives $\mathbf{V}^{(k)} = (V_{11\dots11}, V_{11\dots12}, V_{11\dots22}, \dots)$ of the value function V (evaluated at the symmetric steady state \bar{p}), which we can invert to obtain these derivatives as a function of the profit derivatives $\mathbf{\Pi}^{(k)} = (\Pi_{11\dots11}, \dots)$ and derivatives of the policy function (there are $k + 1$ such equations in the case of $n = 2$ firms).

We can then compute $\mathbf{\Pi}^{(k)}$ as a function of \bar{p} and own- and cross-superelasticities of the demand function d of order up to k .

Combining the solution $\mathbf{V}^{(k)}$ with the $k - 1$ th-derivative of the FOC (A.17) gives us a sequence of equations that must be satisfied at a steady state

$$F^k \left(\bar{p}, g'(\bar{p}), g''(\bar{p}), \dots, g^{(k)}(\bar{p}); \epsilon_{(0)}, \epsilon_{(1)}, \epsilon_{(2)}, \dots, \epsilon_{(k)} \right) = 0$$

where F^k is linear in $\tilde{\epsilon}_{(k)}$. Thus we can construct recursively a unique sequence $\tilde{\epsilon}_{(k)}$

starting from $k = m + 1$, using

$$F^{m+1} \left(\bar{p}, g', \dots, g^{(m-1)}, 0, 0; \epsilon_{(1)}, \epsilon_{(2)}, \dots, \tilde{\epsilon}_{(m+1)} \right) = 0$$

$$F^{m+2} \left(\bar{p}, g', \dots, g^{(m-1)}, 0, 0, 0; \epsilon_{(1)}, \epsilon_{(2)}, \dots, \tilde{\epsilon}_{(m+1)}, \tilde{\epsilon}_{(m+2)} \right) = 0$$

and so on. Below we show that for $n \geq 3$ there are indeed enough degrees of freedom to make the equations F^m, F^{m+1}, \dots independent.

Define $\tilde{\varphi}$ as

$$\tilde{\varphi}(x) = \sum_{k=0}^{\infty} \frac{\tilde{\varphi}^{(k)}(1)}{k!} (x-1)^k$$

where $\tilde{\varphi}^{(k+1)}(1)$ is characterized by $(\epsilon_{(1)}, \dots, \epsilon_{(m)}, \tilde{\epsilon}_{(m+1)}, \dots, \tilde{\epsilon}_{(k)})$ through the same computations as in Appendix C. Given this construction, $\bar{p}, g', \dots, g^{(m-1)}$ are pinned down by $(\epsilon_{(1)}, \dots, \epsilon_{(m)})$ as the solution to the system of equations F^k for $k = 1, \dots, m$.

The main potential impediment to the proof is that demand integrability (e.g., demand functions being generated by actual utility functions) imposes restrictions on higher-order elasticities that would prevent us from constructing the sequence $\tilde{\epsilon}$. Indeed, in Appendix C we saw that with $n = 2$ firms, general Kimball demand functions cannot generate superelasticities beyond those arising from CES demand. We now show that as long as $n \geq 3$, this is not the case, by proving that the number of elasticities exceeds the number of restrictions.

Formally, we want to compute $\#_n(m)$, the number of cross-elasticities of order m , that is derivatives

$$\frac{\partial^m \log d^1(p)}{\partial^{i_1} \log p_1 \partial^{i_2} \log p_2 \dots \partial^{i_n} \log p_n}$$

where

$$0 \leq i_1, \dots, i_n \leq m$$

$$i_1 + \dots + i_n = m$$

as functions of the own- m th-elasticity $\underbrace{\epsilon_{11 \dots 1}^1}_{m \text{ times}}$, and compare $\#_n(m)$ to the number of restrictions imposed by demand integrability and symmetry arguments.

By Schwarz symmetry, in a smooth MPE, we can always invert 2 indices in the derivatives. Moreover, from the viewpoint of firm 1 (whose demand d^1 we're differentiating), firms 2 and 3 are interchangeable. For instance, in the case of $n = 3$ firms

and order of differentiation $m = 3$, these symmetries reduce the number of potential elasticities $n^m = 27$ to only 6 elasticities

$$\epsilon_{111}^1, \epsilon_{112}^1, \epsilon_{122}^1, \epsilon_{123}^1, \epsilon_{222}^1, \epsilon_{223}^1.$$

Denote

$$q_n(M)$$

the number of partitions of an integer M into n non-negative integers. For $M \geq n$ we have

$$q_n(M) = p_n(M + n)$$

where $p_n(M)$ is the number of partitions of an integer M into n positive integers. We can see this by writing, starting from a partition of M into n non-negative integers i_1, \dots, i_n :

$$M + n = (i_1 + 1) + \dots + (i_n + 1)$$

We can then compute $p_j(M)$ using the recurrence formula

$$p_j(M) = \underbrace{p_j(M - j)}_{\text{partitions for which } i_k \geq 2 \text{ for all } k} + \underbrace{p_{j-1}(M - 1)}_{\text{partitions for which } i_k = 1 \text{ for some } k}$$

Lemma 1. For any $n \geq 1$ and $m \geq 1$ the number of elasticities of order m is

$$\#_n(m) = \sum_{k=0}^m q_{n-1}(m - k) \quad (\text{A.18})$$

hence $\#_n(m + 1) = \#_n(m) + q_{n-1}(m + 1)$.

Proof. Firm 1 is special, so we need to count the number of times we differentiate with respect to $\log p_1$, which generates the sum over k . Then we get each term in the sum by counting partitions of $m - k$ into $n - 1$ non-negative integers. \square

Next, we want to count the reduction in the number of degrees of freedom imposed by economic restrictions. Our restrictions are

$$\Phi(p) = \sum_j p_j d^j(p) = 0 \quad \forall p \quad (\text{A.19})$$

$$d_j^i(p) = d_i^j(p) \quad \forall p, \forall i, j \quad (\text{A.20})$$

The first equation is the budget constraint. The second equation is the Slutsky symmetry condition (constant returns to scale allow to go from Hicksian to Marshallian elasticities). Note that Φ defined in (A.19) is symmetric, unlike the demand function d^1 we are using to compute elasticities. Therefore Φ 's derivatives give us fewer restrictions than what we need in (A.18), leaving room for restrictions to come from the Slutsky equation.

We need to differentiate these two equations to obtain independent equations that relate the m th-cross-elasticities to the m th-own-elasticity. The number of restrictions coming from derivatives of Φ at order m is simply the number $q_n(m)$ of partitions of m into n non-negative integers. Denote $b_n(m)$ the number of restrictions we have from derivatives of the Slutsky equation. The initial equation $d_2^1 = d_1^2$ is irrelevant at a symmetric steady state; it only starts mattering once we differentiate it. We actually do not need to compute $b_n(m)$ exactly. The following lemma shows that there are always enough degrees of freedom $\#_n(m)$ to construct the Kimball aggregator in 9:

Lemma 2. *For $n \geq 3$ and any m we have*

$$q_n(m) + b_n(m) + 1 \leq \#_n(m) \tag{A.21}$$

Proof. We know by hand that (A.21) holds for $m = 1, 2$ so take $m \geq 3$. Then all the Slutsky conditions can be written as starting with

$$d_{12\dots}^1 = \dots$$

hence we have

$$b_n(m) \leq \#_n(m-2) = \#_n(m) - p_{n-1}(n+m-1) - p_n(n+m-2)$$

hence the number of equations is bounded by

$$q_n(m) + b_n(m) \leq p_n(n+m) + \#_n(m) - p_{n-1}(n+m-1) - p_n(n+m-2)$$

Then we have (A.21) if

$$\begin{aligned}
& p_n(n+m) < p_{n-1}(n+m-1) + p_n(n+m-2) \\
\Leftrightarrow & p_{n-1}(n+m-1) + p_n(m) < p_{n-1}(n+m-1) + p_n(n+m-2) \\
\Leftrightarrow & p_n(m) < p_n(n+m-2)
\end{aligned}$$

which holds for $n \geq 3$. □

Note that so far we have considered general CRS demand functions. Restricting attention to the Kimball class makes the inequality (A.21) bind, meaning that we can parametrize all the cross-elasticities of order m using the own-elasticity of order m .

What fails in the knife-edge case $n = 2$? Slutsky symmetry imposes too many restrictions: at $m = 2$ we only have 3 elasticities $\epsilon_{11}^1, \epsilon_{12}^1, \epsilon_{22}^1$ and also 3 restrictions, so we can solve out all the superelasticities as functions of ϵ_1^1 , which prevents us from constructing the Kimball aggregator in Proposition 9.

E Model Solution

We apply the solution method described in Appendix D to derive analytical expressions in the case $m = 2$.

E.1 Symmetric Firms

We first solve the linear system in $\{V_j^i, V_{ii}^i, V_{ij}^i, V_{jj}^i, V_{jk}^i\}$ obtained from envelope conditions

$$\begin{aligned}
(\rho + \lambda) V_j^i &= \Pi_j^i + \lambda(n-2) V_j^i \beta \\
(\rho + \lambda) V_{ii}^i &= \Pi_{ii}^i + \lambda(n-1) (V_{jj}^i \beta^2 + 2V_{ij}^i \beta) \\
(\rho + 2\lambda) V_{ij}^i &= \Pi_{ij}^i + \lambda(n-2) (V_{jj}^i \beta^2 + V_{ij}^i \beta + V_{jk}^i \beta) \\
(\rho + \lambda) V_{jj}^i &= \Pi_{jj}^i + \lambda(n-2) (V_{jj}^i \beta^2 + 2V_{jk}^i \beta) + \lambda (V_{ii}^i \beta^2 + 2V_{ij}^i \beta) \\
(\rho + 2\lambda) V_{jk}^i &= \Pi_{jk}^i + \lambda(n-3) (V_{jj}^i \beta^2 + 2V_{jk}^i \beta) + \lambda (V_{ii}^i \beta^2 + 2V_{ij}^i \beta)
\end{aligned}$$

Injecting the solution into the derivative of the first-order condition

$$V_{ii}^i \beta + V_{ij}^i = 0$$

yields an equation

$$0 = A_{ii} \Pi_{ii}^i(\bar{p}) + A_{ij} \Pi_{ij}^i(\bar{p}) + A_{jj} \Pi_{jj}^i(\bar{p}) + A_{jk} \Pi_{jk}^i(\bar{p}) \quad (\text{A.22})$$

with coefficients

$$\begin{aligned} A_{ii} = & \beta((\beta + 1)\lambda^3 (\beta^2 (-2n^2 + 9n - 10) + \beta^3(n - 2) + 6\beta(n - 2) - 4) \\ & - \lambda^2 \rho (\beta^3 (n^2 - 5n + 6) + \beta^2 (2n^2 - 15n + 22) + \beta(24 - 9n) + 8) \\ & + \lambda \rho^2 (\beta^2(n - 2) + \beta(3n - 8) - 5) - \rho^3) \end{aligned} \quad (\text{A.23a})$$

$$\begin{aligned} A_{ij} = & -2(\beta + 1)\lambda^3 (-2\beta^3 (n^2 - 3n + 2) + \beta^4(n - 1) + 2\beta^2(n - 1) - \beta(n - 2) + 1) \\ & + \lambda^2 \rho (\beta^4 (-2n^2 + 7n - 5) - 4\beta^3 (n^2 - 4n + 3) + 3\beta^2 n - 4\beta(n - 3) + 5) \\ & + \lambda \rho^2 (\beta^2 n - 2\beta(n - 3) + 4) + \rho^3 \end{aligned} \quad (\text{A.23b})$$

$$\begin{aligned} A_{jj} = & \beta^2 \lambda ((\beta + 1)\lambda^2 (2(\beta^2 + 3\beta + 2) + \beta(\beta + 1)n^2 - (3\beta^2 + 7\beta + 2)n) \\ & + \lambda \rho (4\beta^2 + 10\beta + \beta(\beta + 1)n^2 - (5\beta^2 + 9\beta + 3)n + 6) \\ & + \rho^2(\beta - (\beta + 1)n + 2)) \end{aligned} \quad (\text{A.23c})$$

$$\begin{aligned} A_{jk} = & -\beta \lambda (n - 2)((\beta + 1)\lambda^2 (-\beta + \beta^3(n - 1) + 3\beta^2(n - 1) + 1) \\ & + \lambda \rho (2\beta^3(n - 1) + \beta^2(3n - 4) + 2) + \rho^2) \end{aligned} \quad (\text{A.23d})$$

Finally $\bar{p}^3 \Pi_{ii}^i(\bar{p})$, $\bar{p}^3 \Pi_{ij}^i(\bar{p})$, $\bar{p}^3 \Pi_{jj}^i(\bar{p})$, $\bar{p}^3 \Pi_{jk}^i(\bar{p})$ are all linear functions of \bar{p} and W . Therefore, multiplying (A.22) by $\frac{\bar{p}^3}{W}$ we get a linear equation in μ which can be solved to obtain a function

$$\mu = \mu(B, \omega, \epsilon, \Sigma, n, \lambda/\rho). \quad (\text{A.24})$$

Equation (A.24) together with the sufficient statistic formula (9)

$$B = B(\mu, \omega, \epsilon, n, \lambda/\rho)$$

form a system of two equations in the two unknowns μ and β .

E.2 Heterogeneous Firms

The demand faced by firm i is

$$c_i = \frac{1}{\bar{\xi}_i} d^i(\tilde{p}_i, \tilde{p}_{-i})$$

where d^i is the demand function from the symmetric case ($\bar{\xi}_i = 1$ for all i) and $\tilde{p}_j = p_j / \bar{\xi}_j$ is the normalized price of good j . As a result the nominal profit of firm i can be written as

$$\Pi^i(t) = \tilde{p}_i(t) d^i(\tilde{p}_i(t), \tilde{p}_{-i}(t)) - W(t) f^{-1} \left(\frac{d^i(\tilde{p}_i(t), \tilde{p}_{-i}(t))}{\bar{\xi}_i z_i} \right) \quad (\text{A.25})$$

where d^i is the previous demand function from the symmetric firms model, and $\tilde{p}_j = p_j / \bar{\xi}_j$ is the normalized price of good j . If $\bar{\xi}_i z_i = 1$, the model with normalized prices is isomorphic to one with symmetric firms.

Suppose as in Section 5.2 that there are two types of firms, a and b , with $n = n_a + n_b$. a and b firms can differ permanently in their productivity z , their demand shifters $\bar{\xi}$, or both. With two types we need to solve for six unknowns: two steady state prices $\{p_a, p_b\}$ and four slopes $\{\beta_a^a, \beta_b^a, \beta_a^b, \beta_b^b\}$ where β_j^i is the slope of the reaction of a firm of type i to the price change of a firm of type j . The envelope conditions for firms of type a are

$$\begin{aligned} (\rho + \lambda) V_i^{i,a} &= \Pi_i^{i,a} + \lambda (n_a - 1) V_{j_a}^{i,a} \beta_a^a + \lambda n_b V_{j_b}^{i,a} \beta_a^b \\ (\rho + \lambda) V_{j_a}^{i,a} &= \Pi_{j_a}^{i,a} + \lambda (n_a - 2) V_{j_a}^{i,a} \beta_a^a + \lambda n_b V_{j_b}^{i,a} \beta_a^b \\ (\rho + \lambda) V_{j_b}^{i,a} &= \Pi_{j_b}^{i,a} + \lambda (n_a - 1) V_{j_a}^{i,a} \beta_a^a + \lambda (n_b - 1) V_{j_b}^{i,a} \beta_a^b \end{aligned}$$

and, in the locally linear equilibrium:

$$\begin{aligned}
(\rho + \lambda) V_{ii}^{i,a} &= \Pi_{ii}^{i,a} + \lambda (n_a - 1) \left[V_{jaja}^{i,a} (\beta_a^a)^2 + 2V_{ija}^{i,a} \beta_a^a \right] + \lambda n_b \left[V_{jbjb}^{i,a} (\beta_a^b)^2 + 2V_{ijb}^{i,a} \beta_a^b \right] \\
(\rho + 2\lambda) V_{ija}^{i,a} &= \Pi_{ija}^{i,a} + \lambda (n_a - 2) \left[V_{jaja}^{i,a} (\beta_a^a)^2 + V_{jaka}^{i,a} \beta_a^a + V_{ija}^{i,a} \beta_a^a \right] + \lambda n_b \left[V_{jbjb}^{i,a} (\beta_a^b)^2 + V_{jakb}^{i,a} \beta_a^b + V_{ijb}^{i,a} \beta_a^b \right] \\
(\rho + 2\lambda) V_{ijb}^{i,a} &= \Pi_{ijb}^{i,a} + \lambda (n_a - 1) \left[V_{jaja}^{i,a} \beta_a^a \beta_b^a + V_{jakb}^{i,a} \beta_a^a + V_{ija}^{i,a} \beta_b^a \right] + \lambda (n_b - 1) \left[V_{jbjb}^{i,a} \beta_a^b \beta_b^b + V_{jbkb}^{i,a} \beta_a^b + V_{ijb}^{i,a} \beta_b^b \right] \\
(\rho + \lambda) V_{jaja}^{i,a} &= \Pi_{jaja}^{i,a} + \lambda \left[V_{ii}^{i,a} (\beta_a^a)^2 + 2V_{ija}^{i,a} \beta_a^a \right] + \lambda (n_a - 2) \left[V_{jaja}^{i,a} (\beta_a^a)^2 + 2V_{jaka}^{i,a} \beta_a^a \right] + \lambda n_b \left[V_{jbjb}^{i,a} (\beta_a^b)^2 + 2V_{jakb}^{i,a} \beta_a^b \right] \\
(\rho + 2\lambda) V_{jakb}^{i,a} &= \Pi_{jakb}^{i,a} + \lambda \left[V_{ii}^{i,a} \beta_a^a \beta_b^a + V_{ijb}^{i,a} \beta_a^a + V_{ija}^{i,a} \beta_b^a \right] + \lambda (n_a - 2) \left[V_{jaja}^{i,a} \beta_a^a \beta_b^a + V_{jakb}^{i,a} \beta_a^a + V_{jaka}^{i,a} \beta_b^a \right] \\
&\quad + \lambda (n_b - 1) \left[V_{jbjb}^{i,a} \beta_a^b \beta_b^b + V_{jbkb}^{i,a} \beta_a^b + V_{jkbj}^{i,a} \beta_b^b \right] \\
(\rho + 2\lambda) V_{jaka}^{i,a} &= \Pi_{jaka}^{i,a} + \lambda \left[V_{ii}^{i,a} (\beta_a^a)^2 + 2V_{ija}^{i,a} \beta_a^a \right] + \lambda (n_a - 3) \left[V_{jaja}^{i,a} (\beta_a^a)^2 + 2V_{jaka}^{i,a} \beta_a^a \right] + \lambda n_b \left[V_{jbjb}^{i,a} (\beta_a^b)^2 + 2V_{jakb}^{i,a} \beta_a^b \right] \\
(\rho + \lambda) V_{jbjb}^{i,a} &= \Pi_{jbjb}^{i,a} + \lambda \left[V_{ii}^{i,a} (\beta_b^a)^2 + 2V_{ijb}^{i,a} \beta_b^a \right] + \lambda (n_a - 1) \left[V_{jaja}^{i,a} (\beta_b^a)^2 + 2V_{jaka}^{i,a} \beta_b^a \right] + \lambda (n_b - 1) \left[V_{jbjb}^{i,a} (\beta_b^b)^2 + 2V_{jbkb}^{i,a} \beta_b^b \right] \\
(\rho + 2\lambda) V_{jbkb}^{i,a} &= \Pi_{jbkb}^{i,a} + \lambda \left[V_{ii}^{i,a} (\beta_b^a)^2 + 2V_{ijb}^{i,a} \beta_b^a \right] + \lambda (n_a - 1) \left[V_{jaja}^{i,a} (\beta_b^a)^2 + 2V_{jaka}^{i,a} \beta_b^a \right] + \lambda (n_b - 2) \left[V_{jbjb}^{i,a} (\beta_b^b)^2 + 2V_{jbkb}^{i,a} \beta_b^b \right]
\end{aligned}$$

We can use these 11 envelope conditions to solve linearly for $\{V_i^{i,a}, V_{ja}^{i,a}, V_{jb}^{i,a}, V_{ii}^{i,a}, \dots\}$, and then inject the solution into the first-order conditions

$$\begin{aligned}
V_i^{i,a} &= 0 \\
V_{ii}^{i,a} \beta_a^a + V_{ija}^{i,a} &= 0 \\
V_{ii}^{i,a} \beta_b^a + V_{ijb}^{i,a} &= 0
\end{aligned}$$

The same steps for firms of type b give us 3 more equations.

F Calibration to Pass-Through Evidence

In Section 3 we use evidence on own-cost pass-through from Amiti, Itskhoki and Konings (2019) (henceforth AIK) to calibrate how the superelasticity Σ varies with concentration. We describe the procedure in more detail here.

In the presence of permanent shocks to marginal costs mc_j , when firm i adjusts its price it sets

$$\log p_i - \log \bar{p}_i = \alpha (\log mc_i - \log \bar{c}_i) + B \frac{\sum_{j \neq i} \log p_j - \log \bar{p}_j}{n - 1} + \gamma \sum_{j \neq i} (\log c_j - \log \bar{c}_j) \tag{A.26}$$

where the coefficients

$$\alpha = \frac{\partial g^i}{\partial mc_i}, \quad B = (n-1) \frac{\partial g^i}{\partial p_j}, \quad \gamma = \frac{\partial g^i}{\partial mc_j}$$

can be computed as before using our envelope conditions applied to a generalization of the Bellman equation (A.1) that allows for permanent cost shocks:

$$(\rho + n\lambda) V^i(p, mc) = \Pi^i(p, mc_i) + \lambda \sum_j V^i(g^j(p_{-j}, mc), p_{-j}, mc). \quad (\text{A.27})$$

Unlike in static models of oligopoly (see Remark 2 below) γ is non-zero in general: although competitor j 's cost c_j does not affect firm i 's current profits, it will affect how firm j sets its price p_j in the future, which is relevant for firm i 's future payoffs. Anticipating this, firm i will already respond itself to c_j when it gets to reset its price. The coefficients must satisfy the homogeneity restriction

$$\alpha + B + (n-1)\gamma = 1,$$

which says that if all firms' marginal costs increase by 1% then all firms' prices also increase by 1%.

Rewrite (A.26) in vector form as

$$\Delta \tilde{\mathbf{p}} = (\alpha I + \gamma S) \Delta \tilde{\mathbf{m}}\mathbf{c} + \beta S \Delta \tilde{\mathbf{p}}$$

where $S = J - I$ and J is the matrix with 1's everywhere, $\Delta \tilde{\mathbf{p}} = [\log p_i - \log \bar{p}_i]'$, $\Delta \tilde{\mathbf{m}}\mathbf{c} = [\log mc_i - \log \bar{m}c_i]'$. The following result describes the mapping from the parameters α, B in (A.26) to the regression estimates $\hat{\alpha}, \hat{B}$ in (12).

Proposition 10. *There exist unique scalars $\hat{\alpha}, \hat{B}$ such that for all vectors $\Delta \mathbf{m}\mathbf{c}$*

$$\Delta \tilde{p}_i = \hat{\alpha} \Delta \tilde{m}c_i + \hat{B} \frac{\sum_{j \neq i} \Delta \tilde{p}_j}{n-1}$$

for all i , namely

$$\hat{\alpha} = \frac{n\alpha + B - 1}{\alpha + B + n - 2} \quad (\text{A.28})$$

$$\hat{B} = \frac{(n-1)(1-\alpha)}{\alpha + B + n - 2} \quad (\text{A.29})$$

thus they satisfy $\hat{\alpha} + \hat{\beta} = 1$.

Proof. We need for all Δc

$$\Delta \tilde{\mathbf{p}} = \hat{\alpha} \Delta \tilde{\mathbf{m}} c + \hat{\beta} S \Delta \tilde{\mathbf{p}}$$

that is

$$(I - \hat{\beta} S) (I - \beta S)^{-1} (\alpha I + \gamma S) = \hat{\alpha} I$$

where $\beta = \frac{B}{n-1}$, $\hat{\beta} = \frac{\hat{B}}{n-1}$. Using $M = (I - \beta S)^{-1} = \sum_{k \geq 0} \beta^k S^k$ this is equivalent to

$$\begin{aligned} \sum_{k \geq 0} \beta^k [S^k - \hat{\beta} S^{k+1}] (\alpha I + \gamma S) &= \hat{\alpha} I \\ \sum_{k \geq 0} \beta^k [\alpha S^k + \gamma S^{k+1} - \alpha \hat{\beta} S^{k+1} - \gamma \hat{\beta} S^{k+2}] &= \hat{\alpha} I \\ \alpha M + \frac{\gamma}{\beta} (M - I) - \alpha \frac{\hat{\beta}}{\beta} (M - I) - \gamma \frac{\hat{\beta}}{\beta^2} (M - I - \beta S) &= \hat{\alpha} I \\ \alpha \beta M + (\gamma - \alpha \hat{\beta} - \gamma \hat{\beta} / \beta) (M - I) + \gamma \hat{\beta} S &= \hat{\alpha} \beta I \end{aligned}$$

Multiplying by $I - \beta S$ this becomes

$$\begin{aligned} \alpha \beta I + (\gamma - \alpha \hat{\beta} - \gamma \hat{\beta} / \beta) \beta S + \gamma \hat{\beta} (S - \beta S^2) &= \hat{\alpha} \beta (I - \beta S) \\ (\gamma \beta - \alpha \beta \hat{\beta} + \hat{\alpha} \beta^2) S - \gamma \hat{\beta} \beta S^2 &= (\hat{\alpha} - \alpha) \beta I \end{aligned}$$

Using

$$J^2 = nJ$$

(recall that J is the matrix with ones everywhere) we have

$$S^2 = (n-1)I + (n-2)S$$

Therefore $\hat{\alpha}, \hat{\beta}$ must satisfy

$$(\gamma \beta - \alpha \beta \hat{\beta} + \hat{\alpha} \beta^2 - \gamma \hat{\beta} \beta (n-2)) S = [(\hat{\alpha} - \alpha) \beta + \gamma \hat{\beta} \beta (n-1)] I$$

which can only be true if both sides are zero, that is (after replacing γ using the ho-

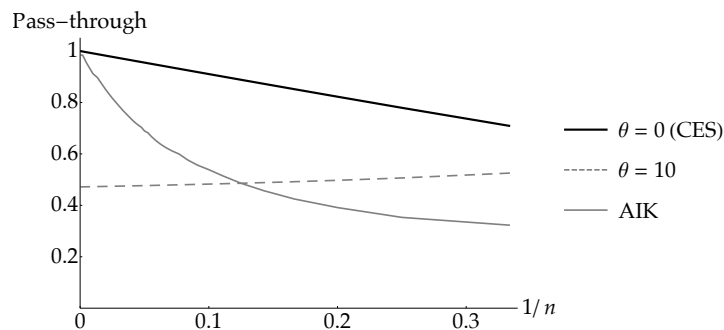


Figure F1: Pass-through $\hat{\alpha}$ as a function of market share $1/n$.

mogeneity restriction):

$$\hat{\alpha} = \frac{n\alpha + B - 1}{\alpha + B + n - 2}$$

$$\hat{B} = \frac{(n-1)(1-\alpha)}{\alpha + B + n - 2}$$

□

Amiti, Itskhoki and Konings (2019) show that the empirical behavior of $\hat{\alpha}$ as a function of market share is well approximated by

$$\hat{\alpha} \approx \frac{1}{1 + \frac{(\eta-1)(1-s)s(\eta-\omega)}{\omega(\eta-1)-s(\eta-\omega)}}$$

with $\eta = 10$ and $\omega = 1$. Therefore in a sector with n firms we set as target the corresponding pass-through $\hat{\alpha}_n = \frac{1}{1+9/n}$. Then, fixing other parameters (e.g., η, λ, ρ), for each (θ, n) we can compute α and B and solve for θ_n that sets allows to match $\hat{\alpha}_n$.

Figure F1 shows the resulting pass-through as a function of market share $1/n$ under this “AIK” calibration, contrasting with the case of fixed $\theta = 0$ (CES) and fixed $\theta = 10$.

G Other Comparative Statics

Changes in Preference Parameters η and θ . Changes in η and θ affect both the steady state markup μ and the half-life of the price level following monetary shocks.

Figure G1 shows the half-life as a function of the steady state markup, when variation in markups is produced through variation in the within-sector elasticity of sub-

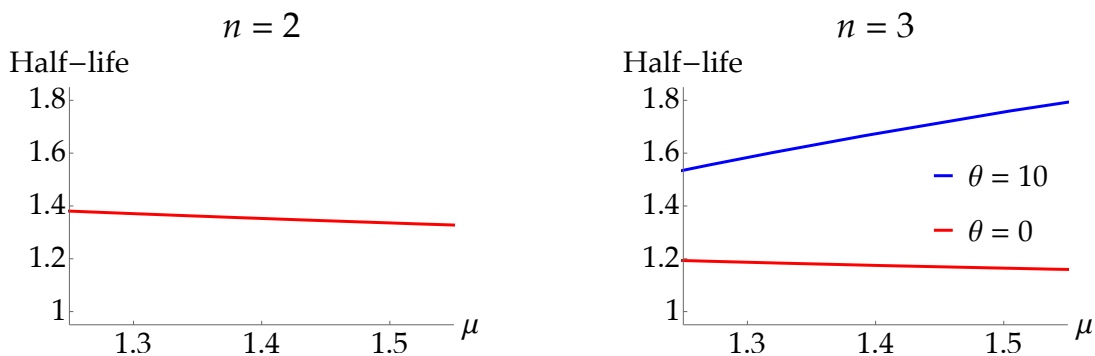


Figure G1: Half-life as a function of steady state markup μ when η varies.

stitution η ; higher η implies lower markups. The effect on the half-life is ambiguous, however, except in the special case $n = 2$ in which there is always a negative relation between the markup and the half-life.¹ In particular, as soon as there are at least $n = 3$ firms, the value of θ matters. When $\theta = 0$ (CES), we have the same negative relation as in the duopoly case, but with a high enough value of θ , the half-life becomes negatively related to the steady state markup. We explain these patterns in Section 4.

We argued that under dynamic oligopoly, markups are not fully determined by demand elasticities. Figure G2 shows the half-life as a function of the steady state markup, when variation in markups is produced through the superelasticity parameter θ in an example with $n = 3$ firms. Higher θ implies higher markups, even though the demand elasticity ϵ is unchanged throughout. As we vary θ , all the objects appearing in the right-hand side of (9) remain fixed except μ , hence this experiment yields a transparent application of the formula showing how B and the half-life increase with μ .

Changes in Discount Rates and Price Stickiness. The discount rate ρ and the frequency of price changes λ can also affect the steady state markup (and therefore the slope B). These two parameters only enter through the ratio ρ/λ , so a higher frequency is isomorphic to a lower discount rate and we focus the discussion on λ .

Figure G3 shows that markups increase with λ , especially when n is low. This

¹In Appendix B we show that for any homothetic preferences, ϵ and Σ are the same as under CES when there are only $n = 2$ symmetric firms, whether the cross-sector aggregator has unit elasticity ω or not. This means that θ is irrelevant when $n = 2$, as can be seen in Figure 2, where all the curves coincide when $n = 2$. When n is above 2, however, knowing the markup is not enough to infer the slope, which is why formula (9) also requires information on demand elasticities.

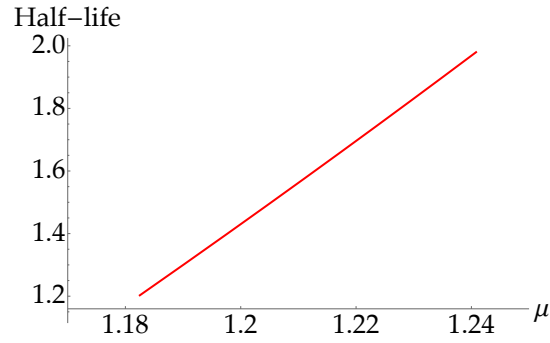


Figure G2: Half-life as a function of steady state markup μ when θ varies ($n = 3$).

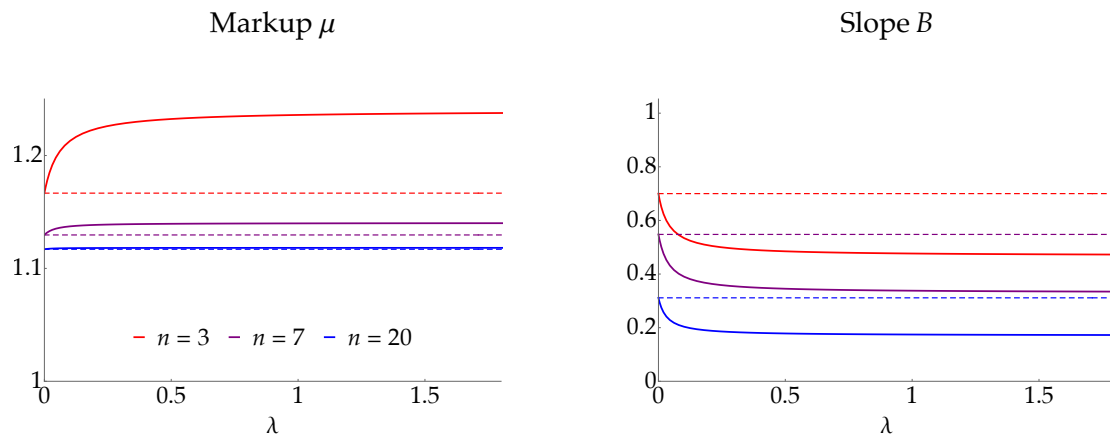


Figure G3: Steady state markup μ and slope B as a function of frequency of price changes λ under “AIK” calibration. Dashed horizontal lines correspond to the static Bertrand-Nash equilibrium (μ^{Nash} and B^{Nash}).

shows once again that equilibrium markups are complex objects that depend on many features of the environment beyond demand elasticities. In the limit of infinitely sticky prices $\lambda \rightarrow 0$, firms play the one-shot best-response, and so the Markov equilibrium coincides with the static Bertrand-Nash equilibrium, both in terms of steady state markup and reaction functions, which is apparent in Figure G3. Interestingly, the limit of infinitely frequent price changes $\lambda/\rho \rightarrow \infty$ does *not* equal the frictionless (flexible price) model, in which firms would play the static Bertrand-Nash equilibrium at each instant. For instance, when $n = 3$ (in red), the static markup is $\mu^{\text{Nash}} = 1.17$ while the steady state markup converges to $\mu = 1.24$ as $\lambda \rightarrow \infty$. For higher n , the gap between the Nash markup and the $\lambda \rightarrow \infty$ limit becomes negli-

ble.²

H Solution of the Naive Model

The quadratic approximation of profit Π^i of firm i around the naive steady state which is the static Nash p^{Nash} writes

$$\pi^i(z_i, z_{-i}) = BQ_i + CQ_i^2 + Dz_iQ_i + Ez_i^2 + FR_i$$

where $z_j = \log p_j - \log p^{\text{Nash}}$ for each j and

$$Q_i = \sum_{j \neq i} z_j$$

$$R_i = \sum_{j \neq i} z_j^2$$

There is no term Az_i because we are approximate around the Nash price p^{Nash} where $\Pi_i^i = 0$ for all i . The most important coefficients D and E are

$$D = \Pi_{ij} \left(p^{\text{Nash}} \right)$$

$$E = \frac{\Pi_{ii}}{2} \left(p^{\text{Nash}} \right)$$

We look for a symmetric equilibrium where each resetting firm j sets

$$z_j^* = \beta Q_j.$$

Then between s and $s + \Delta s$ we have

$$\mathbf{E}_t Q_i(s + \Delta s) = (1 - (n - 1)\lambda\Delta) \mathbf{E}_t Q_i(s) + \lambda\Delta \mathbf{E}_t \sum_{j \neq i} [Q_i(s) - p_j(s) + \beta Q_j(s)]$$

²This discontinuity in markups in the limit of flexible prices or very patient firms has been noted in other contexts, such as the alternating moves model of Maskin and Tirole (1988) and the model with quadratic Rotemberg adjustment costs in Jun and Vives (2004). A recent empirical IO literature, e.g., Brown and MacKay (2021), finds that algorithms allowing for fast repricing do lead to higher markups.

hence taking the limit $\Delta s \rightarrow 0$

$$\frac{d}{ds} \mathbf{E}_t Q_i(s) = \lambda \left\{ \beta \sum_{j \neq i} \mathbf{E}_t Q_j(s) - \mathbf{E}_t Q_i(s) \right\}$$

thus the variable $Z(s) = \sum_i \mathbf{E}_t Q_i(s)$ follows

$$\frac{d}{ds} Z(s) = -\lambda [1 - \beta(n-1)] Z(s)$$

Therefore, by symmetry

$$\mathbf{E}_t Q_i(s) = Q_i(t) e^{-\lambda[1-\beta(n-1)](s-t)}$$

When it resets, firm i chooses $z_i^*(t)$ such that

$$\max_{z_i^*(t)} \mathbf{E}_t \left[\int_t^\infty e^{-(\lambda+\rho)(s-t)} \pi^i(z_i^*(t), z_i(t+s)) ds \right]$$

The FOC is

$$\begin{aligned} z_i^*(t) &= - \frac{\int_t^\infty e^{-(\lambda+\rho)(s-t)} D \mathbf{E}_t [Q_i(s)] ds}{\int_t^\infty e^{-(\lambda+\rho)s} 2E ds} \\ &= - \frac{\int_t^\infty e^{-(\lambda+\rho)(s-t)} \left(D Q_i(t) e^{-\lambda(1-(n-1)\beta)(s-t)} \right) ds}{\int_t^\infty e^{-(\lambda+\rho)(s-t)} 2E ds} \\ &= - \frac{D(\lambda + \rho)}{2E [\lambda + \rho + \lambda(1 - (n-1)\beta)]} Q_i(t) \end{aligned}$$

Therefore $B = (n-1)\beta$ solves

$$B = \frac{B^{\text{Nash}}}{1 + \frac{\lambda}{\rho+\lambda} [1 - B]} \quad (\text{A.30})$$

where the ratio $B^{\text{Nash}} = \frac{(n-1)\Pi_{ij}}{-\Pi_{ii}}$ is the slope of the static best response to a simultaneous price change by all firms $j \neq i$ in a static model. We need B^{Nash} to be strictly lower than 1 for a static symmetric Nash equilibrium to exist. (A.30) shows that the slope of the dynamic naive best response at a stable steady state is always smaller than the slope of the static best response B^{Nash} and is decreasing in λ/ρ . The stable

root in $(0, 1)$ is

$$B^{\text{Naive}} = \left(\frac{\rho + 2\lambda}{2\lambda} \right) \left[1 - \sqrt{1 - 4 \frac{\lambda(\rho + \lambda)}{(\rho + 2\lambda)^2} B^{\text{Nash}}} \right].$$

I Derivation of the Oligopolistic Phillips Curve

Consider the general non-stationary versions of the Bellman equation (A.1) and the first-order condition (A.2):

$$(R_t + n\lambda) V^i(p, t) = V_t^i(p, t) + \Pi^i(p, MC_t, Z_t) + \lambda \sum_j V^i(g^j(p_{-j}, t), p_{-j}, t) \quad (\text{A.31})$$

$$V_t^i(g^i(p_{-i}, t), p_{-i}, t) = 0 \quad (\text{A.32})$$

Nominal profits are given by

$$\Pi^i(p, MC, Z) = ZD^i(p) [p_i - MC]$$

where Z is an aggregate demand shifter that can depend arbitrarily on C_t and P_t .³

Define $\alpha(t)$ as the solution to

$$g^i(\alpha(t), \alpha(t), \dots, \alpha(t), t) = \alpha(t).$$

This is the price that each firm would set if all the firms were resetting at the same time. α is the counterpart of the reset price in the standard New Keynesian model.

To obtain the dynamics of α from (A.31), we start by deriving time-varying envelope conditions evaluated at the symmetric price $p_1 = p_2 = \dots = p_n = \alpha(t)$. After applying symmetry and using Proposition 9 to make the strategies approximately linear in the neighborhood of the steady state, the non-linear first-order and second-order envelope conditions of the non-stationary game imply the following partial

³In Section 2, conditions (5) ensured a constant Z_t .

differential equations (PDEs)

$$0 = V_{it}^i + \Pi_i^i + \lambda (n - 1) V_j^i \beta \quad (\text{A.33a})$$

$$(i_t + \lambda) V_j^i = V_{jt}^i + \Pi_j^i + \lambda (n - 2) V_j^i \beta \quad (\text{A.33b})$$

$$(i_t + \lambda) V_{ii}^i = V_{iit}^i + \Pi_{ii}^i + \lambda (n - 1) (V_{jj}^i \beta^2 + 2V_{ij}^i \beta) \quad (\text{A.33c})$$

$$(i_t + 2\lambda) V_{ij}^i = V_{ijt}^i + \Pi_{ij}^i + \lambda (n - 2) (V_{jj}^i \beta^2 + V_{jk}^i \beta + \beta V_{ij}^i) \quad (\text{A.33d})$$

$$(i_t + \lambda) V_{jj}^i = V_{jjt}^i + \Pi_{jj}^i + \lambda (n - 2) (V_{jj}^i \beta^2 + 2\beta V_{jk}^i) + \lambda (V_{ii}^i \beta^2 + 2\beta V_{ij}^i) \quad (\text{A.33e})$$

$$(i_t + 2\lambda) V_{jk}^i = V_{jkt}^i + \Pi_{jk}^i + \lambda (n - 3) (V_{jj}^i \beta^2 + 2\beta V_{jk}^i) + \lambda (V_{ii}^i \beta^2 + 2\beta V_{ij}^i) \quad (\text{A.33f})$$

Denote the functions

$$W_i^i(t) = V_i^i(\alpha(t), \dots, \alpha(t), t), \quad W_{ii}^i(t) = V_{ii}^i(\alpha(t), \dots, \alpha(t), t)$$

and so on for all derivatives of the value function V^i . We can transform the system (A.33) into a system of ordinary differential equations in the functions $W_i^i(t), W_j^i(t)$, and so on. The partial derivatives with respect to time such as

$$V_{it}^i = \frac{\partial V_i^i}{\partial t}(\alpha(t), \dots, \alpha(t), t)$$

in equations (A.33) can be mapped to corresponding total derivatives of W functions

$\dot{W}_{it}^i = \frac{dW_{it}^i}{dt}$ using

$$\begin{aligned} V_{it}^i &= \dot{W}_i^i - \left[V_{ii}^i + \sum_{j \neq i} V_{ij}^i \right] \dot{\alpha} \\ V_{jt}^i &= \dot{W}_j^i - \left[V_{ij}^i + V_{jj}^i + \sum_{k \neq i,j} V_{jk}^i \right] \dot{\alpha} \\ V_{iit}^i &= \dot{W}_{ii}^i - \left[V_{iii}^i + \sum_{j \neq i} V_{ij}^i \right] \dot{\alpha} \\ V_{ijt}^i &= \dot{W}_{ij}^i - \left[V_{iij}^i + V_{ijj}^i + \sum_{k \neq i,j} V_{ijk}^i \right] \dot{\alpha} \\ V_{jjt}^i &= \dot{W}_{jj}^i - \left[V_{ijj}^i + V_{jjj}^i + \sum_{k \neq i,j} V_{jjk}^i \right] \dot{\alpha} \\ V_{jkt}^i &= \dot{W}_{jk}^i - \left[V_{ijk}^i + V_{jjk}^i + V_{jkk}^i + \sum_{l \neq i,j,k} V_{jkl}^i \right] \dot{\alpha} \end{aligned}$$

where the third derivatives of V at the steady state come from the third-order envelope conditions of the stationary game, solving the linear system:

$$\begin{aligned} (\rho + \lambda) V_{iii}^i &= \Pi_{iii}^i + \lambda (n - 1) \left\{ V_{jjj}^i \beta^3 + 3V_{ijj}^i \beta^2 + 3V_{iij}^i \beta \right\} \\ (\rho + 2\lambda) V_{iij}^i &= \Pi_{iij}^i + \lambda (n - 2) \left\{ V_{jjj}^i \beta^3 + 2V_{ijj}^i \beta^2 + V_{jjk}^i \beta^2 + 2V_{ijk}^i \beta + V_{iij}^i \beta \right\} \\ (\rho + 2\lambda) V_{ijj}^i &= \Pi_{ijj}^i + \lambda (n - 2) \left\{ V_{jjj}^i \beta^3 + 2\beta^2 V_{jjk}^i + \beta^2 V_{iij}^i + 2\beta V_{ijk}^i + \beta V_{jjk}^i \right\} \\ (\rho + 3\lambda) V_{ijk}^i &= \Pi_{ijk}^i + \lambda (n - 3) \left\{ V_{jjj}^i \beta^3 + 2\beta^2 V_{jjk}^i + \beta^2 V_{iij}^i + 2\beta V_{ijk}^i + \beta V_{jkl}^i \right\} \\ (\rho + \lambda) V_{jjj}^i &= \Pi_{jjj}^i + \lambda (n - 2) \left\{ \beta^3 V_{jjj}^i + 3\beta^2 V_{jjk}^i + 3\beta V_{jjk}^i \right\} \\ &\quad + \lambda \left\{ \beta^3 V_{iii}^i + 3\beta^2 V_{iij}^i + 3\beta V_{iij}^i \right\} \\ (\rho + 2\lambda) V_{jjk}^i &= \Pi_{jjk}^i + \lambda (n - 3) \left\{ \beta^3 V_{jjj}^i + 3\beta^2 V_{jjk}^i + \beta V_{jjk}^i + 2\beta V_{jkl}^i \right\} \\ &\quad + \lambda \left\{ \beta^3 V_{iii}^i + 3\beta^2 V_{iij}^i + \beta V_{iij}^i + 2\beta V_{ijk}^i \right\} \\ (\rho + 3\lambda) V_{jkl}^i &= \Pi_{jkl}^i + \lambda (n - 4) \left\{ \beta^3 V_{jjj}^i + 3\beta^2 V_{jjk}^i + 3\beta V_{jkl}^i \right\} \\ &\quad + \lambda \left\{ \beta^3 V_{iii}^i + 3\beta^2 V_{iij}^i + 3\beta V_{ijk}^i \right\} \end{aligned}$$

Importantly, to approximate the second derivatives of V^i , we need to solve for the

third derivatives of V^i around the steady state by applying the envelope theorem one more time.

Imposing symmetry again, the following non-linear system of ODEs in the functions $(\alpha, \beta, W_j^i, W_{ii}^i, W_{ij}^i, W_{jj}^i, W_{jk}^i)$ holds exactly (omitting the time argument):

$$0 = - [W_{ii}^i + (n-1) W_{ij}^i] \dot{\alpha} + \Pi_i^i + \lambda (n-1) W_j^i \beta \quad (\text{A.35a})$$

$$(i_t + \lambda) W_j^i = \dot{W}_j^i - [W_{ij}^i + W_{jj}^i + (n-2) W_{jk}^i] \dot{\alpha} + \Pi_j^i + \lambda (n-2) W_j^i \beta \quad (\text{A.35b})$$

$$0 = W_{ii}^i \beta + W_{ij}^i \quad (\text{A.35c})$$

$$(i_t + \lambda) W_{ii}^i = \dot{W}_{ii}^i - [V_{iii}^i + (n-1) V_{iij}^i] \dot{\alpha} + \Pi_{ii}^i + \lambda (n-1) (W_{ij}^i \beta^2 + 2W_{ij}^i \beta) \quad (\text{A.35d})$$

$$(i_t + 2\lambda) W_{ij}^i = \dot{W}_{ij}^i - [V_{iij}^i + V_{ijj}^i + (n-2) V_{ijk}^i] \dot{\alpha} + \Pi_{ij}^i + \lambda (n-2) (W_{jj}^i \beta^2 + W_{jk}^i \beta + W_{ij}^i \beta) \quad (\text{A.35e})$$

$$(i_t + \lambda) W_{jj}^i = \dot{W}_{jj}^i - [V_{ijj}^i + V_{jjj}^i + (n-2) V_{jjk}^i] \dot{\alpha} + \Pi_{jj}^i + \lambda (n-2) (W_{ij}^i \beta^2 + 2\beta W_{jk}^i) + \lambda (W_{ii}^i \beta^2 + 2\beta W_{ij}^i) \quad (\text{A.35f})$$

$$(i_t + 2\lambda) W_{jk}^i = \dot{W}_{jk}^i - [V_{ijk}^i + V_{jjk}^i + V_{jkk}^i + (n-3) V_{jkl}^i] \dot{\alpha} + \Pi_{jk}^i + \lambda (n-3) (W_{ij}^i \beta^2 + 2\beta W_{jk}^i) + \lambda (W_{ii}^i \beta^2 + 2\beta W_{ij}^i) \quad (\text{A.35g})$$

Next, we linearize system (A.35) around a symmetric steady state $\bar{\alpha} = \alpha(\infty)$ with zero inflation (and steady state values of aggregate variables \bar{C}, \bar{P}). Let lower case variables denote log-deviations, e.g., $a(t) = \log \alpha(t) - \log \bar{\alpha}$, and write nominal marginal cost as

$$p(t) + k(t)$$

where $k(t)$ is the log-deviation of the real marginal cost. Profit derivatives such as $\Pi_i^i(t)$ in (A.35) are evaluated at the moving price $\alpha(t)$, hence become once linearized⁴

$$\pi_i^i(t) = \bar{\alpha} \left[\Pi_{ii}^i + (n-1) \Pi_{ij}^i \right] a(t) + \bar{M}C \Pi_{i,MC}^i (p(t) + k(t)) + \Pi_i^i (z_c c(t) + z_p p(t))$$

$$\pi_j^i(t) = \bar{\alpha} \left[\Pi_{ij}^i + \Pi_{jj}^i + (n-2) \Pi_{jk}^i \right] a(t) + \bar{M}C \Pi_{j,MC}^i (p(t) + k(t)) + \Pi_j^i (z_c c(t) + z_p p(t))$$

$$\pi_{ii}^i(t) = \bar{\alpha} \left[\Pi_{iii}^i + (n-1) \Pi_{iij}^i \right] a(t) + \bar{M}C \Pi_{ii,MC}^i (p(t) + k(t)) + \Pi_{ii}^i (z_c c(t) + z_p p(t))$$

$$\pi_{ij}^i(t) = \bar{\alpha} \left[\Pi_{iij}^i + \Pi_{ijj}^i + (n-2) \Pi_{ijk}^i \right] a(t) + \bar{M}C \Pi_{ij,MC}^i (p(t) + k(t)) + \Pi_{ij}^i (z_c c(t) + z_p p(t))$$

$$\pi_{jj}^i(t) = \bar{\alpha} \left[\Pi_{ijj}^i + \Pi_{jjj}^i + (n-2) \Pi_{jjk}^i \right] a(t) + \bar{M}C \Pi_{jj,MC}^i (p(t) + k(t)) + \Pi_{jj}^i (z_c c(t) + z_p p(t))$$

$$\pi_{jk}^i(t) = \bar{\alpha} \left[\Pi_{ijk}^i + 2\Pi_{jjk}^i + (n-3) \Pi_{jkl}^i \right] a(t) + \bar{M}C \Pi_{jk,MC}^i (p(t) + k(t)) + \Pi_{jk}^i (z_c c(t) + z_p p(t))$$

where $\bar{\Pi}_i^i, \bar{\Pi}_{ii}^i$ etc. denote steady state values.

⁴It is more convenient to linearize and not log-linearize profit derivatives, but we use the notation $\pi_i^i(t) = \Pi_i^i(t) - \bar{\Pi}_i^i$.

This yields the system of 6 linear ODEs in $(a(t), w_j^i(t), w_{ii}^i(t), w_{ij}^i(t), w_{jj}^i(t), w_{jk}^i(t))$

$$\begin{aligned}
& [V_{ii}^i + (n-1)V_{ij}^i] \dot{a}(t) = \frac{1}{\bar{\alpha}} \pi_i^i(t) + \lambda(n-1) \frac{V_j^i \beta}{\bar{\alpha}} [w_j^i(t) + b(t)] \\
(\rho + \lambda) w_j^i(t) + R_t - \rho &= \dot{w}_j^i(t) - \bar{\alpha} \left[\frac{V_{ij}^i + V_{jj}^i + (n-2)V_{jk}^i}{V_j^i} \right] \dot{a}(t) + \frac{1}{V_j^i} \pi_j^i(t) + \lambda(n-2) \beta [w_j^i(t) + b(t)] \\
(\rho + \lambda) w_{ii}^i(t) + R_t - \rho &= \dot{w}_{ii}^i(t) - \frac{\bar{\alpha}}{V_{ii}^i} [V_{iii}^i + (n-1)V_{iij}^i] \dot{a}(t) + \frac{1}{V_{ii}^i} \pi_{ii}^i(t) \\
&\quad + \lambda(n-1) \left\{ \frac{V_{jj}^i \beta^2}{V_{ii}^i} [w_{jj}^i(t) + 2b(t)] + \frac{2V_{ij}^i \beta}{V_{ii}^i} [w_{ij}^i(t) + b(t)] \right\} \\
(\rho + 2\lambda) w_{ij}^i(t) + R_t - \rho &= \dot{w}_{ij}^i(t) - \frac{\bar{\alpha}}{V_{ij}^i} [V_{iij}^i + V_{ijj}^i + (n-2)V_{ijk}^i] \dot{a}(t) + \frac{1}{V_{ij}^i} \pi_{ij}^i(t) \\
&\quad + \lambda(n-2) \left\{ \frac{V_{jj}^i \beta^2}{V_{ij}^i} [w_{jj}^i(t) + 2b(t)] + \frac{V_{jk}^i \beta}{V_{ij}^i} [w_{jk}^i(t) + b(t)] + \beta [w_{ij}^i(t) + b(t)] \right\} \\
(\rho + \lambda) w_{jj}^i(t) + R_t - \rho &= \dot{w}_{jj}^i(t) - \frac{\bar{\alpha}}{V_{jj}^i} [V_{ijj}^i + V_{jjj}^i + (n-2)V_{jjk}^i] \dot{a}(t) + \frac{1}{V_{jj}^i} \pi_{jj}^i(t) \\
&\quad + \lambda(n-2) \left\{ \frac{V_{jj}^i \beta^2}{V_{jj}^i} [w_{jj}^i(t) + 2b(t)] + \frac{2V_{jk}^i \beta}{V_{jj}^i} [w_{jk}^i(t) + b(t)] \right\} \\
&\quad + \lambda \left\{ \frac{V_{ii}^i \beta^2}{V_{jj}^i} [w_{ii}^i(t) + 2b(t)] + \frac{2V_{ij}^i \beta}{V_{jj}^i} [w_{ij}^i(t) + b(t)] \right\} \\
(\rho + 2\lambda) w_{jk}^i(t) + R_t - \rho &= \dot{w}_{jk}^i(t) - \frac{\bar{\alpha}}{V_{jk}^i} [V_{ijk}^i + V_{jjk}^i + V_{jkk}^i + (n-3)V_{jkl}^i] \dot{a}(t) + \frac{1}{V_{jk}^i} \pi_{jk}^i(t) \\
&\quad + \lambda(n-3) \left\{ \frac{V_{jj}^i \beta^2}{V_{jk}^i} [w_{jj}^i(t) + 2b(t)] + \frac{2V_{jk}^i \beta}{V_{jk}^i} [w_{jk}^i(t) + b(t)] \right\} \\
&\quad + \lambda \left\{ \frac{V_{ii}^i \beta^2}{V_{jk}^i} [w_{ii}^i(t) + 2b(t)] + \frac{2V_{ij}^i \beta}{V_{jk}^i} [w_{ij}^i(t) + b(t)] \right\}
\end{aligned}$$

In general there are thus 6 ODEs because β may be time-dependent hence $b(t) \neq 0$. But note that if $b(t) = 0$ then the system becomes block-recursive and we can solve separately the first two equations in a and w_j^i . From the optimality conditions we have

$$\dot{\beta} = -\dot{\alpha} [W_{iij}^i [1 - (n-1)\beta] + (n-1)W_{ijj}^i - \beta W_{iii}^i]$$

Using our perturbation argument we can show that there exists a third-order cross-elasticity ϵ_{iij}^i such that at the steady state

$$V_{iij}^i [1 - (n-1)\beta] + (n-1)V_{ijj}^i - \beta V_{iii}^i = 0 \quad (\text{A.36})$$

where $V_{ijj}, V_{ijj}, V_{iii}$ are solutions to the system (A.34). Thus in what follows we consider β as constant for the first-order dynamics to simplify expressions, although we could solve the larger system without this assumption.

The last step is to replace the single “reset price” variable $a(t)$ with two variables, the aggregate price level $p(t)$ and inflation $\pi(t) = \dot{p}(t)$ using our aggregation result that inflation follows

$$\pi(t) = \lambda [1 - (n - 1) \beta(t)] [\log \alpha(t) - \log P(t)].$$

After log-linearization we have

$$a(t) = \frac{\pi(t)}{\lambda [1 - (n - 1) \beta]} + p(t).$$

Therefore, we obtain in matrix form that the vector

$$\mathbf{Y}(t) = \left(\pi(t), p(t), w_j^i(t) \right)'$$

solves the linear differential equation

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{Z}_k k(t) + \mathbf{Z}_c c(t) + \mathbf{Z}_R [R(t) - \rho]$$

where $\mathbf{A} \in \mathbb{R}^{3 \times 3}$, $\mathbf{Z}_k, \mathbf{Z}_c, \mathbf{Z}_R \in \mathbb{R}^3$ collect the terms above (evaluated at the steady state), with boundary conditions $\lim_{t \rightarrow \infty} \mathbf{Y}(t) = 0$. The solution is given by

$$\mathbf{Y}(t) = - \int_0^\infty e^{s\mathbf{A}} \{ \mathbf{Z}_k k(t+s) + \mathbf{Z}_c c(t+s) + \mathbf{Z}_R [R(t+s) - \rho] \} ds$$

where $e^{s\mathbf{A}} = \sum_{k=0}^\infty \frac{s^k \mathbf{A}^k}{k!}$ denotes the matrix exponential of $s\mathbf{A}$. Proposition 8 then follows by taking the first component of \mathbf{Y} .

To obtain the scalar higher-order ODE for π , let $[\mathbf{M}]_i$ and $[\mathbf{M}]_{xy}$ denote the i th line and the (x, y) element of a generic matrix \mathbf{M} respectively. Let $\mathbf{B}(t) = \mathbf{Z}_k k(t) + \mathbf{Z}_c c(t) + \mathbf{Z}_r [r(t) - \rho]$. Iterating $\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}(t)$, we have for all $k \geq 1$

$$\mathbf{Y}^{(k)}(t) = \mathbf{A}^k \mathbf{Y}(t) + \sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t).$$

Taking the first line for each $k = 1, \dots, K = 3$, we have K equations

$$\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 = [\mathbf{A}^k]_1 \mathbf{Y}(t)$$

which we can each rewrite as

$$\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 - [\mathbf{A}^k]_{11} \pi(t) = \sum_{i=2}^K [\mathbf{A}^k]_{1i} y_i(t)$$

Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_{12} & \dots & \mathbf{A}_{1n} \\ [\mathbf{A}^2]_{12} & & [\mathbf{A}^2]_{1n} \\ \vdots & & \vdots \\ [\mathbf{A}^n]_{12} & \dots & [\mathbf{A}^n]_{1n} \end{pmatrix} \in \mathbb{R}^{K \times (K-1)}$$

Take any vector $\alpha^\pi = \left(\alpha_j^\pi \right)_{j=1}^K$ in $\ker \mathbf{M}'$ (whose dimension is at least 1), i.e., such that $\mathbf{M}' \gamma^\pi = 0 \in \mathbb{R}^{K-1}$. Then

$$\sum_{k=1}^K \alpha_k^\pi \left(\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 - [\mathbf{A}^k]_{11} \pi(t) \right) = 0.$$

and we can define $\alpha_0^\pi = -\sum_{k=1}^K \alpha_k^\pi [\mathbf{A}^k]_{11}$. This simplifies to

$$\begin{aligned} \ddot{\pi} &= (\mathbf{A}_{\pi\pi} + \mathbf{A}_{ww}) \ddot{\pi} \\ &+ (\mathbf{A}_{\pi p} + \mathbf{A}_{\pi w} \mathbf{A}_{w\pi} - \mathbf{A}_{\pi\pi} \mathbf{A}_{ww}) \dot{\pi} \\ &+ (\mathbf{A}_{\pi w} \mathbf{A}_{wp} - \mathbf{A}_{\pi p} \mathbf{A}_{ww}) \pi \\ &+ \mathbf{A}_{\pi w} \dot{\mathbf{B}}_w + \ddot{\mathbf{B}}_\pi - \mathbf{A}_{ww} \dot{\mathbf{B}}_\pi \end{aligned} \tag{A.37}$$

I.1 One-time shocks

Given (19) we can guess and verify that $x = \psi_x e^{-\xi t}$ for all variables $x \in \{\pi, k, c, R - \rho\}$ and solve for the coefficients ψ_x using the system

$$\begin{aligned} \psi_\pi \left(\gamma_0^\pi - \gamma_1^\pi \xi + \gamma_2^\pi \xi^2 - \gamma_3^\pi \xi^3 \right) &= \psi_k \left(\gamma_0^k - \gamma_1^k \xi + \gamma_2^k \xi^2 \right) \\ &+ \psi_c \left(\gamma_0^c - \gamma_1^c \xi + \gamma_2^c \xi^2 \right) \\ &+ (\psi_R - \psi_\pi) \left(\gamma_0^r - \gamma_1^r \xi + \gamma_2^r \xi^2 \right) \\ -\xi \psi_c &= \sigma^{-1} (\psi_R - \psi_\pi - \epsilon_0^r) \\ \psi_R &= \phi_\pi \psi_\pi + \epsilon_0^m + (1 - \kappa) \epsilon_0^r \end{aligned}$$

Thus

$$\begin{aligned} \psi_c &= \frac{1}{\sigma \xi} (\psi_\pi (1 - \phi_\pi) + \kappa \epsilon_0^r - \epsilon_0^m) \\ \psi_k &= \psi_c (\chi + \sigma) \end{aligned}$$

and

$$\begin{aligned} \psi_\pi \left(\gamma_0^\pi - \gamma_1^\pi \xi + \gamma_2^\pi \xi^2 - \gamma_3^\pi \xi^3 \right) &= \frac{1}{\sigma \xi} (\kappa \epsilon_0^r - \epsilon_0^m - \psi_\pi (\phi_\pi - 1)) \left[(\chi + \sigma) \left(\gamma_0^k - \gamma_1^k \xi + \gamma_2^k \xi^2 \right) + \left(\gamma_0^c - \gamma_1^c \xi + \gamma_2^c \xi^2 \right) \right] \\ &+ (\epsilon_0^m + (1 - \kappa) \epsilon_0^r + \psi_\pi (\phi_\pi - 1)) \left(\gamma_0^r - \gamma_1^r \xi + \gamma_2^r \xi^2 \right) \end{aligned}$$

which yields

$$\psi_\pi = \frac{\frac{\kappa \epsilon_0^r - \epsilon_0^m}{\sigma \xi} \left[(\chi + \sigma) \left(\gamma_0^k - \gamma_1^k \xi + \gamma_2^k \xi^2 \right) + \left(\gamma_0^c - \gamma_1^c \xi + \gamma_2^c \xi^2 \right) \right] + (\epsilon_0^m + (1 - \kappa) \epsilon_0^r) \left(\gamma_0^r - \gamma_1^r \xi + \gamma_2^r \xi^2 \right)}{\gamma_0^\pi - \gamma_1^\pi \xi + \gamma_2^\pi \xi^2 - \gamma_3^\pi \xi^3 + (\phi_\pi - 1) \left[\frac{(\chi + \sigma) \left(\gamma_0^k - \gamma_1^k \xi + \gamma_2^k \xi^2 \right) + \left(\gamma_0^c - \gamma_1^c \xi + \gamma_2^c \xi^2 \right)}{\sigma \xi} - \left(\gamma_0^r - \gamma_1^r \xi + \gamma_2^r \xi^2 \right) \right]}$$

J Additional Figures

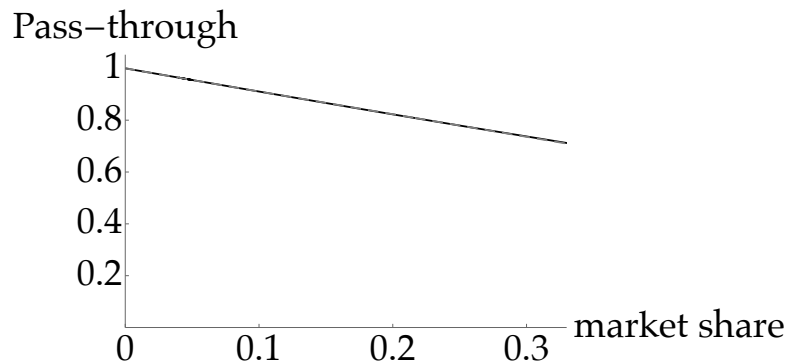


Figure J1: Pass-through $\hat{\alpha}$ as a function of market share: symmetric vs. heterogeneous firms.

Note: Black line: market share varies through the number $n = 2, 3, \dots$ of symmetric firms (black). Gray dashed line: market share varies through heterogeneity in productivity among a fixed number $n = 10$ of firms. The two lines lie almost exactly on top of each other. Nested CES preferences with $\eta = 10$, $\omega = 1$.

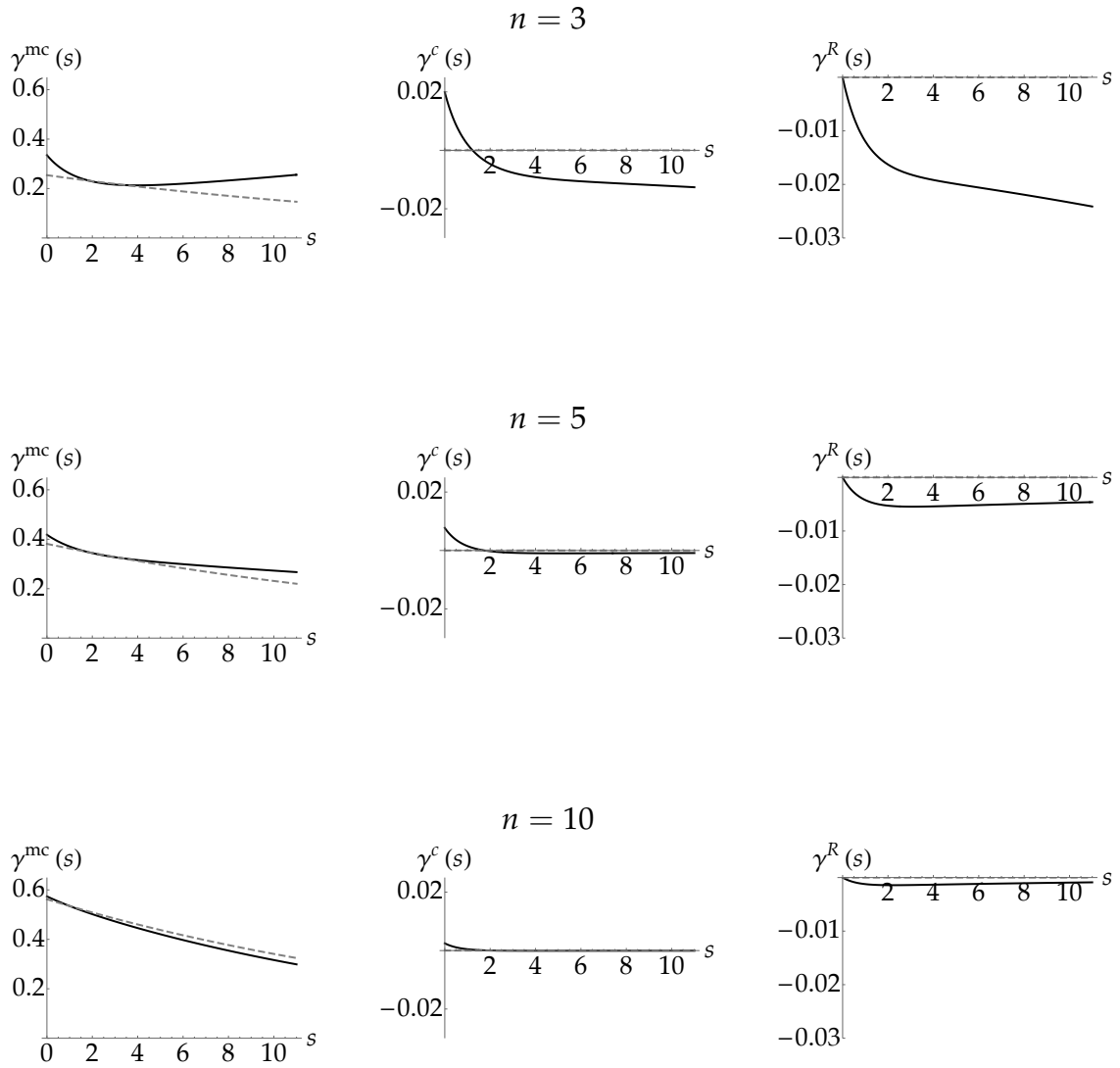


Figure J2: Green functions $\gamma^{mc}(s), \gamma^c(s), \gamma^R(s)$ for different numbers of firms n .
 Note: AIK calibration. Solid black line: Strategic oligopoly. Dashed gray line: Naive model.

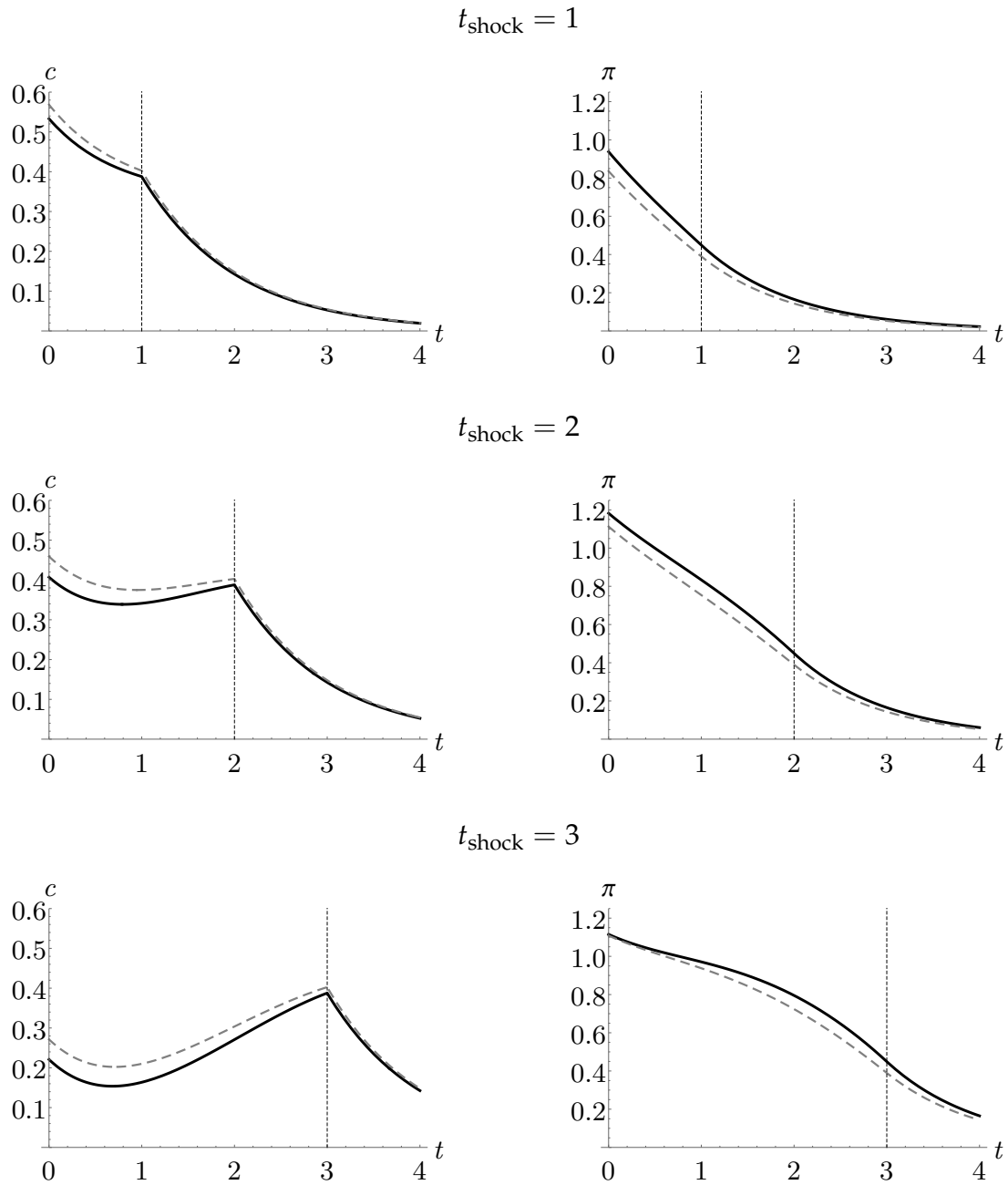


Figure J3: Impulse responses for consumption and inflation following date-0 news about monetary policy shock happening at t_{shock} indicated by the vertical line.

Note: $n = 3$ firms with AIK calibration. Solid black line: Strategic oligopoly. Dashed gray line: Naive model. c and π denote log-deviations from steady state values in %.

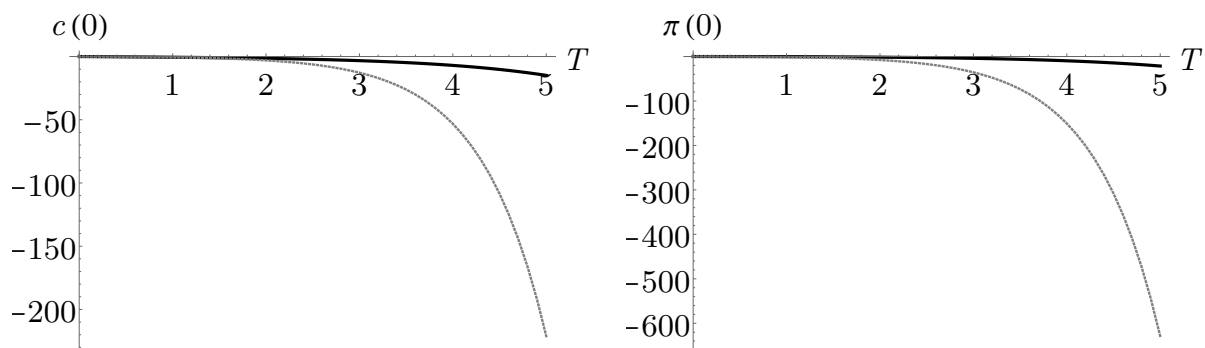


Figure J4: Date-0 consumption and inflation in a liquidity trap lasting from $t = 0$ to $t = T$, for different values of T .

Note: Solid black line: $n = 3$. Dotted gray line: $n = \infty$. c and π denote log-deviations from steady state values in %.

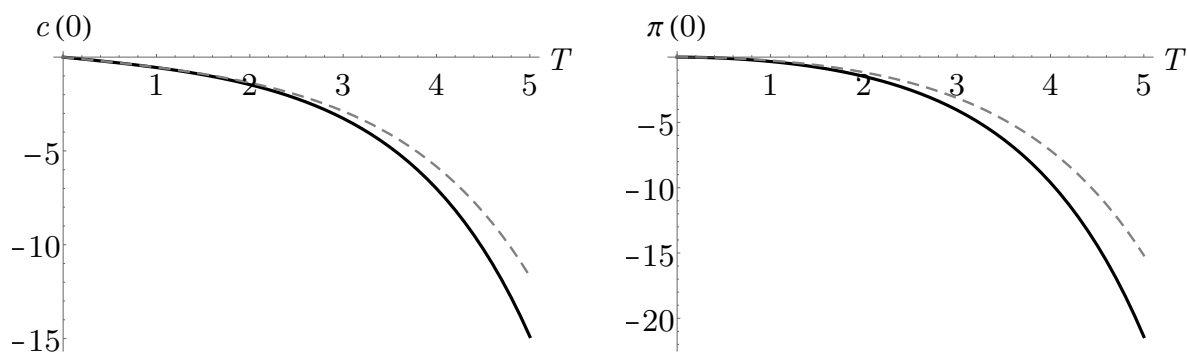


Figure J5: Date-0 consumption and inflation in a liquidity trap lasting from $t = 0$ to $t = T$, for different values of T .

Note: $n = 3$ firms with AIK calibration. Solid black line: Strategic oligopoly. Dashed gray line: Naive model. c and π denote log-deviations from steady state values in %.

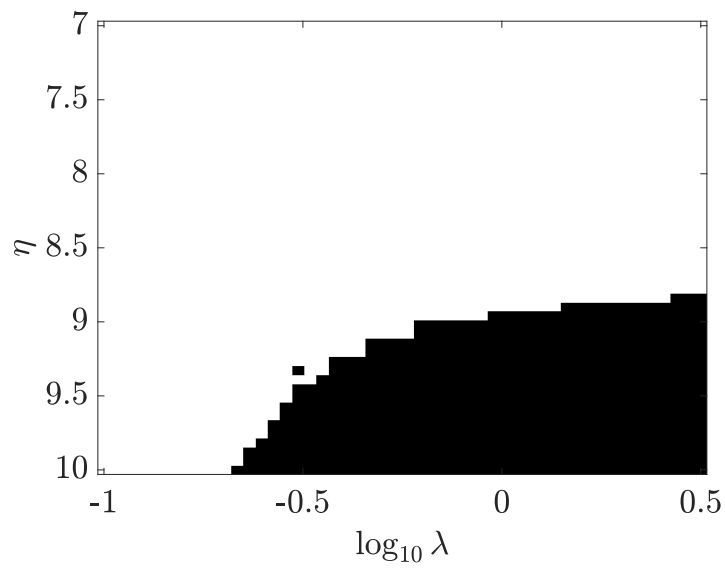


Figure J6: In white: convergence of value function iteration algorithm towards a monotone MPE in (λ, η) space, with $n = 2$ firms.